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THE COEFFICIENTS OF NEVANLINNA'S PARAMETRIZATION ARE NOT IN H^p

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ABSTRACT. We construct an example of a Pick-Nevanlinna interpolation problem such that the coefficients of its Nevanlinna's parametrization are not in H^p, for $p > 0$.

1. INTRODUCTION

Let D be the unit disc in the complex plane and let $H^p(D)$, $0 < p \leq \infty$, be the usual Hardy spaces on D.

We consider the following classical Pick-Nevanlinna interpolation problem: Given two sequences of numbers $\{z_n\}$, $\{w_n\}$ in D, find all analytic functions $f \in H^\infty(D)$ satisfying

$$(*) \qquad ||f||_\infty = \sup\{|f(z)|: z \in D\} \leq 1 \text{ and } f(z_n) = w_n, \qquad n = 1, 2, \ldots .$$

Pick and Nevanlinna found necessary and sufficient conditions in order that such an analytic function exists. If E denotes the set of all analytic functions on D satisfying $(*)$, Nevanlinna showed that in the case where E consists of more than one element, there is a parametrization of the form:

$$E = \left\{ f \in H^\infty(D): f = \frac{p\varphi + q}{r\varphi + s}, \varphi \in H^\infty(D), ||\varphi||_\infty \leq 1 \right\}$$

where p, q, r, s are certain analytic functions on D depending on $\{z_n\}$ and $\{w_n\}$. It is known that p, q, r, s are in the Smirnov class $N^+(D)$. Furthermore, p, q, r, s belong to $H^p(D)$ if and only if s is in $H^p(D)$.

For details and proofs of results above, see [1, pp. 50, 165] and [3, p. 491].

In [2, p. 205] it is claimed that s belongs to $H^2(D)$. Recently, Stray [3] asked for a complex analytic proof of this result. In this note we show that this result is false. Indeed, we will give an example of a Pick-Nevanlinna interpolation

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problem such that the function s appearing in its Nevanlinna's parametrization belongs to no $H^p(D)$ for $p > 0$.

In a private communication, D. Sarason told us that he already knew the fact that s belongs to $H^2(D)$ was false.

2. Construction of the example

Let us choose $\{c_n\}$ a sequence of positive numbers such that $\sum_{n=1}^{\infty} c_n \log(1/c_n) < +\infty$ and $\sum_{n=1}^{\infty} c_n^q = +\infty$ for each $q < 1$ (for instance, $c_n = n^{-1}(\log(n))^{-3}$ satisfies these conditions).

Take a sequence of points $e^{i\theta_n}$ converging to 1, so that the arcs $I_n = \{e^{it} : \theta_n - c_n/2 < t < \theta_n + c_n/2\}$ will be pairwise disjoint and consecutive. Put $z_n = (1 - c_n)e^{i\theta_n}$.

Claim. There exists $h \in H^\infty(D)$, $\|h\|_\infty \le 1$ so that $\int_0^{2\pi} \log(1 - |h(e^{i\theta})|)\, d\theta > -\infty$ and $1 - |h(z_n)| \le C(\alpha)(1 - |z_n|)^\alpha$ for each $\alpha < 1$, where $C(\alpha)$ is a constant depending on α.

Proof of the claim. Put $A = \{e^{i\theta_n}, 1\}$ and $g(e^{it}) = \operatorname{dist}(e^{it}, A)$. Write $u(z) = P_z(g)$, the Poisson integral of g, and let $v(z)$ be the harmonic conjugate of $u(z)$.

Take $h(z) = \exp(-u(z) - iv(z))$. Then h is analytic on D and, since g is positive, one has $\|h\|_\infty \le 1$.

Also,

$$\int_0^{2\pi} \log(1 - |h(e^{i\theta})|)\, d\theta = \int_0^{2\pi} \log(1 - e^{-g(e^{i\theta})})\, d\theta \ge C_2 + C_1 \sum_{n=1}^{\infty} c_n(\log(c_n) - 1) > -\infty,$$

C_1 and C_2 some constants, because $\sum_{n=1}^{\infty} c_n \log(1/c_n) < +\infty$.

Furthermore:

$$\begin{aligned} 1 - |h(z_n)| &= 1 - \exp(-P_{z_n}(g)) \le P_{z_n}(g) \\ &= [P_{z_n}(g) - g(e^{i\theta_n})] \le C(\alpha)|z_n - e^{i\theta_n}|^\alpha = C(\alpha)(1 - |z_n|)^\alpha \end{aligned}$$

for each $\alpha < 1$, because the Poisson integral of a Lip_α function on the unit circle is in Lip_α of the closed unit disc, for $0 < \alpha < 1$. So we have proved the claim.

Let h be a function satisfying the conditions of the claim. Put $w_n = h(z_n)$, $n = 1, 2, \ldots$ and consider the following Pick-Nevanlinna interpolation problem:

(*) Find all analytic functions $f \in H^\infty(D)$ satisfying $\|f\|_\infty \le 1$ and $f(z_n) = w_n$, $n = 1, 2, \ldots$.

Since h solves $(*)$ and $\int_0^{2\pi} \log(1-|h(e^{i\theta})|)\,d\theta > -\infty$, the function

$$h(z) + B(z)\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1-|h(e^{i\theta})|)\,d\theta\right),$$

where B is the Blaschke product with zeros $\{z_n\}$, also solves $(*)$. Therefore, $(*)$ has more than one solution. Then, the set E of all solutions of $(*)$ can be parametrized as:

$$E = \left\{f \in H^\infty(D): f = \frac{p\varphi+q}{r\varphi+s},\ \varphi \in H^\infty(D) \text{ and } \|\varphi\|_\infty \le 1\right\}.$$

Suppose now that $s \in H^p(D)$ for some $p > 0$, and let us arrive at a contradiction.

Choosing $\varphi \equiv 0$ in the parametrization, one has $q/s \in H^\infty(D)$ and $q/s(z_n) = w_n$, $n = 1, 2, \ldots$. It is well known that $1/|s(z)|^2 \le 1 - |q/s(z)|^2$ for $z \in D$ (see Lemma 3 in [3]). So

$$(1) \qquad \begin{aligned} \frac{1}{|s(z_n)|^2} &\le 1 - \left|\frac{q}{s}(z_n)\right|^2 = 1 - |w_n|^2 \\ &= 1 - |h(z_n)|^2 \le 2C(\alpha)(1-|z_n|)^\alpha \qquad \text{for each } \alpha < 1. \end{aligned}$$

Since the arcs $\{I_n\}$ are pairwise disjoint, the sequence $\{z_n\}$ is an interpolating sequence of $H^\infty(D)$ (see [4, p. 77]). Applying Carleson's theorem (see [1, p. 63]), one gets

$$\sum_{n=1}^\infty (1-|z_n|)|s(z_n)|^p < +\infty.$$

But using (1) for any fixed $\alpha < 1$,

$$\begin{aligned} \sum_{n=1}^\infty (1-|z_n|)|s(z_n)|^p &\ge 2^{-p/2}C(\alpha)^{-p/2}\sum_{n=1}^\infty (1-|z_n|)^{1-p\alpha/2} \\ &= 2^{-p/2}C(\alpha)^{-p/2}\sum_{n=1}^\infty c_n^{1-p\alpha/2} = +\infty \end{aligned}$$

because $\sum_{n=1}^\infty c_n^q = +\infty$ for each $q < 1$. This gives us the contradiction.

Therefore, $s \notin H^p(D)$ for each $p > 0$.

References

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