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THE CORONA PROPERTY FOR BOUNDED ANALYTIC FUNCTIONS IN SOME BESOV SPACES

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ABSTRACT. In this paper, the corona theorem for the algebra of bounded analytic functions in the unit disc which are in the Besov space B_p, $1 < p < \infty$, is proved.

Let Δ be the open unit disc in the complex plane and let H^∞ be the Banach space of all bounded analytic functions on Δ with the norm

$$\|f\|_\infty = \sup\{|f(z)|: z \in \Delta\}.$$

For $1 < p < \infty$, let B_pA be the class of all analytic functions f on Δ such that

$$\|f\|^p_{B_p(\Delta)} = \frac{1}{\pi}\int_\Delta |f'(z)|^p (1-|z|)^{p-2}\,dm(z) < \infty.$$

It is easy to see that $H^\infty \cap B_pA$ is a Banach algebra with the norm $\|f\| = \|f\|_\infty + \|f\|_{B_p(\Delta)}$. In this note we consider the corona problem for this algebra.

Let $\mathscr{M}$ be the maximal ideal space of $H^\infty \cap B_pA$ endowed with the Gelfand topology. It is clear that Δ is naturally embedded in $\mathscr{M}$. The corona problem consists of knowing if Δ is dense in $\mathscr{M}$. Here we answer this question in the affirmative. As is known (see [2, p. 191]), this turns out to be equivalent to the following result.

Theorem. *Let $1 < p < \infty$. Given $f_1, \dots, f_n \in H^\infty \cap B_pA$ such that*

$$\max_j |f_j(z)| \geq \delta > 0, \qquad z \in \Delta \tag{1}$$

there exist $g_1, \dots, g_n \in H^\infty \cap B_pA$ such that

$$f_1 g_1 + \cdots + f_n g_n = 1. \tag{2}$$

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Note that for $p = 2$, $H^\infty \cap B_2 A$ is the space of bounded analytic functions with finite Dirichlet integral. So our result contains the answer to a question of [3].

Proof of the theorem. By a normal families argument, we can assume that the corona data $f_1, \dots, f_n$ are analytic on a neighborhood of the closed unit disk, and we have to find analytic functions $g_1, \dots, g_n$ satisfying (2) and

$$\|g_i\| \le C \qquad i = 1, \dots, n,$$

where C is a constant depending on δ, $\|f_1\|, \dots, \|f_n\|$.

It is clear that

$$\varphi_i(z) = \overline{f_i(z)} \Big/ \sum_{i=1}^{n} |f_i(z)|^2$$

is a nonanalytic solution of (2).

As in the case of H^∞ (see [2, Chapter VIII]), our problem is equivalent to solving, with bounds, the following equations. For $1 \le j$, $k \le n$, find $b_{j,k}$ such that

$$\bar\partial b_{j,k} = \varphi_j \bar\partial \varphi_k \quad \text{in } \Delta \tag{3}$$

with

$$\|b_{j,k}\|_{L^\infty(\mathbf{T})} + \int_\Delta |\nabla b_{j,k}(z)|^p (1 - |z|)^{p-2}\, dm(z) \le C, \tag{4}$$

where C is a constant depending on δ, $\|f_1\|, \dots, \|f_n\|$.

It is sufficient to deal with an equation $\bar\partial b = g$ where $g = \varphi_j \bar\partial \varphi_k$. Applying (1), a calculation (see [2, p. 326]) gives

$$|g(z)| \le M \sum_{j=1}^{n} |f'_j(z)|, \tag{5}$$

where M is a constant depending on δ.

In order to find a solution of (3) with bounded $L^\infty(\mathbf{T})$ norm it suffices to show that $|g(z)|\, dm(z)$ is a Carleson measure (see [2, p. 320]). Let us see that this is true.

Put $Q_h = \{z \in \Delta\colon |z| \ge 1 - h \text{ and } \theta - h \le \operatorname{Arg} z \le \theta + h\}$. From (5) one has

$$\begin{aligned}\int_{Q_h} |g(z)|\,dm(z) &\le M\sum_{j=1}^{n}\int_{Q_h} |f_j'(z)|\,dm\\ &= M\sum_{j=1}^{n}\int_{Q_h} |f'_j(z)|(1-|z|)^{1-2/p}(1-|z|)^{-(1-2/p)}\,dm(z)\\ &\le M\sum_{j=1}^{n}\left[\int_{Q_h} |f_j'(z)|^p(1-|z|)^{p-2}\,dm(z)\right]^{1/p}\\ &\quad\times\left[\int_{Q_h}(1-|z|)^{-(1-2/p)p/(p-1)}\,dm(z)\right]^{(p-1)/p}\\ &\le (p-1)M\sum_{j=1}^{n}\left(\int_{\Delta}|f_j'(z)|^p(1-|z|)^{p-2}\,dm(z)\right)^{1/p} h\\ &\le (p-1)Mn\sup_i\|f_i\|\cdot h.\end{aligned}$$

And so, (3) can be solved by means of bounded functions and one has

$$\inf\{\|H\|_{L^\infty(\mathbf{T})}: H \text{ solves } (3)\} \le C. \tag{6}$$

In order to obtain a solution of (3) bounded with respect to the norm

$$\|b\|^p_{B_p(\Delta)} = \int_\Delta |\nabla b(z)|^p(1-|z|)^{p-2}\,dm(z)$$

let us take

$$H_0(z) = \frac{1}{\pi}\int_\Delta \frac{g(\xi)}{\xi - z}\,dm(\xi).$$

One has $\bar\partial H_0 = g$ in Δ. So, applying (5),

$$\int_\Delta |\bar\partial H_0(z)|^p(1-|z|)^{p-2}\,dm(z) \le C.$$

Furthermore, ∂H_0 is the Beurling transform of g. Since $(1-|z|)^{p-2}$ is an A_p weight for $1 < p < \infty$ (see [1, p. 411]), one has

$$\begin{aligned}&\int_\Delta |\partial H_0(z)|^p(1-|z|)^{p-2}\,dm(z)\\ &\qquad\le K(p)\int_\Delta |g(z)|^p(1-|z|)^{p-2}\,dm(z) \le K(p)C\end{aligned}$$

because of (5). So

$$\|H_0\|_{B_p(\Delta)} \le C. \tag{7}$$

Nevertheless, the problem is to solve the $\bar\partial$ equation (3) by means of a function b satisfying simultaneously the two bounds

$$\|b\|_{L^\infty(\mathbf{T})} \le C \quad\text{and}\quad \|b\|_{B_p(\Delta)} \le C.$$

To do this, for $1 < p < \infty$, let us consider the Besov class $B_p(\mathbf{T})$ formed by those functions in $L^p(\mathbf{T})$ such that

$$\|f\|^p_{B_p(\mathbf{T})} = \int_{-\pi}^{\pi} \frac{1}{h^2} \int_{-\pi}^{\pi} |f(e^{i(t+h)}) - f(e^{it})|^p \, dt \, dh < \infty.$$

If $f \in L^p(\mathbf{T})$ and u denotes its Poisson integral, it is well known (see [5, p. 152]) that there exist an absolute constant M such that

$$\begin{aligned} M^{-1} \int_\Delta |\nabla u(z)|^p (1-|z|)^{p-2} \, dm(z) &\le \|f\|_{B_p(\mathbf{T})} \\ &\le M \int_\Delta |\nabla u(z)|^p (1-|z|)^{p-2} \, dm(z). \end{aligned} \tag{8}$$

Claim. $\|H_0\|_{B_p(\mathbf{T})} \le C$.

Of course, since H_0 is not harmonic, the claim cannot be deduced automatically from (7) and (8). Assume the claim is true and let us finish the proof of the theorem.

Since $\bar\partial H_0 = g$ and (6), one has

$$\inf\{\|H_0 - F\|_\infty : F \in \mathrm{BMOA}\} = \inf\{\|H\|_\infty : H \text{ solves (3)}\} \le C,$$

where BMOA is the space of analytic functions on Δ with boundary values of bounded mean oscillation.

Peller and Hruscev proved that $B_p(\mathbf{T})$ has the best approximation property, for $1 < p < \infty$ (see [4, p. 103]). So, there exists a unique $F_0 \in \mathrm{BMOA}$ satisfying

$$\|H_0 - F_0\|_\infty = \inf\{\|H_0 - F\|_\infty : F \in \mathrm{BMOA}\} \le C$$

and furthermore

$$\|F_0\|_{B_p(\mathbf{T})} \le K \|H_0\|_{B_p(\mathbf{T})}. \tag{9}$$

Therefore $H_0 - F_0$ is a solution of the $\bar\partial$ equation (3), satisfying $\|H_0 - F_0\|_\infty \le C$. Now, apply (7), (8), (9), and the claim to get

$$\begin{aligned} &\left(\int_\Delta |\nabla(H_0 - F_0)(z)|^p (1-|z|)^{p-2} \, dm(z) \right)^{1/p} \\ &\quad \le \left(\int_\Delta |\nabla H_0(z)|^p (1-|z|)^{p-2} \, dm(z) \right)^{1/p} \\ &\qquad + 2 \left(\int_\Delta |F_0'(z)|^p (1-|z|)^{p-2} \, dm(z) \right)^{1/p} \\ &\quad \le C. \end{aligned}$$

So $H_0 - F_0$ satisfies (3) and (4).

Proof of the claim. First of all, we remark that

$$\|H_0\|_{\mathrm{BMO}(\mathbf{T})} \le C$$

with the constant C depending only on the data of the corona problem. Because of [6, Theorem 1.1.2.], one only has to check that $|\nabla H_0(z)|\,dm(z)$ is a Carleson measure with norm only depending on δ, $\|f_1\|, \dots, \|f_n\|$, and in fact, this has been done in the proof of (6).

To prove the claim, we have to show that

$$\int_{-\pi}^{\pi} \frac{1}{h^2} \int_{-\pi}^{\pi} |H_0(e^{i(s+h)}) - H_0(e^{is})|^p \, ds\, dh \le C. \tag{10}$$

Since $\|H_0\|_{\mathbf{BMO(T)}} \le C$, one has $\int_{-\pi}^{\pi} |H_0(e^{i\theta})|^p \, d\theta \le AC$ where A is an absolute constant. Therefore, by symmetry on h, in order to prove (10), it suffices to verify

$$\int_{0}^{1/2} \frac{1}{h^2} \int_{-\pi}^{\pi} |H_0(e^{i(s+h)}) - H_0(e^{is})|^p \, ds\, dh \le C. \tag{11}$$

Let us just reproduce a proof of the second inequality in (8) and let us see that the harmonicity is not used.

Take $r = 1 - h$ and let $\partial H_0/\partial n$, $\partial H_0/\partial \theta$ be the derivatives of H_0 with respect to the radius and the argument. We have

$$\begin{aligned}
&|H_0(e^{i(s+h)} - H_0(e^{is})| \\
&\quad \le |H_0(e^{i(s+h)}) - H_0(re^{i(s+h)})| + |H_0(re^{i(s+h)}) - H_0(re^{is})| \\
&\qquad + |H_0(re^{is}) - H_0(e^{is})| \\
&\quad \le \int_r^1 \left|\frac{\partial H_0}{\partial n}(\xi e^{i(s+h)})\right| d\xi + \int_0^h \left|\frac{\partial H_0}{\partial \theta}(re^{i(s+\varphi)})\right| d\varphi + \int_r^1 \left|\frac{\partial H_0}{\partial n}(\xi e^{is})\right| d\xi.
\end{aligned}$$

Apply Minkowski integral inequality [5, p. 271], to get

$$\begin{aligned}
&\left(\int_{-\pi}^{\pi} |H_0(e^{i(s+h)}) - H_0(e^{is})|^p \, ds\right)^{1/p} \\
&\quad \le \int_r^1 \left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial n}(\xi e^{i(s+h)})\right|^p ds\right)^{1/p} d\xi \\
&\qquad + \int_0^h \left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial \theta}(re^{i(s+\varphi)})\right| ds\right)^{1/p} d\varphi \\
&\qquad + \int_r^1 \left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial n}(\xi e^{is})\right|^p ds\right)^{1/p} d\xi \\
&\quad = 2\int_r^1 \left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial n}(\xi e^{is})\right|^p ds\right)^{1/p} d\xi + h\left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial \theta}(re^{is})\right|^p ds\right)^{1/p} \\
&\quad = (I) + (II).
\end{aligned}$$

Changing to planar coordinates and applying (7), one gets

$$\begin{aligned}\int_0^{1/2} \frac{1}{h^2}(II)^p\,dh &= \int_0^{1/2} h^{p-2} \int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial \theta}(re^{is})\right|^p ds\,dh \\ &= \int_0^{1/2} h^{p-2} \int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial \theta}((1-h)e^{is})\right|^p ds\,dh \\ &\le 2\int_\Delta |\nabla H_0(z)|^p (1-|z|)^{p-2}\,dm(z) \le C.\end{aligned}$$

For the term (I), put $x = 1-\xi$ and apply Hardy's inequality ([5, p. 272]) to obtain

$$\begin{aligned}\int_0^{1/2} \frac{1}{h^2}(I)^p\,dh &= 2^p \int_0^{1/2} \frac{1}{h^2}\left[\int_r^1 \left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial n}(\xi e^{is})\right|^p ds\right)^{1/p} d\xi\right]^p dh \\ &= 2^p \int_0^{1/2} \frac{1}{h^2}\left[\int_0^h \left(\int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial n}((1-x) e^{is})\right|^p ds\right)^{1/p} dx\right]^p dh \\ &\le 2^p K(p) \int_0^{1/2} h^{-2+p} \int_{-\pi}^{\pi} \left|\frac{\partial H_0}{\partial n}((1-h)e^{is})\right|^p ds\,dh \\ &\le 2^{p+1} K(p) \int_\Delta |\nabla H_0(z)|^p (1-|z|)^{p-2}\,dm(z) \le C\end{aligned}$$

because of (7).

This gives (10) and therefore we have proved the claim. This completes the proof of the theorem. □

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References

1. J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
2. John B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
3. Kwang-Nan Chow and David Protas, *The maximal ideal space of bounded, analytic, Dirichlet finite functions*, Arch. Math. (Basel) **31** (1978/79), 301.
4. V. V. Peller and S. V. Hruscev, *Hankel operators, best approximations, and stationary Gaussian processes*, Russian Math. Surveys **37** (1982), 103.
5. Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.
6. N. Th. Varopoulos, *BMO functions and the $\bar\partial$-equation*, Pacific J. Math. **71** (1977), 223.

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