FROM LOCAL TO GLOBAL ASYMPTOTIC BEHAVIOUR OF ORTHOGONAL POLYNOMIALS

R. BESSONOV AND A. NICOLAU

Abstract. Let $\{\varphi_n^*\}$ be the sequence of reflected orthogonal polynomials on the unit circle $\mathbb{T}$ generated by a measure μ of Szegő class, and let D_μ be the Szegő function of μ. We prove the uniform Cesàro asymptotics

$$\sup_{z\in\Gamma_\zeta}\left(\frac{1}{n}\sum_{k=0}^{n-1}\Bigl||\varphi_k^*(z)D_\mu(z)|^2-1\Bigr|\right)\to 0, \qquad n\to\infty,$$

for almost all Stolz angles Γ_ζ, $\zeta\in\mathbb{T}$. This extends a well-known asymptotic result of Máté, Nevai, and Totik (1991) from the local scale $O(1/n)$ near $\mathbb{T}$ to the global scale $O(1)$. We also study asymptotic behavior of arguments of orthogonal polynomials and extend a classical theorem due to Grenander and Szegő using a new technique. As an application, we derive global asymptotic results for polynomial reproducing kernels under various assumptions on the orthogonality measure.

1. Introduction

Let μ be a probability measure on the unit circle $\mathbb{T}=\{z: \ |z|=1\}$. Consider its Radon-Nikodym decomposition, $\mu = w\,dm+\mu_{\mathbf{s}}$. Here and in what follows m is the Lebesgue measure on $\mathbb{T}$ normalized by $m(\mathbb{T})=1$, the function w belongs to $L^1(\mathbb{T})=L^1(m)$, and $\mu_{\mathbf{s}}$ is the singular part of μ. The measure μ is said to belong to the Szegő class $\mathrm{Sz}(\mathbb{T})$ if $\log w\in L^1(\mathbb{T})$, or, equivalently,

$$\int_{\mathbb{T}}\log w\,dm > -\infty. \tag{1.1}$$

Measures of Szegő class play a prominent role in the theory of orthogonal polynomials [32], [30], [31]. With each $\mu\in\mathrm{Sz}(\mathbb{T})$ one can associate the unique outer function D_μ in the open unit disk $\mathbb{D}=\{z:|z|<1\}$ satisfying $|D_\mu|^2=w$ almost everywhere on $\mathbb{T}$ and such that $D_\mu(0)>0$. This function is called the Szegő function of μ. Let $\{\varphi_n\}$ be the sequence of orthogonal polynomials in $L^2(\mu)$, $\deg\varphi_n=n$, and let φ_n^* be the corresponding reflected polynomials defined by $\varphi_n^*(z)=z^n\overline{\varphi_n(1/\bar z)}$. We use the standard normalization

$$\|\varphi_n\|_{L^2(\mu)}=1, \qquad \varphi_n^*(0)>0, \qquad n\geqslant 0. \tag{1.2}$$

Szegő theorem determines the asymptotic behavior of reflected polynomials φ_n^* in the open unit disk $\mathbb{D}$. It says that

$$\sup_{n\geqslant 0}\varphi_n^*(0)<\infty \quad\iff\quad \mu\in\mathrm{Sz}(\mathbb{T}),$$

and, moreover, for every $\mu\in\mathrm{Sz}(\mathbb{T})$ we have

$$\lim_{n\to\infty}\varphi_n^*(z)D_\mu(z)=1, \tag{1.3}$$

uniformly on compact subsets of $\mathbb{D}$. See Theorem 2.4.1 and Theorem 2.7.15 in Simon book [30] for the proof. One of the most studied questions in the theory of

2020 *Mathematics Subject Classification.* 42C05.
Key words and phrases. Szegő class, uniform asymptotics, universality, Cesàro convergence.

orthogonal polynomials asks under which assumptions on the orthogonality measure μ, relation (1.3) holds almost everywhere on the unit circle $\mathbb{T}$. Let us briefly describe known results (the reader familiar with the history of the problem can go directly to Section 2 for the results of the present paper).

1.1. **Regular measures, uniform convergence.** When studying uniform convergence of $\{\varphi_n^*\}$ on $\mathbb{T}$, it is natural to require smoothness properties on the measure μ. Let us state apparently the best known result in this direction. Let $\mu \in \mathrm{Sz}(\mathbb{T})$ be such that $\mu = w\,dm$ and $\log w$ is a continuous function in $\mathbb{T}$. Assume its modulus of continuity $\omega(t) = \sup\{|\log w(\xi) - \log w(z)| : |\xi - z| \leqslant t\}$ satisfies the following Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty. \tag{1.4}$$

Then (1.3) holds uniformly on $\mathbb{T}$. Geronimus and Golinski [16] attributed this result to Grenander and Szegő [19]. A proof can be found in Golinski [17] and a local version, that is, considering the modulus of continuity in an arc, in Badkov [4]. It is worth mentioning that the condition on the continuity of $\log w$ is not sufficient and (1.4) is really needed. Actually Golinski [18] proved that there exist weights w such that $\log w$ is a continuous function on $\mathbb{T}$ whose modulus of continuity satisfies $\omega(t)\log(t) \to 0$ as $t \to 0$, such that $\{\varphi_n^*\}$ does not converge uniformly on $\mathbb{T}$.

1.2. **Steklov problem for weights uniformly separated from zero.** Given $0 < \delta < 1$, a probability measure $\mu = w\,dm$ is in the Steklov class S_δ if $w \geqslant \delta$ at almost every point in $\mathbb{T}$. To emphasize the role of the measure μ, the orthonormal polynomials with respect to the measure μ will be denoted by $\varphi_n(z, \mu)$. In 1921, Steklov raised the conjecture of deciding if $\sup_n |\varphi_n(z, \mu)| < \infty$ for any $z \in \mathbb{T}$ provided $\mu \in S_\delta$. This conjecture was solved in the negative by Rahmanov [29] leading to the problem of finding the right asymptotics of $\|\varphi_n(z, \mu)\|_\infty = \sup\{|\varphi_n(z, \mu)| : z \in \mathbb{T}\}$. Consider $M(n, \delta) = \sup\{\|\varphi_n(z, \mu)\|_\infty : \mu \in S_\delta\}$. Aptekarev, Denisov and Tulyakov [3] proved the remarkable result that fixed $0 < \delta < 1$, $M(n, \delta)$ is comparable to $\sqrt{n}$. Later Denisov [11] considered positive measures $\mu = w\,dm$ with w, $1/w \in L^\infty(\mathbb{T})$ and proved that there exists a constant $C > 0$ such that $\sup\{\|\varphi_n(z, \mu)\|_\infty : 1 \leqslant w \leqslant 1 + \varepsilon\} \leqslant Cn^{C\varepsilon}$ if $0 < \varepsilon < 1$ and $\sup\{\|\varphi_n(z, \mu)\|_\infty : 1 \leqslant w \leqslant T\} \leqslant Cn^{1/2 - C/T}$ if $T > 2$. Moreover these estimates are sharp up to the value of the constant C. See also Ambroladze [2] for continuous weights, Denisov and Rush [10] for BMO weights, and Alexis, Aptekarev, Denisov [1] for Muckenhoupt A_p weights.

1.3. **Averaged convergence, general Szegő measures.** In 1991, Máté, Nevai, and Totik [23] proved a fundamental result on Christoffel functions generated by general measures μ of Szegő class $\mathrm{Sz}(\mathbb{T})$. In the equivalent language of orthogonal polynomials, it establishes Cesàro convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi_k^*(z) D_\mu(z)|^2 = 1 \tag{1.5}$$

for Lebesgue almost every point z on the unit circle $\mathbb{T}$. In 2008, Findley [13] showed that (1.5) holds almost everywhere on an open arc $I \subset \mathbb{T}$ if we assume that the measure μ is regular in the sense of Ullman and the *local* Szegő condition holds, i.e., $\log w \in L^1(I)$. Originally, the study of asymptotic behaviour of Christoffel functions was motivated by the problem of Cesàro summability of Fourier series generated by orthogonal polynomials [22], in particular, by the previous work of Freud [14]. Surprisingly, (1.5) turned out to be very important in the seemingly unrelated area of universality properties of orthogonal polynomials. The first application of (1.5) in this context is due to Lubinsky [21], it fast became a standard method. Totik [33] discusses a general idea of usage of (1.5) in universality problems and gives another proof of Findley's result. A recent breakthrough by Eichinger, Lukić, and Simanek [12] showed that Szegő condition (1.1) is not really needed for universality, this phenomenon is much more general. Still, under the Szegő condition we know *the scale* at which universality holds ($\sim 1/n$), and this knowledge follows directly from (1.5), see the discussion after Theorem 1.2 in [12]. In the Szegő case we also know a qualitative estimate for *the rate of convergence* of universality limits, see [5]. This information is not available in the general non-Szegő case due to a usage of compactness in [12].

1.4. **Assumptions on the side of recurrence coefficients.** In the next portion of convergence results, assumptions on the orthogonality measure μ are formulated in terms of the so-called recurrence coefficients. Let us recall that orthogonal polynomials generated by a probability measure μ with an infinite support on $\mathbb{T}$ satisfy the following recurrence relation:

$$\varphi_{n+1}(z) = \frac{z\varphi_n(z) - \overline{a_n}\varphi_n^*(z)}{\sqrt{1-|a_n|^2}}, \quad n \geqslant 0, \tag{1.6}$$

for all $z \in \mathbb{C}$ and some numbers $a_n \in \mathbb{D}$, $n \geqslant 0$, that are called recurrence coefficients of μ. It is also known that any sequence $\{a_n\}_{n\geqslant 0} \subset \mathbb{D}$ is the sequence of recurrence coefficients of a unique probability measure μ with an infinite support on $\mathbb{T}$. See Sections 1.5, 1.7 in [30] for the proofs of these facts. Heuristically, small recurrence coefficients $\{a_n\}$ give rise to "regular" measures μ, Szegő functions D_μ, and reflected orthogonal polynomials φ_n^*. For example, we have $\mu = m$, $D_\mu = 1$, $\varphi_n^* = 1$, $n \geqslant 0$, for identically zero recurrence coefficients $\{a_n\}_{n\geqslant 0}$. It is therefore natural to ask about the almost sure convergence property

$$\lim_{n\to\infty} \varphi_n^*(z) D_\mu(z) = 1 \text{ for Lebesgue almost every } z \in \mathbb{T}, \tag{1.7}$$

in terms of the size of the sequence $\{a_n\}_{n\geqslant 0}$. Using formula (2.4.35) in Simon [30] for polynomials $P = \varphi_k$, one can see that

$$(\overline{D_\mu(0)^{-1}} D_\mu(z)^{-1}, \varphi_k)_{L^2(w\,dm)} = \overline{\varphi_k(0)}, \qquad k \geqslant 0.$$

Denote by S a subset of $\mathbb{T}$ such that $m(S) = 1$, $\mu_{\mathbf{s}}(S) = 0$, and let χ_S be the characteristic function of S. It is known that $\chi_S D_\mu^{-1}$ can be approximated in $L^2(\mu)$ by analytic polynomials, see formula (2.4.34) in [30]. It follows that $\chi_S D_\mu^{-1}$ belongs to the closed linear span of the orthonormal sequence $\{\varphi_k\}_{k\geqslant 0}$ in $L^2(\mu)$. Then basic Fourier analysis tells us that

$$\chi_S(z)\overline{D_\mu(0)^{-1}} D_\mu(z)^{-1} = \sum_{n=0}^{\infty} \overline{\varphi_k(0)}\varphi_k(z), \qquad z \in \mathbb{T}, \tag{1.8}$$

in the sense of convergence in $L^2(\mu)$. Moreover, we have

$$\overline{\varphi_n^*(0)}\varphi_n^*(z) - \overline{\varphi_n(0)}\varphi_n(z) = \sum_{k=0}^{n-1} \overline{\varphi_k(0)}\varphi_k(z), \qquad z \in \mathbb{C}, \quad n \geqslant 1,$$

see (2.2.42) in [30]. From here and the Szegő asymptotic relation (1.3) at $z = 0$ we see that (1.7) holds if and only if the orthonormal series

$$\sum_{k=0}^{\infty} c_k \varphi_k \tag{1.9}$$

converges pointwise almost everywhere on the unit circle $\mathbb{T}$ for the special choice of Fourier coefficients $c_k = \overline{\varphi_k(0)}$. The problem of pointwise convergence of orthogonal series was studied in much detail by Menshov. One of his most striking results [25] says that for each $\varepsilon > 0$ and any square-summable sequence of coefficients $\{c_k\}_{k\geqslant 0}$ the condition

$$\sum_{k:\, 0<|c_k|<1} |c_k|^2 \left(\log\log\frac{1}{|c_k|}\right)^{2+\varepsilon}\left(\log\frac{1}{|c_k|}\right)^2 < \infty$$

implies the pointwise convergence of (1.9) almost everywhere on $\mathbb{T}$. In particular, the above condition holds if

$$\sum_{k\geqslant 0} |c_k|^p < \infty$$

for some $0 < p < 2$. Another theorem of Menshov [24] (obtained independently by Rademacher [28]) says that condition

$$\sum_{k\geqslant 1} |c_k|^2 \log^2 k < \infty,$$

is also sufficient for the pointwise convergence of (1.9) almost everywhere on $\mathbb{T}$. Using the fact that the product $\prod_{k=0}^{\infty}(1-|a_k|^2)$ converges to a positive number for every $\mu \in \mathrm{Sz}(\mathbb{T})$ (this is another version of the Szegő theorem, see Theorem 2.7.14 in [30]), and the formula

$$\varphi_{k+1}(0) = -\frac{\overline{a_k}}{\sqrt{\prod_{j=0}^{k}(1-|a_j|^2)}}, \qquad k \geqslant 0,$$

from Section 1.5 in [30], we see from Menshov theorems that any of the assumptions

$$\sum_{k:\, a_k\neq 0} |a_k|^2 \left(\log\log\frac{1}{|a_k|}\right)^{2+\varepsilon}\left(\log\frac{1}{|a_k|}\right)^2 < \infty \quad \text{for some} \quad \varepsilon > 0, \tag{1.10}$$

$$\sum_{k\geqslant 0} |a_k|^p < \infty \quad \text{for some} \quad 0 < p < 2, \tag{1.11}$$

$$\sum_{k\geqslant 1} |a_k|^2 \log^2 k < \infty, \tag{1.12}$$

implies (1.7). Similarly, yet another Menshov result [25] implies that assumption

$$\sum_{k\geqslant 3} |a_k|^2 (\log\log k)^2 < \infty$$

guarantees the Cesàro convergence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi_k^*(z) D_\mu(z) = 1$$

almost everywhere on the unit circle $\mathbb{T}$. The reader can find more information on Menshov's works and their subsequent development in the review paper by Ul'yanov [34].

1.5. **A theorem by Poltoratski and Denisov's counterexample.**

Theory of orthogonal polynomials on the unit circle (OPUC) is strongly related to the spectral theory of self-adjoint differential operators with simple spectrum. Let us mention two important papers in that field related to our work. In [26], a nonlinear Carleson theorem for Dirac operators is proved, i.e., a continuous version of the statement

$$\lim_{n\to\infty} |\varphi_n^*(z) D_\mu(z)| = 1 \ \text{ for every } \mu \in \mathrm{Sz}(\mathbb{T}) \text{ and Lebesgue almost every } z \in \mathbb{T}.$$

An error in [26] was found by Gevorg Mnatsakanyan. A new variant of the proof is avaliable in the form of arXiv preprint [27].

In recent preprints [8], [9], Denisov provides a counterexample to a strong version of the nonlinear Carleson theorem. He actually constructs [8] a measure $\mu \in \mathrm{Sz}(\mathbb{T})$ such that (1.7) fails.

2. Main results

Consider a signed Borel measure μ of finite variation on the unit circle $\mathbb{T}$. Let us denote by $\mathcal{P}(\mu, z)$ the Poisson integral of μ at $z \in \mathbb{D}$, given by

$$\mathcal{P}(\mu, z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi), \qquad z \in \mathbb{D}.$$

For $u \in L^1(\mathbb{T})$ and $\mu = u\, dm$, we set $\mathcal{P}(u, z) = \mathcal{P}(\mu, z)$. Consider now a probability measure $\mu = w\, dm + \mu_{\mathbf{s}}$ on $\mathbb{T}$ that belongs to the Szegő class $\mathrm{Sz}(\mathbb{T})$. Recall that the latter means $\log w \in L^1(\mathbb{T})$. Our analysis of asymptotic behaviour of orthogonal polynomials will be largely connected with the following entropy function of μ:

$$\mathcal{K}(\mu, z) = \log \mathcal{P}(\mu, z) - \mathcal{P}(\log w, z), \qquad z \in \mathbb{D}.$$

By Jensen inequality, $\mathcal{K}(\mu, z) \geqslant 0$ for every $z \in \mathbb{D}$. Take $\zeta \in \mathbb{T}$ and consider the Stolz angle with vertex at ζ, i.e., the convex hull

$$\Gamma_\zeta = \mathrm{conv}\big(\{z \in \mathbb{C} : \ |z| \leqslant 1/2\},\ \{\zeta\}\big).$$

Here the constant $1/2$ can be replaced by any constant $r \in (0, 1)$ without essential changes in the forthcoming statements and their proofs. Properties of the Poisson kernel imply that for every $\mu \in \mathrm{Sz}(\mathbb{T})$ we have

$$\lim_{\substack{z \to \zeta, \\ z \in \Gamma_\zeta}} \mathcal{K}(\mu, z) = 0 \tag{2.1}$$

for Lebesgue almost every $\zeta \in \mathbb{T}$. Sometimes we will also require more regularity from the measure μ than just the membership in the Szegő class $\mathrm{Sz}(\mathbb{T})$. Take $\zeta \in \mathbb{T}$ and set $z_n(\zeta) = (1 - 2^{-n})\zeta$ for $n \geqslant 0$. Note that $\{z_n(\zeta)\}$ is a sequence in Γ_ζ approaching the point ζ exponentially fast. As we will check (see the discussion after Proposition 2.4 below), both conditions

$$\sum_{n \geqslant 0} \mathcal{K}(\mu, z_n(\zeta)) < \infty, \tag{2.2}$$

$$\sum_{n \geqslant 0} \sqrt{\mathcal{K}(\mu, z_n(\zeta))} < \infty, \tag{2.3}$$

are weaker than the Dini condition (1.4) or its local version on an arc containing ζ. In particular, assumptions (2.2), (2.3) do not force w to be continuous near ζ. Observe also that (2.2) and (2.3) are the discrete versions of conditions

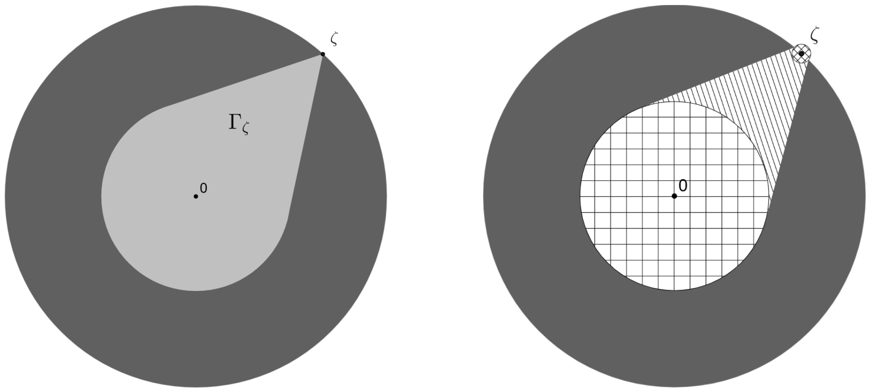
$$\int_0^1 \frac{\mathcal{K}(\mu, r\zeta)}{1-r}\, dr < \infty, \qquad \int_0^1 \frac{\sqrt{\mathcal{K}(\mu, r\zeta)}}{1-r}\, dr < \infty.$$

Our main results are the following theorems.

Theorem 2.1. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that $|D_\mu|$ has a non-zero finite non-tangential limit at ζ. If (2.1) holds at ζ, then*

$$\sup_{z \in \Gamma_\zeta} \left(\frac{1}{n} \sum_{k=0}^{n-1} \Big| |\varphi_k^*(z) D_\mu(z)|^2 - 1 \Big| \right) \to 0, \qquad n \to \infty. \tag{2.4}$$

In particular, (2.4) holds for Lebesgue almost every $\zeta \in \mathbb{T}$ for any $\mu \in \mathrm{Sz}(\mathbb{T})$.



(A) Stolz angle Γ_ζ (B) Tiling the domain of convergence

FIGURE 1. Different asymptotic methods work in different regions. On compact subsets of $\mathbb{D}$, the classical Szegő argument applies. Near the unit circle, at disks of radii $\sim 1/n$, one can use Máté, Nevai, and Totik approach. We prove the uniform Cesàro asymptotics in the whole Stolz angle Γ_ζ. See also Theorem 4.8 for the convergence in the bigger domain shown in Figure 1b.

Theorem 2.2. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that D_μ has a non-zero finite non-tangential limit at ζ. If (2.2) holds at ζ, then*

$$\sup_{z\in\Gamma_\zeta}\left(\frac{1}{n}\sum_{k=0}^{n-1}\left|\varphi_k^*(z)D_\mu(z)-1\right|\right)\to 0, \qquad n\to\infty. \tag{2.5}$$

Theorem 2.3. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that (2.3) holds at ζ. Then*

$$\sup_{z\in\Gamma_\zeta}|\varphi_n^*(z)D_\mu(z)-1| = 0, \qquad n\to\infty. \tag{2.6}$$

Let us make some remarks on Theorems 2.1–2.3. First, we would like to note that Theorem 2.1 implies Máté, Nevai, and Totik theorem [23], see (1.5). Indeed, the property

$$\sup_{z\in\Gamma_\zeta}\left(\frac{1}{n}\sum_{k=0}^{n-1}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big|\right)\to 0, \qquad n\to\infty, \tag{2.7}$$

is stronger than property

$$\frac{1}{n}\sum_{k=0}^{n-1}|\varphi_k^*(\zeta)D_\mu(\zeta)|^2\to 1, \qquad n\to\infty, \tag{2.8}$$

established in [23] for almost all $\zeta \in \mathbb{T}$. Besides the fact that (2.7) holds globally in the Stolz angle Γ_ζ, see Figure 1, there is another important difference between (2.7), (2.8). Namely, (2.8) tells us that for n large enough *the arithmetic mean* of the sequence $\{|\varphi_k^*(\zeta)D_\mu(\zeta)|^2\}_{1\leqslant k\leqslant n}$ is close to 1, while (2.7) implies that *most of the members* of this sequence are close to 1.

Our second remark concerns sharpness of Theorem 2.1. Based on construction in [8], we conjecture that (2.4) cannot hold with $|\varphi_k^*(z)D_\mu(z)|^2$ replaced by $\varphi_k^*(z)^2D_\mu(z)^2$ for general measures of Szegő class.

Let us now discuss assumptions (2.2), (2.3) in Theorems 2.2, 2.3. Without trying to formulate the most general results, we would like to demonstrate which properties of μ can yield (2.2), (2.3).

Proposition 2.4. *Assume that* $\mu = w\,dm$ *is such that* $u = \log w$ *is bounded on* $\mathbb{T}$ *and its Fourier coefficients* $\hat{u}(k) = (u, z^k)_{L^2(\mathbb{T})}$ *satisfy*

$$\sum_{k\geqslant 1} |\hat{u}(k)|^2 \log k < \infty. \tag{2.9}$$

Then (2.2) *holds at Lebesgue almost every* $\zeta \in \mathbb{T}$.

Proposition 2.5. *Let* $\mu = w\,dm$ *be a probability measure in* $\mathrm{Sz}(\mathbb{T})$ *and let* $\zeta \in \mathbb{T}$. *Assume that* $\log w$ *is bounded at a neighborhood of* ζ *and*

$$\int_{\mathbb{T}} \frac{|\log w(\xi) - \log w(\zeta)|^2}{|\xi - \zeta|}\, dm(\xi) < \infty. \tag{2.10}$$

Then (2.2) *holds at* ζ.

Proposition 2.6. *Let* $\mu = w\,dm$ *be a probability measure in* $\mathrm{Sz}(\mathbb{T})$, *let* $\zeta \in \mathbb{T}$ *and* $I_n = \{\xi \in \mathbb{T} : |\xi - \zeta| < 2^{-n}\}$, $n \geqslant 0$. *Assume that* $\log w$ *is bounded in a neighborhood of* ζ *and*

$$\sum_{n=1}^{\infty} \left(\frac{1}{|I_n|} \int_{I_n} \left|\log w - \log w(\zeta)\right|^2 dm \right)^{1/2} < \infty. \tag{2.11}$$

Then (2.3) *holds at* ζ.

Theorem 2.3 and Proposition 2.6 imply the classical result of Grenander and Szegő and its improvement by Badkov (see Section 1.1). Indeed, for $t = |\xi - \zeta|$ we have

$$|\log w(\xi) - \log w(\zeta)| \lesssim |w(\xi) - w(\zeta)| \leqslant \sup_{|z_1 - z_2| \leqslant t} |w(z_1) - w(z_2)| = \omega(t)$$

if w is positive and continuous near ζ and t is small enough (as usual, we write $g_1 \lesssim g_2$ if $g_1 \leqslant c g_2$ for a constant c that does not depend on the parameters under consideration, i.e., in the above case, on t). Therefore, (2.11) is weaker than the classical Dini condition (1.4):

$$\sum_{n=1}^{\infty} \left(\frac{1}{|I_n|} \int_{I_n} \left|\log w - \log w(\zeta)\right|^2 dm \right)^{1/2} \lesssim \sum_{n=1}^{\infty} \omega(|I_n|) \lesssim$$
$$\lesssim \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} \frac{\omega(t)}{t}\, dt \leqslant \int_0^1 \frac{\omega(t)}{t}\, dt < \infty,$$

or its local version on an arc. Since our estimates for the remainders are qualitative, Theorem 2.3 and Proposition 2.6 imply the uniform convergence $\varphi_n^* D_\mu \to 1$ on any open arc of $\mathbb{T}$ where w is positive and satisfies the Dini condition.

We would like to emphasize that assumptions in Propositions 2.4-2.6 (or, more generally, assumptions (2.1)–(2.3)) are given directly in terms of the measure μ. Therefore, they are much easier to check than assumptions (1.10)–(1.12) in terms of recurrence coefficients when one starts from the orthogonality measure μ. On the other hand, if one starts with recurrence coefficients of μ (such a situation occurs in direct spectral theory of differential operators with simple spectrum), then assumptions (1.10)–(1.12) are much easier than (2.1)–(2.3). From this perspective, (2.1)–(2.3) and (1.10)–(1.12) complement each other.

Next, we apply our results to the study of the asymptotic behaviour of the polynomial reproducing kernels

$$k_{\mu,n}(z_1, z_2) = \frac{\overline{\varphi_n^*(z_2)}\varphi_n^*(z_1) - \overline{\varphi_n(z_2)}\varphi_n(z_1)}{1 - \overline{z_2} z_1} \tag{2.12}$$

generated by a measure $\mu \in \mathrm{Sz}(\mathbb{T})$. It is well-known that these kernels have the following universal asymptotic behaviour

$$k_{\mu,n}(z_1, z_2) = |D_\mu^{-1}(\zeta)|^2 \frac{1 - \bar{z}_2^n z_1^n}{1 - \bar{z}_2 z_1}\big(1 + r_n(z_1, z_2)\big), \tag{2.13}$$

$$\lim_{n\to\infty} \sup\{|r_n(z_1, z_2)| \colon z_{1,2} \in \mathbb{C},\ |z_{1,2} - \zeta| \leqslant A/n\} = 0, \tag{2.14}$$

for every $A \geqslant 0$ and almost every point $\zeta \in \mathbb{T}$. Moreover, a bound for the rate of convergence in (2.14) is known, see [5]. Let us write $g_1 \asymp g_2$ for two functions g_1, g_2 if $g_1 \lesssim g_2$ and $g_2 \lesssim g_1$. Recall that the hyperbolic distance in $\mathbb{D}$ is defined by

$$d_H(z_1, z_2) = \frac{1}{2}\log\frac{1 + \rho(z_1, z_2)^2}{1 - \rho(z_1, z_2)^2}, \qquad \rho(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - \overline{z_2} z_1}\right|. \tag{2.15}$$

Note that $d_H(z_1, z_2) \lesssim 1$ for two points z_1, z_2 in some Stolz angle Γ_ζ if and only if $1 - |z_1| \asymp 1 - |z_2|$. We prove the following extension of the local universality relation (2.13).

Theorem 2.7. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that D_μ has a non-zero finite non-tangential limit at ζ. Take $A \geqslant 0$. Define the function r_n in $\mathbb{D} \times \mathbb{D}$ by*

$$k_{\mu,n}(z_1, z_2) = \overline{D_\mu^{-1}(z_2)} D_\mu^{-1}(z_1)\frac{1 - \bar{z}_2^n z_1^n}{1 - \bar{z}_2 z_1}\big(1 + r_n(z_1, z_2)\big). \tag{2.16}$$

We have

$$\lim_{n\to\infty} \sup_{\substack{z_1, z_2 \in \Gamma_\zeta,\\ d_H(z_1,z_2)\leqslant A}} |r_n(z_1, z_2)| = 0, \quad \textit{if (2.1) holds,} \tag{2.17}$$

$$\lim_{n\to\infty} \sup_{z_1, z_2 \in \Gamma_\zeta} \frac{1}{n}\sum_{k=0}^{n-1}|r_k(z_1, z_2)| = 0, \quad \textit{if (2.2) holds,} \tag{2.18}$$

$$\lim_{n\to\infty} \sup_{z_1, z_2 \in \Gamma_\zeta} |r_n(z_1, z_2)| = 0, \quad \textit{if (2.3) holds.} \tag{2.19}$$

In particular, we have (2.17) for almost every $\zeta \in \mathbb{T}$ for any measure $\mu \in \mathrm{Sz}(\mathbb{T})$.

3. Preliminaries

In this section we recall the basic theory of orthogonal polynomials and collect some results from [5]. Given a probability measure μ on $\mathbb{T}$, its Schur function f is defined by the relation

$$\frac{1 + zf(z)}{1 - zf(z)} = \int_{\mathbb{T}} \frac{1 + \bar{\xi} z}{1 - \bar{\xi} z}\, d\mu(\xi), \qquad z \in \mathbb{D}. \tag{3.1}$$

Taking the real part in (3.1), we get

$$\frac{1 - |zf(z)|^2}{|1 - zf(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\xi} z|^2}\, d\mu(\xi) = \mathcal{P}(\mu, z), \qquad z \in \mathbb{D}. \tag{3.2}$$

Since expressions in (3.2) are positive in $\mathbb{D}$, we have $|zf(z)| < 1$ for every $z \in \mathbb{D}$. In other words, zf is an analytic mapping from $\mathbb{D}$ into itself. By Schwarz lemma and the maximal modulus principle, the same is true about f (with the exceptional case when f is a unimodular constant, it corresponds to the case where μ is the point mass measure concentrated at a point of $\mathbb{T}$). Conversely, any analytic mapping f from $\mathbb{D}$ into itself leads to a probability measure μ on $\mathbb{T}$ defined by formula (3.1). It is not difficult to check that in this correspondence the finite Blaschke products generate probability measures μ supported on finite subsets of $\mathbb{T}$. In what follows we will always exclude this case from consideration assuming that μ is not a finite linear combination of point masses.

Consider a pair $\mu = w\,dm + \mu_{\mathbf{s}}$, f related by (3.1). Fatou's theorem (see Theorem 5.3 in Section I.5 in Garnett [15]) implies that for Lebesgue almost every $\xi \in \mathbb{T}$ we have

$$\frac{1-|f(\xi)|^2}{|1-\xi f(\xi)|^2} = w(\xi). \tag{3.3}$$

Note that $\operatorname{Re}(1-zf(z)) \geqslant 0$ for $z \in \mathbb{D}$, hence the function $1-zf$ is outer in $\mathbb{D}$ (see Corollary 4.8 in Section II.4 in Garnett [15]). Hence, we have $\log|1-\xi f| \in L^1(\mathbb{T})$ and

$$\int_{\mathbb{T}} \log|1-\xi f(\xi)|^2\,dm(\xi) = \log|1-zf(z)|^2\Big|_{z=0} = 0. \tag{3.4}$$

This formula and (3.3) imply

$$\int_{\mathbb{T}} \log(1-|f|^2)\,dm = \int_{\mathbb{T}} \log w\,dm,$$

where both sides are finite or equal to $-\infty$ simultaneously. In other words, we have $\mu \in \operatorname{Sz}(\mathbb{T})$ if and only if $\log(1-|f|^2) \in L^1(\mathbb{T})$.

Let us now recall the definition of Schur's algorithm. Take a probability measure μ on $\mathbb{T}$ with an infinite support and consider the Schur function f of μ. Define the analytic functions $f_n : \mathbb{D} \to \mathbb{D}$ by Schur's algorithm:

$$zf_{n+1}(z) = \frac{f_n(z)-f_n(0)}{1-\overline{f_n(0)}f_n(z)}, \qquad n \geqslant 0, \qquad f_0(z) = f(z), \quad z \in \mathbb{D}, \tag{3.5}$$

and let μ_n be the measures generated by f_n via (3.1). Note that $f_0 = f$ and $\mu_0 = \mu$. Szegő theorem says that the measure μ is in the Szegő class $\operatorname{Sz}(\mathbb{T})$ if and only if $\sum_n |f_n(z)|^2 < \infty$ for any $z \in \mathbb{D}$, and, moreover, in this case

$$\int_{\mathbb{T}} \log w\,dm = \int_{\mathbb{T}} \log(1-|f|^2)\,dm = \sum_{n=0}^{\infty} \log(1-|f_n(0)|^2). \tag{3.6}$$

See Theorem 2.7.14 and Theorem 3.1.4 in Simon [30]. In [6], the following generalization of (3.6) was found:

$$\mathcal{K}(\mu,z) = \sum_{k=0}^{\infty} \log\left(1+(1-|z|^2)\frac{|f_k(z)|^2}{1-|f_k(z)|^2}\right), \qquad z \in \mathbb{D}, \tag{3.7}$$

for every $\mu \in \operatorname{Sz}(\mathbb{T})$. Nonnegativity of summands in this formula implies that the entropy decreases in the Schur algorithm, that is,

$$\mathcal{K}(\mu_{n+1},z) \leqslant \mathcal{K}(\mu_n,z), \qquad z \in \mathbb{D}, \quad n \geqslant 0. \tag{3.8}$$

Let us also mention the useful relation

$$\mathcal{K}(\mu,z) = \log(1-|zf(z)|^2) - \mathcal{P}(\log(1-|f|^2),z), \qquad z \in \mathbb{D}, \tag{3.9}$$

that follows from the definition of $\mathcal{K}(\mu,z)$, formula (3.2) and the fact that $1-zf$ is an outer function in $\mathbb{D}$ (in particular, $\mathcal{P}(\log|1-\xi f(\xi)|^2,z) = \log|1-zf(z)|^2$ for every $z \in \mathbb{D}$). The same formula for μ_n, f_n reads as

$$\mathcal{K}(\mu_n,z) = \log(1-|zf_n(z)|^2) - \mathcal{P}(\log(1-|f_n|^2),z), \qquad z \in \mathbb{D}, \quad n \geqslant 0. \tag{3.10}$$

Further, for $\lambda \in \mathbb{D}$, we define

$$\tilde{\varphi}_{\lambda,n}(z) = \frac{z\varphi_{n-1}(z) - \overline{f_{n-1}(\lambda)}\varphi^*_{n-1}(z)}{\sqrt{1-|f_{n-1}(\lambda)|^2}}, \quad n \geqslant 0, \tag{3.11}$$

where we set $f_{-1} \equiv 0$, $z\varphi_{-1} \equiv 1$, $\varphi^*_{-1} \equiv 1$. Let also $\tilde{\varphi}^*_{\lambda,n}(z) = z^n\overline{\tilde{\varphi}_{\lambda,n}(1/\bar{z})}$ be the corresponding reflected polynomials. Note that

$$\tilde{\varphi}^*_{\lambda,n}(z) = \frac{\varphi^*_{n-1}(z) - zf_{n-1}(\lambda)\varphi_{n-1}(z)}{\sqrt{1-|f_{n-1}(\lambda)|^2}}, \quad n \geqslant 0, \tag{3.12}$$

while the usual reflected polynomials φ_n^* can be written in the form

$$\varphi_n^*(z) = \frac{\varphi_{n-1}^*(z) - zf_{n-1}(0)\varphi_{n-1}(z)}{\sqrt{1-|f_{n-1}(0)|^2}} = \varphi_{0,n}^*(z), \quad n \geqslant 0,$$

as follows from Szegő recurrence (1.6) and Geronimus theorem,

$$a_n = f_n(0), \quad n \geqslant 0,$$

see Theorem 3.1.4 in Simon [30]. In particular, for $n \geqslant 1$ we have

$$\frac{\tilde{\varphi}_{\lambda,n}^*(z)}{\varphi_n^*(z)} \frac{\sqrt{1-|f_{n-1}(\lambda)|^2}}{\sqrt{1-|f_{n-1}(0)|^2}} = \frac{1 - z\overline{f_{n-1}(\lambda)}b_{n-1}(z)}{1 - z\overline{f_{n-1}(0)}b_{n-1}(z)}, \qquad b_{n-1} = \frac{\varphi_{n-1}}{\varphi_{n-1}^*}, \tag{3.13}$$

which yields the estimate

$$\frac{1-|f_{n-1}(\lambda)|}{1+|f_{n-1}(0)|} \leqslant \left|\frac{\tilde{\varphi}_{\lambda,n}^*(z)}{\varphi_n^*(z)}\right| \frac{\sqrt{1-|f_{n-1}(\lambda)|^2}}{\sqrt{1-|f_{n-1}(0)|^2}} \leqslant \frac{1+|f_{n-1}(\lambda)|}{1-|f_{n-1}(0)|}, \tag{3.14}$$

that will allow us to switch between $\tilde{\varphi}_{\lambda,n}^*$ and φ_n^* provided that $f_{n-1}(\lambda)$, $f_{n-1}(0)$ are small. Note that (3.14) trivially holds also for $n = 0$ because $\tilde{\varphi}_{\lambda,0}^* = \varphi_0^* = 1$. The main feature of the polynomials $\{\tilde{\varphi}_{\lambda,n}\}$ is the entropy bound

$$\mathcal{K}(\nu_{\lambda,n}, \lambda) \leqslant \mathcal{K}(\mu, \lambda), \qquad n \geqslant 0, \quad \nu_{\lambda,n} = \frac{dm}{|\tilde{\varphi}_{\lambda,n}^*|^2}, \tag{3.15}$$

see Corollary 4 in [6]. Relation (3.15) could be rewritten in the form

$$\mathcal{P}\left(\left|\frac{\tilde{\varphi}_{\lambda,n}^*(\lambda)}{\tilde{\varphi}_{\lambda,n}^*} - 1\right|^2, \lambda\right) \leqslant e^{\mathcal{K}(\mu,\lambda)} - 1, \tag{3.16}$$

see Lemma 2.3 in [5]. The bound (3.16) has important consequences that we state and prove below in Lemma 3.1 and Lemma 3.2. These lemmas are not specific for orthogonal polynomials as they work for any polynomials that satisfy entropy bounds of type (3.16) with the small enough right hand side. For consistency, we will use notation φ^* for a polynomial without zeroes in the open unit disk. The lemmas will be used later for $\varphi^* = \tilde{\varphi}_{\lambda,n}^*$.

Lemma 3.1. *Let φ^* be a polynomial of degree at most n without zeroes in the open unit disk $\mathbb{D}$, and let $\varphi(z) = z^n\overline{\varphi^*(1/\bar{z})}$ be the corresponding reflected polynomial. Denote by $b = \varphi/\varphi^*$ the Blaschke product generated by φ, φ^*. Fix $\zeta \in \mathbb{T}$, $r \in (0,1)$, $A > 0$, and consider the Stolz angle $\Gamma_\zeta = \mathrm{conv}\big(\{z \in \mathbb{C} : \ |z| \leqslant r\}, \{\zeta\}\big)$. For every $\varepsilon > 0$ there exists $\eta_0 > 0$ depending only on r, A, ε, such that if*

$$\mathcal{P}\left(\left|\frac{\varphi^*(\lambda)}{\varphi^*} - 1\right|^2, \lambda\right) \leqslant \eta_0 \tag{3.17}$$

for some $\lambda \in \Gamma_\zeta$, then there exists $\alpha(\lambda) \in \mathbb{T}$ such that

$$\sup_{z:\, d_H(z,\lambda)\leqslant A} \big|b(z) - \alpha(\lambda)z^n\big| \leqslant \varepsilon, \tag{3.18}$$

where $d_H(z,\lambda)$ is the hyperbolic distance between z, λ, see (2.15).

Proof. Put $E'_\varepsilon(\lambda) = \{\xi \in \mathbb{T} : |\varphi^*(\lambda)/\varphi^*(\xi) - 1| > \varepsilon\}$, $E_\varepsilon(\lambda) = \mathbb{T} \setminus E'_\varepsilon(\lambda)$, and

$$\eta = \mathcal{P}\left(\left|\frac{\varphi^*(\lambda)}{\varphi^*} - 1\right|^2, \lambda\right).$$

We have

$$\sup_{z:\, d_H(z,\lambda)\leqslant A} \int_{E'_\varepsilon(\lambda)} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, dm(\xi) \to 0, \qquad \eta \to 0. \tag{3.19}$$

Since on the set $E_\varepsilon(\lambda)$ the argument of φ^* (to be denoted by $\arg\varphi^*$) is close to the argument of $\varphi^*(\lambda)$ modulo $2\pi\mathbb{Z}$, and since $b(\xi)=\xi^n\cdot e^{-2i\arg\varphi^*(\xi)}$ for all $\xi\in\mathbb{T}$, we have

$$\sup_{\xi\in E_\varepsilon(\lambda)}|b(\xi)-\alpha(\lambda)\xi^n|\lesssim\varepsilon,\qquad \alpha(\lambda)=\frac{\overline{\varphi^*(\lambda)}}{\varphi^*(\lambda)}=e^{-2i\arg\varphi^*(\lambda)}.$$

We now see from (3.19) that

$$\sup_{z:\,d_H(z,\lambda)\leqslant A}\int_{\mathbb{T}}|b(\xi)-\alpha(\lambda)\xi^n|\frac{1-|z|^2}{|1-\bar{\xi}z|^2}\,dm(\xi)\lesssim\varepsilon$$

if η is small enough. This relation and the analyticity of b, z^n imply (3.18) provided that (3.17) holds with $\eta_0>0$ small enough. □

Next lemma was implicitly used in [5] but not proved there. We fill this gap below.

Lemma 3.2. *Let φ^* be a polynomial of degree at most n without zeroes in the open unit disk $\mathbb{D}$, let $\zeta\in\mathbb{T}$, $A_1,A_2>0$. Assume that $\max(A_1,A_2)/n\leqslant 1/2$ and set $\lambda_n=(1-A_1/n)\zeta$. There exists $\eta_0>0$ depending only on A_1, A_2 such that if*

$$\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*}-1\right|^2,\lambda_n\right)\leqslant\eta_0, \tag{3.20}$$

then the polynomial φ^* has no zeroes in $B(\zeta,A_2/n)=\{z\in\mathbb{C}:\ |\zeta-z|\leqslant A_2/n\}$.

Proof. Define

$$\eta=\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*}-1\right|^2,\lambda_n\right).$$

Our first goal is to show that if η is small enough, then φ^* cannot have zeroes in the "ε-strip"

$$\{z\in B(\zeta,A_2/n):\ \big|1-|z|\big|<\varepsilon/n\},$$

for some $\varepsilon>0$. Suppose, on the contrary, that for a given $\varepsilon\in(0,A_2/2)$ there is $z^*\in B(\zeta,A_2/n)$ such that $|1-|z^*||<\varepsilon/n$ and $\varphi^*(z^*)=0$. Since φ^* has no zeroes in $\mathbb{D}$, we have $|z^*|\geqslant 1$. Choose a wide Stolz angle $\Gamma=\Gamma_\zeta^{A_2,\varepsilon}$ with vertex at ζ and aperture sufficiently large such that the Hausdorff distance between $\Gamma\cap B(\zeta,A_2/n)$ and the unit circle $\mathbb{T}$ is comparable to ε/n (see Figure 2). Observe that the point $z_{\lambda_n}=\left(1-\frac{A_2}{n}\right)\frac{z^*}{|z^*|}$ belongs to Γ. Denote by z_ε the point on the boundary of Γ such that $z_\varepsilon/|z_\varepsilon|=z^*/|z^*|$. Since all three points λ_n, z_ε, and z_{λ_n} belong to the same Stolz angle Γ and have comparable distances to the unit circle $\mathbb{T}$, we have

$$\begin{aligned}\left|\frac{\varphi^*(\lambda_n)}{\varphi^*(z_{\lambda_n})}-1\right|&\leqslant\sqrt{\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*}-1\right|^2,z_{\lambda_n}\right)}\\&\lesssim\sqrt{\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*}-1\right|^2,\lambda_n\right)}\lesssim\sqrt{\eta},\\ \left|\frac{\varphi^*(\lambda_n)}{\varphi^*(z_\varepsilon)}-1\right|&\leqslant\sqrt{\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*}-1\right|^2,z_\varepsilon\right)}\\&\lesssim\sqrt{\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*}-1\right|^2,\lambda_n\right)}\lesssim\sqrt{\eta},\end{aligned}$$

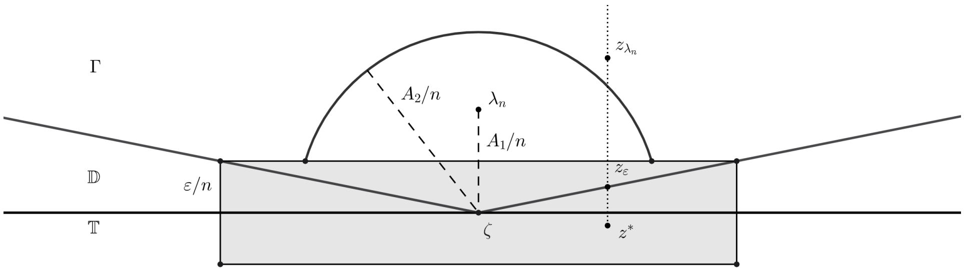


FIGURE 2. Objects appearing in the proof of Lemma 3.2.

with constants depending only on A_1, A_2 and ε. This gives

$$\left|\frac{\varphi^*(z_{\lambda_n})}{\varphi^*(z_\varepsilon)}\right| = 1 + O(\sqrt{\eta}) \tag{3.21}$$

for all $\eta < \eta_{00}$ (again, the value of η_{00} depends solely on A_1, A_2 and ε). At the same time, Lemma 2.3 in [7] implies the estimate

$$\left|\frac{\varphi^*(z_{\lambda_n})}{\varphi^*(z_\varepsilon)}\right| \geqslant \left(\frac{1+r_\varepsilon}{2}\right)^{n-1}\left|\frac{z_{\lambda_n}-z^*}{z_\varepsilon - z^*}\right|, \qquad r_\varepsilon = \left|\frac{z_{\lambda_n}}{z_\varepsilon}\right|,$$

if we factorize $\varphi^* = (z - z^*)p_{n-1}$ and use the fact that the polynomial p_{n-1} has no zeroes in $\mathbb{D}$ (the first usage of such estimates in the context of orthogonal polynomials is due to Máté and Nevai [22]). Observe that

$$\left(\frac{1+r_\varepsilon}{2}\right)^{n-1} \geqslant \left(\frac{1+|z_{\lambda_n}|}{2}\right)^{n-1} \geqslant \left(1-\frac{A_2}{2n}\right)^{n-1} \geqslant c_{A_2},$$

for some constant $c_{A_2} > 0$, and

$$\left|\frac{z_{\lambda_n}-z^*}{z_\varepsilon - z^*}\right| \geqslant \frac{A_2}{\varepsilon}.$$

So, if $\varepsilon > 0$ is so small that

$$\left(\frac{1+r_\varepsilon}{2}\right)^{n-1}\left|\frac{z_{\lambda_n}-z^*}{z_\varepsilon - z^*}\right| \geqslant \frac{A_2 c_{A_2}}{\varepsilon} \geqslant 2,$$

and $\eta_{00} > 0$ is so small that for $\eta \leqslant \eta_{00}$ relation (3.21) gives

$$\left|\frac{\varphi^*(z_{\lambda_n})}{\varphi^*(z_\varepsilon)}\right| \leqslant \frac{3}{2},$$

we obtain a contradiction. In other words, for chosen ε, η_{00} there is no point $z^* \in B(\zeta, A_2/n)$ satisfying assumptions

$$|1 - |z^*|| \leqslant \varepsilon/n, \qquad \varphi^*(z^*) = 0,$$

provided $\eta \leqslant \eta_{00}$. Our next aim is to prove that for this fixed value $\varepsilon > 0$ the polynomial φ^* has no zeroes also in the remaining part of the disk $B(\zeta, A_2/n)$ (in fact, for this we will need to change η_{00} by a smaller constant).

Set $\varphi(z) = z^n\overline{\varphi^*(1/\bar{z})}$ and note that $b = \varphi/\varphi^*$ is a Blaschke product. Lemma 3.1 implies that for every $A > 0$, $\gamma > 0$ there exists $\eta_{01} > 0$ such that if $\eta \leqslant \eta_{01}$, then

$$\sup_{z\colon d_H(z,\lambda)\leqslant A}\Big||b(z)| - |z|^n\Big| \leqslant \gamma. \tag{3.22}$$

For $S \subset \mathbb{D}$, we will denote $S^* = \{z \in \mathbb{C} : 1/\bar{z} \in S\}$. Let us take A so large that the hyperbolic disk $\Omega_n(A) = \{z \in \mathbb{D} :\ d_H(z, \lambda_n) \leqslant A\}$ satisfies

$$\Omega_n(A) \cup \{z \in \mathbb{C} :\ \bigl|1 - |z|\bigr| < \varepsilon/n\} \cup \Omega_n(A)^* \supset B(\zeta, A_2/n). \tag{3.23}$$

Note that the choice of A depends only on parameters ε, A_2 and can be made independent of n. In fact, we have

$$\inf_{z \in \Omega_n(A)} (1 - |z|) \asymp 1/n,$$

which guarantees that γ defined by

$$\gamma = \frac{\inf_{z \in \Omega_n(A)} |z|^n}{2}$$

is bounded below by a constant only depending in A. Then, for this choice of A, γ we use Lemma 3.1 to find $\eta_{01} > 0$ such that (3.22) holds for $\eta \leqslant \eta_{01}$. Note that (3.22) implies that the Blaschke product b has no zeroes in $\Omega_n(A)$. Hence, the same is true about φ, and then the polynomial φ^* has no zeroes in the reflected set $\Omega_n(A)^*$. Finally, (3.23) and the first part of the proof imply that φ^* has no zeroes in the whole disk $B(\zeta, A_2/n)$. We now see that the claim of the lemma holds for $\eta_0 = \min(\eta_{00}, \eta_{01})$. □

The following result is Lemma 2.6 in [5].

Lemma 3.3. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$, $A > 0$, $\zeta \in \mathbb{T}$. Assume that $n \geqslant 2A$ and set $\lambda = (1 - A/2n)\zeta$. There exists a constant $\eta_0 > 0$ depending only on A, such that if $\mathcal{K}(\nu_{\lambda,n}, \lambda) \leqslant \eta_0$, then*

$$\left| \frac{\tilde{\varphi}^*_{\lambda,n}(\lambda)}{\tilde{\varphi}^*_{\lambda,n}(z)} - 1 \right| \leqslant c e^{2A} \sqrt{\mathcal{K}(\nu_{\lambda,n}, \lambda)},$$

for every $z \in \partial\Omega_n$, where Ω_n is a domain in $\mathbb{C}$ with a piece-wise smooth boundary $\partial\Omega_n$ such that $B(\zeta, A/n) \subset \Omega_n \subset B(\zeta, 4A/n)$, $B(\zeta, r) = \{z \in \mathbb{C} :\ |\zeta - z| \leqslant r\}$. Here, the constant c is universal.

Let η_0 be the best possible η_0 in Lemma 3.2, Lemma 3.3 for the choice of parameters $A_1 = 2$, $A_2 = 8$, $A = 2$. In other words, let η_0 be the maximum of the all numbers η_0 for which *both* Lemma 3.2 and Lemma 3.3 work for $A_1 = 2$, $A_2 = 8$, $A = 2$. Take a point $\zeta \in \mathbb{T}$ such that (2.1) holds. Set $\lambda_j = (1 - 1/j)\zeta$ and define

$$n_0 = \min\Bigl\{ n \geqslant 8A :\ e^{\mathcal{K}(\mu, \lambda_j)} - 1 \leqslant \eta_0 \text{ for all } j \geqslant n \Bigr\}. \tag{3.24}$$

We arrive at the following estimates.

Lemma 3.4. *For every $n \geqslant n_0$ and $0 \leqslant k \leqslant n$, we have*

$$\left| \frac{\tilde{\varphi}^*_{\lambda_n,k}(z)}{\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n)} - 1 \right| \lesssim \sqrt{\mathcal{K}(\mu, \lambda_n)}, \qquad z \in B(\zeta, 1/n). \tag{3.25}$$

Consequently, we have

$$\left| \frac{\tilde{\varphi}^*_{\lambda_n,k}(z)}{\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n)} \right|^2 = 1 + O(\sqrt{\mathcal{K}(\mu, \lambda_n)}), \qquad z \in B(\zeta, 1/n), \tag{3.26}$$

where the constant in $O(\cdot)$ does not depend on μ, k, and z.

Proof. Take $n \geqslant n_0$, $0 \leqslant k \leqslant n$. Let us use Lemma 3.2 for $\varphi^* = \tilde{\varphi}^*_{\lambda_n,k}$ with the parameters $\tilde{n} = k$, $\tilde{A}_1 = k/n$, $\tilde{A}_2 = 8k/n$ in place of n, A_1, A_2. We have $\max(\tilde{A}_1, \tilde{A}_2)/\tilde{n} = 8/n \leqslant 8/n_0 \leqslant 1/2$, as required in Lemma 3.2. We also have

$$\lambda_{\tilde{n}} = (1 - \tilde{A}_1/\tilde{n})\zeta = (1 - 1/n)\zeta = \lambda_n.$$

Note that $\tilde{A}_1 \leqslant 1$, $\tilde{A}_2 \leqslant 8$, therefore, the optimal (i.e., largest possible) number $\tilde{\eta}_0$ in Lemma 3.2 for this choice of parameters satisfies $\eta_0 \leqslant \tilde{\eta}_0$. Let us check the main assumption (3.20) in Lemma 3.2:

$$\mathcal{P}\left(\left|\frac{\varphi^*(\lambda_n)}{\varphi^*} - 1\right|^2, \lambda_n\right) = e^{\mathcal{K}(\nu_{\lambda_n,k},\lambda_n)} - 1 \leqslant e^{\mathcal{K}(\mu,\lambda_n)} - 1 \leqslant \eta_0 \leqslant \tilde{\eta}_0. \tag{3.27}$$

Lemma 3.2 now tells us that $\tilde{\varphi}^*_{\lambda_n,k}$ has no zeroes in $B(\zeta, \tilde{A}_2/\tilde{n}) = B(\zeta, 8/n)$. This information will be used at the last part of the proof.

Next, we use Lemma 3.3 for $\tilde{n} = k$, $\tilde{A} = 2k/n$ in place of n, A. With this choice of parameters, we have $\tilde{\lambda} = (1 - \tilde{A}/2\tilde{n})\zeta = \lambda_n$. Moreover,

$$\mathcal{K}(\nu_{\tilde{\lambda},\tilde{n}}, \tilde{\lambda}) \leqslant \mathcal{K}(\mu, \tilde{\lambda}) = \mathcal{K}(\mu, \lambda_n) \leqslant e^{\mathcal{K}(\mu,\lambda_n)} - 1 \leqslant \eta_0 \leqslant \tilde{\eta}_0,$$

for $n \geqslant n_0$. Here $\tilde{\eta}_0$ is the largest possible number in Lemma 3.3 for $\tilde{n}$, $\tilde{A}$, and we have used the fact that $\tilde{A} \leqslant 2$, so $\eta_0 \leqslant \tilde{\eta}_0$. We see that

$$\left|\frac{\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n)}{\tilde{\varphi}^*_{\lambda_n,k}(z)} - 1\right| = \left|\frac{\tilde{\varphi}^*_{\tilde{\lambda},\tilde{n}}(\tilde{\lambda})}{\tilde{\varphi}^*_{\tilde{\lambda},\tilde{n}}(z)} - 1\right| \lesssim e^{2\tilde{A}}\sqrt{\mathcal{K}(\nu_{\tilde{\lambda},\tilde{n}}, \tilde{\lambda})} \lesssim \sqrt{\mathcal{K}(\mu, \lambda_n)}, \tag{3.28}$$

for every $z \in \partial\Omega_{\tilde{n}}$, where $\Omega_{\tilde{n}}$ is a domain in $\mathbb{C}$ with a piece-wise smooth boundary $\partial\Omega_{\tilde{n}}$ such that $B(\zeta, 2/n) \subset \Omega_{\tilde{n}} \subset B(\zeta, 8/n)$. In the last chain of inclusions we have used the fact that $\tilde{A}/\tilde{n} = 2/n$, $4\tilde{A}/\tilde{n} = 8/n$. Then from the maximum modulus principle for the function $1/\tilde{\varphi}^*_{\lambda_n,k}$ in the domain $\Omega_{\tilde{n}}$, we obtain

$$\left|\frac{\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n)}{\tilde{\varphi}^*_{\lambda_n,k}(z)} - 1\right| \lesssim \sqrt{\mathcal{K}(\mu, \lambda_n)}, \qquad z \in \Omega_{\tilde{n}}. \tag{3.29}$$

In other words, (3.28) holds everywhere in $\Omega_{\tilde{n}}$, not just at the boundary $\partial\Omega_{\tilde{n}}$. We used here the fact that $\tilde{\varphi}^*_{\lambda_n,k}$ has no zeroes in $B(\zeta, 8/n)$, hence the function $1/\tilde{\varphi}^*_{\lambda_n,k}$ is analytic in $\Omega_{\tilde{n}} \subset B(\zeta, 8/n)$. Since we also have $\Omega_{\tilde{n}} \supset B(\zeta, 2/n)$, relations (3.25), (3.26) follow from (3.29). □

The bounds (3.25), (3.26) will be among the main ingredients in the proofs of Theorem 2.1 and Theorem 2.2.

4. Proof of Theorem 2.1

Define the nonnegative functions η_n, $\tilde{\eta}_n$ in the open unit disk $\mathbb{D}$ by

$$\eta_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{K}(\mu_k, z), \tag{4.1}$$

$$\tilde{\eta}_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} \log\frac{1}{1 - |f_k(z)|}. \tag{4.2}$$

Lemma 4.1. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$. We have $\sup_{z\in\Gamma_\zeta} \eta_n(z) \to 0$ as $n \to \infty$ for every $\zeta \in \mathbb{T}$ such that (2.1) holds.*

Proof. The estimate (3.8) gives $\eta_n(z) \leqslant \mathcal{K}(\mu, z)$ for any $z \in \mathbb{D}$ and $n \geqslant 1$. Hence, it is sufficient to prove that $\sup\{\mathcal{K}(\mu_n, z) : |z| < r\} \to 0$ as $n \to \infty$, for any $0 < r < 1$. Observe that (3.9) gives

$$\mathcal{K}(\mu_n, z) \leqslant -\mathcal{P}(\log(1 - |f_n|^2), z) \leqslant -\frac{1 + |z|}{1 - |z|}\int_{\mathbb{T}} \log(1 - |f_n|^2)\, dm$$

and one only needs to check that the last integral tends to 0 as $n \to \infty$. In turn, this follows from Szegő formula (3.6) for the pair μ_n, f_n, i.e., from

$$\int_{\mathbb{T}} \log(1 - |f_n|^2)\, dm = \sum_{k=n}^{\infty} \log(1 - |f_k(0)|^2). \tag{4.3}$$

Since by (3.6) we have $\sum_{k\geqslant 0} |f_k(0)|^2 < \infty$ for every $\mu \in \mathrm{Sz}(\mathbb{T})$, the right hand side in (4.3) indeed tends to 0 as $n \to \infty$. □

Lemma 4.2. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$, and let*

$$\Gamma_{\zeta,n} = \Gamma_\zeta \cap \{z \in \mathbb{C} : \, |z - \zeta| \geqslant 1/n\}, \qquad n \geqslant 1.$$

We have $\sup_{z\in\Gamma_{\zeta,n}} \tilde{\eta}_n(z) \to 0$ as $n \to \infty$ for every $\zeta \in \mathbb{T}$ such that (2.1) holds.

Proof. We first deal with $z \in \Gamma_{\zeta,n}$ such that $\mathcal{K}(\mu, z) \leqslant \log 2$. Let us divide the set of indices $I_n = \{k : 0 \leqslant k \leqslant n-1\}$ into two subsets

$$I_{n,s} = \{k \in I_n : \quad |f_k(z)| \leqslant 1/2\}\,, \tag{4.4}$$

$$I_{n,b} = \{k \in I_n : \quad |f_k(z)| > 1/2\}\,. \tag{4.5}$$

For $k \in I_{n,s}$, we have

$$\begin{aligned} \frac{1}{n} \sum_{k\in I_{n,s}} \log \frac{1}{1 - |f_k(z)|} &\lesssim \frac{1}{n} \sum_{k=0}^{n-1} |f_k(z)| \leqslant \sqrt{\frac{1}{n} \sum_{k=0}^{n-1} |f_k(z)|^2} \\ &\lesssim \sqrt{(1 - |z|^2) \sum_{k=0}^{n} |f_k(z)|^2} \lesssim \sqrt{\mathcal{K}(\mu, z)}, \end{aligned} \tag{4.6}$$

where in the last estimate we have used (3.7) and the fact that $x \lesssim \log(1 + x)$ for $x \in [0, 1]$. Next, for $k \in I_{n,b}$ we have

$$\frac{1}{n} \sum_{k\in I_{n,b}} \log \frac{1}{1 - |f_k(z)|} \leqslant \frac{1}{n} \sum_{k\in I_{n,b}} \frac{1 + |f_k(z)|}{1 - |f_k(z)|^2} \lesssim \sum_{k\in I_{n,b}} \frac{(1 - |z|^2)|f_k(z)|^2}{1 - |f_k(z)|^2}.$$

Each summand in the last sum satisfies

$$0 \leqslant \frac{(1 - |z|^2)|f_k(z)|^2}{1 - |f_k(z)|^2} \leqslant 1,$$

for otherwise we cannot have $\mathcal{K}(\mu, z) \leqslant \log 2$, see (3.7). Thus, one can use again the elementary inequality $x \lesssim \log(1 + x)$ for $x \in [0, 1]$ and obtain

$$\frac{1}{n} \sum_{k\in I_{n,b}} \log \frac{1}{1 - |f_k(z)|} \lesssim \sum_{k\in I_{n,b}} \log \left(1 + \frac{(1 - |z|^2)|f_k(z)|^2}{1 - |f_k(z)|^2}\right) \leqslant \mathcal{K}(\mu, z). \tag{4.7}$$

Combining (4.6), (4.7), we arrive at

$$\begin{aligned} \tilde{\eta}_n(z) &= \frac{1}{n} \sum_{k\in I_{n,s}} \log \frac{1}{1 - |f_k(z)|} + \frac{1}{n} \sum_{k\in I_{n,b}} \log \frac{1}{1 - |f_k(z)|} \\ &\lesssim \sqrt{\mathcal{K}(\mu, z)} + \mathcal{K}(\mu, z) \lesssim \sqrt{\mathcal{K}(\mu, z)}, \end{aligned} \tag{4.8}$$

for every $z \in \Gamma_{\zeta,n}$ such that $\mathcal{K}(\mu, z) \leqslant \log 2$. Moreover, the constants in (4.8) are universal. Thus, the result will follow from (2.1) if we show that $\tilde{\eta}_n$ tends to zero uniformly on compact subsets of $\mathbb{D}$. In turn, this will follow if we show that $\max_{|z|\leqslant r}|f_n(z)| \to 0$ as $n\to\infty$ for every $r\in(0,1)$. So, let us take z with $|z|\leqslant r<1$ and estimate

$$|f_n(z)|^2 \leqslant \log\frac{1}{1-|f_n(z)|^2} = -\log\left(1-|\mathcal{P}(f_n,z)|^2\right) \leqslant$$
$$\leqslant -\mathcal{P}\left(\log(1-|f_n|^2),z\right) \lesssim -\frac{1+r}{1-r}\int_{\mathbb{T}}\log(1-|f_n|^2)\,dm,$$

where we used Jensen's inequality for the convex function $x\mapsto -\log(1-x^2)$ and the basic bound for the Poisson kernel. Then, as in the proof of Lemma 4.1,

$$\lim_{n\to\infty}\int_{\mathbb{T}}\log(1-|f_n|^2)\,dm = \lim_{n\to\infty}\sum_{k=n}^{\infty}\log(1-|f_k(0)|^2)=0,$$

and the result follows. □

Lemma 4.3. *Let $\mu = w\,dm+\mu_{\mathbf{s}}$ be a measure in $\mathrm{Sz}(\mathbb{T})$, let D_μ be the corresponding Szegő function, and let $\{\varphi_n\}_{n\geqslant 0}$ be the sequence of orthonormal polynomials generated by μ, see (1.2). We have*

$$\varphi_n^* D_\mu = \frac{O_n}{1-zb_nf_n}, \qquad b_n=\frac{\varphi_n}{\varphi_n^*}, \qquad n\geqslant 0, \tag{4.9}$$

for the outer function O_n defined by the conditions $O_n(0)>0$ and $|O_n|^2 = 1-|f_n|^2$ almost everywhere on $\mathbb{T}$.

Proof. Recall that the reflected polynomials φ_n^* have no zeroes in the closed unit disk, see, e.g., Theorem 1.7.1 in Simon [30]. Therefore, b_n is a finite Blaschke product, $\mathrm{Re}(1-zb_nf_n)\geqslant 0$ in $\mathbb{D}$, and the function $1-zb_nf_n$ is outer (see Corollary 4.8 in Section II.4 in Garnett [15]). From here we see that functions $\varphi_n^*D_\mu$ and $O_n(1-zb_nf_n)^{-1}$ are outer as well. Note that they have positive value at 0. Thus, we only need to check that

$$|\varphi_n^*D_\mu| = \left|\frac{O_n}{1-zb_nf_n}\right|, \tag{4.10}$$

almost everywhere on $\mathbb{T}$. Taking the square and using the definitions of D_μ, O_n, we see that (4.10) is equivalent to

$$|\varphi_n^*|^2 w = \frac{1-|f_n|^2}{|1-zb_nf_n|^2}$$

almost everywhere on $\mathbb{T}$. This formula is due to S. Khrushchev, see Theorem 2 on page 173 in [20]. □

Lemma 4.4. *We have*

$$\frac{1}{n}\sum_{k=0}^{n-1}\Big|\log(1-zb_k(z)f_k(z))\Big| \leqslant \tilde\eta_n(z), \tag{4.11}$$

$$\frac{1}{n}\sum_{k=0}^{n-1}\log|O_k(z)|^{-2} \leqslant \eta_n(z)+\tilde\eta_n(z), \tag{4.12}$$

for every $z\in\mathbb{D}$. In (4.11), we deal with the main branch of the logarithm, $\log 1 = 0$.

Proof. Inequality (4.11) is immediate from the definition of $\tilde\eta_n$ and the estimate

$$\Big|\log(1-zb_k(z)f_k(z))\Big| \leqslant \log\frac{1}{1-|zb_k(z)f_k(z)|} \leqslant \log\frac{1}{1-|f_k(z)|}, \qquad z\in\mathbb{D}.$$

To prove (4.12), we write

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1}\log|O_k(z)|^{-2} &= \frac{1}{n}\sum_{k=0}^{n-1}\mathcal{P}(\log|O_k|^{-2},z)\\ &= -\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{P}(\log(1-|f_k|^2),z)\\ &= \frac{1}{n}\sum_{k=0}^{n-1}\mathcal{K}(\mu_k,z)-\frac{1}{n}\sum_{k=0}^{n-1}\log(1-|zf_k(z)|^2)\\ &\leqslant \eta_n(z)+\tilde{\eta}_n(z).\end{aligned}$$

First equality here holds because O_k is an outer function for each $k \geqslant 0$. In the second line we used the fact that that $|O_k|^2 = 1-|f_k|^2$ on $\mathbb{T}$, $k \geqslant 0$. Equality in the third line follows from (3.10), and the last estimate from $-\log(1-|zf_k(z)|^2) \leqslant -\log(1-|f_k(z)|)$. □

Below we will use the bound

$$\frac{1}{n}\sum_{k=0}^{n-1}|D_\mu(z)\varphi_k^*(z)|^2 \leqslant 1+\delta_n(\mu,z), \qquad z\in\mathbb{D}, \tag{4.13}$$

where

$$\delta_n(\mu,z) = 2\sqrt{\frac{e^{\mathcal{K}(\mu,z)}-1}{n(1-|z|^2)}}+4\frac{e^{\mathcal{K}(\mu,z)}-1}{n(1-|z|^2)}.$$

For the proof of (4.13), see Lemma 2.2 in [7].

Lemma 4.5. *We have*

$$\frac{1}{n}\sum_{k=0}^{n-1}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big| \lesssim \delta_n(\mu,z)+\eta_n(z)+\tilde{\eta}_n(z),$$

for every $z\in\mathbb{D}$.

Proof. Fix $z\in\mathbb{D}$. Let us divide the set of indices $J_n=\{k: 0\leqslant k\leqslant n-1\}$ into two subsets,

$$J_{n,s} = \left\{k\in J_n:\ \Big|\log|\varphi_k^*(z)D_\mu(z)|\Big|\leqslant 1\right\}, \tag{4.14}$$

$$J_{n,b} = \left\{k\in J_n:\ \Big|\log|\varphi_k^*(z)D_\mu(z)|\Big| > 1\right\}. \tag{4.15}$$

For $k\in J_{n,s}$, we have

$$\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big| \lesssim \Big|\log|\varphi_k^*(z)D_\mu(z)|^2\Big| \leqslant \Big|\log|O_k(z)|^2\Big|+\Big|\log|1-zb_k(z)f_k(z)|^2\Big|,$$

where in the second inequality we have used Lemma 4.3. From (4.11), (4.12) we obtain

$$\frac{1}{n}\sum_{k\in J_{n,s}}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big| \lesssim \eta_n(z)+\tilde{\eta}_n(z), \qquad z\in\mathbb{D}. \tag{4.16}$$

To handle indices $k\in J_{n,b}$, we use (4.13). Together with (4.16) it gives

$$\begin{aligned}\frac{1}{n}\sum_{k\in J_{n,b}}|\varphi_k^*(z)D_\mu(z)|^2 &= \frac{1}{n}\sum_{k=0}^{n-1}|D_\mu(z)\varphi_k^*(z)|^2-\frac{1}{n}\sum_{k\in J_{n,s}}|\varphi_k^*(z)D_\mu(z)|^2\\ &\leqslant 1+\delta_n(\mu,z)-\frac{|J_{n,s}|}{n+1}+O(\eta_n(z)+\tilde{\eta}_n(z))\\ &= \frac{|J_{n,b}|}{n+1}+\delta_n(\mu,z)+O(\eta_n(z)+\tilde{\eta}_n(z)).\end{aligned} \tag{4.17}$$

It follows that

$$\frac{1}{n}\sum_{k\in J_{n,b}}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big| \leqslant 2\frac{|J_{n,b}|}{n}+\delta_n(\mu,z)+O(\eta_n(z)+\tilde{\eta}_n(z)). \tag{4.18}$$

Recall that the sets $I_{n,s}$, $I_{n,b}$ were defined in (4.4), (4.5), respectively. We have

$$\frac{|J_{n,b}|}{n}=\frac{|J_{n,b}\cap I_{n,s}|}{n}+\frac{|J_{n,b}\cap I_{n,b}|}{n}\leqslant\frac{|J_{n,b}\cap I_{n,s}|}{n}+\frac{|I_{n,b}|}{n}.$$

Note that

$$\tilde{\eta}_n(z)=\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{1}{1-|f_k(z)|}\geqslant\frac{|I_{n,b}|}{n}\log 2,$$

and if $k\in J_{n,b}\cap I_{n,s}$, Lemma 4.3 gives $\log|O_k(z)|^{-2}\gtrsim 1$. Then (4.12) yields

$$\frac{|J_{n,b}\cap I_{n,s}|}{n}\lesssim\frac{1}{n}\sum_{k=0}^{n-1}\log|O_k(z)|^{-2}\leqslant\eta_n(z)+\tilde{\eta}_n(z).$$

We deduce that

$$\frac{|J_{n,b}|}{n}\lesssim\eta_n(z)+\tilde{\eta}_n(z). \tag{4.19}$$

Taking into account (4.18), this completes the proof. □

Lemma 4.5 together with Lemma 4.1 and Lemma 4.2 will give us the asymptotic relation

$$\sup_{z\in\Gamma_{\zeta,n}}\frac{1}{n}\sum_{k=0}^{n-1}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big|\to 0,\qquad n\to\infty. \tag{4.20}$$

In the remaining piece $\Gamma'_{\zeta,n}:=\Gamma_\zeta\setminus\Gamma_{\zeta,n}$ of the Stolz angle Γ_ζ we will need a different argument. For convenience, let us reproduce the definition of $\Gamma'_{\zeta,n}$ in the following form:

$$\Gamma'_{\zeta,n}=\Gamma_\zeta\cap\{z\in\mathbb{C}:\ |z-\zeta|<1/n\},\qquad n\geqslant 1.$$

We also define

$$R_1(n)=\sup_{z_{1,2}\in\Gamma'_{\zeta,n}}\left|\frac{D_\mu(z_1)}{D_\mu(z_2)}\right|^2, \tag{4.21}$$

$$R_2(n)=\sup_{z\in\Gamma_{\zeta,n}}(\delta_n(\mu,z)+\eta_n(z)+\tilde{\eta}_n(z)). \tag{4.22}$$

It is clear that $R_1(n)\to 1$ and $R_2(n)\to 0$ for every $\zeta\in\mathbb{T}$ such that (2.1) holds and $|D_\mu|$ has a non-zero finite non-tangential limit at ζ.

Lemma 4.6. *Let $\zeta\in\mathbb{T}$. For every $\Lambda\subset\{0\leqslant k\leqslant n-1\}$ we have*

$$\sup_{z\in\Gamma'_{\zeta,n}}\frac{\sum_{k\in\Lambda}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big|}{n}\lesssim\frac{R_1(n)|\Lambda|}{n}+R_1(n)R_2(n), \tag{4.23}$$

for all $n\geqslant 1$.

Proof. Take $z\in\Gamma'_{\zeta,n}$ and consider $\lambda(z)=rz$ where $r\in(1-1/n,1)$ is chosen so that $|\lambda(z)|=1-1/n$. Note that $\lambda(z)\in\Gamma_{\zeta,n}$. Since φ_k^* has no zeroes in $\mathbb{D}$, we have

$$|\varphi_k^*(\lambda(z))|^2\geqslant\left(\frac{1+r}{2}\right)^k|\varphi_k^*(z)|^2\gtrsim|\varphi_k^*(z)|^2,\qquad 0\leqslant k\leqslant n-1,$$

see Lemma 2.3 in [7]. Next, consider the sets $J_{n,s}$, $J_{n,b}$ defined in (4.14), (4.15) with $\lambda(z)$ in place of z. We have

$$\begin{aligned}\sum_{k\in\Lambda}|\varphi_k^*(z)D_\mu(z)|^2 &\lesssim R_1(n)\sum_{k\in\Lambda}|\varphi_k^*(\lambda(z))D_\mu(\lambda(z))|^2\\ &\lesssim R_1(n)\Big(\sum_{k\in\Lambda\cap J_{n,s}}1+\sum_{k\in\Lambda\cap J_{n,b}}|\varphi_k^*(\lambda(z))D_\mu(\lambda(z))|^2\Big)\\ &\lesssim R_1(n)|\Lambda|+R_1(n)\sum_{k\in J_{n,b}}|\varphi_k^*(\lambda(z))D_\mu(\lambda(z))|^2.\end{aligned}$$

Now the claim follows from (4.17), (4.19) applied to $\lambda(z)$ in place of z. □

In the next proof we switch from the polynomials φ_n^* to polynomials $\tilde\varphi_{\lambda,n}^*$ defined in (3.12). The fact that polynomials $\tilde\varphi_{\lambda,n}^*$ have a small distortion near ζ (see (3.26)) will make it possible to get the asymptotics of $\varphi_n^*(z)$ for $z\in\Gamma'_{\zeta,n}$ by using the asymptotics of $\varphi_n^*(\lambda_n)$ at the point $\lambda_n=(1-1/n)\zeta$ in the region $\Gamma_{\zeta,n}$. The latter asymptotics was already found in (4.20).

Given $\lambda\in\mathbb{D}$ and $\varepsilon\in(0,1)$, let us consider the sets $I_{n,s}(\varepsilon)$, $I_{n,b}(\varepsilon)$ defined by

$$I_{n,s}(\varepsilon)=\{0\leqslant k\leqslant n-1:\ \ |f_{k-1}(\lambda)|\leqslant\varepsilon\}\,,\tag{4.24}$$

$$I_{n,b}(\varepsilon)=\{0\leqslant k\leqslant n-1:\ \ |f_{k-1}(\lambda)|>\varepsilon\}\,,\tag{4.25}$$

where, as before $f_{-1}=0$. By construction, see (3.14), for every $k\in I_{n,s}(\varepsilon)$ we have

$$\left|\left|\frac{\tilde\varphi_{\lambda,k}^*(z)}{\varphi_k^*(z)}\right|^2-1\right|\lesssim\varepsilon,\qquad z\in\mathbb{D}.\tag{4.26}$$

Lemma 4.7. *Let $\mu\in\mathrm{Sz}(\mathbb{T})$, $\zeta\in\mathbb{T}$, be such that (2.1) holds and $|D_\mu|$ has a non-zero finite non-tangential limit at ζ, let n_0 be defined by (3.24), $\lambda_n=(1-1/n)\zeta$. For $\varepsilon>0$, consider the set $I_{n,s}(\varepsilon)$ from (4.24) generated by $\lambda=\lambda_n$. We have*

$$\limsup_{n\to\infty}\sup_{z\in\Gamma'_{\zeta,n}}\left(\frac{1}{n}\sum_{k\in I_{n,s}(\varepsilon)}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big|\right)\lesssim\varepsilon.$$

Proof. At first, observe that for every $k\in I_{n,s}(\varepsilon)$ and $z\in\mathbb{D}$ relation (4.26) implies

$$\Big||\varphi_k^*(z)D_\mu(z)|^2-|\tilde\varphi_{\lambda_n,k}^*(z)D_\mu(z)|^2\Big|\lesssim\varepsilon|\varphi_k^*(z)D_\mu(z)|^2.$$

Therefore, we have

$$\sum_{k\in I_{n,s}(\varepsilon)}\Big||\varphi_k^*(z)D_\mu(z)|^2-1\Big|\lesssim\sum_{k\in I_{n,s}(\varepsilon)}\Big||\tilde\varphi_{\lambda_n,k}^*(z)D_\mu(z)|^2-1\Big|+\varepsilon\sum_{k=0}^{n-1}|\varphi_k^*(z)D_\mu(z)|^2.$$

Recall that

$$\frac{1}{n}\sum_{k=0}^{n-1}|\varphi_k^*(z)D_\mu(z)|^2\leqslant 1+\delta_n(\mu,z)$$

by (4.13), and $\lim_{n\to\infty}\sup_{z\in\Gamma'_{\zeta,n}}\delta_n(\mu,z)=0$. Thus, we only need to check that

$$\limsup_{n\to\infty}\sup_{z\in\Gamma'_{\zeta,n}}\left(\frac{1}{n}\sum_{k\in I_{n,s}(\varepsilon)}\Big||\tilde\varphi_{\lambda_n,k}^*(z)D_\mu(z)|^2-1\Big|\right)=0.$$

For $z \in \Gamma'_{\zeta,n}$ and $0 \leqslant k \leqslant n$, $n \geqslant n_0$, with n_0 from (3.24), relation (3.26) gives

$$\frac{1 + O(\sqrt{\mathcal{K}(\mu, \lambda_n)})}{R_1(n)} \leqslant \left| \frac{\tilde{\varphi}^*_{\lambda_n,k}(z) D_\mu(z)}{\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n) D_\mu(\lambda_n)} \right|^2 \leqslant R_1(n) \cdot (1 + O(\sqrt{\mathcal{K}(\mu, \lambda_n)})),$$

where $R_1(n)$ is defined in (4.21), $\lim_{n\to\infty} R_1(n) = 1$. It follows that

$$\sup_{z\in\Gamma'_{\zeta,n}} \Big| |\tilde{\varphi}^*_{\lambda_n,k}(z) D_\mu(z)|^2 - |\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n) D_\mu(\lambda_n)|^2 \Big| \leqslant \varepsilon_n |\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n) D_\mu(\lambda_n)|^2,$$

for some sequence ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$. By triangle inequality, we have

$$\sup_{z\in\Gamma'_{\zeta,n}} \big| |\tilde{\varphi}^*_{\lambda_n,k}(z) D_\mu(z)|^2 - 1 \big| \lesssim \big| |\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n) D_\mu(\lambda_n)|^2 - 1 \big| + \varepsilon_n |\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n) D_\mu(\lambda_n)|^2.$$

Finally, Lemma 4.5 for the point $z = \lambda_n$ gives

$$\frac{1}{n} \sum_{k\in I_{n,s}(\varepsilon)} \big| |\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n) D_\mu(\lambda_n)|^2 - 1 \big| \lesssim \delta_n(\mu, \lambda_n) + \eta_n(\lambda_n) + \tilde{\eta}_n(\lambda_n),$$

and the claim follows from Lemma 4.1, Lemma 4.2, and the fact that $\lambda_n \in \Gamma_{\zeta,n}$. □

Proof of Theorem 2.1. The proof is a combination of previous results. Lemma 4.5 gives

$$\frac{1}{n} \sum_{k=0}^{n-1} \Big| |\varphi^*_k(z) D_\mu(z)|^2 - 1 \Big| \lesssim \delta_n(\mu, z) + \eta_n(z) + \tilde{\eta}_n(z), \qquad z \in \mathbb{D}.$$

Therefore, we have

$$\sup_{z\in\Gamma_{\zeta,n}} \frac{1}{n} \sum_{k=0}^{n-1} \Big| |\varphi^*_k(z) D_\mu(z)|^2 - 1 \Big| \to 0, \qquad n \to \infty,$$

by Lemma 4.1 and Lemma 4.2. To deal with the region $\Gamma'_{\zeta,n} = \Gamma_\zeta \setminus \Gamma_{\zeta,n}$, we take $\varepsilon > 0$, the point $\lambda_n = (1-1/n)\zeta$ and consider the sets $I_{n,s}(\varepsilon)$, $I_{n,b}(\varepsilon)$ from (4.24), (4.25) corresponding to the points λ_n, that is, $I_{n,s}(\varepsilon) = \{0 \leqslant k \leqslant n-1 \colon |f_{k-1}(\lambda_n)| < \varepsilon\}$ and $I_{n,b}(\varepsilon) = \{0 \leqslant k \leqslant n-1 \colon |f_{k-1}(\lambda_n)| \geqslant \varepsilon\}$. Lemma 4.7 gives

$$\limsup_{n\to\infty} \sup_{z\in\Gamma'_{\zeta,n}} \left(\frac{1}{n} \sum_{k\in I_{n,s}(\varepsilon)} \Big| |\varphi^*_k(z) D_\mu(z)|^2 - 1 \Big| \right) \lesssim \varepsilon.$$

Choosing $\Lambda = I_{n,b}(\varepsilon)$ in Lemma 4.6, we obtain

$$\sup_{z\in\Gamma'_{\zeta,n}} \left(\frac{1}{n} \sum_{k\in I_{n,b}(\varepsilon)} \Big| |\varphi^*_k(z) D_\mu(z)|^2 - 1 \Big| \right) \lesssim \frac{R_1(n)|I_{n,b}(\varepsilon)|}{n} + R_1(n) R_2(n).$$

Recall that $R_1(n) \to 1$, $R_2(n) \to 0$ for every $\zeta \in \mathbb{T}$ such that (2.1) holds and $|D_\mu|$ has a non-zero finite non-tangential limit at ζ. The definition (4.2) of $\tilde{\eta}_n$ implies

$$\frac{|I_{n,b}(\varepsilon)|}{n} \log \frac{1}{1-\varepsilon} \leqslant \tilde{\eta}_n(\lambda_n).$$

Since $\lambda_n \in \Gamma_{\zeta,n}$, we have $\limsup_{n\to\infty} \frac{|I_{n,b}(\varepsilon)|}{n} = 0$ by Lemma 4.2. This gives us

$$\limsup_{n\to\infty} \sup_{z\in\Gamma'_{\zeta,n}} \left(\frac{1}{n} \sum_{k\in I_{n,b}(\varepsilon)} \Big| |\varphi^*_k(z) D_\mu(z)|^2 - 1 \Big| \right) = 0.$$

Collecting the estimates, we obtain

$$\limsup_{n\to\infty} \sup_{z\in\Gamma'_{\zeta,n}} \left(\frac{1}{n} \sum_{k=0}^{n-1} \Big| |\varphi^*_k(z) D_\mu(z)|^2 - 1 \Big| \right) \lesssim \varepsilon.$$

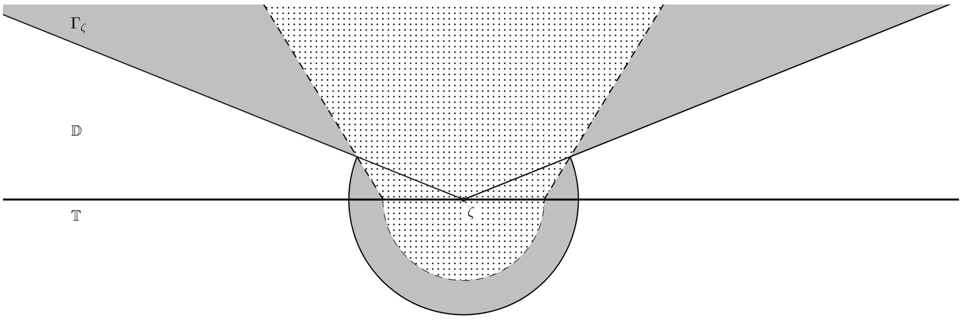


FIGURE 3. The region appearing in Theorem 4.8 (filled with gray) contain the region from Figure 1b (filled here with dots).

The theorem follows by letting $\varepsilon \to 0$. □

We have proved Theorem 2.1 for Stolz angles. The same method gives a bit more general result that we formulate below.

Theorem 4.8. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that (2.1) holds and $|D_\mu|$ has a non-zero finite non-tangential limit at ζ. Fix $A > 0$ and let $B(\zeta, A/n) = \{z \in \mathbb{C} : |z - \zeta| \leqslant A/n\}$. Then*

$$\sup_{z \in \Gamma_\zeta \cup B(\zeta, A/n)} \left(\frac{1}{n} \sum_{k=0}^{n-1} \Big| |\varphi_k^*(z) D_\mu(\tilde{z})|^2 - 1 \Big| \right) \to 0, \qquad n \to \infty, \tag{4.27}$$

where we set $\tilde{z} = \zeta$ for $z \in B(\zeta, A/n)$ and $\tilde{z} = z$ for $z \in \Gamma_\zeta \setminus B(\zeta, A/n)$.

The regions $\Gamma_\zeta \cup B(\zeta, A/n)$ in Theorem 4.8 are shown on Figure 3. Note that the width of the Stolz angle Γ_ζ and the parameter $A > 0$ are arbitrary, hence these regions contain the area shown on Figure 1b. This area is filled with dots on Figure 3.

5. Proof of Theorem 2.2

The proof of Theorem 2.2 uses the same tools as the proof of Theorem 2.1: Khrushchev-type formula (4.9) and the entropy function $\mathcal{K}(\mu, z)$. The latter controls Schur functions of μ that appear in (4.9), see Lemma 4.1 and Lemma 4.2. Recall that we already know the averaged asymptotic behaviour of $|\varphi_n^* D_\mu|$ under assumption (2.1), that is, when $\lim_{r \to 1} \mathcal{K}(\mu, r\zeta) = 0$, see Theorem 2.1. The stronger assumption (2.2), that we reproduce here,

$$\sum_{n \geqslant 0} \mathcal{K}(\mu, z_n(\zeta)) < \infty, \qquad z_n(\zeta) = (1 - 2^{-n})\zeta, \tag{2.2}$$

will allow us to pass from $|\varphi_n^* D_\mu|$ to $\varphi_n^* D_\mu$. For this we utilize the fact that $\varphi_n^* D_\mu$ is an outer function in $\mathbb{D}$ with a positive value at 0, hence $\log \varphi_n^* D_\mu$ is a properly defined analytic function in $\mathbb{D}$ which is completely determined by its real part $\log |\varphi_n^* D_\mu|$ via the operator of harmonic conjugation,

$$\mathcal{Q} : u \mapsto \int_{\mathbb{T}} u(\xi) \, \mathrm{Im} \left(\frac{1 + \bar{\xi} z}{1 - \bar{\xi} z} \right) dm(\xi), \quad z \in \mathbb{D}.$$

Assumption (2.2) will allow us to prove that $\mathrm{Im} \log \varphi_n^* D_\mu = \mathcal{Q}(\log |\varphi_n^* D_\mu|)$ is small in Γ_ζ, hence $\log \varphi_n^* D_\mu$ is close to $\log |\varphi_n^* D_\mu|$, which, in turn, is close to 0. This will lead to the fact that $\varphi_n^* D_\mu$ converges to 1 in the strong Cesàro sense uniformly in Γ_ζ, i.e., to the conclusion of Theorem 2.2.

Let us recall the definition of truncated cones $\Gamma_{\zeta,n} = \{z \in \Gamma_\zeta : |z-\zeta| \geqslant 1/n\}$. We start with a simple lemma.

Lemma 5.1. *Let $u \in L^1(\mathbb{T})$. Then*

$$\lim_{n\to\infty} \frac{\sup_{z\in\Gamma_{\zeta,n}} |(\mathcal{Q}u)(z)|}{n} = 0, \quad \zeta \in \mathbb{T}.$$

Proof. For every $u \in L^1(\mathbb{T})$ and $n \geqslant 1$, we have

$$\frac{\sup_{z\in\Gamma_{\zeta,n}} |(\mathcal{Q}u)(z)|}{n} \leqslant \sup_{z\in\Gamma_{\zeta,n}} \int_{\mathbb{T}} \frac{2|u(\xi)|}{n|1-\bar{\xi}z|}\, dm(\xi) \leqslant 2\|u\|_{L^1(\mathbb{T})}. \tag{5.1}$$

Let $u_\varepsilon \in L^1(\mathbb{T})$ be a function with $\mathcal{Q}u_\varepsilon \in L^\infty(\mathbb{D})$ and such that $\|u-u_\varepsilon\|_{L^1(\mathbb{T})} \leqslant \varepsilon$. Applying (5.1) to $u - u_\varepsilon$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{\sup_{z\in\Gamma_{\zeta,n}} |(\mathcal{Q}u)(z)|}{n} \leqslant &\limsup_{n\to\infty} \frac{\sup_{z\in\Gamma_{\zeta,n}} |(\mathcal{Q}u - \mathcal{Q}u_\varepsilon)(z)|}{n} + \\ &+ \limsup_{n\to\infty} \frac{\sup_{z\in\Gamma_{\zeta,n}} |(\mathcal{Q}u_\varepsilon)(z)|}{n} \leqslant 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the lemma follows. □

Lemma 5.2. *Suppose that $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ are such that (2.2) holds. Then*

$$\lim_{n\to\infty} \sup_{z\in\Gamma_{\zeta,n}} \frac{1}{n}\sum_{k=0}^{n-1} |\operatorname{Im}\log O_k(z)| = 0.$$

Proof. We will use outer functions O_n from Lemma 4.3. Set $\alpha_k(z) = \operatorname{Im}\log O_k(z)$ for $z \in \mathbb{D}$, $k \geqslant 0$. Given a positive integer n, let K_n be the integer part of $\log_2 n$. We first show that there exists a constant $c(\mu,\zeta) > 0$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1} |\alpha_k(z_{j+1}) - \alpha_k(z_j)| \leqslant c(\mu,\zeta)\mathcal{K}(\mu, z_j), \qquad 1 \leqslant j \leqslant K_n, \tag{5.2}$$

where we set $z_j = z_j(\zeta) = (1-2^{-j})\zeta$ for brevity. Fix $1 \leqslant j \leqslant K_n$. We have

$$\begin{aligned}|\alpha_k(z_{j+1}) - \alpha_k(z_j)| &\leqslant \frac{1}{2}\int_{\mathbb{T}} |\log(1-|f_k|^2)| \left|\frac{1+\bar{\xi}z_{j+1}}{1-\bar{\xi}z_{j+1}} - \frac{1+\bar{\xi}z_j}{1-\bar{\xi}z_j}\right| dm \\ &\lesssim \int_{\mathbb{T}} |\log(1-|f_k|^2)| \frac{1-|z_j|^2}{|1-\bar{\xi}z_j|^2} dm \\ &= \mathcal{K}(\mu_k, z_j) - \log(1-|z_j f_k(z_j)|^2).\end{aligned}$$

Last identity is formula (3.10). Since by (3.8) we have $\mathcal{K}(\mu_k, z) \leqslant \mathcal{K}(\mu, z)$ for any $z \in \mathbb{D}$, summing up over k for fixed j and using $|z_j f_k(z_j)| \leqslant |f_k(z_j)|$, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} |\alpha_k(z_{j+1}) - \alpha_k(z_j)| \lesssim \mathcal{K}(\mu, z_j) - \frac{1}{n}\sum_{k=0}^{n-1} \log(1-|f_k(z_j)|^2).$$

As in (4.4) and (4.5), consider the sets $I_{n,s} = \{0 \leqslant k \leqslant n-1 \colon |f_k(z_j)| \leqslant 1/2\}$ and $I_{n,b} = \{0 \leqslant k \leqslant n-1 \colon |f_k(z_j)| > 1/2\}$. Since $j \leqslant K_n$, we have $1/n \lesssim 1-|z_j|^2$ and then

$$-\frac{1}{n}\sum_{k\in I_{n,s}}\log(1-|f_k(z_j)|^2)\asymp\frac{1}{n}\sum_{k\in I_{n,s}}|f_k(z_j)|^2$$
$$\lesssim\sum_{k=0}^{n-1}(1-|z_j|^2)|f_k(z_j)|^2\lesssim\mathcal{K}(\mu,z_j),$$

where in the last estimate we have used (3.7). Next we consider the contribution coming from indices in $I_{n,b}$. Since $c_1(\mu,\zeta)=\sup_{j\geqslant 0}\mathcal{K}(\mu,z_j)$ is finite, formula (3.7) gives

$$\sup_{k,j}\frac{(1-|z_j|^2)|f_k(z_j)|^2}{1-|f_k(z_j)|^2}\leqslant e^{c_1(\mu,\zeta)}.$$

It follows that for $k\in I_{n,b}$ we have

$$-(1-|z_j|^2)\log(1-|f_k(z_j)|^2)\leqslant\frac{1-|z_j|^2}{1-|f_k(z_j)|^2}\leqslant\frac{1}{4}\frac{(1-|z_j|^2)|f_k(z_j)|^2}{1-|f_k(z_j)|^2}\leqslant$$
$$\leqslant c_2(\mu,\zeta)\log\left(1+\frac{(1-|z_j|^2)|f_k(z_j)|^2}{1-|f_k(z_j)|^2}\right),$$

with the constant

$$c_2(\mu,\zeta)=\sup_{0\leqslant y\leqslant e^{c_1(\mu,\zeta)}}\frac{y/4}{\log(1+y)}.$$

Since $1/n\lesssim 1-|z_j|^2$, we deduce from (3.7) the bound

$$-\frac{1}{n+1}\sum_{k\in I_{n,b}}\log(1-|f_k(z_j)|^2)\leqslant c_2(\mu,\zeta)\mathcal{K}(\mu,z_j).$$

This finishes the proof of (5.2). Now using (5.2) and the fact that $\alpha_k(z_0)=\operatorname{Im}\log O_k(0)=0$ for all $k\geqslant 0$, a telescopic sum argument shows that

$$\frac{1}{n}\sum_{k=0}^{n-1}|\alpha_k(z_{j+1})|\leqslant c(\mu,\zeta)\sum_{s=0}^{j}\mathcal{K}(\mu,z_s),\qquad 1\leqslant j\leqslant K_n. \tag{5.3}$$

Next, given $z\in\Gamma_{\zeta,n}$, let $j(z)\geqslant 1$ be the integer such that $|z_{j(z)}|<|z|\leqslant|z_{j(z)+1}|$. Repeating the proof of (5.2), we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1}|\alpha_k(z)-\alpha_k(z_{j(z)})|\leqslant c(\mu,\zeta)\mathcal{K}(\mu,z_{j(z)}),$$

with a possibly larger constant $c(\mu,\zeta)$ that still does not depend on n and $z\in\mathbb{D}$. Then (5.3) gives

$$\frac{1}{n}\sum_{k=0}^{n-1}|\alpha_k(z)|\leqslant c(\mu,\zeta)\sum_{s=0}^{j(z)}\mathcal{K}(\mu,z_s).$$

Recall that measures μ_n, $n\geqslant 0$, were defined using the Schur's algorithm (3.5). Since $\mathcal{K}(\mu_n,z)\leqslant\mathcal{K}(\mu,z)$ for every $n\geqslant 0$, $z\in\mathbb{D}$ by (3.8), our construction gives $c(\mu_n,\zeta)\leqslant c(\mu,\zeta)$ for every $n\geqslant 0$. Now take a large number $\ell\geqslant 0$ and apply the first part of the proof to μ_ℓ in place of μ. We obtain

$$\frac{1}{n}\sum_{k=0}^{n-1}|\alpha_{\ell+k}(z)|\leqslant c(\mu_\ell,\zeta)\sum_{s=0}^{j(z)}\mathcal{K}(\mu_\ell,z_s)\leqslant c(\mu,\zeta)\sum_{s=0}^{\infty}\mathcal{K}(\mu_\ell,z_s). \tag{5.4}$$

Note that the last expression tends to 0 as $\ell\to\infty$. Indeed, assumption (2.2) implies that $\sum_{s=s_0}^{\infty}\mathcal{K}(\mu_\ell,z_s)\leqslant\sum_{s=s_0}^{\infty}\mathcal{K}(\mu,z_s)\to 0$ as $s_0\to\infty$, and we have shown in the proof Lemma 4.1 that $\mathcal{K}(\mu_\ell,z_s)\to 0$ for each fixed point z_s as $\ell\to\infty$ (note that

Lemma 4.1 holds under the assumption (2.1) which is weaker than (2.2)). Since for each fixed $\ell \geqslant 0$ we have

$$\lim_{n\to\infty} \sup_{z\in\Gamma_{\zeta,n}} \frac{1}{n}\sum_{s=0}^{\ell}|\alpha_s(z)| \leqslant \sum_{s=0}^{\ell}\lim_{n\to\infty}\frac{\sup_{z\in\Gamma_{\zeta,n}}|\alpha_s(z)|}{n} = 0,$$

by Lemma 5.1, relation (5.4) implies the claim. □

We are now ready to prove the uniform convergence on the truncated cones $\Gamma_{\zeta,n}$.

Lemma 5.3. *Suppose that $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ are such that (2.2) holds. Then*

$$\lim_{n\to\infty}\sup_{z\in\Gamma_{\zeta,n}}\frac{1}{n}\sum_{k=0}^{n-1}|\varphi_k^*(z)D_\mu(z)-1| = 0.$$

Proof. For $z \in \Gamma_{\zeta,n}$, consider the sets of indices

$$J_{n,s} = \{0 \leqslant k \leqslant n-1 \colon |\varphi_k^*(z)D_\mu(z)| \leqslant 2\},$$
$$J_{n,b} = \{0 \leqslant k \leqslant n-1 \colon |\varphi_k^*(z)D_\mu(z)| > 2\}.$$

By Lemma 4.5, we have

$$\begin{aligned}\frac{1}{n}\sum_{k\in J_{n,b}}\Bigl|\varphi_k^*(z)D_\mu(z)-1\Bigr| &\lesssim \frac{1}{n}\sum_{k=0}^{n-1}\Bigl||\varphi_k^*(z)D_\mu(z)|^2-1\Bigr| \\ &\lesssim \delta_n(\mu,z)+\eta_n(z)+\tilde\eta_n(z),\end{aligned}$$

which by Lemma 4.1 and Lemma 4.2 tends to 0 uniformly on $\Gamma_{\zeta,n}$ as $n\to\infty$. Next, Lemma 4.4 says that

$$\frac{1}{n}\sum_{k=0}^{n-1}\Bigl|\log(1-zb_k(z)f_k(z))\Bigr| \leqslant \tilde\eta_n(z), \quad z\in\mathbb{D},$$
$$\frac{1}{n}\sum_{k=0}^{n-1}\log|O_k(z)|^{-2} \leqslant \eta_n(z)+\tilde\eta_n(z), \quad z\in\mathbb{D}.$$

We also have $\varphi_k^* D_\mu = O_k(1-zb_kf_k)^{-1}$ by Lemma 4.3. Using the elementary estimate

$$|re^{i\theta}-1| \leqslant |r-1|+|e^{i\theta}-1| \lesssim |\log r|+|\theta|,$$

for $0<r\leqslant 2$, $|\theta|\leqslant\pi$, we get for $k\in J_{n,s}$

$$\begin{aligned}|\varphi_k^*D_\mu-1| &\lesssim \bigl|\log|O_k(1-zb_kf_k)^{-1}|\bigr| + \bigl|\operatorname{Im}\log O_k(1-zb_kf_k)^{-1}\bigr|,\\ &\leqslant \bigl|\log|O_k|\bigr| + 2\bigl|\log(1-zb_kf_k)^{-1}\bigr| + |\operatorname{Im}\log O_k|,\\ &\leqslant \log|O_k|^{-2} + 2\bigl|\log(1-zb_kf_k)\bigr| + |\operatorname{Im}\log O_k|.\end{aligned}$$

Then

$$\frac{1}{n}\sum_{k\in J_{n,s}}|\varphi_k^*(z)D_\mu(z)-1| \lesssim \eta_n(z)+\tilde\eta_n(z)+\frac{1}{n}\sum_{k\in J_{n,s}}|\operatorname{Im}\log O_k(z)|,$$

which by Lemma 4.1, Lemma 4.2 and Lemma 5.2 tends to 0 uniformly on $\Gamma_{\zeta,n}$. This finishes the proof. □

Next we focus on the uniform convergence on $\Gamma'_{\zeta,n} = \Gamma_\zeta \setminus \Gamma_{\zeta,n}$.

Lemma 5.4. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta\in\mathbb{T}$ be such that D_μ has a non-zero finite non-tangential limit at ζ. Assume that condition (2.2) holds at ζ. Then*

$$\lim_{n\to\infty}\sup_{z\in\Gamma'_{\zeta,n}}\frac{1}{n}\sum_{k=0}^{n-1}|\varphi_k^*(z)D_\mu(z)-1| = 0.$$

Proof. For $z \in \Gamma'_{\zeta,n}$, let $J_{n,b}(z) = \{0 \leqslant k \leqslant n-1 \colon |\varphi_k^*(z)D_\mu(z)| > 2\}$. Observe that

$$\frac{1}{n}\sum_{k\in J_{n,b}(z)} |\varphi_k^*(z)D_\mu(z) - 1| \lesssim \frac{1}{n}\sum_{k\in J_{n,b}(z)} \big||\varphi_k^*(z)D_\mu(z)|^2 - 1\big|,$$

which by Theorem 2.1 tends to 0 uniformly on Γ_ζ. Set $\lambda_n = (1-n^{-1})\zeta$ for $n \geqslant 1$. Fix $\varepsilon > 0$. Recall that we put $f_{-1}(z) = 0$ for all $z \in \mathbb{D}$. Consider the sets of integers

$$I_{n,s}(\varepsilon) = \{0 \leqslant k \leqslant n-1 \colon |f_{k-1}(\lambda_n)| \leqslant \varepsilon\},$$
$$I_{n,b}(\varepsilon) = \{0 \leqslant k \leqslant n-1 \colon |f_{k-1}(\lambda_n)| > \varepsilon\}.$$

We have

$$\frac{1}{n}\sum_{k\in I_{n,b}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(z)D_\mu(z) - 1| \leqslant \frac{1}{n}\sum_{k\in I_{n,b}(\varepsilon)\setminus J_{n,b}(z)} 3 \leqslant 3\frac{|I_{n,b}|}{n}.$$

The definition (4.2) of $\tilde{\eta}_n$ gives

$$\frac{|I_{n,b}(\varepsilon)|}{n} \lesssim \frac{\tilde{\eta}_n(\lambda_n)}{|\log(1-\varepsilon)|}, \tag{5.5}$$

which by Lemma 4.2, tends to 0 as $n \to \infty$. We see that

$$\lim_{n\to\infty} \sup_{z\in\Gamma'_{\zeta,n}} \frac{1}{n}\sum_{k\in I_{n,b}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(z)D_\mu(z) - 1| = 0 \tag{5.6}$$

for every $\varepsilon > 0$. On the other hand, formula (3.13) gives

$$\left|\frac{\tilde{\varphi}^*_{\lambda_n,k}(z)}{\varphi_k^*(z)} - 1\right| \lesssim \varepsilon, \qquad k \in I_{n,s}(\varepsilon). \tag{5.7}$$

By Lemma 3.4, see (3.25), there exists $n_0(\varepsilon) \geqslant 1$ such that

$$\sup_{z\in\Gamma'_{\zeta,n}} \left|\frac{\tilde{\varphi}^*_{\lambda_n,k}(z)}{\tilde{\varphi}^*_{\lambda_n,k}(\lambda_n)} - 1\right| \lesssim \varepsilon, \quad 0 \leqslant k \leqslant n, \tag{5.8}$$

for every $n \geqslant n_0(\varepsilon)$. Since D_μ has a non-zero finite non-tangential limit at ζ, we also may assume that

$$\sup_{z\in\Gamma'_{\zeta,n}} \left|\frac{D_\mu(z)}{D_\mu(\lambda_n)} - 1\right| \leqslant \varepsilon, \quad n \geqslant n_0(\varepsilon). \tag{5.9}$$

For $\varepsilon > 0$ small enough, a combination of (5.7), (5.8) and (5.9) gives

$$\sup_{n\geqslant n_0(\varepsilon)} \sup_{z\in\Gamma'_{\zeta,n}} \sup_{k\in I_{n,s}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(z)D_\mu(z) - \varphi_k^*(\lambda_n)D_\mu(\lambda_n)| \lesssim \varepsilon.$$

Then,

$$\frac{1}{n}\sum_{k\in I_{n,s}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(z)D_\mu(z) - 1| \lesssim \frac{1}{n}\sum_{k\in I_{n,s}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(\lambda_n)D_\mu(\lambda_n) - 1| + \varepsilon,$$

for $n \geqslant n_0(\varepsilon)$, $z \in \Gamma'_{\zeta,n}$. Since $\lambda_n \in \Gamma_{\zeta,n}$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k\in I_{n,s}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(\lambda_n)D_\mu(\lambda_n) - 1| = 0,$$

for every $\varepsilon > 0$ by Lemma 5.3. Thus, for any small enough $\varepsilon > 0$ we have

$$\limsup_{n\to\infty} \sup_{z\in\Gamma'_{\zeta,n}} \frac{1}{n}\sum_{k\in I_{n,s}(\varepsilon)\setminus J_{n,b}(z)} |\varphi_k^*(z)D_\mu(z) - 1| \lesssim \varepsilon. \tag{5.10}$$

Summing up (5.6), (5.10), and sending $\varepsilon \to 0$, we conclude the proof. □

Proof of Theorem 2.2. The result follows from Lemma 5.3 and Lemma 5.4. □

6. Proof of Theorem 2.3

Lemma 6.1. *Let* $\mu \in \mathrm{Sz}(\mathbb{T})$, $\zeta \in \mathbb{T}$ *be such that* (2.3) *holds. Consider the Schur functions* f_n *of* μ, *see* (3.5). *We have* $\lim_{n\to\infty} \sup_{z\in\Gamma_\zeta} |f_n(z)| = 0$.

Proof. Recall the notation $z_k = z_k(\zeta) = (1-2^{-k})\zeta$, $k \geqslant 0$. We have

$$|f_n(z_{k+1}) - f_n(z_k)| \leqslant \mathcal{P}(|f_n - f_n(z_k)|, z_{k+1}) \lesssim \mathcal{P}(|f_n - f_n(z_k)|, z_k) \lesssim \sqrt{\mathcal{K}(\mu_n, z_k)},$$

see Theorem 2 of [6] for the last estimate. Thus, for every $j \geqslant 1$ we have

$$|f_n(z_j) - f_n(z_0)| \leqslant \sum_{k=0}^{j-1} |f_n(z_{k+1}) - f_n(z_k)| \lesssim \sum_{k=0}^{j-1} \sqrt{\mathcal{K}(\mu_n, z_k)}.$$

Given $z \in \Gamma_\zeta$ pick the positive integer $j(z)$ such that $2^{-j(z)-1} \leqslant 1-|z| < 2^{-j(z)}$. A variant of the previous argument shows $|f_n(z) - f_n(z_{j(z)})| \lesssim \sqrt{\mathcal{K}(\mu_n, z_{j(z)})}$. Hence

$$|f_n(z)| \lesssim \sum_{k=0}^{\infty} \sqrt{\mathcal{K}(\mu_n, z_k)} + |f_n(z_0)|. \tag{6.1}$$

By Szegő theorem, see (3.6), we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} \log(1-|f_n|^2)\,dm = 0. \tag{6.2}$$

It follows that $f_n \to 0$ in Lebesgue measure on $\mathbb{T}$ as $n \to \infty$. In particular, we have $\lim_{n\to\infty} f_n(z) = 0$ for every $z \in \mathbb{D}$. Moreover, (6.2) yields $\lim_{n\to\infty} \mathcal{K}(\mu_n, z) = 0$ for each $z \in \mathbb{D}$, see (3.10) Since $\mathcal{K}(\mu_n, z) \leqslant \mathcal{K}(\mu, z)$ at every $z \in \mathbb{D}$ by (3.8), we now derive from (2.3) that

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=0}^{\infty} \sqrt{\mathcal{K}(\mu_n, z_k)} &= \lim_{\ell\to\infty} \limsup_{n\to\infty} \sum_{k=\ell}^{\infty} \sqrt{\mathcal{K}(\mu_n, z_k)} \\ &\leqslant \lim_{\ell\to\infty} \sum_{k=\ell}^{\infty} \sqrt{\mathcal{K}(\mu, z_k)} = 0, \end{aligned} \tag{6.3}$$

which completes the proof. □

Lemma 6.2. *Let* $\mu \in \mathrm{Sz}(\mathbb{T})$, $\zeta \in \mathbb{T}$ *be such that* (2.3) *holds. Consider the outer functions* O_n, $n \geqslant 0$, *from Lemma 4.3. We have*

$$\lim_{n\to\infty} \sup_{z\in\Gamma_\zeta} |\operatorname{Re} \log O_n(z)| = 0.$$

Proof. Formula (3.9) says that

$$-\log|O_n(z)|^2 = \mathcal{P}(\log(1-|f_n|^2), z) = \mathcal{K}(\mu_n, z) - \log(1-|zf_n(z)|^2), \tag{6.4}$$

for any $z \in \mathbb{D}$ and any $n \geqslant 0$. As we have seen in the proof of Lemma 6.1, $\mathcal{K}(\mu_n, z)$ tends to 0 uniformly on compacts in $\mathbb{D}$. Since $\mathcal{K}(\mu_n, z) \leqslant \mathcal{K}(\mu, z)$ which tends to 0 as z non-tangentially tends to ζ, we obtain

$$\lim_{n\to\infty} \sup_{z\in\Gamma_\zeta} \mathcal{K}(\mu_n, z) = 0. \tag{6.5}$$

By Lemma 6.1, we have $\log(1-|zf_n(z)|^2) \to 0$ uniformly on Γ_ζ. Now formula (6.4) and the fact that $\operatorname{Re}\log O_n = \log|O_n|$ finish the proof. □

Lemma 6.3. *Let* $\mu \in \mathrm{Sz}(\mathbb{T})$, $\zeta \in \mathbb{T}$ *be such that* (2.3) *holds. Consider the outer functions* O_n, $n \geqslant 0$, *from Lemma 4.3. We have*

$$\lim_{n\to\infty} \sup_{z\in\Gamma_\zeta} |\operatorname{Im} \log O_n(z)| = 0. \tag{6.6}$$

Proof. We use again notation $z_k = (1-2^{-k})\zeta$, $k \geqslant 0$. Let us first check that there exists a constant $c(\mu,\zeta)$ depending only on μ, ζ such that

$$|\operatorname{Im}\log O_n(z_{k+1}) - \operatorname{Im}\log O_n(z_k)| \leqslant c(\mu,\zeta)\sqrt{\mathcal{K}(\mu_n, z_k)}, \tag{6.7}$$

for every $k \geqslant 0$, $n \geqslant 0$. Take a constant $\lambda \in \mathbb{R}$. Since the kernel $\operatorname{Im}(1+\bar{\xi}z)(1-\bar{\xi}z)^{-1}$ is odd, we have

$$\operatorname{Im}\log O_n(z) = \operatorname{Im}\frac{1}{2}\int_{\mathbb{T}}(\log(1-|f_n|^2)-\lambda)\frac{1+\bar{\xi}z}{1-\bar{\xi}z}\,dm(\xi), \qquad z \in \mathbb{D}.$$

Then

$$\begin{aligned}
&|\operatorname{Im}\log O_n(z_{k+1}) - \operatorname{Im}\log O_n(z_k)| \leqslant \\
&\leqslant \frac{1}{2}\int_{\mathbb{T}} |\log(1-|f_n(\xi)|^2)-\lambda| \left|\frac{1+\bar{\xi}z_{k+1}}{1-\bar{\xi}z_{k+1}} - \frac{1+\bar{\xi}z_k}{1-\bar{\xi}z_k}\right| dm(\xi) \\
&\lesssim \int_{\mathbb{T}} |\log(1-|f_n(\xi)|^2)-\lambda| \frac{1-|z_k|^2}{|1-\bar{\xi}z_k|^2}\,dm(\xi).
\end{aligned}$$

Now, for the special choice $\lambda = \mathcal{P}(\log(1-|f_n|^2), z_k)$, we have

$$\int_{\mathbb{T}} |\log(1-|f_n|^2)-\lambda| \frac{1-|z_k|^2}{|1-\bar{\xi}z_k|^2}\,dm \lesssim \max(\mathcal{K}(\mu_n,z_k), \sqrt{\mathcal{K}(\mu_n,z_k)}), \tag{6.8}$$

by Lemma 3 in [6] (see also page 12 in [6]). Since

$$\sup_{n,\,k\geqslant 0} \mathcal{K}(\mu_n, z_k) \leqslant \sup_{z\in\Gamma_\zeta} \mathcal{K}(\mu, z) < \infty,$$

there is a constant $c(\mu,\zeta)$ such that the right hand side in (6.8) does not exceed $c(\mu,\zeta)\sqrt{\mathcal{K}(\mu_n,z_k)}$ for every $k \geqslant 0$, $n \geqslant 0$. So, (6.7) is proved. Next, given $z \in \Gamma_\zeta$, pick the positive integer $j(z)$ such that $2^{-j(z)-1} \leqslant 1-|z| < 2^{-j(z)}$. Similarly to (6.7), we have

$$|\operatorname{Im}\log O_n(z) - \operatorname{Im}\log O_n(z_{j(z)})| \leqslant c(\mu,\zeta)\sqrt{\mathcal{K}(\mu_n, z_{j(z)})}. \tag{6.9}$$

Using (6.7), (6.9), and $\operatorname{Im}\log O_n(z_0) = 0$, a telescopic sum argument gives

$$|\operatorname{Im}\log O_n(z)| \leqslant c(\mu,\zeta)\sum_{k=0}^{j(\zeta)}\sqrt{\mathcal{K}(\mu_n,z_k)} \leqslant c(\mu,\zeta)\sum_{k=0}^{\infty}\sqrt{\mathcal{K}(\mu_n,z_k)}. \tag{6.10}$$

Recall that $\lim_{n\to\infty}\sum_{k=0}^{\infty}\sqrt{\mathcal{K}(\mu_n,z_k)} = 0$ under assumption (2.3), see (6.3). We now see that (6.6) follows from (6.10). □

Proof of Theorem 2.3. The claim of the theorem follows immediately from Lemma 6.1, Lemma 6.2, and Lemma 6.3 using formula $\varphi_n^* D_\mu = O_n/(1-zb_nf_n)$ from Lemma 4.3. □

7. Proof of Theorem 2.7

Recall that we study the asymptotic behaviour of $k_{\mu,n}(z_1,z_2)$, $z_{1,2} \in \Gamma_\zeta$, by estimating the function r_n defined by

$$k_{\mu,n}(z_1,z_2) = \overline{D_\mu^{-1}(z_2)}D_\mu^{-1}(z_1)\frac{1-\bar{z}_2^n z_1^n}{1-\bar{z}_2 z_1}\big(1+r_n(z_1,z_2)\big). \tag{7.1}$$

Given $\delta > 0$, $\Delta > 0$, define

$$\Gamma_{\zeta,n}(\delta,\Delta) = \{z \in \Gamma_\zeta:\ \Delta \leqslant |z| \leqslant 1-\delta/n\}, \tag{7.2}$$

$$\Gamma'_{\zeta,n}(\delta) = \{z \in \Gamma_\zeta:\ |z| > 1-\delta/n\}, \tag{7.3}$$

$$\Gamma''_\zeta(\Delta) = \{z \in \Gamma_\zeta:\ |z| < \Delta\}. \tag{7.4}$$

We first consider the "diagonal" case of Theorem 2.7.

Lemma 7.1. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that $|D_\mu|$ has a non-zero finite non-tangential limit at ζ. If (2.1) holds at ζ, we have*

$$k_{\mu,n}(z,z) = |D_\mu^{-1}(z)|^2 \frac{1-|z|^{2n}}{1-|z|^2}\big(1+R_n(z)\big), \tag{7.5}$$

$$\sup_{z\in\Gamma_\zeta} \big|R_n(z)\big| \to 0, \quad n\to\infty. \tag{7.6}$$

Proof. It follows from general Szegő theory (see (1.3) and (2.12)) that for every μ in $\mathrm{Sz}(\mathbb{T})$ we have

$$k_{\mu,n}(z_1,z_2) \to \frac{D_\mu(z_1)^{-1}\overline{D_\mu(z_2)^{-1}}}{1-\bar z_1 z_2}, \quad n\to\infty,$$

on compact subsets of $\mathbb{D}\times\mathbb{D}$. Hence, there exists an increasing sequence $\Delta_n \to 1$ such that

$$\sup_{z\in\Gamma''_\zeta(\Delta_n)} \big|R_n(z)\big| \to 0, \quad n\to\infty,$$

for the functions R_n defined by (7.5). We can also choose a slowly increasing sequence $\delta_n \to \infty$, $\delta_n = o(n)$, such that

$$\sup_{z\in\Gamma'_{\zeta,n}(\delta_n)} \big|R_n(z)\big| \to 0, \quad n\to\infty.$$

Indeed, the existence of such a sequence is a consequence of Theorem 1.1 in [5] and Máté, Nevai, and Totik asymptotic relation (2.8). Note that (2.8) holds under the assumptions of Theorem 2.7 by Theorem 2.1. Thus, to prove (7.6) we need to check that $\sup_{z\in\Gamma_{\zeta,n}(\delta_n,\Delta_n)} |R_n(z)| \to 0$ as $n\to\infty$. Moreover, since $\sup_{z\in\Gamma_{\zeta,n}(\delta_n,\Delta_n)} |z|^{2n} \to 0$ as $n\to\infty$, we actually need to prove that

$$\sup_{z\in\Gamma_{\zeta,n}(\delta_n,\Delta_n)} \big|R_n^*(z)\big| \to 0, \quad n\to\infty, \tag{7.7}$$

for functions R_n^* defined by

$$k_{\mu,n}(z,z) = \frac{|D_\mu^{-1}(z)|^2}{1-|z|^2}\big(1+R_n^*(z)\big).$$

A computation of Fourier coefficients similar to (1.8) leads to the well-known formulas

$$k_{\mu,n}(z_1,z_2) = \sum_{k=0}^{n-1} \varphi_k(z_1)\overline{\varphi_k(z_2)}, \tag{7.8}$$

$$\frac{D_\mu^{-1}(z_1)\overline{D_\mu^{-1}(z_2)}}{1-\bar z_1 z_2} = \sum_{k=0}^{\infty} \varphi_k(z_1)\overline{\varphi_k(z_2)}, \tag{7.9}$$

for $z_1, z_2 \in \mathbb{D}$. In particular,

$$k_{\mu,n}(z,z) \leqslant \frac{|D_\mu^{-1}(z)|^2}{1-|z|^2}, \qquad z\in\mathbb{D}, \tag{7.10}$$

therefore, we just need to estimate $k_{\mu,n}(z,z)$ from below. Let $\mathcal{P}_n$ denote the n-dimensional space of polynomials of degree at most $n-1$. Since $k_{\mu,n}$ is the reproducing kernel, we have

$$\begin{aligned}
k_{\mu,n}(z,z) &= \|k_{\mu,n}(\cdot,z)\|^2_{L^2(\mu)} \\
&= \sup\{|(p,k_{\mu,n}(\cdot,z))_{L^2(\mu)}|^2,\ \ p\in\mathcal{P}_n,\ \|p\|_{L^2(\mu)}\leqslant 1\} \\
&= \sup\{|p(z)|^2,\ \ p\in\mathcal{P}_n,\ \|p\|_{L^2(\mu)}\leqslant 1\} \\
&= \sup\left\{\frac{|p(z)|^2}{\|p\|^2_{L^2(\mu)}},\ \ p\in\mathcal{P}_n\right\}.
\end{aligned}$$

Substituting $p(\xi)=\frac{1-\bar z^n\xi^n}{1-\bar z\xi}$ into the last supremum, we get

$$k_{\mu,n}(z,z)\geqslant\left(\frac{1-|z|^{2n}}{1-|z|^2}\right)^2\cdot\left(\int_{\mathbb{T}}\left|\frac{1-\bar z^n\xi^n}{1-\bar z\xi}\right|^2 d\mu(\xi)\right)^{-1}.$$

Using $|z|^n\to 0$ and $|z|\to 1$ for $z\in\Gamma_{\zeta,n}(\delta_n,\Delta_n)$ when $n\to\infty$, we obtain

$$\begin{aligned}
k_{\mu,n}(z,z) &\geqslant\left(\frac{1}{1-|z|^2}\right)^2\cdot\left(\int_{\mathbb{T}}\frac{1}{|1-\bar z\xi|^2}\,d\mu(\xi)\right)^{-1}\cdot(1+o(1)) \\
&= \frac{1}{1-|z|^2}\cdot\left(\int_{\mathbb{T}}\frac{1-|z|^2}{|1-\bar z\xi|^2}\,d\mu(\xi)\right)^{-1}\cdot(1+o(1)) \\
&= \frac{1}{1-|z|^2}\cdot\mathcal{P}(\mu,z)^{-1}\cdot(1+o(1)) \\
&= \frac{1}{1-|z|^2}\cdot|D_\mu(z)|^{-2}\cdot(1+o(1)). \qquad (7.11)
\end{aligned}$$

In the last line we have used the fact that $\mathcal{P}(\mu,z)|D_\mu(z)|^{-2}=e^{\mathcal{K}(\mu,z)}$, $z\in\mathbb{D}$, because D_μ is an outer function. In particular, we have $\mathcal{P}(\mu,z)=|D_\mu(z)|^2(1+o(1))$ in $\Gamma_{\zeta,n}(\delta_n,\Delta_n)$ provided (2.1) holds. Combining (7.10), (7.11), we obtain (7.7) and complete the proof. □

Lemma 7.2. *Let $\mu\in\mathrm{Sz}(\mathbb{T})$ and $\zeta\in\mathbb{T}$ be such that D_μ has a non-zero finite non-tangential limit at ζ. If (2.1) holds at ζ, then for every $A\geqslant 0$ we have*

$$\lim_{n\to\infty}\ \sup_{\substack{z_1,z_2\in\Gamma_\zeta,\\ d_H(z_1,z_2)\leqslant A}}\big|r_n(z_1,z_2)\big|=0$$

for the functions r_n defined in (7.1).

Proof. Let $\Gamma'_{\zeta,n}(\delta)$ be defined by (7.3), and let $\Gamma_{\zeta,n}(\delta)=\Gamma_\zeta(\delta)\setminus\Gamma'_{\zeta,n}(\delta)$. As in the proof of Lemma 7.1, it follows from Theorem 1.1 in [5] that there exists a sequence $\delta_n\to\infty$, $\delta_n=o(n)$, such that

$$\sup_{z_1,z_2\in\Gamma'_{\zeta,n}(\delta_n)}\big|r_n(z_1,z_2)\big|\to 0,\quad n\to\infty,$$

and we only need to check that

$$\sup_{\substack{z_1,z_2\in\Gamma_{\zeta,n}(\delta_n),\\ d_H(z_1,z_2)\leqslant A}}\big|r_n^*(z_1,z_2)\big|\to 0,\quad n\to\infty,$$

for r_n^* defined by

$$k_{\mu,n}(z_1,z_2)=\frac{\overline{D_\mu^{-1}(z_2)}D_\mu^{-1}(z_1)}{1-\bar z_2 z_1}\big(1+r_n^*(z_1,z_2)\big).$$

For this we will use formulas (7.8), (7.9). Cauchy-Schwarz inequality gives

$$\left|\frac{\overline{D_\mu^{-1}(z_2)}D_\mu^{-1}(z_1)}{1-\bar{z}_2 z_1} - k_{\mu,n}(z_1,z_2)\right|^2 \leqslant \Big(\sum_{k=n}^{\infty}|\varphi_k(z_1)|^2\Big)\cdot\Big(\sum_{k=n}^{\infty}|\varphi_k(z_2)|^2\Big). \tag{7.12}$$

By Lemma 7.1, for every $z \in \Gamma_{\zeta,n}(\delta_n)$ we have

$$\sum_{k=n}^{\infty}|\varphi_k(z)|^2 = \frac{|D_\mu^{-1}(z)|^2}{1-|z|^2} - k_{\mu,n}(z,z) = \frac{|D_\mu^{-1}(z)|^2}{1-|z|^2}\tilde{R}_n(z),$$

for some $\tilde{R}_n$ such that $\sup_{z\in\Gamma_{\zeta,n}(\delta_n)}|\tilde{R}_n(z)| \to 0$ as $n \to \infty$. Now (7.12) gives

$$\begin{aligned}\left|\frac{\overline{D_\mu^{-1}(z_2)}D_\mu^{-1}(z_1)}{1-\bar{z}_2 z_1} - k_{\mu,n}(z_1,z_2)\right| &= \frac{|D_\mu^{-1}(z_1)D_\mu^{-1}(z_2)|}{\sqrt{1-|z_1|^2}\sqrt{1-|z_2|^2}}\sqrt{|\tilde{R}_n(z_1)\tilde{R}_n(z_2)|},\\ &\asymp \left|\frac{\overline{D_\mu^{-1}(z_2)}D_\mu^{-1}(z_1)}{1-\bar{z}_2 z_1}\right|\sqrt{|\tilde{R}_n(z_1)\tilde{R}_n(z_2)|},\end{aligned}$$

where we used

$$\sqrt{1-|z_1|^2}\sqrt{1-|z_2|^2} \asymp |1-\bar{z}_2 z_1|$$

for $z_1, z_2 \in \Gamma_\zeta$ such that $d_H(z_1,z_2) \lesssim 1$. In particular, we have

$$|r_n^*(z_1,z_2)| \lesssim \sqrt{|\tilde{R}_n(z_1)\tilde{R}_n(z_2)|},$$

and the result follows from Lemma 7.1. □

Lemma 7.3. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$, $b_n = \varphi_n/\varphi_n^*$ and f_n be the Schur functions of μ defined in (3.5). Then for every $n \geqslant 1$ we have*

$$|b_n(z)| \lesssim |f_{n-1}(z)| + |f_{n-1}(0)| + |z|^n + \sqrt{e^{\mathcal{K}(\mu,z)}-1}, \tag{7.13}$$

provided $\max\{|f_{n-1}(z)|, |f_{n-1}(0)|\} \leqslant 1/2$. *In particular, if (2.1) holds, we have*

$$\lim_{n\to\infty}\sup_{z\in\Gamma_{\zeta,n}(\delta_n)}\frac{1}{n}\sum_{k=0}^{n-1}|b_k(z)| = 0, \tag{7.14}$$

for every $\delta_n \to \infty$, $\delta_n = o(n)$.

Proof. For $z \in \mathbb{D}$, set $\tilde{b}_n = \tilde{\varphi}_{z,n}/\tilde{\varphi}_{z,n}^*$, see (3.11), (3.12). Note that the definition of $\tilde{b}_n$ depends on the choice of the reference point z. Thus, $\tilde{b}_{z,n}$ would be a more accurate notation for this function. However, we will deal with $\tilde{b}_n$ only at the same point z, therefore, the short notation $\tilde{b}_n$ should not lead to misunderstanding. By definition, we have

$$\begin{aligned}|\tilde{b}_n(z) - b_n(z)| &= \left|\frac{zb_{n-1}(z)-\overline{f_{n-1}(z)}}{1-zb_{n-1}(z)f_{n-1}(z)} - \frac{zb_{n-1}(z)-\overline{f_{n-1}(0)}}{1-zb_{n-1}(z)f_{n-1}(0)}\right|\\ &\leqslant \frac{2|f_{n-1}(0)|+2|f_{n-1}(z)|}{(1-|f_{n-1}(0)|)(1-|f_{n-1}(z)|)}.\end{aligned}$$

It follows that $|\tilde{b}_n(z)-b_n(z)| \lesssim |f_{n-1}(0)|+|f_{n-1}(z)|$ if $\max\{|f_{n-1}(z)|,|f_{n-1}(0)|\} \leqslant 1/2$. Moreover, we have

$$|\tilde{b}_n(z) - \alpha_z z^n|^2 \lesssim e^{\mathcal{K}(\mu,z)} - 1, \qquad |\alpha_z| = 1,$$

by Lemma 2.5 in [5] and (3.15). Then (7.13) follows. To prove (7.14), we estimate

$$\frac{2}{n}\sum_{k=n/2}^{n}|b_k(z)| \lesssim \sqrt{e^{\mathcal{K}(\mu,z)}-1}+\frac{2}{n(1-|z|)}+ \\ +\frac{2|\{n/2\leqslant k\leqslant n:\ |f_{k-1}(z)|>1/2\}|}{n}+\tilde{\eta}_n(z) \\ \lesssim \sqrt{e^{\mathcal{K}(\mu,z)}-1}+\frac{2}{n(1-|z|)}+\tilde{\eta}_n(z),$$

where $\tilde{\eta}_n(z)$ is defined in (4.2). Then

$$\lim_{n\to\infty}\sup_{z\in\Gamma_{\zeta,n}(\delta_n,\Delta_n)}\frac{2}{n}\sum_{k=n/2}^{n}|b_k(z)|=0,$$

for any sequences $\Delta_n\to 1$, $\delta_n\to\infty$ with $\delta_n=o(n)$, by (2.1) and Lemma 4.2 (for the definition of $\Gamma_{\zeta,n}(\delta_n,\Delta_n)$, see (7.2)). At the same time, we have $\varphi_n^*\to D_\mu^{-1}$, $\varphi_n\to 0$ uniformly on compact subsets of $\mathbb{D}$ by Szegő theorem, hence

$$\lim_{n\to\infty}\sup_{z\in\Gamma_\zeta(\Delta_n)}\frac{2}{n}\sum_{k=n/2}^{n}|b_k(z)|=0$$

for some sequence $\Delta_n\to 1$. Since $\Gamma_\zeta(\Delta_n)\cup\Gamma_{\zeta,n}(\delta_n,\Delta_n)=\Gamma_{\zeta,n}(\delta_n)$, see (7.2)-(7.4), this implies that

$$\mathcal{B}_n:=\sup_{z\in\Gamma_{\zeta,n}(\delta_n)}\frac{2}{n}\sum_{k=n/2}^{n}|b_k(z)|\to 0,\quad n\to\infty.$$

Now given a positive integer N, pick the positive integer n with $2^{n-1}\leqslant N<2^n$. Then

$$\frac{1}{N}\sum_{k=1}^{N}|b_k(z)|\leqslant\frac{2}{2^n}\sum_{j=0}^{n}2^j\mathcal{B}_{2^j}$$

which tends to 0 as $N\to\infty$. This implies (7.14). □

Lemma 7.4. *For every $\varepsilon>0$ and every complex numbers x_k, y_k, such that*

$$\frac{1}{n}\sum_{k=0}^{n-1}\big(|x_k-1|+\big||x_k|^2-1\big|+|y_k-1|+\big||y_k|^2-1\big|\big)<\varepsilon$$

we have

$$\frac{1}{n}\sum_{k=0}^{n-1}|x_ky_k-1|<6\varepsilon.$$

Proof. Define $I_{n,s}=\{0\leqslant k\leqslant n-1\colon |x_k|+|y_k|\leqslant 4\}$ and $I_{n,b}=\{0\leqslant k\leqslant n-1\colon |x_k|+|y_k|>4\}$. We have

$$\frac{1}{n}\sum_{k\in I_{n,s}}|x_ky_k-1|\leqslant\frac{1}{n}\sum_{k\in I_{n,s}}|x_k-1||y_k|+\frac{1}{n}\sum_{k\in I_{n,s}}|y_k-1| \\ \leqslant\frac{4}{n}\sum_{k=0}^{n-1}|x_k-1|+\frac{1}{n}\sum_{k=0}^{n-1}|y_k-1|<4\varepsilon.$$

On the other hand, for $k\in I_{n,b}$ we have either $|x_k|>2$ or $|y_k|>2$, hence

$$|x_ky_k-1|\leqslant 1+\max(|x_k|^2,|y_k|^2), \\ \leqslant\max(|x_k-1|,|y_k-1|)+2\max\big(\big||x_k|^2-1\big|,\big||y_k|^2-1\big|\big), \\ \leqslant 2\big(|x_k-1|+\big||x_k|^2-1\big|+|y_k-1|+\big||y_k|^2-1\big|\big),$$

and our assumption implies the bound $n^{-1}\sum_{k\in I_{n,b}}|x_k y_k - 1| < 2\varepsilon$. □

Lemma 7.5. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that D_μ has a non-zero finite non-tangential limit at ζ. If (2.2) holds at ζ, then we have*

$$\lim_{n\to\infty}\sup_{z_1,z_2\in\Gamma_\zeta}\frac{1}{n}\sum_{k=0}^{n-1}|r_k(z_1,z_2)| = 0$$

for the functions r_n defined in (7.1).

Proof. Given a sequence $\delta_n \to \infty$, $\delta_n = o(n)$, let

$$\Omega_n(\delta_n) = \{(z_1,z_2):\ z_1 \text{ or } z_2 \in \Gamma_{\zeta,n}(\delta_n)\},$$
$$\Omega_n'(\delta_n) = \{(z_1,z_2):\ z_1,z_2 \in \Gamma_\zeta\setminus\Gamma_{\zeta,n}(\delta_n)\}.$$

It follows from Theorem 1.1 in [5] that there exists a sequence $\delta_n \to \infty$, $\delta_n = o(n)$, such that the numbers

$$\mathcal{E}_{1,n} = \sup_{(z_1,z_2)\in\Omega_n'(\delta_n)}|r_n(z_1,z_2)|$$

tend to 0 as $n\to\infty$. Therefore, we need to estimate $r_n(z_1,z_2)$ for the pairs (z_1,z_2) in $\Omega_n(\delta_n)$. Set $b_n = \varphi_n/\varphi_n^*$. Using the formula (2.12) written in the form

$$k_{\mu,n}(z_1,z_2) = \varphi_n^*(z_1)\overline{\varphi_n^*(z_2)}\frac{1-b_n(z_1)\overline{b_n(z_2)}}{1-\bar z_2 z_1}, \qquad b_n = \frac{\varphi_n}{\varphi_n^*},$$

we express funtions r_n in (7.1) in the form

$$r_n(z_1,z_2) = \frac{(\overline{D_\mu(z_2)}D_\mu(z_1)\varphi_n^*(z_1)\overline{\varphi_n^*(z_2)}-1)(1-b_n(z_1)\overline{b_n(z_2)})}{1-\bar z_2^n z_1^n} + \\ + \frac{\bar z_2^n z_1^n - b_n(z_1)\overline{b_n(z_2)}}{1-\bar z_2^n z_1^n},$$

by means of an algebraic manipulation. For $(z_1,z_2)\in\Omega_n(\delta_n)$ and $n/2\leqslant k\leqslant n$, we have $|1-\bar z_2^k z_1^k| \asymp 1$. Consider the numbers

$$\begin{aligned}\mathcal{E}_{2,n} &= \sup_{(z_1,z_2)\in\Omega_n(\delta_n)}\frac{2}{n}\sum_{n/2\leqslant k\leqslant n}|r_k(z_1,z_2)| \\ &\lesssim \sup_{z_1,z_2\in\Gamma_\zeta}\frac{2}{n}\sum_{n/2\leqslant k\leqslant n}|\overline{D_\mu(z_2)}D_\mu(z_1)\varphi_k^*(z_1)\overline{\varphi_k^*(z_2)}-1| + \\ &\quad + \sup_{(z_1,z_2)\in\Omega_n(\delta_n)}\frac{2}{n}\sum_{n/2\leqslant k\leqslant n}|z_1^k\bar z_2^k| + \sup_{(z_1,z_2)\in\Omega_n(\delta_n)}\frac{2}{n}\sum_{n/2\leqslant k\leqslant n}|b_k(z_1)b_k(z_2)|.\end{aligned}$$

Theorem 2.1, Theorem 2.2 and Lemma 7.4 imply that

$$\lim_{n\to\infty}\sup_{z_1,z_2\in\Gamma_\zeta}\frac{1}{n}\sum_{0\leqslant k\leqslant n-1}|\overline{D_\mu(z_2)}D_\mu(z_1)\varphi_k^*(z_1)\overline{\varphi_k^*(z_2)}-1| = 0.$$

The last relation, the fact that $\delta_n\to\infty$, and Lemma 7.3 give $\mathcal{E}_{2,n}\to 0$ (recall that b_n is a Blaschke product, so $|b_n|<1$ in $\mathbb{D}$ and one can apply Lemma 7.3 to the point z ($= z_1$ or z_2) lying in $\Gamma_{\zeta,n}(\delta_n)$ for estimating $\frac{2}{n}\sum_{n/2\leqslant k\leqslant n}|b_k(z_1)b_k(z_2)|$). Combining this fact with $\mathcal{E}_{1,n}\to 0$, we see that the quantities

$$\mathcal{E}_{3,n} = \sup_{z_1,z_2\in\Gamma_\zeta}\frac{2}{n}\sum_{n/2\leqslant k\leqslant n}|r_k(z_1,z_2)|$$

also tend to zero as $n \to \infty$. At the same time, for every integer $\ell \geqslant 1$ we have

$$\sup_{z_1, z_2 \in \Gamma_\zeta} \frac{1}{2^\ell} \sum_{1 \leqslant k \leqslant 2^\ell} |r_k(z_1, z_2)| \leqslant \frac{\mathcal{E}_{3,2^\ell}}{2} + \frac{\mathcal{E}_{3,2^{\ell-1}}}{4} + \frac{\mathcal{E}_{3,2^{\ell-2}}}{8} + \ldots + \frac{\mathcal{E}_{3,2}}{2^\ell}.$$

This estimate and the fact that $\mathcal{E}_{3,n} \to 0$ yield the statement. □

Lemma 7.6. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be such that D_μ has a non-zero finite non-tangential limit at ζ. If (2.3) holds at ζ, then we have*

$$\lim_{n \to \infty} \sup_{z_1, z_2 \in \Gamma_\zeta} \left| r_n(z_1, z_2) \right| = 0$$

for the functions r_n defined in (7.1).

Proof. As in the proof of Lemma 7.3, for each $z \in \mathbb{D}$ we set $b_n = \varphi_n/\varphi_n^*$, $\tilde{b}_n = \tilde{\varphi}_{z,n}/\tilde{\varphi}^*_{z,n}$, so that $|\tilde{b}_n(z) - b_n(z)| \lesssim |f_{n-1}(0)| + |f_{n-1}(z)|$ if $|f_{n-1}(z)|, |f_{n-1}(0)|$ are less than $1/2$. By Lemma 6.1, the last assumption holds for all $z \in \Gamma_\zeta$ if n is sufficiently large. On the other hand, we have

$$|\tilde{b}_n(z) - \alpha_z z^n|^2 \lesssim e^{\mathcal{K}(\mu, z)} - 1$$

by Lemma 2.5 in [5]. Here,

$$\alpha_z = \frac{\overline{\tilde{\varphi}^*_{z,n}(z)}}{\tilde{\varphi}^*_{z,n}(z)} = \frac{\overline{\varphi^*_n(z)}}{\varphi^*_n(z)}(1 + o(1)) = \frac{\overline{D_\mu(z)^{-1}}}{D_\mu(z)^{-1}}(1 + o(1)),$$

with uniform in Γ_ζ remainders $o(1)$ by Lemma 6.1 and Theorem 2.3. Since D_μ has a finite non-zero limit at ζ, the quantity α_z tends to some $\alpha \in \mathbb{T}$ when z approaches ζ non-tangentially. We also have $b_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$ by the Szegő theorem. It follows that

$$\lim_{n \to \infty} \sup_{z_1, z_2 \in \Gamma_\zeta} |b_n(z_1)\overline{b_n(z_2)} - \bar{z}_2^n z_1^n| = \lim_{n \to \infty} \sup_{z_1, z_2 \in \Gamma_\zeta} |\overline{\alpha z_2^n} \cdot \alpha z_1^n - \bar{z}_2^n z_1^n| = 0.$$

It remains to use the fact that

$$k_{\mu,n}(z_1, z_2) = \varphi^*_n(z_1)\overline{\varphi^*_n(z_2)} \frac{1 - b_n(z_1)\overline{b_n(z_2)}}{1 - \bar{z}_2 z_1},$$

and the uniform asymptotic relation (2.6) in Theorem 2.3. □

Proof of Theorem 2.7. See Lemma 7.2, Lemma 7.5, and Lemma 7.6. □

8. Proofs of Propositions 2.4, 2.5 and 2.6

Proof of Proposition 2.6. Denote $u = \log w$. Since the value $\mathcal{K}(\mu, z)$ of the entropy of μ at any point $z \in \mathbb{D}$ is invariant under multiplication of μ by a constant, we can assume that $u(\zeta) = 0$. Since u is bounded at a neighborhood of ζ and condition (2.3) is local (provided that $u \in L^1(\mathbb{T})$ as we always assume), we can suppose that u is bounded on $\mathbb{T}$ and vanishes on $\mathbb{T} \setminus I_0$. We have

$$\mathcal{P}(w, z) = \mathcal{P}(e^u, z) = 1 + \mathcal{P}(u, z) + O\left(\mathcal{P}(u^2, z)\right),$$

and, since $\mathcal{P}(u, z)^2 \leqslant \mathcal{P}(u^2, z)$,

$$\log \mathcal{P}(w, z) = \mathcal{P}(u, z) + O\left(\mathcal{P}(u^2, z)\right).$$

At the same time, we have $\mathcal{P}(\log w, z) = \mathcal{P}(u, z)$ by definition. Thus, we have

$$\mathcal{K}(\mu, z) \lesssim \mathcal{P}(u^2, z), \quad z \in \mathbb{D}.$$

Standard estimates of the Poisson kernel give (recall that $u = 0$ on $\mathbb{T} \setminus I_0$)

$$\mathcal{P}(u^2, z_n(\zeta)) \lesssim \sum_{k=0}^{n} 2^{2k-n} \int_{I_k} u^2 \, dm.$$

Thus,

$$\sqrt{\mathcal{K}(\mu, z_n(\zeta))} \lesssim \sum_{k=0}^{n} 2^{k-n/2} \left(\int_{I_k} u^2\, dm \right)^{1/2},$$

and hence

$$\sum_{n=0}^{\infty} \sqrt{\mathcal{K}(\mu, z_n(\zeta))} \lesssim \sum_{k=0}^{\infty} 2^{k/2} \left(\int_{I_k} u^2\, dm \right)^{1/2}.$$

The result follows. □

Proof of Proposition 2.5. As in the proof of Proposition 2.6, we denote $u = \log w$, assume that $u(\zeta) = 0$ and u is bounded on $\mathbb{T}$, and then

$$\mathcal{K}(\mu, z) \lesssim \mathcal{P}(u^2, z), \quad z \in \mathbb{D}.$$

We see that

$$\sum_{n=0}^{\infty} \mathcal{K}(\mu, z_n(\zeta)) \lesssim \sum_{n=0}^{\infty} \mathcal{P}(u^2, z_n(\zeta)),$$

and one only needs to use the estimate

$$\sum_{n=0}^{\infty} \frac{1 - |z_n(\zeta)|^2}{|1 - z_n(\zeta)\bar{\xi}|^2} \lesssim \int_0^1 \frac{dr}{|\xi - r\zeta|^2} \lesssim \frac{1}{|\xi - \zeta|}, \quad \xi, \zeta \in \mathbb{T},$$

to complete the proof. □

Proof of Proposition 2.4. We need to show that condition (2.2) holds for almost every point $\zeta \in \mathbb{T}$. Arguing as in the proof of Proposition 2.6, we arrive at the estimate

$$\mathcal{K}(\mu, r\zeta) \lesssim \int_{\mathbb{T}} (u(\xi) - u(\zeta))^2 \frac{1 - r^2}{|\xi - r\zeta|^2}\, dm(\xi), \qquad 0 \leqslant r < 1, \quad \zeta \in \mathbb{T}.$$

Denote by A the Lebesgue measure on $\mathbb{C}$ normalized by $A(\mathbb{D}) = 1$. We have

$$\int_{\mathbb{D}} \frac{\mathcal{K}(\mu, z)}{1 - |z|^2}\, dA(z) = \int_{\mathbb{T}} \int_0^1 \frac{\mathcal{K}(\mu, r\zeta) r}{1 - r^2} dr\, dm(\zeta) \lesssim$$
$$\lesssim \int_{\mathbb{T}} \int_{\mathbb{T}} (u(\xi) - u(\zeta))^2 \int_0^1 \frac{dr}{|\xi - r\zeta|^2}\, dm(\xi)\, dm(\zeta).$$

Since

$$\int_0^1 \frac{dr}{|\xi - r\zeta|^2} \lesssim \frac{1}{|\xi - \zeta|}, \quad \xi, \zeta \in \mathbb{T},$$

we deduce that

$$\int_{\mathbb{D}} \frac{\mathcal{K}(\mu, z)}{1 - |z|^2}\, dA(z) \lesssim \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(u(\xi) - u(\zeta))^2}{|\xi - \zeta|}\, dm(\xi)\, dm(\zeta).$$

Denoting $\zeta = e^{2\pi i\theta}$ and $\xi = e^{2\pi i(\theta + h)}$, the last double integral rewrites as

$$\int_0^1 \frac{1}{|1 - e^{2\pi i h}|} \int_0^1 (u(e^{2\pi i\theta}) - u(e^{2\pi i(\theta+h)}))^2\, d\theta\, dh.$$

The Fourier coefficients of the real-valued function $\theta \mapsto u(e^{2\pi i\theta}) - u(e^{2\pi i(\theta+h)})$ with respect to the basis $\{e^{2\pi i k\theta}\}_{k\in\mathbb{Z}}$ in $L^2[0,1]$ are $(1 - e^{2\pi i k h})\hat{u}(k)$, $k \in \mathbb{Z}$. Then, Parseval's identity gives

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(u(\xi) - u(\zeta))^2}{|\xi - \zeta|}\, dm(\xi)\, dm(\zeta) = \sum_{k=-\infty}^{+\infty} |\hat{u}(k)|^2 \int_0^1 \frac{|1 - e^{2\pi i k h}|^2}{|1 - e^{2\pi i h}|} dh$$
$$\asymp \sum_{k=-\infty}^{+\infty} |\hat{u}(k)|^2 \log(2 + |k|).$$

Hence assumption (2.9) implies

$$\int_{\mathbb{D}} \frac{\mathcal{K}(\mu, z)}{1-|z|^2}\, dA(z) < \infty.$$

Fubini theorem then gives

$$\int_{\mathbb{T}}\int_{\Gamma_\zeta} \frac{\mathcal{K}(\mu, z)}{(1-|z|^2)^2} dA(z)\, dm(\zeta) = \int_{\mathbb{D}} \frac{\mathcal{K}(\mu, z)}{(1-|z|^2)^2} m(\{\zeta\in\mathbb{T} : z \in \Gamma_\zeta\}) dA(z)$$
$$\lesssim \int_{\mathbb{D}} \frac{\mathcal{K}(\mu, z)}{1-|z|^2}\, dA(z),$$

from where we deduce that

$$\int_{\Gamma_\zeta} \frac{\mathcal{K}(\mu, z)}{(1-|z|^2)^2}\, dA(z) < \infty,$$

for almost every $\zeta \in \mathbb{T}$. Hence condition (2.2) holds for almost every $\zeta \in \mathbb{T}$. □

Acknowledgment. The authors are grateful to S. Denisov for drawing their attention to the paper [34].

The work of RB is supported by the Slovenian Research Agency ARIS (grants J1-70033 and P1-0291). The work of AN is supported by the Spanish Ministerio de Ciencia e Innovación (grant PID2021-123151NB-I00), by the Generalitat de Catalunya (grant 2021 SGR 00071), and by the Spanish Research Agency through the María de Maeztu Program (CEX2020-001084-M).

References

[1] M. Alexis, A. Aptekarev, and S. Denisov. Continuity of weighted operators, Muckenhoupt A_p weights, and Steklov problem for orthogonal polynomials. *Int. Math. Res. Not. IMRN*, 8:5935–5972, 2022. 2

[2] M. U. Ambroladze. On the possible rate of growth of polynomials that are orthogonal with a continuous positive weight. *Mat. Sb.*, 182(3):332–353, 1991. 2

[3] A. Aptekarev, S. Denisov, and D. Tulyakov. On a problem by Steklov. *J. Amer. Math. Soc.*, 29(4):1117–1165, 2016. 2

[4] V. M. Badkov. Asymptotic behavior of orthogonal polynomials. *Mat. Sb. (N.S.)*, 109(151)(1):46–59, 165, 1979. 2

[5] R. Bessonov. On rate of convergence for universality limits. *Integral Equations Operator Theory*, 96(1):Paper No. 6, 20, 2024. 3, 8, 10, 11, 13, 28, 29, 31, 32, 33

[6] R. Bessonov and S. Denisov. Zero sets, entropy, and pointwise asymptotics of orthogonal polynomials. *J. Funct. Anal.*, 280(12):Paper No. 109002, 38, 2021. 9, 10, 26, 27

[7] R. V. Bessonov. Entropy function and orthogonal polynomials. *J. Approx. Theory*, 272:Paper No. 105650, 16, 2021. 12, 17, 19

[8] S. Denisov. Pointwise behavior of SU(1,1) nonlinear Fourier transform. *arXiv:2605.25108*, 2026. 5, 6

[9] S. Denisov. The strong version of nonlinear carleson conjecture fails. *arXiv:2605.01658*, 2026. 5

[10] S. Denisov and K. Rush. Orthogonal polynomials on the circle for the weight w satisfying conditions $w, w^{-1} \in$ BMO. *Constr. Approx.*, 46(2):285–303, 2017. 2

[11] S. A. Denisov. On the growth of polynomials orthogonal on the unit circle with a weight w that satisfies $w, w^{-1} \in L^\infty(\mathbb{T})$. *Mat. Sb.*, 209(7):71–105, 2018. 2

[12] B. Eichinger, M. Lukić, and B. Simanek. An approach to universality using Weyl m-functions. *Ann. of Math. (2)*, 203(2):471–510, 2026. 3

[13] E. Findley. Universality for locally Szegő measures. *J. Approx. Theory*, 155(2):136–154, 2008. 2

[14] G. Freud. Über die starke (C, 1)-summierbarkeit von orthogonalen polynomreihen. *Acta Math. Acad. Sci. Hungar.*, 3:83–88, 1952. 2

[15] J. B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1981. 9, 16

[16] Ja. L. Geronimus and B. L. Golinskii. Asymptotic formulas for orthogonal polynomials. *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 1:141–163, 1965. 2

[17] B. L. Golinskii. Speed of convergence of a sequence of orthogonal polynomials to the limit function. *Ukrain. Mat. Ž.*, 19(4):11–28, 1967. 2

[18] B. L. Golinskii. Two fundamental conditions for the asymptotic representation of polynomials that are orthogonal on the unit circle. *Mat. Zametki*, 15:847–855, 1974. 2

[19] U. Grenander and G. Szegő. *Toeplitz forms and their applications*. California Monographs in Mathematical Sciences. University of California Press, Berkeley-Los Angeles, 1958. 2

[20] S. Khrushchev. Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in $L^2(\mathbb{T})$. *J. Approx. Theory*, 108(2):161–248, 2001. 16

[21] D. Lubinsky. A new approach to universality limits involving orthogonal polynomials. *Ann. of Math. (2)*, 170(2):915–939, 2009. 3

[22] A. Mate and P. Nevai. Bernstein's inequality in L^p for $0 < p < 1$ and (C, 1) bounds for orthogonal polynomials. *Ann. of Math. (2)*, 111(1):145–154, 1980. 2, 12

[23] A. Máté, P. Nevai, and V. Totik. Szegő's extremum problem on the unit circle. *Ann. of Math. (2)*, 134(2):433–453, 1991. 2, 6

[24] D. Menchoff. Sur les séries de fonctions orthogonales. *Fundamenta Mathematicae*, 4(1):82–105, 1923. 4

[25] D. Menchoff. Sur les séries de fonctions orthogonales. *Fundamenta Mathematicae*, 8(1):56–108, 1926. 4

[26] A. Poltoratski. Pointwise convergence of the non-linear Fourier transform. *Ann. of Math. (2)*, 199(2):741–793, 2024. 4

[27] A. Poltoratski. Pointwise convergence of the non-linear Fourier transform. *arXiv:2103.13349v13*, 2025. 4

[28] H. Rademacher. Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen. *Math. Ann.*, 87(1-2):112–138, 1922. 4

[29] E. A. Rahmanov. Estimates of the growth of orthogonal polynomials whose weight is bounded away from zero. *Mat. Sb. (N.S.)*, 114(156)(2):269–298, 335, 1981. 2

[30] B. Simon. *Orthogonal polynomials on the unit circle. Part 1*, volume 54, Part 1 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Classical theory. 1, 3, 4, 9, 10, 16

[31] B. Simon. *Szegő's theorem and its descendants*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 2011. Spectral theory for L^2 perturbations of orthogonal polynomials. 1

[32] G. Szegő. *Orthogonal polynomials*, volume Vol. XXIII of *Amer. Math. Soc. Colloquium Publications*. Amer. Math. Soc., Providence, RI, fourth edition, 1975. 1

[33] V. Totik. Universality under Szegő's condition. *Canad. Math. Bull.*, 59(1):211–224, 2016. 3

[34] P. L. Ul'yanov. Development of D.E.Men'shov's results on the theory of orthogonal series. *Uspekhi Mat. Nauk*, 47(5(287)):45–66, 207, 1992. 4, 35

R. Bessonov: Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska ulica 19, 1000 Ljubljana
Email address: roman.bessonov@fmf.uni-lj.si

A. Nicolau: Departament de Matemàtiques, Universitat Autònoma de Barcelona, and Centre de Recerca Matemàtica, 08193 Barcelona
Email address: artur.nicolau@uab.cat