Extreme values of the Riemann zeta function

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Riemann zeta function

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Meromorphic continuation:

$$\zeta(s) = \frac{1}{s-1} + A(s), \qquad \Re(s) > 1,$$

$A(s)$ is an entire function.

Symmetry

Functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

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$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / x}$$

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$$\mathcal{M}\theta(s) = \int_0^\infty x^{s-1}\theta(x)dx$$

Riemann hypothesis

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$\zeta(s) = 0$, for $s = -2, -4, \ldots$.
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Riemann hypothesis: All nontrivial zeroes are on the critical line.

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For any $\epsilon > 0$

$$|\zeta(\frac{1}{2} + it)| = o(t^{\epsilon}), \qquad t \to \infty.$$

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Bourgain (2016, JAMS):

$$|\zeta(\frac{1}{2} + it)| = O(t^{13/84+\epsilon}).$$

Approximate formulae

$$\zeta(\frac{1}{2}+it)=\sum_{n=1}^{N} n^{-1/2-it}-\frac{N^{1/2-it}}{1/2-it}+O(N^{-1/2}),\quad |t|<N.$$

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Should be a lot of cancelations!
Why for any large $a \in \mathbb{N}$ and $t < a^{1/\epsilon}$

$$a^{it}+(a+1)^{it}+\ldots+(2a)^{it}=o(a^{1/2+\epsilon})?$$

Asymmetry

Asymmetry

It seems that

$$\sum_{n=1}^{N} n^{-1/2-it}$$

attains “small values” on $[N/2, N]$, but doesn't attain “large values”!

Lower bounds

Montgomery; Balasubramanian and Ramachandra, 1977: $\exists$ large T with

$$|\zeta(1/2 + iT)| \geq \exp\left(c\sqrt{\frac{\log T}{\log\log T}}\right).$$

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Theorem 1.(B, Seip; ArXiv, 2015)
$\exists$ large T with

$$|\zeta(1/2+iT)| \geq \exp\left(c\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\right),$$

where $c = 1/\sqrt{2} + o(1)$.

What is the truth?

Farmer–Gonek–Hughes (2007) have conjectured, by use of random matrix theory, that the right bound is

$$\exp\left((1/\sqrt{2} + o(1))\sqrt{\log T \log\log T}\right).$$

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Example:

$$|\zeta(1/2+i*3.9246764...*10^{31})| \approx 16244.86526$$

For this particular T

$$\exp\left((1/\sqrt{2})\sqrt{\log T \log\log T}\right) \approx 264964.$$

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Then for some $t \in [-T, T]$, $|\zeta(1/2 + it)| \gg M_1/M_2$.

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2. Choice of F

$$F = \Big|\sum_{m\in\mathcal{M}'} r(m)m^{it}\Big|^2\Phi\Big(\frac{\log T}{T}t\Big),$$

where $\Phi(t) = e^{-t^2/2}$.

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$$M_1 = \frac{\sqrt{2\pi}\, T}{\log T} \sum_{m,n \in \mathcal{M}'} r(m) r(n) \Phi\Big(\frac{T}{\log T} \log \frac{m}{n}\Big) \ + \ \text{small terms.}$$

$$M_2 = \frac{\sqrt{2\pi}\, T}{\log T} \sum_{m,n \in \mathcal{M}'} \sum_{k \le T} \frac{r(m) r(n)}{\sqrt{k}} \Phi\Big(\frac{T}{\log T} \log \frac{km}{n}\Big) \ + \ \text{small terms.}$$

Sketch of the proof

3. Optimization problem
$|\mathcal{M}| = N \approx T^{1/2}$,

$$\sum_{m \in \mathcal{M}} f(m)^2 = 1$$

Maximize

$$\sum_{m,n \in \mathcal{M}, m=kn} \frac{f(n)f(m)}{\sqrt{k}}.$$

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Answer:

$$\exp\left(\sqrt{\frac{\log N \log\log\log N}{\log\log N}}\right).$$

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f is a multiplicative function, that is $f(mn) = f(n)f(m)$ for $(m,n) = 1$ supported on square-free numbers,

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$$p \leq \log N \exp((\log_2 N)^{1-o(1)}).$$

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$\mathcal{M}$ is the set where the "mass" of f lives.

Related question

What is the maximum of

$$\sum_{m,n\in\mathcal{M}} c_m c_n \frac{(m,n)}{\sqrt{mn}},$$

where

$$\sum_{m\in\mathcal{M}} c_n^2 = 1?$$

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They are almost the same! Reason: (m,n) is a certain inner product.

Related question

Example: If $\mathcal{M}$ are all divisors of $p_1 \ldots p_\ell$ then

$$\frac{1}{|\mathcal{M}|} \sum_{m,n \in \mathcal{M}} \frac{(m,n)}{\sqrt{mn}} = \prod_{j=1}^{\ell} (1 + p_j^{-1/2})$$

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- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.

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- It is enough to consider square free numbers
- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.
- Divisor closed extremal sets $\mathcal{M}$ enjoy the following completeness property: If $n \in \mathcal{M}$, $p|n$, $p' < p$, then either $p'|n$ or $p'n/p \in \mathcal{M}$.

Related question

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Theorem 2.(B, Seip, 2015)

$$\frac{1}{N} \sup_{1 \le n_1 \ldots < n_N} \sum_{k,\ell=1}^{N} \frac{(n_k, n_\ell)}{\sqrt{n_k n_\ell}} \approx \exp\left(A \sqrt{\frac{\log N \log\log\log N}{\log\log N}} \right),$$

where $1 \le A < 7$.
Other tools: Bohr correspondence, multiplicative functions,
Chauchy–Shwarz inequality

Questions

Questions

- Is there a better choice of resonator, allowing for improvements?
- Could we apply resonators for the upper bounds?

THANK YOU!