L^2-Betti numbers of one-relator groups

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Abstract

We determine the L^2-Betti numbers of all one-relator groups and all surface-plus-one-relation groups. We also obtain some information about the L^2-cohomology of left-orderable groups, and deduce the non-L^2 result that, in any left-orderable group of homological dimension one, all two-generator subgroups are free.

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1 Notation and background

Let G be a (discrete) group, fixed throughout the article.

We use $\mathbb{R} \cup \{-\infty, \infty\}$ with the usual conventions; for example, $\frac{1}{\infty} = 0$, and $3 - \infty = -\infty$. Let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$. We call $\mathbb{N} \cup \{\infty\}$ the set of *vague cardinals*, and, for each set X, we define its *vague cardinal* $|X| \in \mathbb{N} \cup \{\infty\}$ to be the cardinal of X if X is finite, and to be ∞ if X is infinite.

Mappings of right modules will be written on the left of their arguments, and mappings of left modules will be written on the right of their arguments.

Let $\mathbb{C}[[G]]$ denote the set of all functions from G to $\mathbb{C}$ expressed as formal sums, that is, a function $a\colon G \to \mathbb{C}$, $g \mapsto a(g)$, will be written as $\sum_{g \in G} a(g)g$. Then $\mathbb{C}[[G]]$ has a natural $\mathbb{C}G$-bimodule structure, and contains a copy of $\mathbb{C}G$ as $\mathbb{C}G$-sub-bimodule. For each $a \in \mathbb{C}[[G]]$, we define $\|a\| := (\sum_{g \in G} |a(g)|^2)^{1/2} \in [0, \infty]$, and $\operatorname{tr}(a) := a(1) \in \mathbb{C}$.

Define
$$l^2(G) := \{a \in \mathbb{C}[[G]] : \|a\| < \infty\}.$$
We view $\mathbb{C} \subseteq \mathbb{C}G \subseteq l^2(G) \subseteq \mathbb{C}[[G]]$. There is a well-defined *external multiplication* map
$$l^2(G) \times l^2(G) \to \mathbb{C}[[G]], \quad (a, b) \mapsto a \cdot b,$$
where, for each $g \in G$, $(a \cdot b)(g) := \sum_{h \in G} a(h)b(h^{-1}g)$; this sum converges in $\mathbb{C}$, and, moreover, $|(a \cdot b)(g)| \le \|a\| \, \|b\|$, by the Cauchy-Schwarz inequality. The external multiplication extends the multiplication of $\mathbb{C}G$.

The *group von Neumann algebra of* G, denoted $\mathcal{N}(G)$, is the ring of bounded $\mathbb{C}G$-endomorphisms of the right $\mathbb{C}G$-module $l^2(G)$; see [19, §1.1]. Thus $l^2(G)$ is an $\mathcal{N}(G)$-$\mathbb{C}G$-bimodule. We view $\mathcal{N}(G)$ as a subset of $l^2(G)$ by the map $\alpha \mapsto \alpha(1)$, where 1 denotes the identity element of $\mathbb{C}G \subseteq l^2(G)$. It can be shown that
$$\mathcal{N}(G) = \{a \in l^2(G) \mid a \cdot l^2(G) \subseteq l^2(G)\},$$

and that the action of $\mathcal{N}(G)$ on $l^2(G)$ is given by the external multiplication. Notice that $\mathcal{N}(G)$ contains $\mathbb{C}G$ as a subring and also that we have an induced 'trace map' $\mathrm{tr}\colon \mathcal{N}(G) \to \mathbb{C}$. The elements of $\mathcal{N}(G)$ which are injective, as operators on $l^2(G)$, are precisely the (two-sided) non-zerodivisors in $\mathcal{N}(G)$, and they form a left and right Ore subset of $\mathcal{N}(G)$; see [19, Theorem 8.22(1)].

Let $\mathcal{U}(G)$ denote the *ring of unbounded operators affiliated to* $\mathcal{N}(G)$; see [19, §8.1]. It can be shown that $\mathcal{U}(G)$ is the left, and the right, Ore localization of $\mathcal{N}(G)$ at the set of its non-zerodivisors. For example, it is then clear that,

$$\text{if } x \text{ is an element of } G \text{ of infinite order, then } x-1 \text{ is invertible in } \mathcal{U}(G). \tag{1.0.1}$$

Moreover, $\mathcal{U}(G)$ is a von Neumann regular ring in which one-sided inverses are two-sided inverses, and, hence, one-sided zerodivisors are two-sided zerodivisors; see [19, §8.2].

There is a continuous, additive von Neumann dimension that assigns to every left $\mathcal{U}(G)$-module M a value $\dim_{\mathcal{U}(G)} M \in [0,\infty]$; see Definition 8.28 and Theorem 8.29 of [19]. For example,

$$\text{if } e \text{ is an idempotent element of } \mathcal{N}(G), \text{ then } \dim_{\mathcal{U}(G)} \mathcal{U}(G)e = \mathrm{tr}(e); \tag{1.0.2}$$

see Theorem 8.29 and §§6.1-2 of [19].

Consider any subring Z of $\mathbb{C}$, and any resolution of Z by projective, or, more generally, flat, left ZG-modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Z \longrightarrow 0, \tag{1.0.3}$$

and let $\mathcal{P}$ denote the unaugmented complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0.$$

By Definition 6.50, Lemma 6.51 and Theorem 8.29 of [19], we can define, for each $n \in \mathbb{N}$, the *nth L^2-Betti number of G* as

$$b_n^{(2)}(G) := \dim_{\mathcal{U}(G)} \mathrm{H}_n(\mathcal{U}(G) \otimes_{ZG} \mathcal{P}),$$

where $\mathcal{U}(G)$ is to be viewed as a $\mathcal{U}(G)$-ZG-bimodule. Of course,

$$\mathrm{H}_n(\mathcal{U}(G) \otimes_{ZG} \mathcal{P}) = \mathrm{Tor}_n^{ZG}(\mathcal{U}(G), Z) \simeq \mathrm{Tor}_n^{\mathbb{Z}G}(\mathcal{U}(G), \mathbb{Z}) = \mathrm{H}_n(G;\mathcal{U}(G)),$$

where, for the purposes of this article, it will be convenient to understand that $\mathrm{H}_n(G;-)$ applies to *right* G-modules. Thus the L^2-Betti numbers do not depend on the choice of Z, nor on the choice of $\mathcal{P}$.

1.1 Remark. If G contains an element of infinite order, then (1.0.1) implies that $\mathcal{U}(G) \otimes_{ZG} Z = 0$, and $\mathcal{U}(G) \otimes_{ZG} P_1 \longrightarrow \mathcal{U}(G) \otimes_{ZG} P_0 \longrightarrow 0$ is exact, and $\mathrm{H}_0(G;\mathcal{U}(G)) = 0$, and $b_0^{(2)}(G) = 0$. □

1.2 Remarks. In general, there is little relation between the nth L^2-Betti number, $b_n^{(2)}(G) = \dim_{\mathcal{U}(G)} \mathrm{H}_n(G;\mathcal{U}(G)) \in [0,\infty]$, and the nth (ordinary) Betti number,

$$b_n(G) := \dim_{\mathbb{Q}} \mathrm{H}_n(G;\mathbb{Q}) \in [0,\infty].$$

We say that G is *of type* FL if, for $Z = \mathbb{Z}$, there exists a resolution (1.0.3) such that all the P_n are finitely generated free left $\mathbb{Z}G$-modules and all but finitely many of the P_n are 0.

If G is of type FL, then it is easy to see that the L^2*-Euler characteristic*

$$\chi^{(2)}(G) := \sum_{n\geq 0}(-1)^n b_n^{(2)}(G)$$

is equal to the (ordinary) *Euler characteristic*

$$\chi(G) := \sum_{n\geq 0}(-1)^n b_n(G).$$

We say that G is *of type* VFL if G has a subgroup H of finite index such that H is of type FL. In this event, the (ordinary) *Euler characteristic* of G is defined as $\chi(G) := \frac{1}{[G:H]}\chi(H)$; this is sometimes called the virtual Euler characteristic. Here again, $\chi^{(2)}(G) = \chi(G)$; see [19, Remark 6.81]. □

2 Summary of results

In outline, the article has the following structure. More detailed definitions can be found in the appropriate sections.

In Section 3, we prove a useful technical result about $\mathcal{U}(G)$ for special types of groups.

In Section 4, we calculate the L^2-Betti numbers of one-relator groups. Let us describe the results.

For any element x of a group G, we define the *exponent* of x in G, denoted $\exp_G(x)$, as the supremum in $\mathbb{Z} \cup \{\infty\}$ of the set of those integers m such that x equals the mth power of some element of G. Then $\exp_G(x)$ is a nonzero vague cardinal. We write $G/\langle\!\langle\, x \,\rangle\!\rangle$ to denote the quotient group of G modulo the normal subgroup of G generated by x.

Suppose that G has a one-relator presentation $\langle X \mid r\rangle$. Thus r is an element of the free group F on X, and $G = F/\langle\!\langle\, r \,\rangle\!\rangle$.

Set $d := |X| \in [0, \infty]$, $m := \exp_F(r) \in [1, \infty]$, and $\chi := 1 - d + \frac{1}{m} \in [-\infty, 1]$.

It is known that if $d < \infty$ then G is of type VFL and $\chi(G) = \chi$. If $d = \infty$, then G is not finitely generated and $\chi = -\infty$; here we *define* $\chi(G) = -\infty$, which is non-standard, but it is reasonable.

In general, $\max\{\chi(G), 0\} = \frac{1}{|G|}$.

In Theorem 4.2, we will show that,

$$\text{for } n \in \mathbb{N}, \quad b_n^{(2)}(G) = \begin{cases} \max\{\chi(G), 0\} & \text{if } n = 0, \\ \max\{-\chi(G), 0\} & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \tag{2.0.1}$$

Lück [19, Example 7.19] gave some results and conjectures concerning the L^2-Betti numbers of torsion-free one-relator groups, and (2.0.1) shows that the conjectured statements are true.

In Section 5, we calculate the L^2-Betti numbers of an arbitrary surface-plus-one-relation group $G = \pi_1(\Sigma)/\langle\!\langle\, \alpha \,\rangle\!\rangle$. Here Σ is a connected orientable surface, and α is an element of the fundamental group, $\pi_1(\Sigma)$. The surface-plus-one-relation groups were introduced and studied by Hempel [12], and further investigated by Howie [15]; these authors called the groups 'one-relator surface groups', but we are reluctant to adopt this terminology.

If Σ is not closed, then $\pi_1(\Sigma)$ is a countable free group, see [20], and G is a countable one-relator group. In light of Theorem 4.2, we may assume that Σ is a closed surface.

Let g denote the genus of the closed surface Σ, and let $m = \exp_{\pi_1(\Sigma)}(\alpha)$. It is not difficult to deduce from known results that G is of type VFL and

$$\chi(G) = \begin{cases} 1 & \text{if } g = 0, \\ 0 & \text{if } g = 1, \\ 2 - 2g + \frac{1}{m} & \text{if } g \geq 2. \end{cases}$$

Then $\chi(G) \in (-\infty, 1]$ and $\max\{\chi(G), 0\} = \frac{1}{|G|}$. In Section 5, we will show that (2.0.1) is also valid for surface-plus-one-relation groups.

For any group G, $b_0^{(2)}(G) = \frac{1}{|G|}$; see [19, Theorem 6.54(8)(b)]. It is obvious that if G is finite then $b_n^{(2)}(G) = 0$ for all $n \geq 1$. Thus, in essence, the foregoing results assert that if G is an infinite one-relator group, or an infinite surface-plus-one-relation group, then

$$b_n^{(2)}(G) = \begin{cases} -\chi(G) & \text{if } n = 1, \\ 0 & \text{if } n \neq 1, \end{cases}$$

and we emphasize that, in this case, we understand that $\chi(G) = -\infty$ if G is not finitely generated.

In Section 6, we consider a variety of situations where Z is a nonzero ring and there exists some positive integer n such that $P_n = ZG^2$ in a projective ZG-resolution (1.0.3) of ${}_{ZG}Z$. For example, this happens for two-generator groups and for two-relator groups.

Thus, in Corollary 6.8, we recover Lück's result [19, Theorem 7.10] that all the L^2-Betti numbers of Thompson's group F vanish; see [6] for a detailed exposition of the definition and main properties of F.

2.1 Definitions. Recall that G is *left orderable* if there exists a total order $\leq$ of G which is left G-invariant, that is, whenever $g, x, y \in G$ and $x \leq y$, then $gx \leq gy$. One then says that $\leq$ is a *left order* of G. The reverse order is also a left order. Since every group is isomorphic to its opposite through the inversion map, we see that 'left-orderable' is a short form for 'one-sided-orderable'.

A group is said to be *locally indicable* if every finitely generated subgroup is either trivial or has an infinite cyclic quotient. Burns and Hale [5] showed that every locally indicable group is left orderable. This often provides a convenient way to prove that a given group is left orderable.

Recall that the *cohomological dimension of G with respect to a ring Z*, denoted $\operatorname{cd}_Z G$, is the least $n \in \mathbb{N}$ such that $P_{n+1} = 0$ in some projective ZG-resolution (1.0.3) of ${}_{ZG}Z$. The *cohomological dimension* of G, denoted $\operatorname{cd} G$, is $\operatorname{cd}_{\mathbb{Z}} G$. A classic result of Stallings and Swan says that the groups of cohomological dimension at most one are precisely the free groups.

Similarly, the *homological dimension of G with respect to a ring Z*, denoted $\operatorname{hd}_Z G$, is the least $n \in \mathbb{N}$ such that $P_{n+1} = 0$ in some flat ZG-resolution (1.0.3) of ${}_{ZG}Z$. The *homological dimension* of G, denoted $\operatorname{hd} G$, is $\operatorname{hd}_{\mathbb{Z}} G$. □

We understand that Robert Bieri, in the 1970's, first raised the question as to whether the groups of homological dimension at most one are precisely the locally free groups. Notice that a locally free group has homological dimension at most one, since the augmentation ideal of a locally free group is a directed union of finitely generated free left submodules. Recently, in [16], it was proved that if the homological dimension of G is at most one and G satisfies the Atiyah conjecture (or, more generally, the group ring $\mathbb{Z}G$ embeds in a one-sided Noetherian ring), then G is locally free. In Corollary 6.12, we show that if G is locally indicable, or, more generally, left orderable, and the homological dimension of G is at most one, then every *two-generator* subgroup of G is free.

Finally, in Proposition 6.13, we calculate the first three L^2-Betti numbers of an arbitrary left-orderable two-relator group of cohomological dimension at least three.

2.2 Notation. We will frequently consider maps between free modules over a ring U, and we will use the following format.

Let X and Y be sets.

By an $X \times Y$ *row-finite matrix* over U we mean a function $(u_{x,y})\colon X \times Y \to U$, $(x, y) \mapsto u_{x,y}$ such that, for each $x \in X$, $\{y \in Y \mid u_{x,y} \neq 0\}$ is finite.

We write $\oplus_X U$ to denote the direct sum of copies of U indexed by X. If $n \in \mathbb{N}$, and $X = \{1, \ldots, n\}$, we identify $X = n$ and also write $\oplus_n U$ as U^n. An element of $\oplus_X U$ will be viewed as a $1 \times X$ row-finite matrix (u_{1x}) over U. Then $\oplus_X U$ is a left U-module in a natural way.

A map $\oplus_X U \to \oplus_Y U$ of left U-modules will be thought of as right multiplication by a row-finite $X \times Y$ matrix $(u_{x,y})$ in a natural way, and we will write $\oplus_X U \xrightarrow{(u_{x,y})} \oplus_Y U$. □

3 Preliminary results about $\mathcal{U}(G)$

For $a = \sum_{g\in G} a(g)g \in \mathbb{C}[[G]]$, we let $a^* = \sum_{g\in G} \overline{a(g^{-1})}g$ where $\overline{z}$ indicates the complex conjugate of z. This involution restricts to $\mathbb{C}(G)$ and $\mathcal{N}(G)$, and extends in a unique way to $\mathcal{U}(G)$. Furthermore, if $a, b \in \mathcal{N}(G)$, then $(ab)^* = b^*a^*$ and $a^*a = 0$ if and only if $a = 0$.

In Sections 4 and 5, we shall see that the narrow hypotheses of the following result hold whenever G is a one-relator group or a surface-plus-one-relation group.

3.1 Theorem. *Suppose that G has a normal subgroup H such that H is the semidirect product $F \rtimes C$ of a free subgroup F by a finite subgroup C, and that G/H is locally indicable, or, more generally, left orderable.*

Let $m = |C|$, and let $e = \frac{1}{m} \sum\limits_{c\in C} c \in \mathbb{C}G$.

Then the following hold.

(i) *Each torsion subgroup of G embeds in C.*

(ii) *Each nonzero element of $e\mathbb{C}Ge$ is invertible in $e\mathcal{U}(G)e$.*

(iii) *For all $x \in \mathcal{U}(G)e$ and $y \in e\mathbb{C}G$, if $xy = 0$ then $x = 0$ or $y = 0$.*

Proof. (i) Each torsion subgroup of G lies in H and has trivial intersection with F, and therefore embeds in C.

(ii) Notice that e is a projection, that is, e is idempotent and $e^* = e$. Clearly, $\mathrm{tr}(e) = \frac{1}{m}$. Also, $e\mathcal{U}(G)e$ is a ring and $e\mathbb{C}Ge$ is a subring of $e\mathcal{U}(G)e$. Moreover, in $e\mathcal{U}(G)e$, one-sided inverses are two-sided inverses.

Let $a \in e\mathbb{C}Ge - \{0\}$. We want to show that a is left invertible in $e\mathcal{U}(G)e$.

Let T be a transversal for the right (or left) H-action on G, and suppose that T contains 1. Write $a = t_1a_1 + \cdots + t_na_n$ where the t_i are distinct elements of T, and, for each i, $a_i \in \mathbb{C}(H)e - \{0\}$.

Let $\preceq$ be a left order for G/H. We may assume that $t_1H \prec \cdots \prec t_nH$. To show that a is left invertible in $e\mathcal{U}(G)e$, it suffices to show that $(ea_1^*t_1^{-1}e)a$ is left invertible in $e\mathcal{U}(G)e$. On replacing a with $(ea_1^*t_1^{-1}e)a = a_1^*t_1^{-1}a$, we see that we may assume that $t_1 = 1$ and $a_1 \in e\mathbb{C}He - \{0\}$.

By (i), m is the least common multiple of the orders of the finite subgroups of H. Now the strong Atiyah conjecture holds for H; see [18] or [19, Chapter 10]. Hence $\dim_{\mathcal{U}(H)} \mathcal{U}(H)a_1 \geq \frac{1}{m} = \mathrm{tr}(e)$. Of course, $\mathcal{U}(H)a_1 \subseteq \mathcal{U}(H)e$, and thus $\dim_{\mathcal{U}(H)} \mathcal{U}(H)a_1 \leq \dim_{\mathcal{U}(H)} \mathcal{U}(H)e = \mathrm{tr}(e)$. Hence $\dim_{\mathcal{U}(H)} \mathcal{U}(H)a_1 = \mathrm{tr}(e)$.

Also, $\mathcal{U}(H)(a_1 + 1 - e) = \mathcal{U}(H)a_1 \oplus \mathcal{U}(H)(1 - e)$. Hence

$$\begin{aligned}\dim_{\mathcal{U}(H)} \mathcal{U}(H)(a_1 + 1 - e) &= \dim_{\mathcal{U}(H)} \mathcal{U}(H)a_1 + \dim_{\mathcal{U}(H)} \mathcal{U}(H)(1 - e)\\ &= \mathrm{tr}(e) + \mathrm{tr}(1 - e) = 1.\end{aligned}$$

This implies that $a_1 + 1 - e$ is invertible in $\mathcal{U}(H)$. The $*$-dual of [17, Theorem 4] now implies that $a + 1 - e = 1(a_1 + 1 - e) + t_2a_2 + \cdots + t_na_n$ is invertible in $\mathcal{U}(G)$. It is then straightforward to show that a is invertible in $e\mathcal{U}(G)e$.

(iii) Suppose that $y \neq 0$. Then $x^*xyy^* = 0$, $yy^* \in e\mathbb{C}Ge - \{0\}$ and $x^*x \in e\mathcal{U}(G)e$. By (ii), yy^* is invertible in $e\mathcal{U}(G)e$. Hence $x^*x = 0$ and $x = 0$. □

3.2 Remark. The above proof shows that the conclusions of Theorem 3.1(ii) and (iii) hold under the following hypotheses: H is a normal subgroup of G; G/H is left orderable; the strong Atiyah conjecture holds for H; and, e is a nonzero projection in $\mathbb{C}H$ such that $\frac{1}{\mathrm{tr}(e)}$ is the least common multiple of the orders of the finite subgroups of H. □

The degenerate case of Theorem 3.1(ii) where $H = F = C = 1$ follows directly from [17, Theorem 2].

3.3 Theorem. *If G is locally indicable, or, more generally, left orderable, then every nonzero element of $\mathbb{C}G$ is invertible in $\mathcal{U}(G)$.* □

4 One-relator groups

We shall now calculate the L^2-Betti numbers of one-relator groups.

4.1 Notation. Suppose that G is a one-relator group, and let $\langle X \mid r\rangle$ be a one-relator presentation of G.

Here r is an element of the free group F on X and $G = F/\langle\!\langle r \rangle\!\rangle$.

Let $m = \exp_F(r)$ and let $d = |X|$. These are vague cardinals. Here $m \neq 0$; moreover, $m = \infty$ if and only if $r = 1$, in which case $G = F$.

If $m < \infty$, then $r = q^m$ for some $q \in F$. Let c denote the image of q in G, and let $C = \langle c \rangle \leq G$. Then C has order m. Let $e = \frac{1}{m}\sum_{x\in C} x \in \mathbb{C}G$.

If $m = \infty$, we define $e = 0 \in \mathbb{C}G$.

In any event e is a projection and $\mathrm{tr}(e) = \frac{1}{m}$.

There is an exact sequence of left $\mathbb{Z}G$-modules

$$\begin{aligned} 0 \longrightarrow \oplus_X \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad & \text{if } m = \infty, \\ 0 \longrightarrow \mathbb{Z}[G/C] \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad & \text{if } d = 1 \text{ and } m < \infty, \\ 0 \longrightarrow \mathbb{Z}[G/C] \longrightarrow \oplus_X \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad & \text{if } d \geq 2 \text{ and } m < \infty. \end{aligned}$$

see [7], specifically, Lemma 6.21 and $(*)$ on p. 167 in the proof of Theorem 6.22. In all cases, there is then an exact sequence of left $\mathbb{C}G$-modules

$$0 \longrightarrow \mathbb{C}Ge \xrightarrow{(a_{1,x})} \oplus_X \mathbb{C}G \xrightarrow{(b_{x,1})} \mathbb{C}G \longrightarrow \mathbb{C} \longrightarrow 0; \tag{4.1.1}$$

for each $x \in X$, $b_{x,1}$ is the image of $x - 1$ in $\mathbb{C}G$, and $a_{1,x}$ is the left Fox derivative $\frac{\partial r}{\partial x} = (me)\frac{\partial q}{\partial x} \in e\mathbb{C}G$.

If $d < \infty$, then G is of type VFL and

$$\chi(G) = 1 - d + \frac{1}{m} \in (-\infty, 1]; \tag{4.1.2}$$

see Theorem 6.22 and Corollary 6.15 of [7], for the cases where $m < \infty$ and $m = \infty$, respectively.

In the case where $d = \infty$, that is, G is a non-finitely-generated one-relator group, we *define* $\chi(G) := -\infty$. This is non-standard, but it extends (4.1.2).

It is easy to verify that $\frac{1}{|G|} = \max\{\chi(G), 0\}$. In fact, by abelianizing G, we see that G is finite if and only if either $d = 1$ and $m < \infty$, or $d = 0$ (and hence $m = \infty$). □

We shall now prove the following.

4.2 Theorem. *If G is a one-relator group, then, for $n \in \mathbb{N}$,*

$$b_n^{(2)}(G) = \begin{cases} \max\{\chi(G), 0\} \ (= \frac{1}{|G|}) & \text{if } n = 0, \\ \max\{-\chi(G), 0\} & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \tag{4.2.1}$$

Proof. Suppose that Notation 4.1 holds.

Unaugmenting (4.1.1) and applying $\mathcal{U}(G) \otimes_{\mathbb{C}G} -$ gives

$$0 \longrightarrow \mathcal{U}(G)e \xrightarrow{(a_{1,x})} \oplus_X \mathcal{U}(G) \xrightarrow{(b_{x,1})} \mathcal{U}(G) \longrightarrow 0; \tag{4.2.2}$$

the homology of (4.2.2) is then $\mathrm{H}_*(G;\mathcal{U}(G))$.

We claim that

$$\text{if } y \in \mathcal{U}(G)e - \{0\} \text{ and } a \in e\mathbb{C}G - \{0\}, \text{ then } ya \neq 0. \tag{4.2.3}$$

This is vacuous if $m = \infty$.

If $m < \infty$, let H denote the normal subgroup of G generated by c. Then $G/H = \langle X \mid q \rangle$ is a torsion-free one-relator group. Hence G/H is locally indicable by [3, Theorem 3], [13, Theorem 4.2] or [14, Corollary 3.2]. Also H is the free product of certain G-conjugates of C, by [11, Theorem 1]. By mapping each of these conjugates of C isomorphically to C, we obtain an epimorphism $H \twoheadrightarrow C$. Applying [8, Proposition I.4.6] to this epimorphism, we see that its kernel F is free. Clearly, $H = F \rtimes C$. Now (4.2.3) holds by Theorem 3.1(iii).

Since $(a_{1,x})$ is injective in (4.1.1), either $e = 0$ or there is some $x_0 \in X$ such that $a_{1,x_0} \neq 0$. It follows from (4.2.3) that $(a_{1,x})$ is injective in (4.2.2), and hence $\mathrm{H}_2(G;\mathcal{U}(G)) = 0$. On taking $\mathcal{U}(G)$-dimensions, we find $b_2^{(2)}(G) = 0$, and $\dim_{\mathcal{U}(G)} \mathrm{im}((a_{1,x})) = \frac{1}{m}$.

If either $d \geq 2$, or $d = 1$ and $m = \infty$, then, by abelianizing, we see that there is some $x_1 \in X$ whose image in G has infinite order. By (1.0.1), we see that $(b_{x,1})$ is surjective in (4.2.2), and hence $\mathrm{H}_0(G;\mathcal{U}(G)) = 0$. On taking $\mathcal{U}(G)$-dimensions, we find that $b_0^{(2)}(G) = 0$, $\dim_{\mathcal{U}(G)} \mathrm{im}((b_{x,1})) = 1$, and $\dim_{\mathcal{U}(G)} \ker((b_{x,1})) = d - 1$. Now

$$b_1^{(2)}(G) = \dim_{\mathcal{U}(G)} \ker((b_{x,1})) - \dim_{\mathcal{U}(G)} \mathrm{im}((a_{1,x})) = d - 1 - \frac{1}{m} = -\chi(G).$$

Thus (4.2.1) holds.

This leaves the cases where either $d = 0$ or $d = 1$ and $m < \infty$. Here G is finite cyclic, and again (4.2.1) holds. □

5 Surface-plus-one-relation groups

We next calculate the L^2-Betti numbers for an arbitrary surface-plus-one-relation group $G = \pi_1(\Sigma)/\langle\!\langle \alpha \rangle\!\rangle$, where Σ is a connected orientable surface, possibly with boundary and not necessarily compact, and $\langle\!\langle \alpha \rangle\!\rangle$ is the normal closure of a single element $\alpha \in \pi_1(S)$.

By the results of the previous section, we may assume that the implicit presentation of G has more than one relator. As explained in Section 2, Σ must a closed surface. Let g denote the genus of Σ. Then $g \in \mathbb{N}$ and

$$\pi_1(\Sigma) = \langle \quad x_1, x_2, \ldots, x_{2g-1}, x_{2g} \quad | \quad [x_1, x_2][x_3, x_4] \cdots [x_{2g-1}, x_{2g}] \quad \rangle,$$

where $[x, y]$ denotes $xyx^{-1}y^{-1}$. Since this is a one-relator presentation, we have $\alpha \neq 1$. In particular, g is nonzero. The non one-relator cases are included in the following.

5.1 Theorem. *Let Σ be a closed orientable surface of genus at least one, let $S = \pi_1(\Sigma)$, let α be a nontrivial element of S, and let $G = S/\langle\!\langle \alpha \rangle\!\rangle$.*

Let g denote the genus of Σ, let $m = \exp_S(\alpha)$, and let Q be a nonzero ring in which $\frac{1}{m}$ is defined, that is, if $m < \infty$ then $mQ = Q$. Then the following hold.

(i) *G is of type* VFL *and* $\chi(G) = \min\{2 - 2g + \frac{1}{m}, 0\} = \begin{cases} 0 & \text{if } g = 1, \\ 2 - 2g + \frac{1}{m} & \text{if } g \geq 2. \end{cases}$

(ii) $\mathrm{cd}_Q\, G = \min\{2, g\} = \begin{cases} 1 & \text{if } g = 1, \\ 2 & \text{if } g \geq 2. \end{cases}$

(iii) *For $n \in \mathbb{N}$,* $b_n^{(2)}(G) = -\delta_{n,1}\chi(G) = \begin{cases} -\chi(G) & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$

Proof. We break the proof up into a series of lemmas and summaries of notation.

5.2 Notation. As in [8, Examples I.3.5(v)], the expression $S_1 *_{S_0} s$ will denote an HNN extension, where it is understood that S_1 is a group, S_0 is a subgroup of S_1 and s is an injective group homomorphism $s\colon S_0 \to S_1$, $a \mapsto a^s$. The image of this homomorphism is denoted S_0^s. □

5.3 Lemma (Hempel). *If $g \geq 2$, then there exists an HNN-decomposition $S = S_1 *_{S_0} s$ such that S_1 is a free group, α lies in S_1, and the normal subgroup of S_1 generated by α intersects both S_0 and S_0^s trivially.*

Hence, $G = S/\langle\!\langle \alpha \rangle\!\rangle$ has a matching HNN-decomposition $S/\langle\!\langle \alpha \rangle\!\rangle = S_1/\langle\!\langle \alpha \rangle\!\rangle *_{S_0} s$.

Proof. This was implicit in the proof of [12, Theorem 2.2], and was made explicit in [15, Proposition 2.1]. □

5.4 Lemma (Hempel). *If $m < \infty$, there exists $\beta \in S$ such that $\beta^m = \alpha$, and the image of β in G has order m.*

Proof. As this is obvious for $g = 1$, we may assume that $g \geq 2$. Thus we have matching HNN-decompositions $S = S_1 *_{S_0} s$ and $G = S/\langle\!\langle \alpha \rangle\!\rangle = S_1/\langle\!\langle \alpha \rangle\!\rangle *_{S_0} s$, as in Lemma 5.3.

Let $m' = \exp_{S_1} \alpha$. Since $\alpha \neq 1$ and S_1 is free, we see that $m' < \infty$. Choose $\beta \in S_1$ such that $\beta^{m'} = \alpha$. Let c denote the image of β in G, and let $C = \langle c \rangle \leq G$. Then C has order m', and every torsion subgroup of $S_1/\langle\!\langle \alpha \rangle\!\rangle$ embeds in C. From the HNN decomposition for G, we see that any finite subgroup of G is conjugate to a subgroup of $S_1/\langle\!\langle \alpha \rangle\!\rangle$, and hence has order dividing m'.

A similar argument shows that for any positive integer i, $S/\langle\!\langle \alpha^i \rangle\!\rangle$ has a matching HNN decomposition, and therefore has a subgroup of order $m'i$ and a subgroup of order i. It follows that if $\alpha = \gamma^j$ for some positive integer j then $S/\langle\!\langle \alpha \rangle\!\rangle$ has a subgroup of order j, and hence j divides m'. It now follows that $m = m' < \infty$. □

5.5 Notation. Let β denote an element of S such that $\beta^m = \alpha$.

Let c denote the image of β in G. Let $C = \langle c \rangle$, a cyclic subgroup of G of order m. Let $e = \frac{1}{m} \sum_{x \in C} x$, an idempotent element of $\mathbb{C}G$ with $\operatorname{tr}(e) = \frac{1}{m}$; we shall also view e as an idempotent element of QG.

Let H denote the normal subgroup of G generated by c; thus, $G/H \simeq S/\langle\!\langle \beta \rangle\!\rangle$. □

5.6 Lemma. (i) *H has a free subgroup F such that $H = F \rtimes C$.*

(ii) *G/H is locally indicable.*

(iii) *Every torsion subgroup of G embeds in C.*

(iv) *If $x \in \mathcal{U}(G)e - \{0\}$ and $y \in e\mathbb{C}G - \{0\}$, then $xy \neq 0$.*

Proof. (i). As this is clear for $g = 1$, we may assume that $g \geq 2$.

By Lemma 5.3 with β in place of α, there exists an HNN-decomposition $S = S_1 *_{S_0} s$ where S_1 is a free group, β lies in S_1, and the normal subgroup of S_1 generated by β intersects both S_0 and S_0^s trivially. Hence α lies in S_1, and the normal subgroup of S_1 generated by α intersects both S_0 and S_0^s trivially. It follows that we can make identifications

$$G = S/\langle\!\langle \alpha \rangle\!\rangle = S_1/\langle\!\langle \alpha \rangle\!\rangle *_{S_0} s \quad \text{and} \quad G/H = S/\langle\!\langle \beta \rangle\!\rangle = S_1/\langle\!\langle \beta \rangle\!\rangle *_{S_0} s.$$

Thus we have matching HNN-decompositions for S, G and G/H.

Let us apply Bass-Serre theory, following, for example, [8, Chapter 1]. Consider the action of H on the Bass-Serre tree for the above HNN-decomposition of G. Then H acts freely on the edges. Let H_0 denote the normal subgroup of $S_1/\langle\!\langle \alpha \rangle\!\rangle$ generated by c. Then H_0 is a vertex stabilizer for the H-action, and the other vertex stabilizers are G-conjugates of H_0. By Bass-Serre theory, or the Kurosh Subgroup Theorem, H is the free product of a free group and various G-conjugates of H_0.

By [11, Theorem 1], H_0 itself is a free product of certain $S_1/\langle\!\langle \alpha \rangle\!\rangle$-conjugates of C.

Thus H is the free product a free group and various G-conjugates of C. If we map each of these G-conjugates of C isomorphically to C, and map the free group to 1, we obtain an epimorphism $H \twoheadrightarrow C$. Applying [8, Proposition I.4.6] to this epimorphism, we see that its kernel F is free. Clearly, $H = F \rtimes C$. This proves (i).

(ii). Since $G/H = S/\langle\!\langle \beta \rangle\!\rangle$ and β is not a proper power in S, G/H is locally indicable by [12, Theorem 2.2].

(iii) and (iv) hold by Theorem 3.1. □

Let us dispose of the case where $g = 1$, which is well known and included only for completeness.

5.7 Lemma. *If $g = 1$, then the following hold.*

(i) *$H = C$ and G/C is infinite cyclic generated by xC for some $x \in G$.*

(ii) $0 \longrightarrow \mathbb{Z}[G/C] \xrightarrow{x-1} \mathbb{Z}[G/C] \longrightarrow \mathbb{Z} \longrightarrow 0$ *is an exact sequence of left $\mathbb{Z}G$-modules.*

(iii) $0 \longrightarrow QGe \xrightarrow{x-1} QGe \longrightarrow Q \longrightarrow 0$ *is an exact sequence of left QG-modules.*

(iv) *$\langle x \rangle$ is an infinite cyclic subgroup of G of finite index, G is of type* VFL, *$\chi(G) = 0$ and $\mathrm{cd}_Q\, G = 1$.*

(v) *The homology of* $0 \longrightarrow \mathcal{U}(G)e \xrightarrow{x-1} \mathcal{U}(G)e \longrightarrow 0$ *is* $\mathrm{H}_*(G;\mathcal{U}(G))$.

(vi) *For each $n \in \mathbb{N}$, $b_n^{(2)}(G) = 0$.* □

5.8 Remark. For $g = 1$, Lemma 5.7(ii) gives the augmented cellular chain complex of a one-dimensional $\underline{\mathrm{E}}(G)$ which resembles the real line. □

5.9 Notation. Henceforth we assume that $g \geq 2$.

Let $X = \{x_1, x_2, \ldots, x_{2g-1}, x_{2g}\}$, let F be the free group on X, and let $r_1 = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \in F$. Then $S = \langle X \mid r_1 \rangle$.

Let q_2 be any element of F which maps to β in S, and let $r_2 = q_2^m$. Then $G = \langle X \mid r_1, r_2 \rangle$.

For $i \in \{1, 2\}$, $j \in \{1, \ldots, 2g\}$, we set $a_{i,j} := \frac{\partial r_i}{\partial x_j} \in \mathbb{Z}G$, the left Fox derivatives, and $b_{j,1} := x_j - 1 \in \mathbb{Z}G$.

Notice that $me = \sum_{x \in C} x \in \mathbb{Z}G$ and $a_{2,j} = \frac{\partial r_2}{\partial x_j} = (me)\frac{\partial q_2}{\partial x_j}$. □

5.10 Lemma (Howie). *The sequence of left $\mathbb{Z}G$-modules*

$$0 \longrightarrow \mathbb{Z}G \oplus \mathbb{Z}[G/C] \xrightarrow{(a_{i,j})} \mathbb{Z}G^{2g} \xrightarrow{(b_{j,1})} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad (5.10.1)$$

is exact.

Proof. Howie [15, Theorem 3.5] describes a $K(G,1)$, and it is straightforward to give it a CW-structure as follows.

We take a $K(S,1)$ with one zero-cell, $2g$ one-cells, and a two-cell which is a $2g$-gon, and then the exact sequence of left $\mathbb{Z}S$-modules arising from the augmented cellular chain complex of the universal cover of the $K(S,1)$ is

$$0 \longrightarrow \mathbb{Z}S \xrightarrow{(a_{1,j})} \mathbb{Z}S^{2g} \xrightarrow{(b_{j,1})} \mathbb{Z}S \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where we view the $a_{1,j}$ and $b_{j,1}$ as elements of $\mathbb{Z}S$.

We take a $K(C,1)$ with one cell in each dimension such that the infinitely repeating exact sequence of left $\mathbb{Z}C$-modules arising from the augmented cellular chain complex of the universal cover of the $K(C,1)$ is

$$\cdots \longrightarrow \mathbb{Z}C \xrightarrow{me} \mathbb{Z}C \xrightarrow{c-1} \mathbb{Z}C \xrightarrow{me} \mathbb{Z}C \xrightarrow{c-1} \mathbb{Z}C \longrightarrow \mathbb{Z} \longrightarrow 0.$$

By [15, Theorem 3.5], we get a $K(G,1)$ by melding the one-skeleton of our $K(C,1)$ into the one-skeleton of our $K(S,1)$ in the natural way; the attaching map of the two-cell at the homology level is then $(a_{2,j})$. The exact sequence of left $\mathbb{Z}G$-modules arising from the augmented cellular chain complex of the three-skeleton of the universal cover of the $K(G,1)$ is

$$\mathbb{Z}G \xrightarrow{(0,1-c)} \mathbb{Z}G^2 \xrightarrow{(a_{i,j})} \mathbb{Z}G^{2g} \xrightarrow{(b_{j,1})} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The lemma now follows easily. □

We now imitate the proof of [11, Theorem 2].

5.11 Lemma. *G is of type* VFL *and* $\chi(G) = 2 - 2g + \frac{1}{m}$.

Proof. Let p be a prime divisor of m. It was shown in [1] that S is residually a finite p-group; see [10, Theorem B] for an alternative proof. Hence there exists a finite p-group $P = P(p)$ and a homomorphism $S \to P$ whose kernel does not contain $\beta^{\frac{m}{p}}$, and we assume that P has smallest possible order. The centre $\mathrm{Z}(P)$ of P is nontrivial. By minimality of P, $\beta^{\frac{m}{p}}$ lies in the kernel of the composite $S \twoheadrightarrow P \twoheadrightarrow P/\mathrm{Z}(P)$. Thus $\beta^{\frac{m}{p}}$, and β^m, are mapped to $\mathrm{Z}(P)$. By minimality of P, β^m is mapped to 1 in P.

By considering the direct product of such $P(p)$, one for each prime divisor p of m, we find that there is a finite quotient of S in which the image of β has order exactly m.

Hence there exists a normal subgroup N of G such that N has finite index in G and $N \cap C = \{1\}$. It follows that N acts freely on G/C. The number of orbits is

$$|N\backslash(G/C)| = |N\backslash G/C| = |(N\backslash G)/C| = [G : N]/m,$$

where the last equality holds since C acts freely on $N\backslash G$, on the right.

Now (5.10.1) is a resolution of $\mathbb{Z}$ by free left $\mathbb{Z}N$-modules. Thus N is of type FL, and, in particular, N is torsion-free. It is now a simple matter to calculate $\chi(G)$ $(= \frac{1}{[G:N]}\chi(N))$. □

Together Lemma 5.7(iv) and Lemma 5.11 give Theorem 5.1(i).

By Lemma 5.10, the following is clear.

5.12 Corollary. *The sequence of left QG-modules*

$$0 \longrightarrow QG \oplus QGe \xrightarrow{(a_{i,j})} QG^{2g} \xrightarrow{(b_{j,1})} QG \longrightarrow Q \longrightarrow 0$$

is exact. □

5.13 Lemma. $\operatorname{cd}_Q G = 2$.

Proof. By Corollary 5.12, $\operatorname{cd}_Q G \leq 2$. It remains to show that $\operatorname{cd}_Q G > 1$. Let us suppose that $\operatorname{cd}_Q G \leq 1$ and derive a contradiction.

By Notation 5.5 and Lemma 5.6(ii), H is the (normal) subgroup of G generated by the elements of finite order. By Dunwoody's Theorem [8, Theorem IV.3.13], G is the fundamental group of a graph of finite groups; by [8, Proposition I.7.11], H is the normal subgroup of G generated by the vertex groups. From the presentation of G as in [8, Notation I.7.1], it can be seen that G/H is a free group.

Since $G/H = S/\langle\!\langle \beta \rangle\!\rangle$, the abelianization of G/H has $\mathbb{Z}$-rank $2g$ or $2g-1$. Thus the rank of the free group G/H is $2g$ or $2g-1$. Hence $\chi(S/\langle\!\langle \beta \rangle\!\rangle)$ is $1-2g$ or $2-2g$.

But $\chi(S/\langle\!\langle \beta \rangle\!\rangle) = 3-2g$ by Lemma 5.11. This is a contradiction. □

Together Lemma 5.7(iv) and Lemma 5.13 give Theorem 5.1(ii).

By Corollary 5.12 with $Q = \mathbb{C}$, the following is clear.

5.14 Corollary. *The homology of*

$$0 \longrightarrow \mathcal{U}(G) \oplus \mathcal{U}(G)e \xrightarrow{(a_{i,j})} \mathcal{U}(G)^{2g} \xrightarrow{(b_{j,1})} \mathcal{U}(G) \longrightarrow 0$$

is $\mathrm{H}_*(G;\mathcal{U}(G))$. □

We now come to the subtle part of the argument.

5.15 Lemma. $\mathcal{U}(G) \oplus \mathcal{U}(G)e \xrightarrow{(a_{i,j})} \mathcal{U}(G)^{2g}$ *is injective.*

Proof. Let (u,v) be an arbitrary element of the kernel. Thus, $(u,v) \in \mathcal{U}(G) \oplus \mathcal{U}(G)e$ and

$$\text{for each } j \in \{1,\ldots,2g\}, \quad ua_{1,j} + va_{2,j} = 0 \text{ in } \mathcal{U}(G). \tag{5.15.1}$$

Consider first the case where u does not lie in $v\mathbb{C}G$. We shall obtain a contradiction.

We form the right $\mathbb{C}G$-module $W = \mathcal{U}(G)/(v\mathbb{C}G)$, and let $w = u + v\mathbb{C}G \in W$. By (5.15.1),

$$\text{for each } j \in \{1,\ldots,2g\}, \quad wa_{1,j} = 0 \text{ in } W. \tag{5.15.2}$$

Let $K = \{x \in G \mid wx = w\}$. Clearly, K is a subgroup of G.

We claim that $K = G$; it suffices to show that $\{x_1,\ldots,x_{2g}\} \subseteq K$.

We will show by induction that, if $j \in \{0,1,\ldots,g\}$, then $\{x_1,\ldots,x_{2j}\} \subseteq K$. This is clearly true for $j=0$. Suppose that $j \in \{1,\ldots,g\}$ and that it is true for $j-1$. We will show it is true for j. Let $k = [x_1,x_2]\cdots[x_{2j-3},x_{2j-2}]$; then k lies in K by the induction hypothesis. Recall that $r_1 = [x_1,x_2]\cdots[x_{2g-1},x_{2g}]$. By (5.15.2) and Notation 5.9,

$$0 = wa_{1,2j-1} = w\frac{\partial r_1}{\partial x_{2j-1}} = wk(1 - x_{2j-1}x_{2j}x_{2j-1}^{-1})$$

and

$$0 = wa_{1,2j} = w\frac{\partial r_1}{\partial x_{2j}} = wkx_{2j-1}(1 - x_{2j}x_{2j-1}^{-1}x_{2j}^{-1}).$$

Since $K = \{x \in G \mid w(1-x) = 0\}$, we see that K contains

$$k(x_{2j-1}x_{2j}x_{2j-1}^{-1})k^{-1} \text{ and } (kx_{2j-1})(x_{2j}x_{2j-1}^{-1}x_{2j}^{-1})(kx_{2j-1})^{-1}.$$

Thus K contains

$$x_{2j-1}x_{2j}x_{2j-1}^{-1} \text{ and } x_{2j-1}(x_{2j}x_{2j-1}^{-1}x_{2j}^{-1})x_{2j-1}^{-1},$$

and it follows easily that K contains $x_{2j}^{-1}x_{2j-1}^{-1}$, x_{2j-1} and x_{2j}. This completes the proof by induction.

Hence, $K = G$, and w is fixed under the right G-action on W. Thus, the subset $u + v\mathbb{C}G$ of $\mathcal{U}(G)$ is closed under the right G-action on $\mathcal{U}(G)$. We denote the set $u + v\mathbb{C}G$ viewed as right G-set by $(u + v\mathbb{C}G)_G$. Notice that $u + v\mathbb{C}G$ does not contain 0.

By Lemma 5.6(iv), the surjective map $e\mathbb{C}G \to v\mathbb{C}G$, $x \mapsto vx$, is either injective or zero. In either event, $v\mathbb{C}G$ is a projective right $\mathbb{C}G$-module. By the left-right dual of [9, Corollary 5.6] there exists a right G-tree with finite edge stabilizers and vertex set $(u+v\mathbb{C}G)_G$. It follows that there exists a (left) G-tree T with finite edge stabilizers and vertex set ${}_G(u + v\mathbb{C}G)^* \subseteq {}_G(\mathcal{U}(G) - \{0\})$.

Each vertex stabilizer for T is torsion, by (1.0.1), and hence embeds in C, by Lemma 5.6(iii). By [8, Theorem IV.3.13], $\operatorname{cd}_Q G \le 1$ which contradicts Lemma 5.13; in essence, T is a one-dimensional $\underline{E}(G)$. Alternatively, one can use T to prove that $b_2^{(2)}(G) = 0$ and deduce that $(u, v) = (0, 0)$, which is also a contradiction.

Thus u lies in $v\mathbb{C}G$, and there exists $y \in e\mathbb{C}G$ such that $u = vy$.

We consider first the case where $v \neq 0$. For each $j \in \{1, \ldots, 2g\}$,

$$v(ya_{1,j} + a_{2,j}) = ua_{1,j} + va_{2,j} = 0$$

by (5.15.1), and, by Lemma 5.6(iv), $0 = ya_{1,j} + a_{2,j} = ya_{1,j} + ea_{2,j}$. Hence, (y, e) lies in the kernel of $\mathbb{C}G \oplus \mathbb{C}Ge \xrightarrow{(a_{i,j})} \mathbb{C}G^{2g}$; since this map is injective by Corollary 5.12, we see $e = 0$, which is a contradiction.

Thus $v = 0$, and hence $u = 0$. □

By Lemma 5.15 and Remark 1.1 it is straightforward to obtain the following.

5.16 Lemma. *The $\mathcal{U}(G)$-dimensions of the kernel and the image of the map $\mathcal{U}(G) \oplus \mathcal{U}(G)e \xrightarrow{(a_{i,j})} \mathcal{U}(G)^{2g}$ are 0 and $1 + \frac{1}{m}$, respectively.*

The $\mathcal{U}(G)$-dimensions of the image and the kernel of the map $\mathcal{U}(G)^{2g} \xrightarrow{(b_{j,1})} \mathcal{U}(G)$ are 1 and $2g-1$, respectively.

For $n \in \mathbb{N}$, $b_n^{(2)}(G) = \begin{cases} (2g-1) - (1 + \frac{1}{m}) & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$ □

Together Lemma 5.7(vi) and Lemma 5.16 give Theorem 5.1(iii). This completes the proof of Theorem 5.1. □

6 Left-orderable groups

Throughout this section we will frequently make the following assumption.

6.1 Hypotheses. There exist nonzero rings Z and U such that ZG is a subring of U and each nonzero element of ZG is invertible in U.

This holds, for example, if G is locally indicable, or, more generally, left orderable, with Z being any subring of $\mathbb{C}$, and U being $\mathcal{U}(G)$, by Theorem 3.3.

Notice that ZG has no nonzero zerodivisors, and hence G is torsion free. □

6.2 Lemma. *Let U be a ring, and let X and Y be sets.*

Let A and B be nonzero row-finite matrices over U in which each nonzero entry is invertible, such that A is $X \times 2$, B is $2 \times Y$, and the product AB is the zero $X \times Y$ matrix.

Then $\oplus_X U \xrightarrow{A} U^2 \xrightarrow{B} \oplus_Y U$ is an exact sequence of free left U-modules.

Moreover, U^2 has a left U-basis v_1, v_2 such that $\ker B = \operatorname{im} A = Uv_1$ and B induces an isomorphism $Uv_2 \simeq \operatorname{im} B$.

Proof. Write $A = (a_{x,i})$ and $B = (b_{i,y})$.

There exists $x_0 \in X$ such that $(a_{x_0,1}, a_{x_0,2}) \neq (0,0)$. We take $v_1 = (a_{x_0,1}, a_{x_0,2})$. Clearly $Uv_1 \subseteq \operatorname{im} A \subseteq \ker B$. Without loss of generality, there exists $y_0 \in Y$ such that b_{1,y_0} is invertible in U. We take $v_2 = (1,0)$.

Since $AB = 0$, $a_{x_0,1}b_{1,y_0} + a_{x_0,2}b_{2,y_0} = 0$. Thus $a_{x_0,1} = -a_{x_0,2}b_{2,y_0}b_{1,y_0}^{-1}$. Hence $a_{x_0,2}$ cannot be zero, and is therefore invertible.

Hence v_1, v_2 is a basis of U^2, and $b_{2,y_0}b_{1,y_0}^{-1} = -a_{x_0,2}^{-1}a_{x_0,1}$.

Consider any $(a_1, a_2) \in \ker B$. Then $a_1 b_{1,y_0} + a_2 b_{2,y_0} = 0$, and

$$\begin{aligned}(a_1, a_2) &= (-a_2 b_{2,y_0} b_{1,y_0}^{-1}, a_2) = a_2(-b_{2,y_0}b_{1,y_0}^{-1}, 1) \\ &= a_2(a_{x_0,2}^{-1}a_{x_0,1}, 1) = a_2 a_{x_0,2}^{-1}(a_{x_0,1}, a_{x_0,2}) = a_2 a_{x_0,2}^{-1} v_1 \in Uv_1,\end{aligned}$$

as desired. Finally, $Uv_2 \simeq (Uv_1 + Uv_2)/Uv_1 = U^2/\ker B \simeq \operatorname{im} B$. □

6.3 Remark. We see from the proof that the hypotheses that A and B are nonzero and every nonzero entry in A and B is invertible can be replaced with the hypotheses that some element of the first row of B is invertible, and some element of the second column of A is invertible.

There are other variations, but the stated form is most convenient for our purposes. □

6.4 Proposition. *Suppose that Hypotheses* 6.1 *hold, and suppose that there exists a positive integer n and a resolution* (1.0.3) *of Z by projective left ZG-modules such that $P_n = ZG^2$. Then either the map $P_{n+1} \to P_n$ in* (1.0.3) *is the zero map or* $\mathrm{H}_n(G; U) = 0$.

Proof. We may assume that $P_{n+1} \to P_n$ is nonzero. Then we have an exact sequence

$$P_{n+1} \to P_n \to P_{n-1}, \tag{6.4.1}$$

and we want to deduce that

$$U \otimes_{ZG} P_{n+1} \to U \otimes_{ZG} P_n \to U \otimes_{ZG} P_{n-1} \tag{6.4.2}$$

remains exact.

This is clear if $P_n \to P_{n-1}$ is the zero map. Thus we may assume that the maps in (6.4.1) are nonzero.

By adding a suitable ZG-projective summand to P_{n+1} with a zero map to P_n, we may assume that P_{n+1} is ZG-free without affecting the images. Similarly, we may assume that P_{n-1} is ZG-free without affecting the kernels. Thus we may assume that we have specified ZG-bases of P_{n+1}, P_n and P_{n-1}, and that the maps in (6.4.1) are represented by nonzero matrices over ZG.

The maps in (6.4.2) are then represented by nonzero matrices over U with all coefficients lying in ZG. Now we may apply Lemma 6.2 to deduce that (6.4.2) is exact, as desired. □

6.5 Remark. In Proposition 6.4, if we replace the hypothesis $P_n = ZG^2$ with the hypothesis $P_n = ZG^1$, then it is easy to see that at least one of the maps $P_{n+1} \to P_n$, $P_n \to P_{n-1}$ is necessarily the zero map. □

Applying Proposition 6.4 with $U = \mathcal{U}(G)$, together with Theorem 3.3, we obtain the following two results.

6.6 Corollary. *Let G be a left-orderable group, and let Z be a subring of $\mathbb{C}$. Suppose that there exists a positive integer n and a resolution* (1.0.3) *of Z by projective left ZG-modules such that $P_n = ZG^2$. Then either $\operatorname{cd}_Z G \le n$ or $b_n^{(2)}(G) = 0$.* □

6.7 Corollary. *If G is a left-orderable group, and there exists an exact $\mathbb{C}G$-sequence of the form*

$$\cdots \xrightarrow{\partial_3} \mathbb{C}G^2 \xrightarrow{\partial_2} \mathbb{C}G^2 \xrightarrow{\partial_1} \mathbb{C}G^2 \xrightarrow{\partial_0} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0 \qquad (6.7.1)$$

in which all the ∂_n are nonzero, then all the $b_n^{(2)}(G)$ are zero.

Proof. Since ∂_0 is nonzero, we see that G is nontrivial. Since G is torsion-free, $b_0^{(2)}(G) = 0$. For $n \ge 1$, $b_n^{(2)}(G) = 0$ by Proposition 6.4. □

6.8 Corollary (Lück [19, Theorem 7.10]). *All the L^2-Betti numbers of Thompson's group F vanish.*

Proof. This follows from Corollary 6.7 since F is orderable, see [6], and has a resolution as in (6.7.1), see [4]. □

We now look at situations where we can deduce that a two-generator group is free.

6.9 Proposition. *Suppose that Hypotheses* 6.1 *hold. The following are equivalent.*

(a) *G is a two-generator group, and $\mathrm{H}_1(G; U) \simeq U$.*

(b) *G is a two-generator group, and $\mathrm{H}_1(G; U) \ne 0$.*

(c) *G is free of rank two.*

Proof. (a) ⇒ (b) is obvious.

(b) ⇒ (c). Let $\{x, y\}$ be a generating set of G. Then we have an exact sequence of left ZG-modules

$$\oplus_R ZG \longrightarrow ZG^2 \xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} ZG \longrightarrow Z \longrightarrow 0,$$

where R is the set of relators which have a nonzero left Fox derivative in ZG. By Proposition 6.4 with $n = 1$, we see that R is empty, and that the augmentation ideal is left ZG-free on $x - 1$ and $y - 1$.

A result of Bass-Nakayama [21, Proposition 1.6] then says that G is freely generated by x and y. This can be seen geometrically, as follows. Let $\Gamma = \Gamma(G, \{x, y\})$ denote the Cayley graph of G with respect to the subset $\{x, y\}$. The above exact sequence is precisely the augmented cellular Z-chain complex of Γ. It is then straightforward to show that Γ is a tree, and that G is freely generated by x and y.

(c) $\Rightarrow$ (a) is straightforward. □

6.10 Corollary. *The following are equivalent.*

(a) *G is a two-generator left-orderable group and $b_1^{(2)}(G) \neq 0$.*

(b) *G is free of rank two.* □

6.11 Theorem. *Suppose that Hypotheses* 6.1 *hold. If* $\operatorname{hd}_Z G \leq 1$ *then every two-generator subgroup of G is free.*

Proof. Since the hypotheses pass to subgroups, we may assume that G itself is generated by two elements, and it remains to show that G is free.

We calculate $\mathrm{H}_*(G; U)$ in the case where G is not free.

By Hypotheses 6.1, G is torsion free. As in Remark 1.1, if $\mathrm{H}_0(G; U) \neq 0$, then G is free of rank zero. Thus we may assume that $\mathrm{H}_0(G; U) = 0$.

By Proposition 6.9, if $\mathrm{H}_1(G, U) \neq 0$, then G is free of rank two. Thus we may assume that $\mathrm{H}_1(G; U) = 0$.

Since $\operatorname{hd}_Z G \leq 1$, $\mathrm{H}_n(G; U) = 0$ for all $n \geq 2$.

In summary, we may assume that $\mathrm{H}_*(G; U) = 0$.

By [2, Theorem 4.6(b)], since G is countable and $\operatorname{hd}_Z G \leq 1$, we have $\operatorname{cd}_Z G \leq 2$; in essence, the augmentation ideal ω of ZG is a countably-related flat left ZG-module, hence the projective dimension of ${}_{ZG}\omega$ is at most one. Since G is a two-generator group, we have a resolution of Z by projective left ZG-modules

$$0 \longrightarrow P \longrightarrow ZG^2 \longrightarrow ZG \longrightarrow Z \longrightarrow 0.$$

Since $\mathrm{H}_*(G; U) = 0$, we have an exact sequence of projective left U-modules

$$0 \longrightarrow U \otimes_{ZG} P \longrightarrow U^2 \longrightarrow U \longrightarrow 0.$$

This sequence splits, and we see that ${}_U(U \otimes_{\mathbb{Z}G} P)$ is finitely generated.

Hence ${}_{ZG}P$ is finitely generated, by the following standard argument. Let R be a set such that P is a ZG-summand of $\oplus_R ZG$, that is, P is a ZG-submodule of $\oplus_R ZG$ and we have a ZG-linear retraction of $\oplus_R ZG$ onto P. We may assume that R is minimal, that is, for each $r \in R$, the image of P under projection onto the rth coordinate is nonzero. Then $U \otimes_{ZG} P$ is a U-submodule of $\oplus_R U$, and here also R is minimal. Since ${}_U(U \otimes_{\mathbb{Z}G} P)$ is finitely generated, R is finite, as desired.

Now ${}_{ZG}Z$ has a resolution by finitely generated projective left ZG-modules. By [2, Theorem 4.6(c)], $\operatorname{cd}_Z G \leq 1$; in essence, ${}_{ZG}\omega$ is finitely related and flat, and is therefore projective. Since G is torsion free, G is free by Stallings' Theorem; see Remark II.2.3(ii) (or Corollary IV.3.14) in [8]. □

6.12 Corollary. *Suppose that G is locally indicable, or, more generally, that G is left orderable. If* $\operatorname{hd} G \leq 1$ *then every two-generator subgroup of G is free.* □

We now turn from two-generator groups to two-relator groups.

6.13 Proposition. *Suppose that G is left orderable, that G has a presentation $\langle X \mid R\rangle$ with $|R| = 2$, and that $\operatorname{cd} G \geq 3$.*

Then $b_0^{(2)}(G) = 0$, $b_1^{(2)}(G) = |X| - 2$, and $b_2^{(2)}(G) = 0$.

Proof. The given presentation of G yields an exact sequence of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow \oplus_Y \mathbb{Z}G \xrightarrow{A} \mathbb{Z}G^2 \xrightarrow{B} \oplus_X \mathbb{Z}G \xrightarrow{C} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Then $H_*(G,\mathcal{U}(G))$ is the homology of the sequence

$$\cdots \longrightarrow \oplus_Y \mathcal{U}(G) \xrightarrow{A} \mathcal{U}(G)^2 \xrightarrow{B} \oplus_X \mathcal{U}(G) \xrightarrow{C} \mathcal{U}(G) \longrightarrow 0. \tag{6.13.1}$$

Since G is left orderable, G is torsion free. Since $\operatorname{cd} G \neq 0$, G is non-trivial. Hence G has an element of infinite order. By Remark 1.1, $b_0^{(2)}(G) = 0$ and the $\mathcal{U}(G)$-dimension of $\ker C$ in (6.13.1) is $|X| - 1$.

Since G is left orderable, all nonzero elements of $\mathbb{C}G$ are invertible in $\mathcal{U}(G)$ by Theorem 3.3. Since $\operatorname{cd} G \geq 3$, $b_2^{(2)}(G) = 0$ by Corollary 6.6. Moreover, by Lemma 6.2, the $\mathcal{U}(G)$-dimension of $\operatorname{im} B$ in (6.13.1) is one.

Finally, $b_1^{(2)}$ is the difference between the $\mathcal{U}(G)$-dimensions of $\ker C$ and $\operatorname{im} B$ in (6.13.1), that is, $|X|-2$. Of course, the hypotheses clearly imply that $|X| \geq 2$. □

Suppose that G is a left-orderable two-relator group. We know the first three L^2-Betti numbers of G if $\operatorname{cd} G \geq 3$ by Proposition 6.13. If $\operatorname{cd} G \leq 1$, then G is free, and again one knows the L^2-Betti numbers. There remains the case where $\operatorname{cd} G = 2$; here all we know are the L^2-Betti numbers of torsion-free surface-plus-one-relation groups; these groups are left-orderable by [12, Theorem 2.2] and they are clearly two-relator groups.

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