On free-group algorithms that sandwich a subgroup between free-product factors

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Abstract. Let F be a finite-rank free group and Z be a finite subset of F. We give topology-free proofs for two algorithms that yield sub-bases E'' and E' of F satisfying $\langle E''\rangle \leqslant \langle Z\rangle \leqslant \langle E'\rangle$ that minimize the value $|E'|-|E''|$.

Here, the subgroup $\langle E'\rangle$ is uniquely determined, and Richard Stong showed that a special basis thereof is produced by J. H. C. Whitehead's cut-vertex algorithm. Stong's proof used bi-infinite paths in a Cayley tree and sub-surfaces of a handlebody. We give a new proof that uses edge-cuts of the Cayley tree that are induced by edge-cuts of a Bass–Serre tree.

A. Clifford and R. Z. Goldstein used Whitehead's three-manifold techniques to give an algorithm that determines whether or not there exists a basis of F that meets $\langle Z\rangle$. We replace the topology with the cut-vertex algorithm, and obtain a slightly simpler Clifford–Goldstein algorithm that yields a basis B of F that maximizes the value $|B\cap\langle Z\rangle|$.

2010 Mathematics Subject Classification. Primary: 20E05; Secondary: 20E36, 20E08.

Key words. Sub-bases of free groups. Free-product factors. Cut-vertex algorithm. Cut-vertex lemma. Clifford–Goldstein algorithm.

1 Introduction

1.1 Definitions. For any set A, we let $\langle A\mid\ \rangle$ denote the free group on A. By a *basis* of $\langle A\mid\ \rangle$, we mean a free-generating set of $\langle A\mid\ \rangle$. By a *sub-basis* of $\langle A\mid\ \rangle$ we mean a subset of a basis of $\langle A\mid\ \rangle$.

For any subset S of $\langle A\mid\ \rangle$, we let $\langle S\rangle$ denote the subgroup of $\langle A\mid\ \rangle$ generated by S. We let $A_{|S}$ denote the $\subseteq$-smallest subset of A such that $S\subseteq\langle A_{|S}\rangle$. We denote by $\mathrm{C\scriptsize L}(S)$, or $\mathrm{C\scriptsize L}_{\langle A\mid\ \rangle}(S)$, the intersection of all the free-product factors of $\langle A\mid\ \rangle$ that contain S.

In general, $\mathrm{C\scriptsize L}(S)$ is not necessarily a free-product factor of $\langle A\mid\ \rangle$, but it is if A is finite. See [1] and Review 2.8 below, respectively.

1.2 Hypotheses. Throughout, let E be a finite set and Z be a finite subset of $\langle E\mid\ \rangle$.

1.3 History. • In [11, Theorem, p. 52], J. H. C. Whitehead showed that a graph-based, length-reducing procedure that we shall call the cut-vertex algorithm determines whether or not Z is a sub-basis (of $\langle E\mid\ \rangle$). The word 'algorithm' is being used here in the loose sense of a procedure with choices that has various possible outputs all of which have a specified property.

One form of the cut-vertex algorithm accepts as input E and Z, and returns as output a basis D of $\langle E \mid \ \rangle$ such that, in the terminology introduced in Notation 2.3 below, the graph $\mathrm{W{\scriptsize H}}(Z \text{ rel } D_{|Z})$ has no Whitehead cut-vertices. Whitehead used a three-manifold model to prove that Z is a sub-basis if and only if Z is one of the sub-bases contained in $D \cup D^{-1}$; see [11, Lemma, p. 51]. Richard Stong proved the more general result that if $(H_i)_{i\in I}$ is any family of subgroups of $\mathrm{C{\scriptsize L}}(Z)$ such that $\mathrm{C{\scriptsize L}}(Z) = \ast_{i\in I} H_i$ and $Z \subseteq \bigcup_{i\in I} H_i$, then D contains a basis of each H_i; see (1)⇒(3) of [10, Theorem 10]. In the case where Z is a sub-basis, we have $\mathrm{C{\scriptsize L}}(Z) = \langle Z\rangle = \ast_{z\in Z}\langle\{z\}\rangle$ and $Z \subseteq \bigcup_{z\in Z}\langle\{z\}\rangle$, whence D contains a basis of each $\langle\{z\}\rangle$, either $\{z\}$ or $\{z^{-1}\}$, and $Z \subseteq D \cup D^{-1}$. In his proof, Stong used bi-infinite paths in a Cayley tree and sub-surfaces of a handlebody that are homologous to an essential disk.

• A. Clifford and R. Z. Goldstein revisited Whitehead's three-manifold techniques, and constructed an algorithm that determines whether or not some element of $\langle Z\rangle$ forms a sub-basis of $\langle E \mid \ \rangle$ and, if so, returns such an element; see [2].

1.4 Outline. In Section 2, we use edge-cuts of a Cayley tree induced by edge-cuts of a Bass–Serre tree to prove Stong's beautiful result.

In Section 3, we present the form of the cut-vertex algorithm that we will use. By the end of the section, we will have given a topology-free version of Stong's detailed proof that the cut-vertex algorithm yields a special basis of $\mathrm{C{\scriptsize L}}(Z)$.

In Section 4, we restructure the Clifford–Goldstein algorithm, and replace the topology with the cut-vertex algorithm. The revised Clifford–Goldstein algorithm then yields a basis B of $\langle E \mid \ \rangle$ which maximizes $|B \cap \langle Z\rangle\,|$. The cut-vertex algorithm has been largely overshadowed by Whitehead's general-purpose algorithm [12, Theorem 3], but here the former carries out a task that the latter, as yet, can not.

In order to streamline the exposition, we do not digress to consider the cyclic-word analogues.

2 Bass–Serre proofs of cut-vertex lemmas

2.1 Definitions. We use the terminology of [3]. In particular, a *graph* is a set given as the disjoint union of two sets, called the *vertex-set* and the *edge-set*, together with an *initial-vertex map* and a *terminal-vertex map*, each of which maps the edge-set to the vertex-set.

• Let S be a set. We let $\mathbb{K}(S)$ denote the graph which has vertex-set S and edge-set $S^{\times 2} \coloneqq S\times S$, where an edge (x, y) has initial vertex x and terminal vertex y.

• Let F be a group and A be a subset of F. We let $\mathrm{Cayley}(F, A)$ denote the graph with vertex-set F and edge-set $F\times A$, where each edge $(g, a) \in F\times A$ has initial vertex g and terminal vertex ga; we shall sometimes write $\mathrm{edge}(g \xrightarrow{\bullet a} ga)$ to denote the pair (g, a) viewed as an edge. In a natural way, $\mathrm{Cayley}(F, A)$ is an F-graph.

• Let F be a group, I be a set, and $(H_i)_{i\in I}$ be a family of subgroups of F. We let $\mathrm{BassSerre}(F, (H_i)_{i\in I})$ denote the graph whose vertex-set is the disjoint union of the set F together with the sets F/H_i, $i \in I$, and whose edge-set is $F \times I$, where each edge $(g,i) \in F \times I$ has initial vertex g and terminal vertex gH_i; we shall sometimes write edge$(g \xrightarrow{\bullet H_i} gH_i)$ to denote the pair (g,i) viewed as an edge. In a natural way, BassSerre$(F,(H_i)_{i\in I})$ is an F-graph.

2.2 Review. In [7], Bass and Serre developed a powerful theory of groups acting on trees, which we shall now apply following [3, I.7]. Here, we have a graph of groups $(\mathrm{G}(\ ), Y)$, together with the resulting fundamental group G and standard G-graph T. The corner-stone of Bass–Serre theory is the fact that T is a tree; the proof given in [3, I.7.6] uses universal properties rather than normal-form expressions. We shall be interested in two cases, the first of which is classical.

• Let A be a set. Take Y to be the one-vertex graph with vertex-set $\{v_0\}$ and edge-set A. For each $y \in Y$, set $\mathrm{G}(y) := \{1\}$. Here, we may make the identifications $G = \langle A \mid \ \rangle$ and $T = \mathrm{Cayley}(G, A)$; see Definitions 2.1.

• Let $(H_i)_{i\in I}$ be a family of groups. Take Y to be the star-graph with vertex-set $(v_j \mid j \in I \cup \{0\})$, $0 \notin I$, and edge-set I, where each $i \in I$ has initial vertex v_0 and terminal vertex v_i. For each $i \in I$, set $\mathrm{G}(v_i) := H_i$ and $\mathrm{G}(i) := \mathrm{G}(v_0) := \{1\}$. Here, we may make the identifications $G = \mathop{\ast}\limits_{i\in I} H_i$ and $T = \mathrm{BassSerre}(G,(H_i)_{i\in I})$; see Definitions 2.1.

2.3 Notation. Recall Hypotheses 1.2.

• For $e \in E$, we write $e^{\pm 1} := \{e, \overline{e}\}$, where $\overline{e} := e^{-1}$. We set $E^{\pm 1} := E \cup E^{-1}$, where $E^{-1} := \{\overline{e} \mid e \in E\}$. We shall be interested in the graph $\mathbb{K}(E^{\pm 1} \cup \{1\})$, which has basepoint 1 and an inversion map on the vertex-set. (Whitehead employs a graph with vertex-set $E^{\pm 1} \cup \{O\}$ in which a finite set of edges joins any vertex to any other vertex.)

• Consider any $s \in \langle E \mid \ \rangle$, and let $e_1 e_2 \cdots e_n$ represent the reduced $E^{\pm 1}$-expression for s.

Notice that $E_{|\{s\}} = \bigcup\limits_{i=1}^{n} (E \cap e_i^{\pm 1})$ and $E_{|Z} = \bigcup\limits_{z\in Z} E_{|\{z\}}$.

We write $||s||_E := n$ and $||Z||_E := \sum\limits_{z\in Z} ||z||_E$. We shall write a product gh as $g{\cdot}h$ when we wish to indicate that $||gh||_E = ||g||_E + ||h||_E$.

We set $e_0 := e_{n+1} := 1$ and $\textsc{edges}(s \text{ rel } E) := \{\, (\overline{e}_i, e_{i+1})\,\}_{i=0}^{n}$, a set of edges of $\mathbb{K}(E^{\pm 1} \cup \{1\})$. The key point is that if $(e', e'') \in \textsc{edges}(s \text{ rel } E)$, then there exist $g', g'' \in \langle E \mid \ \rangle$ such that $s = \overline{g}'{\cdot}\overline{e}'{\cdot}e''{\cdot}g''$, $g' = 1$ if $e' = 1$, and $g'' = 1$ if $e'' = 1$.

• Let S be a subset of $\langle E \mid \ \rangle$. We let $\textsc{Wh}(S \text{ rel } E)$ denote the subgraph

$$E^{\pm 1} \cup \{1\} \cup \bigcup_{s\in S} \textsc{edges}(s \text{ rel } E)$$

of $\mathbb{K}(E^{\pm 1} \cup \{1\})$. Set $W := \textsc{Wh}(S \text{ rel } E)$. A vertex $e_\star$ of W is said to be a *Whitehead cut-vertex of* W if $e_\star \neq 1$ and the graph obtained from W by removing $e_\star$ and all the edges incident to $e_\star$ is not connected. (Whitehead employs the phrase 'a cut vertex other than O'.)

The basepointed star-graph $\textsc{Wh}(E \text{ rel } E)$ has no Whitehead cut-vertices. If $S \supseteq E$, then W has no Whitehead cut-vertices. Theorem 2.4 will give a partial converse to the latter implication.

If W itself is not connected, then $E^{\pm 1} \neq \emptyset$ and each element of $E^{\pm 1}$ is a Whitehead cut-vertex of W, since the set of valence-zero vertices of W is closed under inversion.

We now give a new proof of the (1)⇒(3) portion of [10, Theorem 10]; the case where each free-product factor is cyclic is [11, Lemma, p. 51].

2.4 The Stong–Whitehead theorem. *For each finite set E and free-product factorization $\langle E \mid \ \rangle = \mathop{\ast}\limits_{i\in I} H_i$ such that $\bigcup\limits_{i\in I} H_i \not\supseteq E$, the graph* $\mathrm{W{\small H}}((\bigcup\limits_{i\in I} H_i) \text{ rel } E)$ *has a Whitehead cut-vertex.*

Proof. Set $F := \langle E \mid \ \rangle = \ast_{i\in I} H_i$. Then F lies in the vertex-sets of the two F-trees $T := \mathrm{Cayley}(F, E)$ and $U := \mathrm{BassSerre}(F, (H_i)_{i\in I})$; see Review 2.2.

We consider first U. Let $\mathrm{link}_U(1)$ denote the set of U-edges incident to the U-vertex 1. Let $\mathrm{star}_U(1)$ denote the set of components of the forest $U - \mathrm{link}_U(1)$. For each U-vertex v, there exists a unique component $\kappa(v) \in \mathrm{star}_U(1)$ such that $v \in \kappa(v)$. For U-vertices v and w, we let $U[v,w]$ denote the $\subseteq$-smallest subtree of U that contains $\{v,w\}$; then $\kappa(v) \neq \kappa(w)$ if and only if $1 \in U[v,w]$ and $\{v,w\} \neq \{1\}$.

In T now, let $\delta := \{\mathrm{edge}(g \xrightarrow{\bullet e} ge) \in F \times E \mid \kappa(g) \neq \kappa(ge)\}$. An arbitrary T-edge $(g,e) \in F \times E$ lies in δ if and only if $1 \in U[g, ge]$, or, equivalently, $\overline{g} \in U[1,e]$. Since E is nonempty and finite, it is clear that δ is nonempty and finite. Hence, there exists a pair $(g_\delta, e_\delta) \in F \times E^{\pm 1}$ satisfying $||g_\delta e_\delta||_E = ||g_\delta||_E + 1$ and $\kappa(g_\delta) \neq \kappa(g_\delta e_\delta)$ such that $||g_\delta||_E$ has the largest possible value.

We now show that $g_\delta \neq 1$. By hypothesis, there exists some $e_0 \in E - \bigcup_{i\in I} H_i$. Since $e_0 \neq 1$, there exists a unique $U[1, e_0]$-neighbour of 1, necessarily $1H_{i_0}$ for some $i_0 \in I$. Since $e_0 \neq 1H_{i_0}$, there exists a unique $U[1H_{i_0}, e_0]$-neighbour of $1H_{i_0}$, necessarily some $h_0 \in H_{i_0} - \{1\}$. Now $h_0 \in U[1, e_0]$, $1 \in U[\overline{h}_0, \overline{h}_0 e_0]$, $\kappa(\overline{h}_0) \neq \kappa(\overline{h}_0 e_0)$, and $||g_\delta||_E \geqslant \min\{||\overline{h}_0||_E, ||\overline{h}_0 e_0||_E\}$. We know $e_0 \in E - \{h_0\}$ and $h_0 \in H_{i_0} - \{1\}$. Hence, $1 \notin \{\overline{h}_0, \overline{h}_0 e_0\}$ and $g_\delta \neq 1$.

There exists a unique $e_\star \in E^{\pm 1}$ such that $||g_\delta e_\star||_E = ||g_\delta||_E - 1$. Clearly, $e_\star \notin \{1, e_\delta\}$.

In T, define $\mathrm{link}_T(1)$ and $\mathrm{star}_T(1)$ as for U. For each $e \in E^{\pm 1} \cup \{1\}$, there exists a unique component $[e] \in \mathrm{star}_T(1)$ such that $e \in [e]$. Then the map $E^{\pm 1} \cup \{1\} \to \mathrm{star}_T(1)$, $e \mapsto [e]$, is bijective. Notice that $1 \in g_\delta[e_\star]$ and $\delta \subseteq g_\delta \,\mathrm{link}_T(1) \cup g_\delta[e_\star]$.

Fix an edge (e', e'') of $\mathrm{W{\small H}}(\bigcup_{i\in I} H_i \text{ rel } E)$. Then there exist $j \in I$, $h \in H_j$, and $g', g'' \in F$ such that $h = \overline{g}' {\cdot} \overline{e}' {\cdot} e'' {\cdot} g''$, $g' = 1$ if $e' = 1$, and $g'' = 1$ if $e'' = 1$. Here, $e' {\cdot} g' \in [e']$, even when $[e'] = \{1\}$, for then $g' = 1$. Similarly, $e'' {\cdot} g'' \in [e'']$. Also, $e'g'H_j = e''g''H_j$, since $e'g'h = e''g''$.

Let W' denote the graph that is obtained from $\mathrm{W{\small H}}(\bigcup_{i\in I} H_i \text{ rel } E)$ by removing $e_\star$ and its incident edges. Suppose that $(e', e'') \in W'$. As $e' \neq e_\star$, we see that $g_\delta[e'] \cap \delta = \emptyset$, and, by the definition of δ, κ is constant on the vertex-set of the tree $g_\delta[e']$. Also, $1 \notin g_\delta[e']$. As $\{g_\delta e', g_\delta e' g'\} \subseteq g_\delta[e']$, we see that $\kappa(g_\delta e') = \kappa(g_\delta e' g')$. Also, $g_\delta e' g' \neq 1$, $\mathrm{edge}(g_\delta e' g' \xrightarrow{\bullet H_j} g_\delta e' g' H_j) \notin \mathrm{link}_U(1)$, and $\kappa(g_\delta e' g') = \kappa(g_\delta e' g' H_j)$. Thus, $\kappa(g_\delta e') = \kappa(g_\delta(e' g' H_j))$. It follows that

$$\kappa(g_\delta e') = \kappa(g_\delta(e' g' H_j)) = \kappa(g_\delta(e'' g'' H_j)) = \kappa(g_\delta e'').$$

We now see that the map $e \mapsto \kappa(g_\delta e)$ is constant on the vertex-set of each component of W'. Since 1 and e_δ are vertices of W' such that $\kappa(g_\delta 1) \neq \kappa(g_\delta e_\delta)$, W' is not connected. Hence, $e_\star$ is a Whitehead cut-vertex of $\mathrm{W{\small H}}(\bigcup_{i\in I} H_i \text{ rel } E)$. □

2.5 Corollary. *With Hypotheses* 1.2*, suppose that* $\text{Wh}(Z \text{ rel } E)$ *has no Whitehead cut-vertices. If* $(H_i)_{i\in I}$ *is any family of subgroups of* $\langle E\mid\ \rangle$ *such that* $\langle E\mid\ \rangle = \ast_{i\in I} H_i$ *and* $Z \subseteq \bigcup_{i\in I} H_i$*, then, for each* $j \in I$*,* E *contains a basis of* H_j*. Also,* $\text{Cl}(Z) = \langle E\mid\ \rangle$*.*

Proof. As it contains $\text{Wh}(Z \text{ rel } E)$, $\text{Wh}(\bigcup_{i\in I} H_i \text{ rel } E)$ has no Whitehead cut-vertices. By the contrapositive of Theorem 2.4, $E \subseteq \bigcup_{i\in I} H_i$. We have a retraction $\langle E\mid\ \rangle \to H_j$ which carries $\bigcup_{i\in I-\{j\}} H_i$ to $\{1\}$; since E is mapped to $(E \cap H_j) \cup \{1\}$, we see that $E \cap H_j$ generates H_j, freely.

Since $\text{Wh}(Z \text{ rel } E)$ has no valence-zero vertices, $E - E_{|Z} = \emptyset$. If $Z \subseteq H_j$, then $E = E_{|Z} \subseteq E_{|H_j} = E \cap H_j \subseteq H_j$. It follows that $\text{Cl}(Z) = \langle E\mid\ \rangle$. □

2.6 Review. We sketch, with some minor modernizations, the proof by Schreier [8, p. 179] that subgroups of free groups are free. The case of finitely generated subgroups had been proved earlier by Nielsen [6, Sætning I].

Let A be a set, $F := \langle A\mid\ \rangle$, H be a subgroup of F, and $T := \text{Cayley}(F, A)$, viewed as an H-tree; see Review 2.2. The vertices of the *Schreier graph* $H\backslash T$ are the cosets Hg, $g \in F$, the basepoint is $H1$, and we write

$$\text{edge}(v \xrightarrow{\bullet a} va) := (v, a) \in (H\backslash F) \times A.$$

The graph $H\backslash T$ is connected. Let $\pi(H\backslash T, H1)$ denote the fundamental group of $H\backslash T$ at the basepoint $H1$. Each (reduced) $H\backslash T$-path from $H1$ to itself will be viewed as a (reduced) $A^{\pm 1}$-expression for some element of H; for example, we view $(H1 \xrightarrow{\bullet a_1} Ha_1 \xleftarrow{\bullet a_2} Ha_1\overline{a}_2 \xrightarrow{\bullet a_3} Ha_1\overline{a}_2a_3 = H1)$ as an $A^{\pm 1}$-expression $a_1\overline{a}_2a_3$ for an element of H. In this way, we may identify $\pi(H\backslash T, H1)$ with H.

Choose a maximal subtree Y' of $H\backslash T$, and let Y'' denote the complement of Y' in $H\backslash T$; then Y'' is a set of edges. Each element y'' of Y'' determines the element of $\pi(H\backslash T, H1)$ that travels in Y' from $H1$ to the initial vertex of y'', travels along y'', and then travels in Y' from the terminal vertex of y'' to $H1$. By letting y'' range over Y'', we get a subset S of $\pi(H\backslash T, H1)$. By collapsing the tree Y' to a vertex, we find that S freely generates $\pi(H\backslash T, H1)$ $(= H)$.

For each $a \in A \cap H$, it is clear that $\text{edge}(H1 \xrightarrow{\bullet a} Ha = H1)$ is not in the tree Y', and, hence, $a \in S$. Thus, $A \cap H \subseteq S$. (I am indebted to Clifford and Goldstein for this and other illuminating observations.)

The vertices and edges involved in S form a connected basepointed subgraph of $H\backslash T$ that we shall denote $\text{core}(H \text{ rel } A)$. An alternative description is that $\text{core}(H \text{ rel } A)$ consists of those vertices and edges that are involved in the reduced $H\backslash T$-paths from $H1$ to itself. Thus, $\pi(\text{core}(H \text{ rel } A), H1) = H$ and $\text{core}(H \text{ rel } A)$ is the $\subseteq$-smallest subgraph of $H\backslash T$ with this property.

2.7 Whitehead's cut-vertex lemma. *With Hypotheses* 1.2*, if* $\text{Wh}(Z \text{ rel } E_{|Z})$ *has no Whitehead cut-vertices and* Z *is a sub-basis of* $\langle E\mid\ \rangle$*, then* $Z \subseteq E^{\pm 1}$*.*

Proof. By hypothesis, Z is contained in some basis A of $\langle E\mid\ \rangle$. By Review 2.6, $A \cap \langle E_{|Z}\rangle$ is contained in some basis S of $\langle E_{|Z}\rangle$. Now $\langle E_{|Z}\rangle = \ast_{s\in S}\langle\{s\}\rangle$ and $Z \subseteq A \cap \langle E_{|Z}\rangle \subseteq S \subseteq \bigcup_{s\in S}\langle\{s\}\rangle$. By Corollary 2.5, $E_{|Z}$ contains a basis of each $\langle\{s\}\rangle$, necessarily $\{s\}$ or $\{\overline{s}\}$. Thus, $E^{\pm 1} \supseteq (E_{|Z})^{\pm 1} \supseteq S \supseteq Z$. □

The final topic of this section is Stong's generalization of Lemma 2.7.

2.8 Review. • It is clear from an argument of Kurosch [5, p. 651] that if H and K are subgroups of a group F such that K is a free-product factor of F, then $H \cap K$ is a free-product factor of H and, therefore, K is a free-product factor of each subgroup intermediate between K and F. We shall sketch a Bass–Serre-theoretic proof, although for our purposes the case $F = \langle E \mid \ \rangle$ and the graph-theoretic techniques of Stallings [9] would suffice.

Say $F = K{*}L$, and view $\mathrm{BassSerre}(F, (K, L))$ as an H-tree; see Review 2.2. Then the vertex $1K$ can be extended to a fundamental H-transversal. In the resulting graph of groups, $H \cap K$ is one of the vertex-groups and all of the edge-groups are trivial. By another result of Bass and Serre, $H \cap K$ is a free-product factor of H. See, for example, [3, I.4.1].

• With Hypotheses 1.2, let $\mathcal{S}$ denote the set of all the free-product factors of $\langle E \mid \ \rangle$ that contain Z. Thus, $\langle E_{|Z} \rangle \in \mathcal{S}$. Let H be an element of $\mathcal{S}$ of smallest possible rank. For each $K \in \mathcal{S}$, $H \cap K$ is a free-product factor of H, by Kurosch's result. Hence, $H \cap K \in \mathcal{S}$, and, by the minimality of the rank, $H = H \cap K$. By definition, $\mathrm{C{\scriptstyle L}}(Z)$ is the intersection of all the elements of $\mathcal{S}$. Thus, $\mathrm{C{\scriptstyle L}}(Z) = H$. It follows that the bases of $\mathrm{C{\scriptstyle L}}(Z)$ are the smallest-cardinality sets of the form $B_{|Z}$ for B a basis of $\langle E \mid \ \rangle$. Also, $\mathrm{C{\scriptstyle L}}(Z)$ is a free-product factor of $\langle E \mid \ \rangle$ and of the intermediate subgroup $\langle E_{|Z} \rangle$. Thus, $\mathrm{C{\scriptstyle L}}_{\langle E_{|Z}\rangle}(Z) = \mathrm{C{\scriptstyle L}}_{\langle E \mid \ \rangle}(Z)$.

Corollary 2.5 gives the following.

2.9 Stong's cut-vertex lemma. *With Hypotheses* 1.2*, if* $\mathrm{W{\scriptstyle H}}(Z \text{ rel } E_{|Z})$ *has no Whitehead cut-vertices, then* $E_{|Z}$ *is a basis of* $\mathrm{C{\scriptstyle L}}(Z)$*; moreover, if* $(H_i)_{i \in I}$ *is any family of subgroups of* $\mathrm{C{\scriptstyle L}}(Z)$ *such that* $\mathrm{C{\scriptstyle L}}(Z) = \ast_{i \in I} H_i$ *and* $Z \subseteq \bigcup_{i \in I} H_i$*, then* $E_{|Z}$ *contains a basis of each* H_i*.* □

3 A formalized cut-vertex algorithm

This technical section gives elementary definitions and arguments that formalize part of Whitehead's discussion [11, pp. 50–52] of cut-vertices and free-group automorphisms.

We first introduce a subgraph of $\mathbb{K}(E^{\pm 1} \cup \{1\})$ which is expressed as the union of two subgraphs with exactly one vertex and one edge in common. We then recall Whitehead's associated free-group automorphism whose inverse will be applied advantageously to elements compatible with the subgraph.

3.1 Notation. With Hypotheses 1.2, we let $\mathcal{P}$ denote the set of pairs $({}_0E, e_\star)$ such that $e_\star \in {}_0E \subseteq E^{\pm 1}$. Whenever any $P \in \mathcal{P}$ is specified, it will be understood that the following notation applies.

We write $({}_0E, e_\star) \coloneqq P$ and ${}_1E \coloneqq (E^{\pm 1} - {}_0E) \cup \{e_\star\}$.

For each $(\alpha, \beta) \in \{0,1\}^{\times 2}$, we write ${}_\alpha E_\beta \coloneqq {}_\alpha E \cap ({}_\beta E)^{-1}$.

We define $\mathrm{W{\scriptstyle H}}(P) \coloneqq \mathbb{K}({}_0E \cup \{1\}) \cup \mathbb{K}({}_1E)$, a subgraph of $\mathbb{K}(E^{\pm 1} \cup \{1\})$. We write $\mathrm{W{\scriptstyle H}}_0(P) \coloneqq \mathbb{K}({}_0E \cup \{1\})$ and $\mathrm{W{\scriptstyle H}}_1(P) \coloneqq \mathbb{K}({}_1E)$, subgraphs with union $\mathrm{W{\scriptstyle H}}(P)$ and intersection $\mathbb{K}(\{e_\star\})$.

Let $\chi\colon E^{\pm 1} \to \{0,1\}$ be the characteristic map of ${}_1E$, $e \mapsto \chi(e) \coloneqq |\{e\} \cap {}_1E|$. Set $\gamma \coloneqq \gamma_P \coloneqq \chi(\overline{e}_\star) \in \{0,1\}$ and $d_\star \coloneqq e_\star^{2\gamma - 1} \in e_\star^{\pm 1}$. Let $\varphi \coloneqq \varphi_P\colon g \mapsto g^\varphi$ denote the automorphism of $\langle E|\ \rangle$ that fixes $d_\star$ and maps e to $d_\star^{\chi(e)}\cdot e\cdot \overline{d}_\star^{\chi(\overline{e})}$ for each $e \in E^{\pm 1} - d_\star^{\pm 1}$.

3.2 Observations. With Notation 3.1, fix $P \in \mathcal{P}$.

(i) Let $e \in E^{\pm 1}$ and $(\alpha, \beta) \in \{0,1\}^{\times 2}$. There are three possibilities.

(1) If $e^{\pm 1} \neq e_\star^{\pm 1}$, then $e \in {}_\alpha E_\beta$ if and only if $(\alpha,\beta) = (\chi(e), \chi(\overline{e}))$. Here, $d_\star^\alpha e \overline{d}_\star^\beta = e^\varphi$.

(2) If $e = e_\star$, then $e \in {}_\alpha E_\beta$ if and only $\beta = \gamma$. Here,
either $(\alpha,\beta) = (\gamma,\gamma)$, whence $d_\star^\alpha e \overline{d}_\star^\beta = e = e^\varphi$,
or $(\alpha,\beta) = (1{-}\gamma,\gamma)$, whence $d_\star^\alpha e \overline{d}_\star^\beta = d_\star^{1-\gamma} d_\star^{2\gamma-1} \overline{d}_\star^\gamma = 1$.

(3) If $e = \overline{e}_\star$, then $e \in {}_\alpha E_\beta$ if and only if $\alpha = \gamma$. Here,
either $(\alpha,\beta) = (\gamma,\gamma)$, whence $d_\star^\alpha e \overline{d}_\star^\beta = e = e^\varphi$,
or $(\alpha,\beta) = (\gamma, 1{-}\gamma)$, whence $d_\star^\alpha e \overline{d}_\star^\beta = d_\star^\gamma d_\star^{1-2\gamma} \overline{d}_\star^{1-\gamma} = 1$.

(ii) For each $e \in E^{\pm 1}$, there exists a unique $(\alpha,\beta) \in \{0,1\}^{\times 2}$ such that $e \in {}_\alpha E_\beta$ and $d_\star^\alpha e \overline{d}_\star^\beta = e^\varphi$, by (i).

(iii) Let $z \in \langle E|\ \rangle$. Let $e_1 e_2 \cdots e_n$ represent the reduced $E^{\pm 1}$-expression for z, and set $e_0 \coloneqq e_{n+1} \coloneqq 1$.

Suppose that $\mathrm{W\textsc{h}}(\{z\} \text{ rel } E) \subseteq \mathrm{W\textsc{h}}(P)$. For each $i \in \{0,1,\ldots,n\}$, there exists a unique $\alpha_i \in \{0,1\}$ such that $(\overline{e}_i, e_{i+1}) \in \mathrm{W\textsc{h}}_{\alpha_i}(P)$. Here, $\alpha_0 = \alpha_n = 0$. For each $i \in \{1,2,\ldots,n\}$, $e_i \in {}_{\alpha_{i-1}}E_{\alpha_i}$, and then, by (i), $d_\star^{\alpha_{i-1}} e_i \overline{d}_\star^{\alpha_i} \in \{e_i^\varphi, 1\}$. It follows that $(d_\star^{\alpha_0} e_1 \overline{d}_\star^{\alpha_1})(d_\star^{\alpha_1} e_2 \overline{d}_\star^{\alpha_2}) \cdots (d_\star^{\alpha_{n-1}} e_n \overline{d}_\star^{\alpha_n})$ is an $((E^\varphi)^{\pm 1} \cup \{1\})$-expression for z. Thus, $||z^{\overline{\varphi}}||_E = ||z||_{E^\varphi} \leqslant n = ||z||_E$.

Suppose further that $e_\star$ has positive valence in $\mathrm{W\textsc{h}}(\{z\} \text{ rel } E) \cap \mathrm{W\textsc{h}}_{1-\gamma}(P)$. Then there exists some $j \in \{0,1,\ldots,n\}$ such that $\alpha_j = 1{-}\gamma$ and $e_\star \in \{\overline{e}_j, e_{j+1}\}$. If $e_\star = \overline{e}_j$, then $j \geqslant 1$ and, by (i)(3), $d_\star^{\alpha_{j-1}} e_j \overline{d}_\star^{\alpha_j} = 1$. If $e_\star = e_{j+1}$, then $j \leqslant n{-}1$ and, by (i)(2), $d_\star^{\alpha_j} e_{j+1} \overline{d}_\star^{\alpha_{j+1}} = 1$. In both cases, $||z^{\overline{\varphi}}||_E = ||z||_{E^\varphi} < n = ||z||_E$.

We now come to the essence of the cut-vertex algorithm.

3.3 Algorithm. With Notation 3.1, the *cut-vertex subroutine* [11, p. 51] has the following structure.
INPUT: a Whitehead cut-vertex d of $\mathrm{W\textsc{h}}(Z \text{ rel } E_{|Z})$.
OUTPUT: $P \in \mathcal{P}$ such that $\mathrm{W\textsc{h}}(Z \text{ rel } E) \subseteq \mathrm{W\textsc{h}}(P)$ and $||Z^{\overline{\varphi}_P}||_E < ||Z||_E$.
PROCEDURE. Find the component W of $\mathrm{W\textsc{h}}(Z \text{ rel } E_{|Z})$ that contains $\{1\}$, find $V \coloneqq W \cap (E_{|Z})^{\pm 1}$, and search for some $c \in V{-}V^{-1}$. There are two cases.

Case 1: $V{-}V^{-1} = \emptyset$.

Here, $V = V^{-1}$, $Z \subseteq \langle V \rangle$, $V = (E_{|Z})^{\pm 1}$, and $\mathrm{W\textsc{h}}(Z \text{ rel } E_{|Z}) = W$, which is connected. Find the graph W' that is obtained from W by deleting d and its incident edges. By hypothesis, W' is not connected. Find the component W_0' of W' that contains $\{1\}$. Set $P \coloneqq ((W_0' \cap E^{\pm 1}) \cup \{d\}, d) \in \mathcal{P}$. Then $\mathrm{W\textsc{h}}(Z \text{ rel } E) \subseteq \mathrm{W\textsc{h}}(P)$, and d has positive valence in $\mathrm{W\textsc{h}}(Z \text{ rel } E) \cap \mathrm{W\textsc{h}}_\alpha(P)$ for each $\alpha \in \{0,1\}$. It follows from Observations 3.2(iii) that $||Z^{\overline{\varphi}_P}||_E < ||Z||_E$. Return P and stop.

Case 2: $c \in V{-}V^{-1}$.

Set $P \coloneqq (V, c) \in \mathcal{P}$. It can be seen that $\mathrm{W\textsc{h}}(Z \text{ rel } E) \subseteq \mathrm{W\textsc{h}}(P)$. Here, $\gamma_P = |\{\overline{c}\}{-}(V{-}\{c\})| = 1$. Now $\mathrm{W\textsc{h}}(Z \text{ rel } E) \cap \mathrm{W\textsc{h}}_0(P)$ is the component W of $\mathrm{W\textsc{h}}(Z \text{ rel } E_{|Z})$ that contains $\{c, 1\}$. As c has positive valence in W, it follows from Observations 3.2(iii) that $||Z^{\overline{\varphi}_P}||_E < ||Z||_E$. Return P and stop. □

We shall use the following result in the next section.

3.4 Corollary. *With Notation* 3.1, *if* Z *is a sub-basis of* $\langle E \mid \enspace \rangle$ *and* $Z \not\subseteq E^{\pm 1}$, *then, for some* $P \in \mathcal{P}$, $\mathrm{W{\small H}}(Z \text{ rel } E) \subseteq \mathrm{W{\small H}}(P)$ *and* $||Z^{\overline{\varphi}_P}||_E < ||Z||_E$.

Proof. By the contrapositive of Lemma 2.7, $\mathrm{W{\small H}}(Z \text{ rel } E_{|Z})$ has a Whitehead cut-vertex, and then Algorithm 3.3 gives the desired conclusion. □

3.5 Algorithm. Recall Hypotheses 1.2, and let $\mathrm{Aut}\langle E \mid \enspace \rangle$ denote the group of automorphisms of $\langle E \mid \enspace \rangle$.

Set $\Phi := 1 \in \mathrm{Aut}\langle E \mid \enspace \rangle$ and $Z' := Z \; (= Z^{\overline{\Phi}})$.

↓

→ Construct $\mathrm{W{\small H}}(Z' \text{ rel } E_{|Z'})$.

↓

Search for a Whitehead cut-vertex d of $\mathrm{W{\small H}}(Z' \text{ rel } E_{|Z'})$.

↓

Does such a d exist? $\xrightarrow{\text{No}}$ Return (Φ, Z') and stop.

↓ Yes

Algorithm 3.3 yields a $\varphi \in \mathrm{Aut}\langle E \mid \enspace \rangle$ such that $||Z'^{\overline{\varphi}}||_E < ||Z'||_E$.

↓

← Reset $\Phi := \varphi{\cdot}\Phi$ and $Z' := Z'^{\overline{\varphi}} \; (= Z^{\overline{\Phi}})$, thereby decreasing $||Z'||_E$.

Figure 1: Mock flow chart for Whitehead's cut-vertex algorithm.

Figure 1 presents a form of Whitehead's *cut-vertex algorithm* [11, p. 51] which finds some $\Phi \in \mathrm{Aut}\langle E \mid \enspace \rangle$ such that $\mathrm{W{\small H}}(Z^{\overline{\Phi}} \text{ rel } E_{|Z^{\overline{\Phi}}})$ has no Whitehead cut-vertices, and returns $(\Phi, Z^{\overline{\Phi}})$.

By Lemma 2.7, Z is a sub-basis of $\langle E \mid \enspace \rangle$ if and only if $Z \cap Z^{-1} = \emptyset$ and $Z^{\overline{\Phi}} \subseteq E^{\pm 1}$, and then $(E^{\Phi} - Z^{-1}) \cup Z$ is a basis of $\langle E \mid \enspace \rangle$.

Knowing $(\Phi, Z^{\overline{\Phi}})$, one can find $(E_{|Z^{\overline{\Phi}}})^{\Phi}$, which, by Lemma 2.9, is a basis of $\mathrm{C{\small L}}(Z)$ that contains a basis of each constituent of each free-product factorization $\mathrm{C{\small L}}(Z) = \mathop{\ast}\limits_{i \in I} H_i$ for which $Z \subseteq \bigcup\limits_{i \in I} H_i$. In particular, $|E_{|Z^{\overline{\Phi}}}|$ is smallest-possible over $\mathrm{Aut}\langle E \mid \enspace \rangle$. □

3.6 Remarks. Algorithm 3.5 determines whether or not $\mathrm{C{\small L}}(Z) = \langle E \mid \enspace \rangle$ in any specific case.

A finitely generated group G is said *to have at most one end* if, for some/each finite generating set S of G, no graph obtained from $\mathrm{Cayley}(G, S)$ by deleting a finite set of edges has two infinite components; Freudenthal [4, Satz 3] showed that 'some' and 'each' are interchangeable here. If the group $\langle E \mid Z \rangle$ has at most one end, then $\mathrm{C{\small L}}(Z) = \langle E \mid \enspace \rangle$, since the contrapositive is easily seen to hold.

4 A strengthened Clifford–Goldstein algorithm

Clifford and Goldstein [2] produced an ingenious algorithm which determines whether or not there exists some element of $\langle Z\rangle$ which forms a sub-basis of $\langle E \mid \ \rangle$ and, if so, returns such an element. They used Whitehead's three-manifold techniques to construct a sufficiently large finite set of finitely generated subgroups of $\langle E \mid \ \rangle$ whose elements of sufficiently bounded E-length give the desired information.

In this section, we restructure their argument, replace the topology with Corollary 3.4, and obtain a less complicated, more powerful algorithm which yields a basis B of $\langle E \mid \ \rangle$ which maximizes $|B \cap \langle Z\rangle|$. In particular, $B \cap \langle Z\rangle = \emptyset$ if and only if no basis of $\langle E \mid \ \rangle$ meets $\langle Z\rangle$. We construct a smaller sufficiently large finite set of finitely generated subgroups of $\langle E \mid \ \rangle$ whose intersections with E give the desired information.

4.1 Review. With Hypotheses 1.2, we sketch Stallings' important *core construction* [9, Algorithm 5.4] for the special case which synthesizes the methods of Nielsen and Schreier. The graph $\textsc{core}(\langle Z\rangle \text{ rel } E)$ defined in Review 2.6 is a finite, basepointed, E-labelled graph; we shall suppress the information that the vertices are cosets, and we shall build an isomorphic finite, basepointed, E-labelled graph $\textsc{modelcore}(\langle Z\rangle \text{ rel } E)$ that has an abstract set as vertex-set.

For each $z \in Z-\{1\}$, we create a circle, we divide it into $||z||_E$ segments by adding $||z||_E$ vertices, we choose one of the vertices to be the basepoint, and we orient and E-label the segments in such a way that the reduced $E^{\pm 1}$-expression for z can be read off the circle-graph in one direction starting from the basepoint. We next create a basepoint, and attach to it each of our circle-graphs at its basepoint. Here, and henceforth, each edge has an expression of the form $\text{edge}(v \xrightarrow{\bullet e} w)$ with v, w vertices and $e \in E$, but, for the moment, the expression need not determine the edge. We identify any distinct pair of edges having expressions $\text{edge}(v \xrightarrow{\bullet e} w)$ and $\text{edge}(v' \xrightarrow{\bullet e} w')$ where $v = v'$ or $w = w'$ or both; identifying the edges entails identifying w with w' or v with v' or neither, respectively. When no such pair of distinct edges is left, the procedure has yielded a basepointed E-labelled graph isomorphic to $\textsc{core}(\langle Z\rangle \text{ rel } E)$. Here, any expression $\text{edge}(v \xrightarrow{\bullet e} w)$ does determine an edge, and, moreover, we may define formal products $ve \coloneqq w$ and $w\bar{e} \coloneqq v$. □

The following is the key construction, extracted from [2, Theorem 1].

4.2 Notation. With Notation 3.1, fix $P \in \mathcal{P}$. Set $F \coloneqq \langle E \mid \ \rangle$.

We first construct a map ψ from the edge-set of $T \coloneqq \text{Cayley}(F, E)$ to the edge-set of $T' \coloneqq \text{Cayley}(F, E^{\varphi})$ by defining, for each T-edge $(g, e) \in F \times E$, $(\text{edge}(g \xrightarrow{\bullet e} ge))^{\psi} \coloneqq \text{edge}(g\bar{d}_{\star}^{\alpha} \xrightarrow{\bullet e^{\varphi}} ge\bar{d}_{\star}^{\beta})$ for the unique $(\alpha, \beta) \in \{0, 1\}^{\times 2}$ such that $e \in {}_{\alpha}E_{\beta}$ and $e^{\varphi} = d_{\star}^{\alpha} e \bar{d}_{\star}^{\beta}$; see Observations 3.2(ii). The map ψ does not act on vertices. It is clear that ψ is a map of F-sets.

Let H be a finitely generated subgroup of F. Then ψ induces a map from the edge-set of $H\backslash T$ to the edge-set of $H\backslash T'$. The image of the edge-set of $\textsc{core}(H \text{ rel } E)$ under this induced map is the edge-set of a unique subgraph X

of $H\backslash T'$ with the full vertex-set, $H\backslash F$. Let $K \coloneqq \pi(X, H1) \leqslant \pi(H\backslash T', H1) = H$, where we view $(H\backslash T')$-paths as $(E^{\varphi})^{\pm 1}$-expressions. We set $\partial_P H \coloneqq K^{\overline{\varphi}} \leqslant H^{\overline{\varphi}}$.

Recall that MODELCORE(H rel E) was constructed in Review 4.1; we shall be viewing ∂_P as a graph operation that converts MODELCORE(H rel E) into MODELCORE($\partial_P H$ rel E).

4.3 Lemma. *With Notation 4.2, the following hold for* $\partial_P H \leqslant H^{\overline{\varphi}_P}$.

(i) MODELCORE($\partial_P H$ rel E) *may be constructed from* MODELCORE(H rel E) *algorithmically.*

(ii) *The number of edges of* CORE($\partial_P H$ rel E) *is at most the number of edges of* CORE(H rel E).

(iii) *For each* $h \in H$, *if* $\text{Wh}(\{h\} \text{ rel } E) \subseteq \text{Wh}(P)$, *then* $h^{\overline{\varphi}_P} \in \partial_P H$.

(iv) *If* C *is any sub-basis of* $\langle E \mid \ \rangle$ *such that* $C \subseteq H$ *and* $C \not\subseteq E^{\pm 1}$, *then there exists some* $P' \in \mathcal{P}$ *such that* $C^{\overline{\varphi}_{P'}} \subseteq \partial_{P'} H$ *and* $\|C^{\overline{\varphi}_{P'}}\|_E < \|C\|_E$.

Proof. (i). Since $K^{\overline{\varphi}} = \partial_P H$, there is a natural graph isomorphism that maps CORE(K rel E^{φ}) to CORE($\partial_P H$ rel E), changing each $Kg \xrightarrow{\bullet e^{\varphi}} Kg(e^{\varphi})$ to $K^{\overline{\varphi}} g^{\overline{\varphi}} \xrightarrow{\bullet e} K^{\overline{\varphi}} g^{\overline{\varphi}} e$. Hence, there is a natural graph isomorphism that maps MODELCORE(K rel E^{φ}) to MODELCORE($\partial_P H$ rel E), changing each $v \xrightarrow{\bullet e^{\varphi}} w$ to $v \xrightarrow{\bullet e} w$; the labels on the non-basepoint vertices are irrelevant or non-existent. It remains to construct MODELCORE(K rel E^{φ}) algorithmically.

For each vertex v of MODELCORE(H rel E) for which no formal product $v\overline{d}_{\star}$ is defined, we create a valence-zero vertex called $v\overline{d}_{\star}$. In MODELCORE(H rel E) adorned with these valence-zero vertices, we simultaneously replace each edge($v \xrightarrow{\bullet e} w$) with edge($v\overline{d}_{\star}^{\alpha} \xrightarrow{\bullet e^{\varphi}} w\overline{d}_{\star}^{\beta}$) for the unique $(\alpha, \beta) \in \{0,1\}^{\times 2}$ such that $e \in {}_{\alpha}E_{\beta}$ and $e^{\varphi} = d_{\star}^{\alpha} e \overline{d}_{\star}^{\beta}$. In the resulting finite graph, we then keep only the component that has the basepoint. We next successively delete non-basepoint, valence-one vertices and their unique incident edges, while possible. When this is no longer possible, we have completed the algorithmic construction of MODELCORE(K rel E^{φ}).

(ii). There exist bijective maps first from the edge-set of CORE($\partial_P H$ rel E) to the edge-set of CORE(K rel E^{φ}) and then to a subset of the edge-set of CORE(H rel E).

(iii). Let $e_1 e_2 \cdots e_n$ represent the reduced $E^{\pm 1}$-expression for h. By Observations 3.2(iii), there exists a map $\{0, 1, \ldots, n\} \to \{0, 1\}$, $i \mapsto \alpha_i$, such that $\alpha_0 = \alpha_n = 0$ and, for $i \in \{1, 2, \ldots, n\}$, $e_i \in {}_{\alpha_{i-1}}E_{\alpha_i}$ and $d_{\star}^{\alpha_{i-1}} e_i \overline{d}_{\star}^{\alpha_i} \in \{e_i^{\varphi}, 1\}$. We view h as a reduced $H\backslash T$-path from $H1$ to itself, which we may write in $H\backslash \operatorname{Cayley}(F, E^{\pm 1})$ as

$$H1 \xrightarrow{\bullet e_1} He_1 \xrightarrow{\bullet e_2} He_1 e_2 \xrightarrow{\bullet e_3} \cdots \xrightarrow{\bullet e_n} He_1 e_2 \cdots e_n = Hh = H1.$$

The $H\backslash T$-path stays within the subgraph CORE(H rel E). Let us change each $He_1 \cdots e_i$ to $He_1 \cdots e_i \overline{d}_{\star}^{\alpha_i}$ and each $He_1 \cdots e_{i-1} \xrightarrow{\bullet e_i} He_1 \cdots e_{i-1} e_i$ to

$$He_1 \cdots e_{i-1} \overline{d}_{\star}^{\alpha_{i-1}} \xrightarrow{\bullet d_{\star}^{\alpha_{i-1}} e_i \overline{d}_{\star}^{\alpha_i}} He_1 \cdots e_{i-1} e_i \overline{d}_{\star}^{\alpha_i},$$

which corresponds to an edge, inverse edge, or equality in the graph X of Notation 4.2. We thus obtain an X-path from $H1$ to itself that reads an $((E^{\varphi})^{\pm 1} \cup \{1\})$-expression for h. Hence, $h \in \pi(X, H1) = K$, and $h^{\overline{\varphi}} \in \partial_P H$.

(iv). By Corollary 3.4, there exists $P' \in \mathcal{P}$ such that $\|C^{\overline{\varphi}_{P'}}\|_E < \|C\|_E$ and $\text{Wh}(C \text{ rel } E) \subseteq \text{Wh}(P')$. By (iii), $C^{\overline{\varphi}_{P'}} \subseteq \partial_{P'} H$. □

We now give a variant of a construction of Clifford and Goldstein [2, p. 609].

4.4 Notation. With Notation 3.1, let $\mathcal{F}$ denote the set of all finitely generated subgroups of $\langle E \mid \ \rangle$, and let Γ denote the graph whose vertex-set is $\mathcal{F}$ and whose edge-set is $\mathcal{F} \times \mathcal{P}$, where each edge $(H, P) \in \mathcal{F} \times \mathcal{P}$ has initial vertex H and terminal vertex the finitely generated subgroup $\partial_P H$ defined in Notation 4.2.

Clearly $\langle Z \rangle \in \mathcal{F}$. Let $\langle Z \rangle\blacktriangleleft$ denote the subgraph of Γ that radiates out from $\langle Z \rangle$, that is, $\langle Z \rangle\blacktriangleleft$ is the smallest subgraph of Γ that has $\langle Z \rangle$ as a vertex and is closed in Γ under the operation of adding to each vertex H each outgoing edge (H, P) and its terminal vertex $\partial_P H$.

For $n \geqslant 0$, we associate with each element $(P_i)_{i=1}^n$ of $\mathcal{P}^{\times n}$ the oriented $\langle Z \rangle\blacktriangleleft$-path with edge-sequence $((H_{i-1}, P_i))_{i=1}^n$ and vertex-sequence $(H_i)_{i=0}^n$, where $H_0 = \langle Z \rangle$ and $H_i = \partial_{P_i} H_{i-1}$ for $i = 1, 2, \ldots, n$. To simplify notation, we shall say that $(P_i)_{i=1}^n$ itself is an oriented $\langle Z \rangle\blacktriangleleft$-path with initial vertex $\langle Z \rangle$.

To be able to recognize when two vertices are equal, we think of a vertex H of $\langle Z \rangle\blacktriangleleft$ as the graph MODELCORE(H rel E). We shall see that we are interested in finding a vertex of $\langle Z \rangle\blacktriangleleft$ which has the largest possible one-vertex subgraph at the basepoint.

4.5 Theorem. *With Notation* 4.4*, the following hold.*

(i) $\langle Z \rangle\blacktriangleleft$ *has a finite, algorithmically constructible maximal subtree* T_0 *that radiates out from* $\langle Z \rangle$.

(ii) *Let* H' *be a* $\langle Z \rangle\blacktriangleleft$*-vertex,* $(P_i)_{i=1}^n$ *the oriented* T_0*-path from* $\langle Z \rangle$ *to* H'*, and* $E' := E^{\varphi_{P_n} \cdots \varphi_{P_1}}$. *Then* E' *is a basis of* $\langle E \mid \ \rangle$*, and* $|E \cap H'| \leqslant |E' \cap \langle Z \rangle|$.

(iii) *For each basis* E'' *of* $\langle E \mid \ \rangle$*, there exists some* $\langle Z \rangle\blacktriangleleft$*-vertex* H'' *such that* $|E'' \cap \langle Z \rangle| \leqslant |E \cap H''|$.

Proof. (i). By Review 4.1, we may construct MODELCORE($\langle Z \rangle$ rel E). By Lemma 4.3(ii), $\langle Z \rangle\blacktriangleleft$ is finite. By Lemma 4.3(i), we may use a depth-first search to construct a maximal subtree T_0 of $\langle Z \rangle\blacktriangleleft$ that radiates out from $\langle Z \rangle$.

(ii). Here, $E \cap H' = E \cap \partial_{P_n} \cdots \partial_{P_2} \partial_{P_1} \langle Z \rangle \subseteq (E' \cap \langle Z \rangle)^{\overline{\varphi}_{P_1} \overline{\varphi}_{P_2} \cdots \overline{\varphi}_{P_n}}$.

(iii). It follows from Lemma 4.3(iv) that there exists some $n \geqslant 0$ and some $(P_i)_{i=1}^n$ such that $(E'' \cap \langle Z \rangle)^{\overline{\varphi}_{P_1} \overline{\varphi}_{P_2} \cdots \overline{\varphi}_{P_n}} \subseteq E^{\pm 1} \cap \partial_{P_n} \cdots \partial_{P_2} \partial_{P_1} \langle Z \rangle$. □

Notice that the cut-vertex algorithm is being run automatically in the preceding argument.

We now construct a basis B of $\langle E \mid \ \rangle$ which maximizes $|B \cap \langle Z \rangle|$, in theory. However, even just to verify by hand that no basis of $\langle \{x, y\} \mid \ \rangle$ meets $\langle \{x^2, yx^3\overline{y}\} \rangle$ looks quite daunting.

4.6 Algorithm. • With Notation 4.4, construct a (finite) maximal subtree T_0 of $\langle Z \rangle\blacktriangleleft$ that radiates out from $\langle Z \rangle$; see Theorem 4.5(i).

• Find a T_0-vertex H maximizing the number of edges in the one-vertex subgraph at the basepoint of MODELCORE(H rel E), that is, maximizing $|E \cap H|$.

• Find the oriented T_0-path $(P_i)_{i=1}^n$ from $\langle Z \rangle$ to H.

• Return $B := E^{\varphi_{P_n} \cdots \varphi_{P_2} \varphi_{P_1}}$, a basis of $\langle E \mid \ \rangle$ which maximizes $|B \cap \langle Z \rangle|$ by Theorem 4.5(ii),(iii). □

Acknowledgments. I am grateful to all of the following: Spain's Ministerio de Ciencia e Innovación for support through Project MTM2011-25955; Andrew Clifford and Richard Goldstein for email correspondence that was very helpful; and the referee for the speed with which the article was processed and the thought invested in the suggestions, which improved the exposition.

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