# Retracts of vertex sets of trees and the almost stability theorem

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**Abstract**

Let $G$ be a group, let $T$ be an (oriented) $G$-tree with finite edge stabilizers, and let $VT$ denote the vertex set of $T$. We show that, for each $G$-retract $V'$ of the $G$-set $VT$, there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is $V'$. This fact leads to various new consequences of the almost stability theorem.

We also give an example of a group $G$, a $G$-tree $T$ and a $G$-retract $V'$ of $VT$ such that no $G$-tree has vertex set $V'$.

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## 1 Outline

Throughout the article, let $G$ be a group, and let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$. All our $G$-actions will be on the left.

The following extends Definitions II.1.1 of [3] (where $A$ is assumed to have trivial $G$-action).

**1.1 Definition.** Let $E$ and $A$ be $G$-sets.

Let $(E, A)$ denote the set of all functions from $E$ to $A$. An element $v$ of $(E, A)$ has the form $v\colon E \to A$, $e \mapsto v(e)$. There is a natural $G$-action on $(E, A)$ such that $(gv)(e) := g(v(g^{-1}e))$ for all $v \in (E, A)$, $g \in G$, $e \in E$.

Two elements $v$ and $w$ of $(E, A)$ are said to be *almost equal* if the set

$$\{e \in E \mid v(e) \neq w(e)\}$$

is finite. Almost equality is an equivalence relation; the equivalence classes are called the *almost equality classes in* $(E, A)$.

A subset $V$ of $(E, A)$ is said to be *$G$-stable* if $V$ is closed under the $G$-action. In general, a $G$-stable subset is the same as a $G$-subset. $\square$

In this article, we wish to strengthen the following result.

**1.2 The almost stability theorem** [3, Theorem III.8.5]. *If $E$ is a $G$-set with finite stabilizers, and $A$ is a nonempty set with trivial $G$-action, and $V$ is a $G$-stable almost equality class in the $G$-set $(E, A)$, then there exists a $G$-tree with finite edge stabilizers and vertex set $V$.* $\square$

In the light of Bass-Serre theory, the almost stability theorem can be thought of as a broad generalization of Stallings' ends theorem; see [3, Theorem III.2.1].

Let us now recall the notion of a $G$-retract of a $G$-set. The following alters Definition III.1.1 of [3] slightly.

**1.3 Definition.** A *$G$-retract* $U$ of a $G$-set $V$ is a $G$-subset of $V$ with the property that, for each $w \in V - U$, there exists $u \in U$ such that $G_w \leq G_u$, or, equivalently, with the property that there exists a $G$-map, called a *$G$-retraction*, from $V$ to $U$ which is the identity on $U$. □

Chapter IV of [3] collects together a wide variety of consequences of the almost stability theorem 1.2. In some of these applications, the conclusions assert that certain naturally arising $G$-sets are $G$-retracts of vertex sets of $G$-trees with finite edge stabilizers. This leads to the question of whether or not the class of vertex sets of $G$-trees with finite edge stabilizers is closed under taking $G$-retracts. We are now able to answer this in the affirmative; in Section 4 below, we prove that any $G$-retract of the vertex set of a $G$-tree with finite edge stabilizers is itself the vertex set of a $G$-tree with finite edge stabilizers.

In Section 5, we record the resulting generalizations of the almost stability theorem and the applications which are affected. In the most classic example, if $G$ has cohomological dimension one, and $\omega\mathbb{Z}G$ is the augmentation ideal of the group ring $\mathbb{Z}G$, one can deduce that $G$ acts freely on a tree whose vertex set is the $G$-set $1 + \omega\mathbb{Z}G$, and, hence, $G$ is a free group; this is a slightly more detailed version of a theorem of Stallings and Swan.

In Section 6, we record an even more general form of the almost stability theorem in which the $G$-action on $A$ need not be trivial.

In Section 7, we construct a group $G$ and a $G$-retract of a vertex set of a $G$-tree (with infinite edge stabilizers) that is not itself the vertex set of a $G$-tree.

# 2 Operations on trees

Throughout this section we will be working with the following.

**2.1 Hypotheses.** Let $T = (T, V, E, \iota, \tau)$ be a $G$-tree, as in [3, Definition I.2.3].

We write $VT = V$ and $ET = E$, and we view the underlying $G$-set of $T$ as the disjoint union of $V$ and $E$, written $T = V \vee E$. Here $\iota\colon E \to V$ is the *initial vertex* map and $\tau\colon E \to V$ is the *terminal vertex* map. □

We first consider a simple form of retraction, which amplifies Definitions III.7.1 of [3]. Recall that a vertex $v$ of a tree is called a *sink* if every edge of the tree is oriented towards $v$.

**2.2 The compressing lemma.** *Suppose that Hypotheses* 2.1 *hold.*

*Let $E'$ be a $G$-subset of $E$ such that each component of the subforest $T - E'$ of $T$ has a* (*unique*) *sink. Let $V'$ denote the set of sinks of the components of $T - E'$.*

*Let $i\colon E' \to E$ denote the inclusion map, and let $\phi\colon V \to V'$ denote the $G$-retraction which assigns, to each $v \in V$, the sink of that component of $T - E'$ which contains $v$.*

*Then the $G$-graph $T' = (T', V', E', \phi \circ \iota \circ i, \phi \circ \tau \circ i)$ is a $G$-tree.*

Let $E'' = E - E'$ and let $V'' = V - V'$. Then $T - E'$ is the $G$-subforest of $T$ with vertex set $V$ and edge set $E''$. For each $v \in V$, $\phi(v)$ is reached in $T$ by starting at $v$ and travelling as far as possible along edges in $E''$ respecting the orientation. The initial vertex map $\iota\colon E \to V$ induces a bijective map $E'' \to V''$.

We say that $T'$ is obtained from $T$ by *compressing the closures of the elements of $E''$ to their terminal vertices* or by *compressing the components of $T - E'$ to their sinks.*

In applications, we usually first $G$-equivariantly reorient $T$ and then, in the resulting tree, compress a $G$-set of closed edges to their terminal vertices; we then call the combined procedure a *$G$-equivariant compressing operation.*

*Proof of Lemma* 2.2. The map $\phi$ induces a surjective $G$-map $T \to T'$ in which the fibres are the components of $T - E'$. It follows that $T'$ is a $G$-tree. $\square$

We now recall the sliding operation of Rips-Sela [8, p. 59] as generalized by Forester [7, Section 3.6]; see also the Type 1 operation of [6, p. 146]. We find it convenient to express the result and the proof in the notation of [3].

**2.3 The sliding lemma.** *Suppose that Hypotheses* 2.1 *hold.*
*Let $e$ and $f$ be elements of $E$.*
*Suppose that $\tau e = \iota f$, $G_e \le G_f$, and $Gf \cap Ge = \emptyset$.*
*Let $\tau' \colon E \to V$ denote the map given by*

$$e' \mapsto \tau'(e') := \begin{cases} \tau(e') & \text{if } e' \in E - Ge, \\ \tau(gf) & \text{if } e' = ge \text{ for some } g \in G, \end{cases}$$

*for all $e' \in E$.*
*Then the $G$-graph $T' = (T', V, E, \iota, \tau')$ is a $G$-tree.*

Here, we say that $T'$ is obtained from $T$ by *$G$-equivariantly sliding $\tau e$ along $f$ from $\iota f$ to $\tau f$.*

In applications, we usually first $G$-equivariantly reorient $Ge$, or $Gf$, or both, or neither, and then, in the resulting tree, $G$-equivariantly slide $\tau e$ along $f$ from $\iota f$ to $\tau f$, and then reorient back again. We then call the combined procedure a *$G$-equivariant sliding operation.*

*Proof of Lemma* 2.3. It is clear that $T'$ is a $G$-graph.

Let $X$ be the $G$-graph obtained from $T$ by deleting the two edge orbits $Ge \cup Gf$, and then inserting one new vertex orbit $Gv$ and three new edge orbits $Ge' \cup Gf_1 \cup Gf_2$, with $G_{e'} = G_e$, $G_v = G_{f_1} = G_{f_2} = G_f$, and setting

$$\iota(e') := \iota(e), \quad \iota(f_1) := \iota(f) = \tau(e), \quad \iota(f_2) := \tau(e') := \tau(f_1) := v, \quad \tau(f_2) := \tau(f).$$

Thus we are $G$-equivariantly subdividing $f$ into $f_1$ and $f_2$ by adding $v$, and then sliding $\tau e$ along $f_1$ from $\iota f_1$ to $\tau f_1 = v$.

Then $T$ is recovered from $X$ by $G$-equivariantly compressing the closure of $f_1$ to $\iota(f_1)$, and renaming $f_2$ as $f$, $e'$ as $e$. Thus $X$ maps onto $T$ with fibres which are trees. It follows that $X$ is a tree; see [3, Proposition III.3.3].

Also $T'$ is recovered from $X$ by $G$-equivariantly compressing the closure of $f_2$ to $\tau(f_2)$, and renaming $f_1$ as $f$, $e'$ as $e$. By the compressing lemma 2.2, $T'$ is a tree. $\square$

# 3 Filtrations

Throughout this section we will be working with the following.

**3.1 Hypotheses.** Let $T = (T, V, E, \iota, \tau)$ be a $G$-tree, let $U$ be a $G$-retract of the $G$-set $V$, and let $W = V - U$. $\square$

**3.2 Conventions.** We shall use interval notation for ordinals; for example, if $\kappa$ is an ordinal, then $[0, \kappa)$ denotes the set of all ordinals $\alpha$ such that $\alpha < \kappa$.

If we have an ordinal $\kappa$ and a specified map from a set $X$ to $[0, \kappa)$, then we will understand that the following notation applies. Denoting the image of each $x \in X$ by $\mathrm{height}(x) \in [0, \kappa)$, we write, for each $\alpha \in [0, \kappa)$ and each $\beta \in [0, \kappa]$,

$$X[\alpha] := \{x \in X \mid \mathrm{height}(x) = \alpha\} \quad \text{and} \quad X[0, \beta) := \{x \in X \mid \mathrm{height}(x) < \beta\}. \ \square$$

**3.3 Definitions.** Suppose that Hypotheses 3.1 hold.

Let $P(T)$ denote the set of paths in $T$, as in Definitions I.2.3 of [3]. Thus, for each $p \in P(T)$, we have the *initial vertex* of $p$, denoted $\iota p$, the *terminal vertex* of $p$, denoted $\tau p$, the *set of edges which occur in* $p$, denoted $E(p) \subseteq E$, the *length* of $p$, denoted $\mathrm{length}(p) \in \mathbb{N}$, and the $G$*-stabilizer* of $p$, denoted $G_p \le G$.

Let $\kappa$ be an ordinal and let

$$T \to [0,\kappa), \quad x \mapsto \mathrm{height}(x), \tag{3.3.1}$$

be a map. Since $T$ is nonempty, $\kappa$ must be nonzero. As a set, $T = V \cup E$. Thus, for each $\alpha \in [0,\kappa)$, we have $T[\alpha]$, $E[\alpha]$ and $V[\alpha]$, and, for each $\beta \in [0,\kappa]$, we have $T[0,\beta)$, $E[0,\beta)$ and $V[0,\beta)$.

For each $w \in W$, we then define

$$\begin{aligned} P_T(w) := \{p \in P(T) \mid \iota p = w,\, G_p = G_w,\, &\mathrm{height}(\tau p) < \mathrm{height}(w),\\ &\mathrm{height}(E(p)) \subseteq \{\mathrm{height}(w), \mathrm{height}(w)+1\}\}. \end{aligned}$$

We say that (3.3.1) is a $U$*-filtration of* $T$ if all of the following hold:

(3.3.2) for each $\beta \in [0,\kappa]$, $T[0,\beta)$ is a $G$-subforest of $T$;

(3.3.3) $T[0] = U$;

(3.3.4) for each $\alpha \in [1,\kappa)$, $T[\alpha]$ is a $G$-finite $G$-subset of $T$; and,

(3.3.5) for each $w \in W$, $P_T(w)$ is nonempty. □

**3.4 Lemma.** *If Hypotheses* 3.1 *hold, then there exists a* $U$*-filtration of* $T$.

*Proof.* We shall recursively construct a family $(E[\alpha] \mid \alpha \in [0,\kappa))$ of $G$-subsets of $E$, for some nonzero ordinal $\kappa$.

We take $E[0] = \emptyset$.

Suppose that $\gamma$ is a nonzero ordinal, and that we have a family $(E[\alpha] \mid \alpha \in [0,\gamma))$ of $G$-subsets of $E$.

For each $\beta \in [0,\gamma]$, we define

$$E[0,\beta) := \bigcup_{\alpha\in[0,\beta)} E[\alpha] \quad \text{and} \quad V[0,\beta) := \begin{cases} \emptyset & \text{if } \beta = 0,\\ U \cup \iota(E[0,\beta)) \cup \tau(E[0,\beta)) & \text{if } \beta > 0. \end{cases}$$

For each $\alpha \in [0,\gamma)$, we define $V[\alpha] := V[0,\alpha+1) - V[0,\alpha)$. Thus

$$V[0,\beta) = \bigcup_{\alpha\in[0,\beta)} V[\alpha].$$

If $E[0,\gamma) = E$, we take $\kappa = \gamma$ and the construction terminates.

Now suppose that $E[0,\gamma) \subset E$. We shall explain how to choose $E[\gamma]$.

If $\gamma$ is a limit ordinal or 1, we take $E[\gamma]$ to be an arbitrary single $G$-orbit in $E - E[0,\gamma)$.

If $\gamma$ is a successor ordinal greater than 1 then there is a unique $\alpha \in [1,\gamma)$ such that $\gamma = \alpha + 1$, and we want to construct $E[\alpha+1]$. Notice that $V[0,\alpha)$ is a $G$-retract of $V$ because $V[0,\alpha)$ contains $U$. Thus we can $G$-equivariantly specify, for each $w \in V[\alpha]$, a $T$-geodesic $p = p(w)$ from $w$ to an element $v = v(w) \in V[0,\alpha)$ fixed by $G_w$. Since $G_w$ fixes both ends of $p$, $G_w$ fixes $p$. Hence we may assume that $v$ is the first, and hence only, vertex of $p$ that lies in $V[0,\alpha)$. Clearly $G_p$ fixes $w$. Thus $G_w = G_p$. Let $P_{\alpha+1}$ denote the set of edges which occur in the $p(w)$, as $w$

ranges over $V[\alpha]$. Then $P_{\alpha+1} \subseteq E - E[0,\alpha)$, since each element of $E[0,\alpha)$ has *both* vertices in $V[0,\alpha)$. If $P_{\alpha+1} \subseteq E[\alpha]$, we choose $E[\alpha+1]$ to be an arbitrary single $G$-orbit in $E - E[0,\alpha+1)$. If $P_{\alpha+1} \not\subseteq E[\alpha]$, we take $E[\alpha+1] = P_{\alpha+1} - E[\alpha]$. This completes the description of the recursive construction.

We now verify that we have a $U$-filtration of $T$.

It can be seen that, for each ordinal $\gamma$ such that $(E[\alpha] \mid \alpha \in [0,\gamma))$ is defined, the $E[\alpha]$, $\alpha \in [1,\gamma)$, are pairwise disjoint, nonempty, $G$-subsets of $E$. Hence the cardinal of $\gamma$ is at most one more than the cardinal of $E$. Therefore the construction terminates at some stage. This implies that there exists a nonzero ordinal $\kappa$ such that $E[0,\kappa) = E$. Also $V[0,\kappa) = V$, and $(V[\alpha] \mid \alpha \in [0,\kappa))$ gives a partition of $V$. Thus we have an implicit map $T \to [0,\kappa)$ and we denote it by $x \mapsto \text{height}(x)$.

Clearly (3.3.2), (3.3.3) and (3.3.5) hold.

If $\alpha \in [1,\kappa)$ and $E[\alpha]$ is $G$-finite, then either $E[0,\alpha+1) = E$ or $V[\alpha]$, $P_{\alpha+1}$ and $E[\alpha+1]$ are $G$-finite. It follows, by transfinite induction, that $E[\alpha]$ and $V[\alpha]$ are $G$-finite for all $\alpha \in [1,\kappa)$. Thus (3.3.4) holds. □

## 4 The main result

Let us introduce a technical concept which generalizes that of a finite subgroup.

**4.1 Definitions.** A subgroup $H$ of $G$ is said to be *$G$-conjugate incomparable* if, for each $g \in G$, $H^g \subseteq H$ (if and) only if $H^g = H$. This clearly holds if $H$ is finite.

We say that a $G$-set $X$ *has $G$-conjugate-incomparable stabilizers* if, for each $x \in X$, the $G$-stabilizer $G_x$ is a $G$-conjugate-incomparable subgroup, that is, for each $g \in G$, $G_x \subseteq G_{gx}$ (if and) only if $G_x = G_{gx}$. □

Throughout this section we will be working with the following.

**4.2 Hypotheses.** Let $T = (T, V, E, \iota, \tau)$ be a $G$-tree, let $U$ be a $G$-retract of the $G$-set $V$, and let $W = V - U$.

Suppose that the $G$-set $W$ has $G$-conjugate-incomparable stabilizers.

Let $\kappa$ be an ordinal and let

(4.2.1) $\quad \text{height}\colon V \cup E \to [0,\kappa), \quad x \mapsto \text{height}(x),$

be a $U$-filtration of $T$. □

**4.3 Definitions.** Suppose that Hypotheses 4.2 hold.

Let $w \in W$. Define $d_T(w) := \min\{\text{length}(p) \mid p \in P_T(w)\}$. Then $d_T(w)$ is a positive integer and

(4.3.1) $\quad d_T(gw) = d_T(w)$ for all $g \in G$.

For $v_0$, $v_1$ in $V$, we say that $v_1$ is *lower than* $v_0$ if one of the following holds:

(4.3.2) $\quad \text{height}(v_0) > \text{height}(v_1)$;

(4.3.3) $\quad \text{height}(v_0) = \text{height}(v_1) > 0$ and $G_{v_0} < G_{v_1}$; or,

(4.3.4) $\quad \text{height}(v_0) = \text{height}(v_1) > 0$ and $G_{v_0} = G_{v_1}$ and $d_T(v_0) > d_T(v_1)$.

An edge $e$ of $T$ is said to be *problematic* if it joins vertices $v_0$, $v_1$ such that $\text{height}(e) = \text{height}(v_1) = \text{height}(v_0) + 1$. Notice that $\text{height}(e)$ is a successor ordinal and that $v_0$ is lower than $v_1$.

For each $v_0 \in W$, there exists a path

(4.3.5) $\quad v_0, e_1^{\epsilon_1}, v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$ in $P_T(v_0)$ such that $d = d_T(v_0)$.

Here $\mathrm{height}(v_1) \leq \mathrm{height}(v_0) + 1$. We say that $v_0$ is a *problematic* vertex of $T$ if there exists a path as in (4.3.5) such that $\mathrm{height}(v_1) = \mathrm{height}(v_0) + 1$. In this event $\mathrm{height}(e_1) = \mathrm{height}(v_1)$ and $e_1$ is a problematic edge of $T$. $\square$

**4.4 Lemma.** *If Hypotheses* 4.2 *hold, then applying some transfinite sequence of $G$-equivariant sliding operations to $T$ yields a $G$-tree $T' = (T', V, E, \iota', \tau')$ such that* (4.2.1) *is also a $U$-filtration of $T'$ and $T'$ has no problematic vertices.*

*Proof.* We shall construct a family of trees

$$(T_\beta = (T_\beta, V, E, \iota_\beta, \tau_\beta) \mid \beta \in [0, \kappa])$$

such that, for each $\beta \in [0, \kappa]$, (4.2.1) is a $U$-filtration of $T_\beta$, and $T_\beta$ has no problematic vertices in $V[0, \beta)$.

We take $T_0 = T$.

For each successor ordinal $\beta = \alpha + 1 \in [0, \kappa)$, $T_{\alpha+1}$ will be obtained from $T_\alpha$ by altering, if necessary, $\iota_\alpha$ and $\tau_\alpha$ on $E[\alpha + 1]$, as described below.

For each limit ordinal $\beta \in [0, \kappa]$, we let $\iota_\beta$ be given on $E[\alpha]$ by $\iota_\alpha$, for each $\alpha \in [0, \beta)$, and similarly for $\tau_\beta$.

Suppose then that $\beta = \alpha + 1 \in [0, \kappa)$, that we have a tree $T_\alpha = (T_\alpha, V, E, \iota_\alpha, \tau_\alpha)$, and that (4.2.1) is a $U$-filtration of $T_\alpha$, and that $T_\alpha$ has no problematic vertices in $V[0, \alpha)$.

We now describe a crucial *problem-reducing procedure* that can be applied in the case where there exists some $v_0 \in V[\alpha]$ which is a problematic vertex of $T_\alpha$.

Let $d = d_{T_\alpha}(v_0)$. Thus, there exists a path

$$v_0, e_1^{\epsilon_1}, v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$$

in $P_{T_\alpha}(v_0)$ such that $v_1 \in V[\alpha+1]$. Hence, $e_1 \in E[\alpha+1]$. Without loss of generality, let us assume that $\epsilon_1 = -1$.

There exists a least $i \in [2, d]$ such that $v_i \in V[0, \alpha + 1)$. Then

$$\{v_1, \ldots, v_{i-1}\} \subseteq V[\alpha + 1] \quad \text{and, hence,} \quad \{e_1, \ldots, e_i\} \subseteq E[\alpha + 1].$$

We claim that $Ge_1 \cap \bigcup_{j=2}^{i} Ge_j = \emptyset$. Suppose this fails. Then $e_1 \in \bigcup_{j=2}^{i} Ge_j$. Here, $v_0 \in \bigcup_{j=1}^{i} Gv_j$. Since $v_0 \in V[\alpha]$ and $\bigcup_{j=1}^{i-1} Gv_j \subseteq V[\alpha + 1]$ we see that $v_0 \in Gv_i$. Hence $v_i \in V[\alpha]$ and, by (4.3.1), $d_{T_\alpha}(v_i) = d_{T_\alpha}(v_0) = d$. But $G_{v_0} = G_p \subseteq G_{v_i}$. Since $G_{v_0}$ is a $G$-conjugate-incomparable subgroup, $G_{v_0} = G_{v_i}$. It follows that

$$v_i, e_{i+1}^{\epsilon_{i+1}}, v_{i+1}, \ldots, e_d^{\epsilon_d}, v_d$$

lies in $P_{T_\alpha}(v_i)$. Hence $d_{T_\alpha}(v_i) \leq d - i$, which is a contradiction. This proves the claim.

By Lemma 2.3, we can $G$-equivariantly slide $\iota e_1$ along $e_2^{\epsilon_2}$ from $v_1$ to $v_2$, and then $G$-equivariantly slide $\iota e_1$ along $e_3^{\epsilon_3}$ from $v_2$ to $v_3$, and so on, up to $v_i$. We then get a new $G$-tree $T_{\alpha,1} = (T_{\alpha,1}, V, E, \iota_{\alpha,1}, \tau_{\alpha,1})$ by $G$-equivariantly sliding $\iota e_1$ along our path from $v_1$ to $v_i$.

Let $e_1'$ denote $e_1$ viewed as an edge of $T_{\alpha,1}$. Wherever $v_1, e_1, v_0$ occurs in a path in $T_\alpha$, it can be replaced with the sequence

$$v_1, e_2^{\epsilon_2}, v_2, \ldots, v_{i-1}, e_i^{\epsilon_i}, v_i, e_1', v_0$$

to obtain a path in $T_{\alpha,1}$. It is important to note that all the edges involved here lie in $E[\alpha + 1]$. In terms of the free groupoid on $E[\alpha + 1]$, $e_1 = e_2^{\epsilon_2} e_3^{\epsilon_3} \cdots e_i^{\epsilon_i} e_1'$, and we are performing the change-of-basis which replaces $e_1$ with $e_1'$.

It is easy to see that (3.3.2)–(3.3.5) then hold for $T_{\alpha,1}$. Thus (4.2.1) is a $U$-filtration of $T_{\alpha,1}$. Notice that $T_{\alpha,1}$, like $T_\alpha$, has no problematic vertices in $V[0,\alpha)$. We have reduced the number of $G$-orbits of problematic edges in $E[\alpha+1]$.

This completes the description of a problem-reducing procedure.

Since $E[\alpha+1]$ is $G$-finite by (3.3.4), on repeating problem-reducing procedures as often as possible, we find some $m \in \mathbb{N}$, and a sequence

$$T_\alpha = T_{\alpha,0},\, T_{\alpha,1},\, \ldots,\, T_{\alpha,m},$$

such that $T_{\alpha,m}$ has no problematic vertices in $V[0,\alpha) \cup V[\alpha] = V[0,\alpha+1)$. We define $T_{\alpha+1} = (T_{\alpha+1}, V, E, \iota_{\alpha+1}, \tau_{\alpha+1})$ to be $T_{\alpha,m}$. Notice that $\iota_{\alpha+1}$ agrees with $\iota_\alpha$ on $E - E[\alpha+1]$, and similarly for $\tau_{\alpha+1}$.

Continuing this procedure transfinitely, we arrive at a tree $T_\kappa$ which has no problematic vertices. □

**4.5 Lemma.** *If Hypotheses* 4.2 *hold and $T$ has no problematic vertices, then applying some $G$-equivariant compressing operation on $T$ yields a $G$-tree with vertex set $U$.*

*Proof.* We claim that any sequence in $V$ is finite if each term is lower than all its predecessors.

Let $\alpha \in [0,\kappa)$.

If $v_0$, $v_1$ are elements of the same $G$-orbit of $V[\alpha]$, then $v_1$ is not lower than $v_0$, that is, (4.3.2)–(4.3.4) all fail; this follows from (4.3.1) and the fact that $V[\alpha]$ has $G$-conjugate-incomparable stabilizers.

Thus, if $n \in \mathbb{N}$ and $v_1, v_2, \ldots, v_n$ is a sequence in $V[\alpha]$ such that each term is lower than all its predecessors, then $Gv_1, Gv_2, \ldots, Gv_n$ are pairwise disjoint, and $n$ is at most the number of $G$-orbits in $V[\alpha]$. It follows that any sequence in $V[\alpha]$ is finite if each term is lower than all its predecessors. The claim now follows.

Let us $G$-equivariantly reorient $T$ so that, for each edge $e$, $\iota e$ is not lower than $\tau e$.

Let $v_0 \in W$. Let us $G$-equivariantly choose a path

$$v_0, e_1^{\epsilon_1}, v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$$

in $P_T(v_0)$ such that $d = d_T(v_0)$. Then we call $e_1$ the *distinguished edge* associated to $v_0$, and $v_1$ the *distinguished neighbour* of $v_0$.

Let $E''$ denote the set of distinguished edges chosen in this way.

Let us consider the above path for $v_0$. From Definitions 4.3, we see that, since $T$ has no problematic vertices, $\text{height}(v_0) \geq \text{height}(v_1)$. We claim that $v_1$ is lower than $v_0$. The claim is clear if $\text{height}(v_0) > \text{height}(v_1)$ (in which case, $d = 1$), and we may assume that $\text{height}(v_0) = \text{height}(v_1)$ $(> 0)$. Again, the claim is clear if $G_{v_0} < G_{v_1}$, and we may assume that $G_{v_0} = G_{v_1}$. Here $G_{v_1}$ fixes $p$, and the path

$$v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$$

shows that $d_T(v_1) \leq d-1 < d = d_T(v_0)$, and the claim is proved. Hence $\epsilon_1 = 1$.

Thus $\iota$ induces a bijection $E'' \to W$.

Moreover, in travelling along the distinguished edge $e_1$ respecting the orientation, from $v_0$ to its distinguished neighbour $v_1$, we move to a lower vertex.

Thus, starting at any element $v$ of $V$, after travelling a finite number of steps along distinguished edges respecting the orientation, we arrive at a vertex, denoted $\phi(v)$, with no distinguished neighbours, that is, $\phi(v) \in U$.

By Lemma 2.2, compressing the closures of the distinguished edges to their terminal vertices gives a $G$-tree with vertex set $U$ and edge set $E - E''$. □

We now come to our main result. In Section 7, we will see that the $G$-conjugate-incomparability hypotheses cannot be omitted.

**4.6 Theorem.** *Let $T$ be a $G$-tree, and let $U$ be a $G$-retract of the $G$-set $VT$. Suppose that the $G$-set $ET$ has $G$-conjugate-incomparable stabilizers, or, more generally, that the $G$-set $VT - U$ has $G$-conjugate-incomparable stabilizers.*

*Then applying to $T$ some transfinite sequence of $G$-equivariant sliding operations followed by some $G$-equivariant compressing operation yields a $G$-tree $T'$ such that $VT' = U$.*

*Here $ET'$ is a $G$-subset of $ET$, and there exists a $G$-set isomorphism*

$$ET - ET' \simeq VT - VT' = VT - U.$$

*Proof.* For each $w \in VT - U$, there exists $u \in U$ such that $G_w \leq G_u$. If $e$ denotes the first edge in the $T$-geodesic from $w$ to $u$, then $G_e = G_w$. Thus, if $E$ has $G$-conjugate-incomparable stabilizers, then the same holds for $VT - U$.

By Lemma 3.4, we may assume that Hypotheses 4.2 hold. By Lemma 4.4, we may assume that $T$ itself has no problematic vertices. Applying Lemma 4.5, we obtain the result; the final assertion follows from the compression lemma 2.2. □

We record the special case of Theorem 4.6 that is of interest to us.

**4.7 The retraction lemma.** *Let $T$ be a $G$-tree whose edge stabilizers are finite, and let $U$ be any $G$-retract of the $G$-set $VT$. Then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $U$.* □

# 5 The almost stability theorem and applications

We now combine the almost stability theorem 1.2 and the retraction lemma 4.7.

**5.1 Theorem.** *Let $E$ and $A$ be $G$-sets such that $E$ has finite stabilizers and $A$ is nonempty and has trivial $G$-action. If $V$ is a $G$-retract of a $G$-stable almost equality class in $(E, A)$, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $V$.*

*Proof.* Let $\widetilde{V}$ be a $G$-stable almost equality class in $(E, A)$ which contains $V$ as a $G$-retract. By the almost stability theorem 1.2, there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is $\widetilde{V}$. By the retraction lemma 4.7, there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is $V$. □

We now recall Definitions IV.2.1 and IV.2.2 of [3].

**5.2 Definitions.** Let $M$ be a *$G$-module*, that is, an additive abelian group which is also a $G$-set such that $G$ acts as group automorphisms on $M$. Thus a $G$-module is simply a left module over the integral group ring $\mathbb{Z}G$.

If $d\colon G \to M$ is a *derivation*, that is, a map such that $d(xy) = d(x) + xd(y)$ for all $x, y \in G$, then $M_d$ denotes the set $M$ endowed with the $G$-action

$$G \times M \to M, \quad (g, m) \mapsto g \cdot m := gm + d(g) \quad \text{for all } g \in G \text{ and all } m \in M.$$

It is straightforward to show that $M_d$ is a $G$-set. This construction has made other appearances in the literature; see [1, Remarque 4.a.5].

We say that $M$ is an *induced* $G$-module if there exists an abelian group $A$ such that $M$ is isomorphic, as $G$-module, to $AG := \mathbb{Z}G \otimes_{\mathbb{Z}} A$.

We say that $M$ is a *$G$-projective* $G$-module if $M$ is isomorphic, as $G$-module, to a direct summand of an induced $G$-module. □

**5.3 Example.** If $R$ is any ring and $P$ is a projective left $RG$-module, then there exists a free left $R$-module $F$ such that $P$ is isomorphic, as $RG$-module, to an $RG$-summand of

$$RG \otimes_R F = \mathbb{Z}G \otimes_{\mathbb{Z}} R \otimes_R F = \mathbb{Z}G \otimes_{\mathbb{Z}} F = FG.$$

Hence $P$ is $G$-projective. $\square$

The following generalizes Theorem IV.2.5 and Corollary IV.2.8 of [3].

**5.4 Theorem.** *If $P$ is a $G$-projective $G$-module, and $d\colon G \to P$ is a derivation, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $P_d$.*

*Proof.* There exists an abelian group $A$ such that $P$ is isomorphic to a $G$-summand of $AG$. We view $P$ as a $G$-submodule of $AG$. There exists an additive $G$-retraction $\pi\colon AG \to P$.

We view $AG$ as the almost equality class of $(G, A)$ which contains the zero map. Thus $AG$ is a $G$-submodule of $(G, A)$, and we have a derivation

$$d : G \to P \subseteq AG \subseteq (G, A).$$

By a classic result of Hochschild's, there exists $v \in (G, A)$ such that, for all $g \in G$, $d(g) = gv - v$. For example, we can take $v\colon x \mapsto -(d(x))(x)$, for all $x \in G$. See the proof of Proposition IV.2.3 in [3].

Let $U = v + P$ and $V = v + AG$. Then $U \subseteq V \subseteq (G, A)$, and $V$ is the almost equality class which contains $v$. Also, $U$ and $V$ are $G$-stable, since, for each $g \in G$, $gv = v + d(g) \in v + P \subseteq v + AG$. The map

$$V \to U, \quad v + m \mapsto v + \pi(m), \text{ for all } m \in AG,$$

is a $G$-retraction, since, for all $m \in AG$,

$$\begin{aligned} g(v + m) = v + gm + d(g) \quad &\mapsto \quad v + \pi(gm + d(g)) = v + g\pi(m) + d(g) \\ &\qquad\qquad\qquad\qquad\quad = g(v + \pi(m)). \end{aligned}$$

By Theorem 5.1, there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $U$.

The bijective map $P \to U$, $p \mapsto v + p$, is an isomorphism of $G$-sets $P_d \xrightarrow{\sim} U$. Now the result follows. $\square$

**5.5 Remark.** Notice that, in Theorem 5.4, the stabilizer of a vertex $p \in P_d$ is precisely the kernel of the derivation

$$d + \operatorname{ad} p\colon G \to P, \quad g \mapsto d(g) + gp - p = (g - 1)(v + p). \qquad \square$$

The following generalizes Corollary IV.2.10 of [3] and is used in the proof of Lemma 5.16 of [5].

**5.6 Corollary.** *Let $M$ be a $G$-module, let $P$ be a $G$-projective $G$-submodule of $M$, and let $v$ be an element of $M$. If the subset $v + P$ of $M$ is $G$-stable, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $v + P$.*

*Proof.* The inner derivation $\operatorname{ad} v\colon G \to M$ restricts to a derivation $d\colon G \to P$, $g \mapsto gv - v \in P \subseteq M$, for all $g \in G$. The bijective map $P \to v+P$, $p \mapsto v+p$, is then an isomorphism of $G$-sets $P_d \xrightarrow{\sim} v+P$. Now the result follows from Theorem 5.4. $\square$

**5.7 Example.** Let $R$ be a nonzero associative ring, and let $\omega RG$ be the augmentation ideal of the group ring $RG$.

Notice that, in the (left) $G$-set $RG$, the $G$-subset $RG - \{0\}$ has finite stabilizers. The coset $1 + \omega RG$ lies in $RG - \{0\}$ and is $G$-stable. Hence $1 + \omega RG$ is a $G$-set with finite stabilizers.

If $\omega RG$ is projective as left $RG$-module, then, by Corollary 5.6, there exists a $G$-tree $T$ with $VT = 1 + \omega RG$; hence $T$ has finite stabilizers. This sheds some light on the main step in the characterization of groups of cohomological dimension at most one over $R$. See, for example, [3, Theorem IV.3.13]. □

# 6 A more general almost stability theorem

We next want to generalize Theorem 5.1.

The following is similar to Lemma 2.2 of [4], and the proof is straightforward.

**6.1 Lemma.** *Let $E$ and $A$ be $G$-sets such that, for each $e \in E$, $G_e$ acts trivially on $A$.*

*Let $\bar{A}$ denote the $G$-set with the same underlying set as $A$ but with trivial $G$-action.*

*Let $E_0$ be a $G$-transversal in $E$.*

*For each $\phi \in (E, A)$, let $\widehat{\phi} \in (E, \bar{A})$ be defined by $\widehat{\phi}(ge) = g^{-1} \cdot \phi(ge)$ for all $(g, e) \in G \times E_0$, where $\cdot$ denotes the $G$ action on $A$.*

*For each $\psi \in (E, \bar{A})$, let $\widetilde{\psi} \in (E, A)$ be defined by $\widetilde{\psi}(ge) = g \cdot \psi(ge)$ for all $(g, e) \in G \times E_0$.*

*Then*

$$(E, A) \to (E, \bar{A}), \quad \phi \mapsto \widehat{\phi}, \quad \textit{and} \quad (E, \bar{A}) \to (E, A), \quad \psi \mapsto \widetilde{\psi},$$

*are mutually inverse isomorphisms of $G$-sets which preserve almost equality between functions.* □

Combined, Lemma 6.1 and Theorem 5.1 give the most general form that we know of the almost stability theorem.

**6.2 Theorem.** *Let $E$ and $A$ be $G$-sets such that $A$ is nonempty and, for each $e \in E$, $G_e$ is finite and acts trivially on $A$. If $V$ is a $G$-retract of a $G$-stable almost equality class in $(E, A)$, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $V$.* □

For each $e \in E$, if $G_e$ is trivial, then $G_e$ is finite and acts trivially on $A$. It was this case that was useful in [4].

# 7 An example

In this section, we shall give an example of a group $G$ and a retract of a vertex set of a $G$-tree that is not the vertex set of any $G$-tree.

**7.1 Hypotheses.** Let $Y = (Y, \overline{V}, \overline{E}, \overline{\iota}, \overline{\tau})$ be the graph given as follows:

$$\overline{V} = \{\overline{u}, \overline{w}\}, \quad \overline{E} = \{\overline{e}, \overline{f}\}, \quad \overline{\iota}(\overline{e}) = \overline{u}, \quad \overline{\tau}(\overline{e}) = \overline{\iota}(\overline{f}) = \overline{\tau}(\overline{f}) = \overline{w}.$$

Let $Y_0 := (Y_0, \overline{V}, \{\overline{e}\}, \overline{\iota}, \overline{\tau})$ be the unique maximal subtree of $Y$.

We now use the notation of Definitions I.3.1 of [3] to define a graph of groups $(G(-), Y)$ as follows. Let the vertex groups be given by

$$G(\overline{u}) = \langle x, y \mid \quad \rangle, \qquad\qquad G(\overline{w}) = \langle x', y' \mid \quad \rangle.$$

Let the edge groups be given by

$$G(\overline{e}) = \langle x^4,\, xyx,\, y^4 \mid \quad \rangle, \qquad\qquad G(\overline{f}) = \langle x',\, y' \mid \quad \rangle,$$

where we have

$$\begin{aligned} G(\overline{e}) &= \langle x^4,\, xyx,\, y^4 \mid \quad \rangle \;\leq\; \langle x,\, y \mid \quad \rangle = G(\overline{u}) = G(\overline{\iota}\,\overline{e}),\\ G(\overline{f}) &= \langle x',\, y' \mid \quad \rangle = G(\overline{w}) = G(\overline{\iota}\,\overline{f}). \end{aligned}$$

Finally, let the edge-group monomorphisms be given by

$$\begin{array}{rlcl} t_{\overline{e}}\colon G(\overline{e}) = & \langle x^4,\, xyx,\, y^4 \mid \quad \rangle & \to & \langle x',\, y' \mid \quad \rangle = G(\overline{w}) = G(\overline{\tau}\,\overline{e}),\\ & (x^4, xyx, y^4) & \mapsto & (x'^4, x'y'x', y'^4),\\ t_{\overline{f}}\colon G(\overline{f}) = & \langle x', y' \mid \quad \rangle & \to & \langle x',\, y' \mid \quad \rangle = G(\overline{w}) = G(\overline{\tau}\,\overline{f}),\\ & (x', y') & \mapsto & (x'^2, y'^2). \end{array}$$

Using notation whose interpretation we hope is clear, we represent the resulting graph of groups as follows.

$$\begin{tikzcd} \langle x,\, y \mid \quad \rangle \bullet \arrow[d, "{\langle x^4 \mapsto x'^4,\, xyx \mapsto x'y'x',\, y^4 \mapsto y'^4 \mid \quad \rangle}"] \\ \langle x', y' \mid \quad \rangle \bullet \arrow[loop right, "{\langle x' \mapsto x'^2,\, y' \mapsto y'^2 \mid \quad \rangle}"] \end{tikzcd}$$

Let $G$ be the fundamental group of the graph of groups, $\pi(G(-), Y, Y_0)$, as in Definitions I.3.4 of [3]. We shall write $t$ for the element of $G$ that realizes the edge-group monomorphism $t_{\overline{f}}\colon G(\overline{f}) \to G(\overline{w})$; thus

(7.1.1) $\quad G = \langle x, y, x', y', t \mid x^4 = x'^4,\ xyx = x'y'x',\ y^4 = y'^4,\ x'^t = x'^2,\ y'^t = y'^2 \rangle,$

where $x'^t$ denotes $t^{-1}x't$. Here $G(\overline{u})$ and $G(\overline{w})$ are subgroups of $G$; see [3, Corollary I.7.5].

Let $T = (T, V, E, \iota, \tau)$ be the Bass-Serre tree $T(G(-), Y, Y_0)$, as in Notation I.7.1 of [3]. Thus, using $\vee$ to denote disjoint union, we can write

$$V = Gu \vee Gw, \quad G_u = \langle x,\, y \rangle, \quad G_w = \langle x',\, y' \rangle,$$

$$E = Ge \vee Gf, \quad G_e = \langle x^4,\, xyx,\, y^4 \rangle, \quad G_f = \langle x',\, y' \rangle,$$

$$\iota(e) = u, \quad \tau(e) = w, \quad \iota(f) = w, \quad \tau(f) = tw.$$

By Bass-Serre Theory, $T$ is a $G$-tree; see [3, Theorem I.7.6].

For any subset $S$ of $T$, let $S^{xyx}$ denote $\{s \in S \mid (xyx)s = s\}$. □

We shall see that $Gu$ is a retract of a vertex set of a $G$-tree, but is not itself the vertex set of a $G$-tree.

**7.2 Lemma.** *Suppose that Hypotheses* 7.1 *hold. In particular, in $T$, $V = Gu \vee Gw$, $E = Ge \vee Gf$, $\iota(e) = u$, $\tau(e) = w$, $\iota(f) = w$, and $\tau(f) = tw$.*

(i) *In $G$, $x' = x^{4t^{-2}}$ and $y' = y^{4t^{-2}}$.*

(ii) $G = \langle x, y, t \mid x^{4t} = x^8,\ y^{4t} = y^8,\ x^{t^2} y^{t^2} x^{t^2} = x^4 y^4 x^4 \rangle$.

(iii) *In $T$, $G_u = \langle x,\, y \rangle$, $G_w = \langle x^4,\, y^4 \rangle^{t^{-2}}$, $G_e = \langle x^4,\, xyx,\, y^4 \rangle$, $G_f = \langle x^4,\, y^4 \rangle^{t^{-2}}$.*

(iv) *$Gu$ is a $G$-retract of $V$.*

*Proof.* (i). Now $x'^{t^2} = x'^{2t} = x'^4 = x^4$. Thus $x' = x^{4t^{-2}}$. Similarly, $y' = y^{4t^{-2}}$.

(ii). By (7.1.1),

$$\begin{aligned} G &= \langle x, y, x', y', t \mid x^4 = x'^4,\ xyx = x'y'x',\ y^4 = y'^4,\ x'^t = x'^2,\ y'^t = y'^2 \rangle \\ &= \langle x, y, x', y', t \mid x^4 = x'^4,\ xyx = x'y'x',\ y^4 = y'^4, \\ &\qquad\qquad x'^t = x'^2,\ y'^t = y'^2,\ x' = x^{4t^{-2}},\ y' = y^{4t^{-2}} \rangle \\ &= \langle x, y, t \mid x^4 = x^{16t^{-2}},\ xyx = x^{4t^{-2}} y^{4t^{-2}} x^{4t^{-2}},\ y^4 = y^{16t^{-2}}, \\ &\qquad x^{4t^{-1}} = x^{8t^{-2}},\ y^{4t^{-1}} = y^{8t^{-2}} \rangle \\ &= \langle x, y, t \mid x^{4t^2} = x^{16},\ x^{t^2} y^{t^2} x^{t^2} = x^4 y^4 x^4,\ y^{4t^2} = y^{16}, \\ &\qquad x^{4t} = x^8, \qquad\qquad y^{4t} = y^8 \rangle \\ &= \langle x, y, t \mid x^{4t} = x^8,\ x^{t^2} y^{t^2} x^{t^2} = x^4 y^4 x^4,\ y^{4t} = y^8 \rangle. \end{aligned}$$

(iii). $G_f = G_w = \langle x',\, y' \rangle = \langle x^4,\, y^4 \rangle^{t^{-2}}$.

(iv). We have $G_w = \langle x^4,\, y^4 \rangle^{t^{-2}} \leq \langle x,\, y \rangle^{t^{-2}} = G_u^{t^{-2}} = G_{t^2 u}$. Thus $Gu$ is a $G$-retract of $Gu \vee Gw = V$. □

It remains to show that $Gu$ is not the vertex set of any $G$-tree. We shall use a sequence of technical lemmas.

It is straightforward to prove the following, using Lemma 7.2(ii).

**7.3 Lemma.** *Suppose that Hypotheses* 7.1 *hold, and let* $n \in \mathbb{N}$.

(i) *In* $G$, $(xyx)^{t^{n+2}} = (x^4)^{2^n} (y^4)^{2^n} (x^4)^{2^n}$.

(ii) *If* $n \neq 1$, *then, in* $G$, $(xyx)^{t^n} = x^{2^n} y^{2^n} x^{2^n}$. □

The next result concerns the free group of rank two.

**7.4 Lemma.** *Suppose that Hypotheses* 7.1 *hold, let* $n \in \mathbb{N}$, *and let* $g \in G_u$. *In particular,* $G_u = \langle x, y \mid \quad \rangle$.

(i) *If* $x^{2^n} y^{2^n} x^{2^n} \in \langle x^2, y^2 \rangle^g$, *then* $n \neq 0$ *and* $g \in \langle x^2, y^2 \rangle$.

(ii) *If* $x^{2^n} y^{2^n} x^{2^n} \in \langle x^4, xyx, y^4 \rangle^g$, *then* $n \neq 1$ *and* $g \in \langle x^4, xyx, y^4 \rangle$.

*Proof.* Let $T_u = X(G_u, \{x, y\})$, the Cayley graph of $G_u$ with respect to $\{x, y\}$, as in [3, Definitions I.2.1]. Each (oriented) edge of $T_u$ is labelled $x$ or $y$.

Let $H$ be any subgroup of $G_u$; we have in mind the cases $H = \langle x^2, y^2 \rangle$ and $H = \langle x^4, xyx, y^4 \rangle$.

Let $w = x^{2^n} y^{2^n} x^{2^n} \in G_u$.

Let $X := H \backslash T_u$, let $Y := \langle w \rangle \backslash T_u$, and let $Z := G_u \backslash T_u$.

The pullback of the two natural maps $X \to Z$, $Y \to Z$ provides detailed information about all nontrivial subgroups of $G_u$ of the form $\langle w \rangle \cap H^g$; see [2, p. 380]. However, this pullback can be rather cumbersome and we do not require detailed information. For our purposes, special considerations will suffice, as follows.

Define $g^{-1}X := (H^g) \backslash T_u$.

There is a graph isomorphism $X \simeq g^{-1}X$, $Hx \leftrightarrow H^g g^{-1} x$.

The fundamental group of $X$ with basepoint $H1$, $\pi(X, H1)$, is naturally isomorphic to $H$, with the elements of $H$ being read off closed paths based at $H1$.

Similarly, $H^g$ is naturally isomorphic to $\pi(g^{-1}X, H^g 1)$, and this in turn is naturally isomorphic to $\pi(X, Hg)$ via the graph isomorphism $g^{-1}X \simeq X$.

Suppose that $w$ lies in $H^g$. Then $w$ can be read off a closed path in $X$ based at $Hg$. Since $w$ is a cyclically reduced word, the closed path is cyclically reduced. The smallest subgraph of $X$ which contains all the cyclically reduced closed paths in $X$ is called the *core* of $X$, denoted core$(X)$. It follows that the vertex $Hg$ lies in core$(X)$, and that we can start at $Hg$, read $w$ and stay inside core$(X)$.

(i) Suppose that $H = \langle x^2, y^2\rangle$.

Here core$(X)$ has vertex set $\{H1,\ Hx,\ Hy\}$ and labelled-edge set

$$\{(H1, x, Hx), (Hx, x, Hx^2), (H1, y, Hy), (Hy, y, Hy^2)\}$$

with $Hx^2 = Hy^2 = H1$.

We note that $Hxy$ and $Hyx$ are outside core$(X)$.

Notice that $(Hy)x = Hyx$. This lies outside core$(X)$. Thus, $Hg \neq Hy$, since $Hgw$ can be read in core$(X)$. Hence, $Hg \in \{H1, Hx\}$.

Notice that $(H1)(xy) = Hxy$ and $(Hx)(xyx) = Hyx$. These lie outside core$(X)$. Thus $n \neq 0$. Hence, $x^{2^n} \in H$.

Notice that $(Hx)(x^{2^n}y) = Hxy$ lies outside core$(X)$. Thus $Hg \neq Hx$. Hence, $Hg = H1$, that is, $g \in H$.

This proves (i).

(ii). Suppose that $H = \langle x^4, xyx, y^4\rangle$.

Here core$(X)$ has vertex set

$$\{H1\} \cup \{Hx^i,\ Hy^i \mid 1 \leq i \leq 3\}.$$

and labelled-edge set

$$\{(Hx^i, x, Hx^{i+1}),\ (Hy^i, y, Hy^{i+1}) \mid 0 \leq i \leq 3\} \cup \{(Hx, y, Hxy)\},$$

with $Hx^4 = Hy^4 = H1$ and $Hxy = Hx^3$.

We note that $Hxy^2 = Hx^3y$, $Hx^2y$, $Hyx$, $Hy^2x$, and $Hy^3x$ all lie outside core$(X)$.

Consider any $j$ with $1 \leq j \leq 3$. Notice that $(Hy^j)(x) = Hy^jx$. This lies outside core$(X)$. It follows that $Hg \neq Hy^j$. Hence $Hg = Hx^i$ for some $i$ with $0 \leq i \leq 3$.

Notice that $(Hx)(xy) = Hx^2y$, $(Hx^2)(xy) = Hx^3y$, and $(Hx^3)(xyx) = Hyx$. These all lie outside core$(X)$. Thus, if $n = 0$, then $Hg = H1$.

Notice that $(H1)(x^2y) = Hx^2y$, $(Hx)(x^2y) = Hx^3y$, $(Hx^2)(x^2y^2x) = Hy^2x$, and $(Hx^3)(x^2y^2) = Hxy^2$. These all lie outside core$(X)$. Thus $n \neq 1$.

Now suppose that $n \geq 2$. Thus $x^{2^n} = (x^4)^{2^{n-2}} \in H$.

Notice that $(Hx)(x^{2^n}y^2) = Hxy^2$, $(Hx^2)(x^{2^n}y) = Hx^2y$, and $(Hx^3)(x^{2^n}y) = Hx^3y$. These all lie outside core$(X)$. Thus $Hg = H1$.

This proves (ii). $\square$

**7.5 Lemma.** *Suppose that Hypotheses* 7.1 *hold and let* $n \in \mathbb{N}$.

(i) $(t^nG_ue)^{xyx} = \{t^ne\}$ *if* $n \neq 1$.

(ii) $(t^nG_we)^{xyx} = \begin{cases} \{t^ne\} & \textit{if } n \neq 1, \\ \emptyset & \textit{if } n = 1. \end{cases}$

(iii) $(t^nG_wt^{-1}f)^{xyx} = \begin{cases} \{t^{n-1}f\} & \textit{if } n \neq 0, \\ \emptyset & \textit{if } n = 0. \end{cases}$

(iv) $(t^nG_wf)^{xyx} = \{t^nf\}$.

*Proof.* (i). Let $g \in G_u = \langle x, y\rangle$.

Suppose that $n \neq 1$ and that $(xyx)t^nge = t^nge$. Then $(xyx)^{t^ng} \in G_e$. By Lemma 7.3(ii),

$$(x^{2^n}y^{2^n}x^{2^n})^g \in G_e = \langle x^4,\ xyx,\ y^4\rangle.$$

By Lemma 7.4(ii), $g \in \langle x^4, xyx, y^4\rangle = G_e$. Hence $t^n ge = t^n e$. It is now easy to see that (i) holds.

(ii). Let $h \in G_w = \langle x^4, y^4\rangle^{t^{-2}}$. Let $g = h^{t^2} \in \langle x^4, y^4\rangle$.

Suppose that $(xyx)t^n he = t^n he$. Then $(xyx)t^{n+2}gt^{-2}e = t^{n+2}gt^{-2}e$, and $(xyx)^{t^{n+2}gt^{-2}} \in G_e$. By Lemma 7.3(i),

$$((x^4)^{2^n}(y^4)^{2^n}(x^4)^{2^n})^g \in G_e^{t^2} = \langle x^4, xyx, y^4\rangle^{t^2} = \langle (x^4)^4, (x^4)(y^4)(x^4), (y^4)^4\rangle.$$

By Lemma 7.4(ii) with $x^4$ , $y^4$ in place of $x$, $y$, we see that $n \neq 1$ and

$$g \in \langle (x^4)^4, (x^4)(y^4)(x^4), (y^4)^4\rangle = G_e^{t^2}.$$

Hence $h \in G_e$ and $t^n he = t^n e$. It is now clear that (ii) holds.

(iii). Let $h \in G_w = \langle x^4, y^4\rangle^{t^{-2}}$. Let $g = h^{t^2} \in \langle x^4, y^4\rangle$.

Suppose that $(xyx)t^n ht^{-1}f = t^n ht^{-1}f$. Then $(xyx)t^{n+2}gt^{-3}f = t^{n+2}gt^{-3}f$, and $(xyx)^{t^{n+2}gt^{-3}} \in G_f$. By Lemma 7.3(i),

$$((x^4)^{2^n}(y^4)^{2^n}(x^4)^{2^n})^g \in G_f^{t^3} = \langle x^4, y^4\rangle^t = \langle (x^4)^2, (y^4)^2\rangle.$$

By Lemma 7.4(i), with $x^4$ , $y^4$ in place of $x$, $y$, we see that $n \neq 0$ and

$$g \in \langle (x^4)^2, (y^4)^2\rangle = G_f^{t^3}.$$

Hence $h^{t^{-1}} \in G_f$ and $t^n ht^{-1}f = t^{n-1}f$. It is now clear that (iii) holds.

(iv). By Lemma 7.3(i), $(xyx)^{t^n} \in \langle x^4, y^4\rangle^{t^{-2}} = G_f = G_w$. □

**7.6 Lemma.** *Suppose that Hypotheses* 7.1 *hold. Then*

$$V^{xyx} \quad = \quad \{t^n u \mid n \in \mathbb{N} - \{1\}\} \quad \cup \quad \{t^n w \mid n \in \mathbb{N}\}.$$

*Proof.* Let $n \in \mathbb{N}$.

From [3, Definitions I.3.4], we obtain the following.

$$\begin{aligned} \iota^{-1}(t^n u) &= t^n G_u e, & \tau^{-1}(t^n u) &= \emptyset, \\ \iota^{-1}(t^n w) &= t^n G_w f, & \tau^{-1}(t^n w) &= t^n G_w e \,\cup\, t^n G_w t^{-1} f. \end{aligned}$$

By Lemma 7.5(ii), (iii) and (iv), the edges of $T^{xyx}$ incident to $w$ are $e$ and $f$, the edges of $T^{xyx}$ incident to $tw$ are $f$ and $tf$, and, for $n \geq 2$, the edges of $T^{xyx}$ incident to $t^n w$ are $t^n e$, $t^{n-1}f$ and $t^n f$.

Hence, in $T^{xyx}$, the neighbours of $w$ are $u$ and $tw$, the neighbours of $tw$ are $w$ and $t^2 w$, and, for $n \geq 2$, the neighbours of $t^n w$ are $t^n u$, $t^{n-1}w$ and $t^{n+1}w$.

By Lemma 7.5(i), if $n \neq 1$, then the unique edge of $T^{xyx}$ incident to $t^n u$ is $t^n e$, and hence the unique neighbour of $t^n u$ in $T^{xyx}$ is $t^n w$.

The result now follows. □

**7.7 Lemma.** *Suppose that Hypotheses* 7.1 *hold. There exists no $G$-tree with vertex set $Gu$.*

*Proof.* Suppose that there exists a $G$-tree $T'$ with $VT' = Gu$. We will derive a contradiction.

Let $L$ denote the subtree of $T$ with vertex set $\langle t\rangle w$ and edge set $\langle t\rangle f$. Then $L$ is homeomorphic to $\mathbb{R}$ and $t$ acts on $L$ by translation. In particular, $\langle t\rangle$ acts freely on $VT$. Hence, $\langle t\rangle$ acts freely on $VT' \subseteq VT$. As in [3, Proposition I.4.11], there exists a subtree $L'$ of $T'$ homeomorphic to $\mathbb{R}$ on which $t$ acts by translation.

Let $v'$ denote the vertex of $L'$ closest to $u$ in $T'$. It is well known, and easy to prove, that the $T'$-geodesic from $u$ to $t^2u$, denoted $T'[u, t^2u]$, is the concatenation of the four $T'$-geodesics $T'[u, v']$, $T'[v', tv']$, $T'[tv', t^2v']$, and $T'[t^2v', t^2u]$.

By Lemma 7.6, and the fact that $\langle t\rangle$ acts freely on $VT'$,

$$VT'^{xyx} = (Gu)^{xyx} = \{t^n u \mid n \in \mathbb{N} - \{1\}\} = \{t^n u \mid n \in \mathbb{N}\} - \{tu\}. \tag{7.7.1}$$

By (7.7.1), or by direct calculation, $xyx$ fixes $u$, moves $tu$, and fixes $t^2u$. Thus, $xyx$ fixes $T'[u, t^2u]$, and, hence, $xyx$ fixes $v'$, fixes $tv'$, and fixes $t^2v'$.

In particular, $tu \neq tv'$, hence $u \neq v'$, that is, $u \notin L'$.

Since $xyx$ fixes $v'$, we see, by (7.7.1), that $v' = t^n u$ for some $n \in \mathbb{N} - \{1\}$. Hence $u = t^{-n}v' \in t^{-n}L' = L'$. This is a contradiction. □

We now have the desired example.

**7.8 Theorem.** *There exists a group $G$ and a $G$-set $U$ such that $U$ is a $G$-retract of the vertex set of some $G$-tree but $U$ is not the vertex set of any $G$-tree.* □

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