HOROSPHERES AND CONVEX BODIES IN THE n-DIMENSIONAL HYPERBOLIC SPACE

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To the memory of professor L.A. Santaló

Abstract. In the n-dimensional Euclidean space the measure of hyperplanes that intersect a convex domain is equal to the $(n-2)$-integral of mean curvature of the boundary. The same question was treated by Santaló in the hyperbolic case. In non-euclidean geometry the analogue of linear subspaces are not always the totally geodesic and complete hypersurfaces. In some situations horospheres play the role of euclidean hyperplanes in hyperbolic spaces.

Santaló found in dimensions 2 and 3 that the measure of horospheres intersecting a convex domain has the same expression, multiplied by 2, than the measure of hyperplanes intersecting a convex body in the euclidean spaces. Therefore a question arises: is this equality true for every dimension?

Here we find the measure of intersecting horospheres in $\mathbb{H}^n$. For convex bodies it is a linear combination of the integrals of curvature of the boundary, and the analogy with the euclidean case does not hold in higher dimensions.

1. Introduction

One of the first results in *integral geometry* is the Cauchy-Crofton formula. It says that the length of a piecewise differentiable plane curve C can be computed measuring, with its multiplicity, the set of lines intersecting C:

$$\int_{l\cap C\neq\emptyset} \sharp(l\cap C)\,\mathrm{d}l = 2\,\mathrm{length}(C) \tag{1}$$

where $\mathrm{d}l$ is the rigid motions invariant measure of lines in the plane and is given by $\mathrm{d}l = \mathrm{d}p\,\mathrm{d}\theta$ where p is the distance from l to the origin and θ the angle to a fixed direction (see figure 1).

When applied to the boundary of a convex domain Ω, the Cauchy-Crofton formula tells us that the measure of lines intersecting Ω equals the length of the boundary.

There are some generalizations to higher dimensions in euclidean spaces (see [San76]). First, the measure of hyperplanes intersecting a convex domain Ω with C^2 boundary is

$$\int_{L\cap\Omega\neq\emptyset} \mathrm{d}L = M_{n-2}(\partial\Omega) \tag{2}$$

where M_{n-2} is the integral of the $(n-2)$-function of curvature of ∂K

Second, if we see $\sharp(l\cap C)$ as the measure of the intersection $l\cap C$, in higher dimensions this measure corresponds to the volume of the intersection of a hyperplane L

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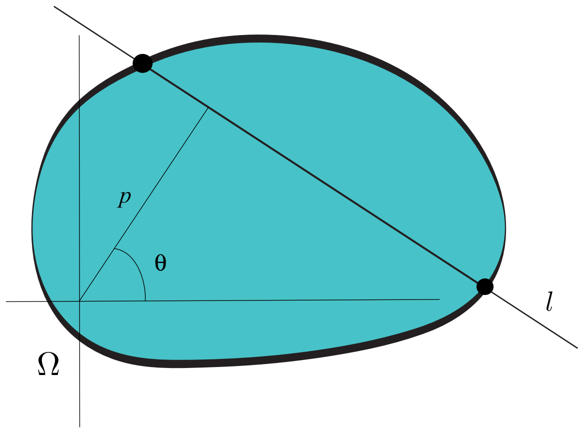

FIGURE 1.

with $\partial\Omega$. In this case

$$\int_{L\cap\partial\Omega\neq\emptyset} \mathrm{vol}_{n-2}(L\cap\partial\Omega)\,\mathrm{d}L = \frac{O_n O_{n-2}}{O_{n-1}O_0}\mathrm{vol}_{n-1}(\partial\Omega) \tag{3}$$

with O_r the volume of r-dimensional euclidean sphere. Third, if we count the intersection points of lines l with $\partial\Omega$ we have

$$\int_{l\cap\partial\Omega\neq\emptyset} \sharp(l\cap\partial\Omega)\,\mathrm{d}l = \frac{O_n}{O_1}\mathrm{vol}_{n-1}(\partial\Omega). \tag{4}$$

Note that the same formulas hold changing $\partial\Omega$ by a compact C^2 hypersurface M.

In the hyperbolic space $\mathbb{H}^n$ there are similar formulas, Cauchy-Crofton and formulas (3) and (4) are valid without change. It is known (see [San76]) that the measure of hyperplanes (complete totally geodesic hypersurfaces) intersecting a convex body is a linear combination of integrals of curvature. For instance in $\mathbb{H}^3$ the measure of hyperplanes intersecting a convex domain is the integral of mean curvature of the boundary minus the volume of the domain, $M_1 - V$.

The intrinsic geometry of horospheres (limit of spheres) is euclidean. Then it is natural to ask whether we have similar formulas to the euclidean ones when we substitute hyperplanes by horospheres.

In the works [San67] and [San68] Santaló started the study, in dimensions 2 and 3, of the measure of horospheres intersecting a given domain. In those cases formulas (1) and (2) are still valid.

In this paper we obtain the measure of horospheres intersecting a convex domain in $\mathbb{H}^n$. Indeed the main theorem is more general

Theorem. *If Ω is a domain in $\mathbb{H}^n$ bounded by an embedded C^2 hypersurface $\partial\Omega$ then*

$$\int_{H\cap\partial\Omega\neq\emptyset} \chi(H\cap\partial\Omega)\mathrm{d}H = 2\sum_{h=0}^{[(n-2)/2]} \binom{n-2}{2h}\frac{1}{2h+1}M_{n-2-2h}(\partial\Omega)$$

where $\chi(H\cap\partial\Omega)$ is the Euler-Poincaré characteristic of $H\cap\partial\Omega$.

From here we conclude that the measure of the horospheres which intersect an h-convex hypersurface (convex with respect to horocycles) M can be expressed as a linear combination of the integral of mean curvatures. It must be noticed than in dimensions 2 and 3 the measure of horosphere are $2M_0$ and $2M_1$ and it seemed

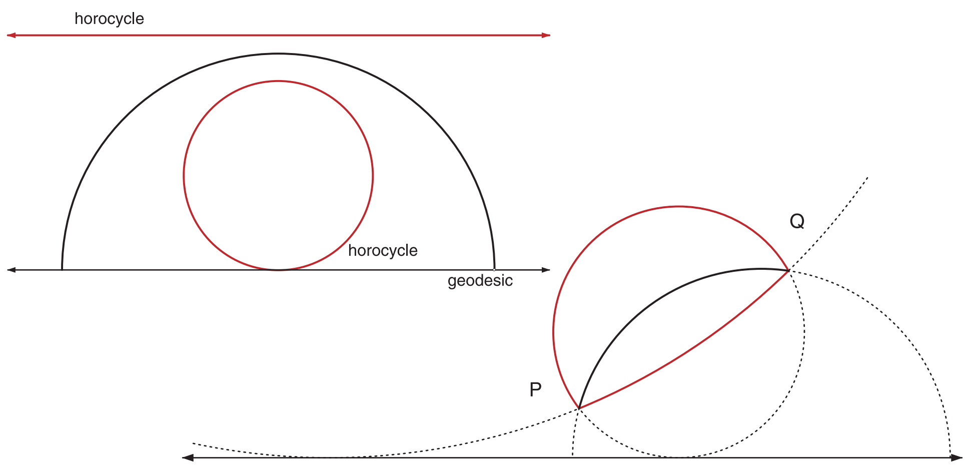

FIGURE 2.

natural to expect the measure of horospheres in higher dimensions to behave like the measure of hyperplanes in the euclidean spaces. The theorem tells us that this is not the case.

In section 2 we review some general notions and fix the notation needed in the rest of the paper. Section 3 is devoted to the definition of an invariant measure for horospheres in the hyperbolic space, with the normalization considered, some key formulas will give the same measure (multiplied by a 2 factor) than the 'linear' case (i.e. the totally geodesic case) (see (11) and (12) in proposition 3.1). Next, in section 4 we prove the main theorem, give examples and discuss some related problems.

2. Preliminaries

The hyperbolic space is the unique simply connected and complete Riemannian manifold of constant curvature -1. We do not consider any particular model but sometimes it will be useful to think in the half-space model, obtained considering the metric $(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2)/x_n^2$ in the half-space $x_n > 0$ in $\mathbb{R}^n$. In geodesic polar coordinates $\{\rho, \theta_1, \dots \theta_n\}$ the hyperbolic arc element has the form

$$ds^2 = d\rho^2 + \sinh^2 \rho (d\theta_1^2 + \cdots + d\theta_{n-1}^2)$$

and the volume element

$$dV = \sinh^{n-1} \rho d\rho \wedge d\Omega_{n-1}$$

where $d\Omega_{n-1}$ represents the element of solid angle corresponding to the direction $\theta_1, \dots \theta_{n-1}$.

Geodesic lines —or simply lines— are, in the half-space model, euclidean semi-circles and straight lines orthogonal to the boundary.

We consider two particular families of hypersurfaces in the hyperbolic space. *Hyperplanes* are complete totally geodesic hypersurfaces. In the half-space we see them as euclidean half spheres with center in $\{x_n = 0\}$ and euclidean hyperplanes orthogonal to $\{x_n = 0\}$. *Horospheres* are limit spheres, given a geodesic $c(t)$ and a fixed point $c(t_0) = p$ consider the hyperbolic sphere with center $c(t)$ passing through p, if the center $c(t)$ tends to infinity we obtain two horospheres, one for every direction of divergence. Horospheres can also be thought as hypersurfaces orthogonal to a family of parallel geodesics. In the half-space model horospheres are seen as euclidean spheres tangent to the boundary and also as the horizontal hyperplanes $x_n = ct$.

Let Σ be a compact hypersurface of class C^2. Consider it oriented by a unit normal $\mathbf{n}$, the second fundamental form h is $h(X,Y) = \langle \nabla_X Y, \mathbf{n}\rangle$. It is a bilinear symmetric form and its eigenvalues $\kappa_1, \dots, \kappa_{n-1}$ are the *principal curvatures*. We define the *functions of mean curvature* $\sigma_i^\Sigma(x)$ to be such that

$$\det(I + th_x) = \sum_{i=0}^{n-1} \binom{n-1}{i} \sigma_i^\Sigma(x) t^i$$

and the integrals of mean curvature M_k are

$$M_k(\Sigma) = \int_\Sigma \sigma_k^\Sigma(x)\, \mathrm{d}x.$$

When the context is clearly defined we shall write only M_k and σ_k. Note that M_0 is the volume of the hypersurface Σ and M_{n-1} is the integral of the Gauss curvature $K = \kappa_1 \dots \kappa_{n-1}$.

Every hypersurface has the intrinsic geometry induced by the immersion in $\mathbb{H}^n$. Hyperplanes L have the geometry of $\mathbb{H}^{n-1}$. The metric induced on horospheres H is flat metric and they are isometric to the $(n-1)$-dimensional euclidean space.

We shall need Gauss-Bonnet formula for hypersurfaces of the euclidean space: if Σ is a compact, orientable hypersurface of class C^2 in the n-dimensional euclidean space and $n-1$ is even we have

$$M_{n-1}(\Sigma) = \frac{1}{2} O_{n-1} \chi(\Sigma)$$

with $\chi(\Sigma)$ the Euler characteristic of Σ. When Σ is the boundary of a domain Ω, for every n

$$M_{n-1}(\partial\Omega) = O_{n-1}\chi(\Omega). \tag{5}$$

3. Density for horospheres

In this section we shall give invariant measures for horospheres in $\mathbb{H}^n$.

We have invariant measures for geodesic planes and for rigid motions, i.e. an inavariant measure for the group of isometries, in $\mathbb{H}^n$. We denote them respectively by $\mathrm{d}L$ and $\mathrm{d}K$. The measure $\mathrm{d}K$ is called *kinematic density*. In [San76] it is proved

Theorem. *Let M^q be a fixed q-dimensional compact manifold in $\mathbb{H}^n$ and N^r an r-dimensional compact manifold moving with kinematic density* $\mathrm{d}K$. *If $r + q - n \geq 0$ and* $\mathrm{vol}_{r+q-n}(M \cap N)$ *denote the $r+q-n$-dimensional volume of the intersection then*

$$\int_{M\cap N \neq \emptyset} \mathrm{vol}_{r+q-n}(M \cap N)\, \mathrm{d}K = \frac{O_n O_{n-1} \dots O_1 O_{r+q-n}}{O_q O_r} \mathrm{vol}_q(M) \mathrm{vol}_r(N) \tag{6}$$

where $\mathrm{vol}_q(M)$ *and* $\mathrm{vol}_r(N)$ *are the volumes of M and N respectively and O_k denote the volume of the k-dimensional euclidean sphere.*

Remark. In formula (6) the rigid motions are always considered as direct rigid motions.

Example 3.1. When $r + q - n = 0$, vol_0 denotes the number of intersection points $\sharp(M \cap N)$ of M and N. For instance, if $M = \Gamma$ is a curve of length L and N a compact hypersurface

$$\int_{\Gamma\cap N\neq\emptyset} \sharp(\Gamma \cap N)\ \mathrm{d}K = \frac{O_n O_{n-1} \dots O_1 O_0}{O_{n-1} O_1} L\, \mathrm{vol}_{n-1}(N).$$

Now, let us consider N to be a sphere S_R of radius R. As rotations leave the sphere invariant and $\mathrm{d}K = \mathrm{d}V\,\mathrm{d}\Omega_{n-1}\cdots\mathrm{d}\Omega_1$, where $\mathrm{d}\Omega_k$ denote the volume element of S^k, we have

$$\begin{aligned}\int_{\Gamma\cap S_R\neq\emptyset} \sharp(\Gamma\cap S_R)\ \mathrm{d}K &= \\ &= \mathrm{vol}(SO(n-1))\int_{\Gamma\cap S_R\neq\emptyset}\sharp(\Gamma\cap S_R)\ \mathrm{d}V \\ &= O_{n-1}\dots O_1\int_{\Gamma\cap S_R\neq\emptyset}\sharp(\Gamma\cap S_R)\ \mathrm{d}V.\end{aligned}$$

Therefore, as the volume of a sphere in $\mathbb{H}^n$ is $O_{n-1}\sinh^{n-1}(R)$ we have that, with respect to the measure $\mathrm{d}V$ of the center of the sphere, the measure of the spheres intersecting a curve Γ of length L with its multiplicity is given by:

$$\int_{\Gamma\cap S_R\neq\emptyset}\sharp(\Gamma\cap S_R)\ \mathrm{d}V = \frac{O_nO_0}{O_1}L\ \sinh^{n-1}R. \tag{7}$$

If dL_{n-1} is the usual measure (see [San76]) for oriented hyperplanes in $\mathbb{H}^n$ then it is known that

$$\int_{\Gamma\cap L_{n-1}\neq\emptyset}\sharp(\Gamma\cap L_{n-1})\,dL_{n-1} = \frac{O_n\,O_0}{O_1}L \tag{8}$$

We modify the measure of spheres of radius R in order to make formulas (7) and (8) equal. To this end we put $\mathrm{d}S_R = \mathrm{d}V/\sinh^{n-1}R$. When R tends to infinity we obtain a measure for horospheres. To see how this measures can be expressed fix an origin O, therefore the measure of spheres of radius R, with O an exterior point, can be written as

$$\mathrm{d}V = \sinh^{n-1}(\rho+R)\,\mathrm{d}\rho\wedge\,\mathrm{d}\Omega_{n-1}$$

where ρ denotes the distance from O to S_R.

When R goes to infinity we obtain a measure for the horospheres leaving O outside of its convex side

$$\mathrm{d}H_+ = e^{(n-1)\rho}\mathrm{d}\rho\wedge\mathrm{d}\Omega_{n-1}, \tag{9}$$

and considering $\mathrm{d}V = \sinh^{n-1}(R-\rho)\mathrm{d}\rho\wedge\mathrm{d}\omega$, for spheres with O an interior point, if R tends to infinity we obtain the measure

$$\mathrm{d}H_- = e^{-(n-1)\rho}\mathrm{d}\rho\wedge\mathrm{d}\Omega_{n-1} \tag{10}$$

of the horospheres leaving O on its convex side.

If $\mathrm{d}H$ denotes the measure for all the horospheres, with the previous definitions we have the formula

$$\int_{\Gamma\cap H\neq\emptyset}\sharp(\Gamma\cap H)\,\mathrm{d}H = \frac{O_n\,O_0}{O_1}L \tag{11}$$

Remark. If $n=2$ we have $\int\sharp(\Gamma\cap H)\,\mathrm{d}H = 4L$ and for dimension $n=3$ we have $\int\sharp(\Gamma\cap H)\,\mathrm{d}H = 2\pi L$. These formulas were given in [San67] and [San68].

If we consider in formula (6) a compact manifold M and a sphere with radius R the same arguments lead to a more general result

Proposition 3.1. *Let M^q be a fixed q-dimensional compact manifold in $\mathbb{H}^n$ and H horospheres with invariant measure* $\mathrm{d}H$ *given by (9) and (10). If* vol_r *denote the r-dimensional volume then*

$$\int_{H\cap M\neq\emptyset}\mathrm{vol}_{q-1}(H\cap M)\mathrm{d}H = \frac{O_nO_{q-1}}{O_q}\mathrm{vol}_q(M). \tag{12}$$

4. Horospheres which intersect convex bodies in hyperbolic space

Suppose M a hyperbolic hypersurface of class at least C^2, given an horosphere H denote by C the intersection $M \cap H$. When H and M are in general position, C is an embedded submanifold of H and also of M.

Using Gauss-Bonnet formula (5) for $Q \cap H$ as a domain of H we have

$$\chi(H \cap Q) = \frac{1}{O_{n-2}} M_{n-2}(C).$$

(remember that H has an intrinsic euclidean structure).

Integrating $\chi(H \cap Q)$ for every H intersecting Q

$$\begin{aligned}\int_{H\cap Q\neq\emptyset} \chi(Q \cap H)\mathrm{d}H &= \frac{1}{O_{n-2}} \int_{H\cap Q\neq\emptyset} M_{n-2}(C)\mathrm{d}H \\ &= \frac{1}{O_{n-2}} \int_{\mathcal{H}} \int_C \sigma^C_{n-2}\, \mathrm{d}x_{n-2}\mathrm{d}H.\end{aligned}$$

Using the next proposition it will be possible to change the order of integration in the last integral.

Proposition 4.1. *Let M be a C^2 hypersurface in $\mathbb{H}^n$, H a horosphere intersecting M in C then*

$$\mathrm{d}x_{n-2} \wedge \mathrm{d}H = \sin\theta\, \mathrm{d}x_{n-1} \wedge \mathrm{d}\Omega_{n-1}. \tag{13}$$

Where $\mathrm{d}x_{n-2}$ and $\mathrm{d}x_{n-1}$ denote the volume elements in C and M respectively, $\mathrm{d}\Omega_{n-1}$ the volume element of S^{n-1}, sphere that gives the normal vectors defining horospheres through x and θ the angle between M and H.

Proof. We consider a quite more general setting. Let M and Σ be hypersurfaces of class C^2 in $\mathbb{H}^n$, M is fixed and Σ is moving with kinematic density $\mathrm{d}K$ (i.e. the volume element of isometries of $\mathbb{H}^n$). Let $C = M \cap \Sigma$, we denote by θ the angle between the normal directions of M and Σ in a point x of C. It is known (see [San76]) that

$$\mathrm{d}C \wedge \mathrm{d}K = \sin^{n-1}\theta\, \mathrm{d}\theta \wedge \mathrm{d}M \wedge \mathrm{d}\Sigma \tag{14}$$

where $\mathrm{d}C = \mathrm{d}x_{n-2} \wedge \mathrm{d}O(n-2)$ with $\mathrm{d}O(n-2)$ the volume element of orthogonal transformations in T_xC and $\mathrm{d}M$ and $\mathrm{d}\Sigma$ the analogous densities for M and Σ.

If we consider Σ a sphere S_R of radius R and write $\mathrm{d}\Sigma = \mathrm{d}\sigma_x \wedge \mathrm{d}\Omega_{n-2} \wedge \cdots \wedge \mathrm{d}\Omega_1$ and $\mathrm{d}K = \mathrm{d}V \wedge \mathrm{d}\Omega_{n-1} \wedge \cdots \wedge \mathrm{d}\Omega_1$ with $\mathrm{d}\sigma_x$ the volume element in S_R at x, $\mathrm{d}V$ the volume element of $\mathbb{H}^n$ in the center of the sphere and $\mathrm{d}\Omega_i$ the volume element of S^i (with the corresponding identifications), then integrating over the orthogonal group we have

$$\mathrm{d}C \wedge \mathrm{d}V = \sin^{n-1}\theta \sinh^{n-1} R\, \mathrm{d}\theta \wedge \mathrm{d}M$$

because $\mathrm{d}\sigma_x = \sinh^{n-1} R \mathrm{d}\Omega_{n-1}$ in hyperbolic polar coordinates.

Normalizing and making R go to infinity we get

$$\mathrm{d}x_{n-2} \wedge \mathrm{d}H = \sin^{n-1}\theta\, \mathrm{d}\theta \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}\Omega_{n-2}.$$

Writing $\mathrm{d}\Omega_{n-1}$ in polar coordinates we have $\mathrm{d}\Omega_{n-1} = \sin^{n-2}\theta\, \mathrm{d}\theta \wedge \mathrm{d}\Omega_{n-2}$, therefore we obtain formula (13). ∎

We have

$$\int_{\mathcal{H}} \int_C \sigma^C_{n-2}\, \mathrm{d}x_{n-2}\mathrm{d}H = \int_M \int_{S^{n-1}} \sigma^C_{n-2}\, \sin\theta\, \mathrm{d}\Omega_{n-1}\mathrm{d}x_{n-2}. \tag{15}$$

Now, let us compare normal curvatures in C with normal curvatures in M.

Lemma 4.1. *Let v be a tangent vector in C then*

$$k_M(v) = \cos\theta\, k_H(v) + \sin\theta\, k_C^H(v)$$

where θ is the angle between the hypersurfaces M and H, $k_M(v)$ and $k_H(v)$ the normal curvatures in M and H and $k_C^H(v)$ the normal curvature in C as a hypersurface of H.

Proof. The vectorial second fundamental forms satisfy $B_C^H = B_C - B_H$. Here $B_C(X,Y)$ is the normal part of $\nabla_X Y$ in $\mathbb{H}^n$. Let $\mathbf{n}$ be the normal of C in H, then $\mathbf{n}\cdot\mathbf{n}_M = \sin\theta$. If a superindex in ∇ means covariant derivative in the corresponding manifold we have $B_C^H(X,Y) = B_M(X,Y) - B_H(X,Y) + \nabla_X^M Y - \nabla_X^C Y$. Multiplying both sides by the normal $\mathbf{n}_M$ we finish the proof of the lemma. □

If $B_M(X,Y) = h_M(X,Y)\cdot\mathbf{n}_M, B_H(X,Y) = h_H(X,Y)\cdot\mathbf{n}_H$ and $B_C^H(X,Y) = h_C^H(X,Y)\cdot\mathbf{n}$, from the lemma and taking into account that for horospheres h_H is the identity I (as a matrix) we have the equality

$$h_C^H = \frac{h_M}{\sin\theta} - \frac{I}{\tan\theta}. \tag{16}$$

when the forms are restricted to tangent vectors of C.

Now we relate σ_{n-2}^C the curvature of C as a hypersurface of H and the symmetric functions of h_M restricted to the tangent space of C. We have that $\sigma_{n-2}^C = \det(h_C^H)$, then

$$\begin{aligned}\sigma_{n-2}^C &= \det\left(\frac{h_M}{\sin\theta} - \frac{I}{\tan\theta}\right) = \\ &= \sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-2-i}\frac{\cos^{n-2-i}\theta}{\sin^{n-2}\theta}\sigma_i^M\end{aligned} \tag{17}$$

Using this expression we compute $\int_{S^{n-1}}\sigma_{n-2}^C\sin\theta \mathrm{d}\Omega_{n-1}$ in (15). In this integral the point x in M is fixed, then we have the direction $\mathbf{n}_M$ normal to M fixed too. S^{n-1} is the space of directions defining H, hence if we use polar coordinates with 'center' $\mathbf{n}_M$ by virtue of formula (17) we have

$$\begin{aligned}\int_{S^{n-1}}\sigma_{n-2}^C\sin\theta\mathrm{d}\theta\mathrm{d}\Omega_{n-1} &= \int_{S^{n-2}}\int_0^\pi \sigma_{n-2}^C\sin\theta\sin^{n-2}\theta\mathrm{d}\Omega_{n-2} \\ &= \int_{S^{n-2}}\int_0^\pi\left(\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-2-i}\frac{\cos^{n-2-i}\theta}{\sin^{n-2}\theta}\sigma_i^M\right)\sin\theta\sin^{n-2}\theta\,\mathrm{d}\theta\,\mathrm{d}\Omega_{n-2}.\end{aligned}$$

For a fixed point in S^{n-2}, when θ moves, the submanifold C changes but not his tangent space, hence σ_i^M is independent of θ. Therefore

$$\begin{aligned}&\int_{S^{n-1}}\sigma_{n-2}^C\sin\theta\mathrm{d}\theta\mathrm{d}\Omega_{n-1} = \\ &= \sum_{i=0}^{n-2}(-1)^{n-2-i}\binom{n-2}{i}\int_{S^{n-2}}\sigma_i^M\left(\int_0^\pi\cos^{n-2-i}\theta\sin\theta\,\mathrm{d}\theta\right)\mathrm{d}\Omega_{n-2} \\ &= \sum_{i=0}^{n-2}(-1)^{n-2-i}\binom{n-2}{i}\frac{2\epsilon(n-1-i)}{n-1-i}\int_{S^{n-2}}{\sigma_i}^M\,\mathrm{d}\Omega_{n-2}\end{aligned}$$

where $\epsilon(n-1-i)$ has value 0 if $n-1-i$ is even and 1 if it is odd.

Now we need to compute $\int_{S^{n-2}}{\sigma_i}^M\,\mathrm{d}\Omega_{n-2}$. This computation can be found for instance in [Lan80] and is a generalization of the well known relation between mean and normal curvature in classical differential geometry of surfaces, i.e. $H = (1/\pi)\int_0^\pi k_n(\theta)\,\mathrm{d}\theta$. We have

$$\int_{\mathbb{R}P^{n-2}} {\sigma_i}^M \,\mathrm{d}\Omega_{n-2} = \mathrm{vol}(\mathrm{G}(n-1,n-2))\,\sigma_i.$$

Note that now σ_i in the right means function of curvature in $x \in M$ for all the directions in M.

Using this relation we have

$$\begin{aligned}
&\int_{S^{n-1}} \sigma^C_{n-2} \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\Omega_{n-1} = \\
&\quad = \sum_{i=0}^{n-2}(-1)^{n-2-i}\binom{n-2}{i}\frac{2\epsilon(n-1-i)}{n-1-i}2\mathrm{vol}(\mathrm{G}(n-1,n-2))\,\sigma_i \\
&\quad = 2\sum_{i=0}^{n-2}(-1)^{n-2-i}\binom{n-2}{i}\frac{\epsilon(n-1-i)}{n-1-i}O_{n-2}\,\sigma_i
\end{aligned}$$

and

$$\begin{aligned}
\int_{H\cap Q\neq\emptyset}\chi(Q\cap H)\mathrm{d}H &= \frac{1}{O_{n-2}}\int_{\mathcal{H}}\int_C \sigma^C_{n-2}\,\mathrm{d}x_{n-2}\mathrm{d}H \\
&= 2\sum_{i=0}^{n-2}(-1)^{n-2-i}\binom{n-2}{i}\frac{\epsilon(n-1-i)}{n-1-i}\,\sigma_i.
\end{aligned}$$

Reordering the indices we have proved the main theorem

Theorem. *If Ω is a domain in $\mathbb{H}^n$ bounded by a embedded hypersurface $\partial\Omega$ then*

$$\int_{H\cap\partial\Omega\neq\emptyset}\chi(H\cap\partial\Omega)\mathrm{d}H = 2\sum_{h=0}^{[(n-2)/2]}\binom{n-2}{2h}\frac{1}{2h+1}M_{n-2-2h}(\partial\Omega)$$

Remark. Considering cases according the parity of n we have

- if $n = 2m+1$
$$\int_{H\cap\partial\Omega\neq\emptyset}\chi(H\cap\partial\Omega)\mathrm{d}H = \frac{2}{n-1}\sum_{k \text{ odd}}\binom{2m}{k}M_k(\partial\Omega)$$
- If $n = 2m$
$$\int_{H\cap\partial\Omega\neq\emptyset}\chi(H\cap\partial\Omega)\mathrm{d}H = \frac{2}{n-1}\sum_{k>2 \text{ even}}\binom{2m}{k+1}M_k(\partial\Omega)$$

When Ω is h-convex we have $\chi(H\cap\partial\Omega) = 1$, therefore we can find the measure of horospheres intersecting a h-convex set.

Corollary 4.1. *The measure of the horospheres which intersect an h-convex hypersurface M can be expressed as a linear combination of the integrals of mean curvature.*

For $\mathbb{H}^2$ and $\mathbb{H}^3$ the formulas obtained in the theorem are

$$\int_{H\cap\Sigma_0\neq\emptyset}\chi(H\cap\partial\Omega)\mathrm{d}H = 2M_0 = 2\,\mathrm{length}(\partial\Omega)$$

for $n = 2$ and

$$\int_{H\cap\partial\Omega\neq\emptyset}\chi(H\cap\partial\Omega)\mathrm{d}H = 2M_1$$

for $n = 3$ which are formulas given in [San67] and [San68] respectively.

Given a convex body K in $\mathbb{H}^n$ using the formulas given in [San76] it can be found the measure of hyperplanes intersecting K. For instance, in the plane this measure equals the length of the boundary and in the space it is $M_1 - \mathrm{vol}(K)$.

If S_R is a $(n-1)$-sphere in $\mathbb{H}^n$ let us compare the measures of hyperplanes L and horospheres H intersecting S_R. We have

$$m(L : L \cap S_R \neq \emptyset) = O_{n-1} \int_0^R \cosh^{n-1} r \, \mathrm{d}r$$

and

$$m(H : H \cap S_R \neq \emptyset) \approx \frac{2^{n-2}}{n-1} \mathrm{vol}(S_R).$$

When R tends to infinity $m(L : L \cap S_R \neq \emptyset)/m(H : H \cap S_R \neq \emptyset)$ tends to 2^{2-n}.

Let us give some possible developments related to the problems trated here. Hyperplanes are hypersurfaces with vanishing normal curvature in every direction and horospheres have normal curvature equal 1. Between hyperplanes and horospheres there are equidistant hypersurfaces, they have normal curvature equal to λ between 0 and 1. It seems an interesting problem to find the measure of equidistants for a given curvature λ that intersect a λ-convex body, i.e. its boundary has normal curvature greater or equal than λ in every direction.

References

[Lan80] Rémi Langevin, *Courbures, feuilletages et surfaces*, Université de Paris-Sud Département de Mathématique, Orsay, 1980, Mesures et distributions de Gauss. [Gaussian measures and distributions], Dissertation, Université Paris-Sud, Orsay, 1980, With an English summary.

[San67] L. A. Santaló, *Horocycles and convex sets in hyperbolic plane*, Arch. Math. (Basel) **18** (1967), 529–533.

[San68] L. A. Santaló, *Horospheres and convex bodies in hyperbolic space*, Proc. Amer. Math. Soc. **19** (1968), 390–395.

[San76] Luis A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1.

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