Mémoire d'habilitation à diriger des recherches
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Contributions à la géométrie systolique locale et globale

My research is devoted to systolic geometry and related topics. In this report, I will present the main results corresponding to the following selected papers:

[1] **Contact geometry and isosystolic inequalities.**
With J.-C. Álvarez Paiva.
Preprint (2011).

[2] **Distribution of the systolic volume of homology classes.**
With I. Babenko.
Preprint (2010).

[3] **Bers' constants for punctured spheres and hyperelliptic surfaces.**
With H. Parlier.
Journal of Topology and Analysis, Vol. 4 (2012), No 3, 271-296.

[4] **Short loop decompositions of surfaces and the geometry of Jacobians.**
With H. Parlier and S. Sabourau
Geometric And Functional Analysis, Vol. 32 (2012), No 1, 37-73.

[5] **A local optimal diastolic inequality on the two-sphere.**
Journal of Topology and Analysis, Vol. 2 (2010), No 1, 109-121.

[6] **Diastolic inequalities and isoperimetric inequalities on surfaces.**
With S. Sabourau.
Annales de l'École Normale Supérieure, Vol. 43 (2010), No 4, 579-605.

[7] **A Zoll counterexample to a geodesic length conjecture.**
With C. Croke and M.G. Katz.
Geometric and Functional Analysis, Vol. 19 (2009), No 1, 1-10.

[8] **Sur la systole de la sphère au voisinage de la métrique standard**.
Geometriae Dedicata, Vol. 121 (2006), No 1, 61-71.

Introduction and statement of the results

We start this report with a short overview of the main results we will present in the sequel. First recall the context. Let M be a closed manifold of dimension m. For a given Riemannian metric g on M, we denote by $\mathrm{sys}(M,g)$ the *systole* defined as the least length of a closed geodesic and by $\mathrm{vol}(M,g)$ the Riemannian volume. When the manifold is non-simply connected, we define the *homotopical systole* denoted by $\mathrm{sys}_\pi(M,g)$ as the least length of a *non-contractible* closed geodesic. Lastly, when the first integral homology group of the manifold is non-trivial, we define the *homological systole* denoted by $\mathrm{sys}_1(M,g)$ as the least length of a *homologically non-trivial* closed geodesic. There exist other notions of systole such as the $\mathbb{Z}_2$-homological systole, the stable systole or the higher dimensional systoles for instance, but they will not be considered in the sequel. It is important to underline that the term systole usually stands for the homotopical systole, but in order to unify our terminology, we fix this vocabulary which was suggested by M. Berger in [Berg00, p.107].

The main topic in systolic geometry is to etablish *systolic* inequalities which are inequalities of the form

$$\mathrm{vol}(M,g) \geq C \cdot \mathrm{sys}_*(M,g)^m,$$

where C is some positive constant and which hold for a large subset of metrics g on M. Here sys_* denotes either the systole, the homotopical systole or the homological systole. If we authorize more general metrics such as Finsler metrics, then the first systolic inequality was discovered by H. Minkowski [Mink96], who proved that the Hausdorff volume of any flat reversible Finsler torus $(\mathbb{T}^m, F)$ of dimension m satisfies the optimal inequality

$$\mathrm{vol}(\mathbb{T}^m, F) \geq \frac{b_m}{2^m} \cdot \mathrm{sys}_1(\mathbb{T}^m, F)^m.$$

Here b_m denotes the Euclidian volume of the unit ball in dimension m. This statement is a reformulation of the celebrated theorem on which is based the geometry of numbers theory, and obviously still holds for the homotopical systole or even the systole. Since, other optimal systolic inequalities have been discovered, see [CroKatz02] for an overview.

Variations around systolic inequalities.

In its celebrated paper [Gro83], M. Gromov proved the following central result in systolic geometry: every essential closed manifold satisfies a systolic inequality for the set of all Riemannian metrics. In the same paper, he also precised the result for closed orientable surfaces of genus γ: a closed Riemannian surface of area equal to its genus γ admits a homologically non-trivial closed geodesic of length at most $\log \gamma$ (up to some universal constant). This bound is known to be optimal since the work [BusSar94] of P. Buser and P. Sarnak.

In the article [4] written in collaboration with H. Parlier and S. Sabourau, we extend Gromov's asymptotic $\log \gamma$ bound on the homological systole of genus γ surfaces: *for any $\lambda \in (0,1)$ there exists a constant C_λ such that every closed Riemannian surface of genus γ whose area is normalized at γ has at least $[\lambda\gamma]$ homologically independent loops of length at most $C_\lambda \log \gamma$.* We construct hyperbolic surfaces showing that our general result is sharp for the number of such loops. We also extend the upper bound obtained by P. Buser and P. Sarnak in [BusSar94] on the minimal norm of nonzero period lattice vectors of Riemann surfaces in their geometric approach of the Schottky problem to almost γ homologically independent vectors. Finally we derive a lower bound on the systolic area of finitely presentable groups with no free factor isomorphic to $\mathbb{Z}$ in terms of their first Betti number which corresponds to a generalization of Gromov's asymptotic $\log \gamma$ bound.

In the article [6] written in collaboration with S. Sabourau, we present another type of generalization of Gromov's result. Loosely speaking, we prove the following result: *any closed Riemannian surface of area equal to its genus γ can be swept out by a family of multi-loops of total length at* *most γ (up to some universal constant).* This *diastolic* inequality, which relies on an upper bound on Cheeger's constant, yields an effective process to find short closed geodesics on the two-sphere, for instance. The diastolic term comes from the fact that a family of multi-loops sweeping out a surface produces a closed geodesic via a minimax process.

Finally in the article [3] written in collaboration with H. Parlier we consider another type of minimax quantity which bounds from below the area of surfaces. Our central result is as follows: *any hyperbolic two-sphere with n cusps admits a decomposition by three-holed spheres all of whose boundary loops are of length at most $\sqrt{n}$ (up to some universal constant).* We produce examples showing that this bound is optimal. These results address to a question asked by P. Buser in [Bus92].

Local versions of systolic inequalities.

Among subsets of metrics for which systolic inequalities are relevant, we find neighbourhoods of a fixed metric (for some reasonnable topology). In [Cro88] C. Croke proved the existence of a systolic inequality for the set of all Riemannian metrics on the two-sphere and conjectured (following E. Calabi) that the corresponding optimal constant C should be $1/(2\sqrt{3})$. This optimal constant corresponds to the flat metric g_c with three conical singularities obtained by gluing two copies of an equilateral triangle along their boundary. In [5], we proved the following local version of this conjecture: *any Riemannian metric g sufficiently C^1-closed from the Calabi-Croke metric g_c outside its singularities satisfies the optimal inequality*

$$\mathrm{vol}(S^2, g) \geq \frac{1}{2\sqrt{3}} \cdot \mathrm{sys}(S^2, g)^2,$$

the equality case being reached only by metrics homothetic to g_c. It is important to remark that such Riemannian metrics g necessarily also have three conical singularities of angle $2\pi/3$. The major ingredient in our proof is a degree 3 ramified cover of the two-sphere by the two-torus, relying the systolic properties of both surfaces.

In the same spirit, we can look for systolic inequalities available for a subset of metrics made of some one-parameter deformations of a fixed metric. The first result in this direction has been proved in [8] and states that the round metric g_0 on the two-sphere satisfies an optimal systolic inequality for natural one-parameter deformations: *for any non-trivial smooth function $\Psi : S^2 \to \mathbb{R}$ such that $\int_{S^2} \Psi dv_{g_0} = 0$ there exists a positive ε such that for $0 < |t| < \varepsilon$*

$$\frac{\mathrm{area}(S^2, (1+t\Psi)^2 \cdot g_0)}{\mathrm{sys}(S^2, (1+t\Psi)^2 \cdot g_0)^2} > \frac{\mathrm{area}(S^2, g_0)}{\mathrm{sys}(S^2, g_0)^2} = \frac{1}{\pi}.$$

In the article [1] written in collaboration with Juan-Carlos Álvarez Paiva, we reformulate systolic geometry in a contact-geometric context. By using the canonical perturbation theory to exploit the large symmetry group that systolic geometry inherits, we prove among others things the following optimal systolic inequality: *any smooth volume-preserving deformation g_t of the canonical metric g_0 on the real projective space $\mathbb{R}P^m$ which does not coincide to all orders to trivial defomations of the form $\phi_t^* g_0$ for some isotopy ϕ_t satisfies*

$$\frac{\mathrm{vol}(\mathbb{R}P^m, g_t)}{\mathrm{sys}(\mathbb{R}P^m, g_t)^m} > \frac{\mathrm{vol}(\mathbb{R}P^m, g_0)}{\mathrm{sys}(\mathbb{R}P^m, g_0)^m}$$

for small non-zero values of t. To appreciate the result, recall that for $m > 2$ the systolic optimality of the standard metric g_0 on $\mathbb{R}P^m$ is an open question since the work of P. Pu in 1952. In this paper, we also characterize contact structures which are critical for the systolic volume: they are exactly the regular contact structures, that is contact forms for which the Reeb flow is periodic (the analog of Zoll metrics in Riemannian geometry). As a consequence, *for any closed manifold, the* *smooth Finsler metrics which are critical for the systolic volume associated to the Holmes-Thompson volume notion are exactly Zoll Finsler metrics.*

The comparaison of the systole with other Riemannian invariants such as the diameter is also a natural and deep question. For a closed non-simply connected Riemannian manifold (M, g), it is straighforward to see that $\mathrm{sys}(M, g) \leq 2 \cdot \mathrm{diam}(M, g)$. But for simply connected manifolds, the question is much more difficult. Until now, the only simply connected manifold for which a universal inequality exists between the systole and the diameter is the two-sphere. More precisely, A. Nabutovsky & R. Rotman [NabRot02] and S. Sabourau [Sab04], improving previous bounds due to C. Croke [Cro88] and M. Maeda [Mae94], proved that for any Riemannian metric g on the two-sphere $\mathrm{sys}(S^2, g) \leq 4 \cdot \mathrm{diam}(S^2, g)$. It was a long-standing conjecture that the best constant in this inequality should be 2, the equality case being reached by the round metric. In the article [7] written in collaboration with C. Croke and M. Katz, we prove that this conjecture is false, even in its local form: *there exists smooth deformations* $\{g_t\}$ *of the round metric by Zoll Riemannian metrics such that* $\mathrm{sys}(S^2, g_t) > 2 \cdot \mathrm{diam}(S^2, g_t)$ *for small non-zero values of* t.

Systolic geometry of homology classes.

Given a pair (G, a) where G is a finitely presentable group and a an integer homology class of G, the *systolic volume* is defined as the least volume of any geometric realisation of the homology class by a pseudomanifold endowed with a polyhedral metric for which the length of curves representing elements of G are at least 1. We know by [Gro83] that an essential manifold M always satisfies a systolic inequality for the homotopical systole over the set of all Riemannian metrics. The best constant $\mathfrak{S}(M)$ involved in such an inequality is called the *systolic constant* and coincides with the systolic volume of the pair $(\pi_1(M), f_*[M])$ where $f : M \to K(\pi_1(M), 1)$ is the classifying map of M into its corresponding Eilenberg-MacLane space. This result shared by I. Babenko [Bab06, Bab08] and M. Brunnbauer [Bru08] determines precisely the topological nature of the systolic constant. In the paper [2] written in collaboration with I. Babenko we study numerous aspects of the systolic volume showing that it is a complex invariant of homology classes of finitely presentable groups. Among other results, we show that *for any dimension* $m \geq 3$ *there exists an infinite sequence of finitely presentable pairwise distinct groups* $\{G_i\}$ *for which at least one irreducible class* $a_i \in H_m(G_i, \mathbb{Z})$ *satisfies* $\mathfrak{S}(G_i, a_i) \leq 1$. So we can not hope finiteness results without additional restrictions. An example of efficient restriction is illustrated by the following result: *the systolic volume of integer multiples* ka *of a fixed homolgy class* a *is a sublinear function in* k. This result is strongly connected with the asymptotic behaviour of the systolic constant under the operation of connected sum. We also prove that *the systolic volume of a homology class is bounded from below by its torsion*, thus addressing to a question of M. Gromov in [Gro96].

1. Geometry of surfaces through the length of closed curves

The first part of this report deals with surfaces. In the same way a taylor determines the geometry of a body by carefully taking some relevant measures, we study the geometry of surfaces through the lengths of some special closed curves. These curves are respectively the diastole, which corresponds to a curve obtained by a minimax process on the one-cycle space, the curves involved in a short decomposition of the surface by elementary topological subsets—the pants—, and short homologically independent curves. We look for bounds on the length of these curves by the area of the surface which is equivalent to estimate their length while the area is normalized. We will present the main results of the article [3], [4], and [6] written in collaboration with H. Parlier and S. Sabourau. In the sequel, all surfaces are supposed to be orientable in order to facilitate the presentation and we refer to [3], [4], and [6] for considerations in the non-orientable case. All Riemannian metrics are supposed to be smooth. Otherwise stated all surfaces are supposed to be closed and connected.

1.1. Diastole.

Using the isomorphism between the second homotopy group of a two-sphere and the fundamental group of its loop space relatively to the subspace of constant loops, G. D. Birkhoff proved the existence of a nontrivial closed geodesic on every Riemannian two-sphere using a minimax argument, see [Bir27]. The length of the shortest closed geodesic obtained by such a minimax process can not be uniformly bounded by the square root of the area as follows from the example of two-spheres with constant area and three spikes arbitrarily long (see [Sab04, Remark 4.10] for further detail). This made remarkable the result of C. Croke [Cro88] that the *systole*—the shortest length of a closed geodesic—of any Riemannian two-sphere is uniformly bounded by the square root of its area. More precisely, any Riemannian metric g on the two-sphere satisfies the systolic inequality

$$\operatorname{sys}(S^2, g) \leq 31\sqrt{\operatorname{area}(S^2, g)}. \tag{1.1}$$

The constant in this inequality is not optimal and has been improved several times, see [NabRot02], [Sab04] and [Rot06].

On non-simply connected closed Riemmanian surfaces, no minimax principle is required to show the existence of a closed geodesic. The minimum of the length functional over all non-contractible loops is always positive and realized by the length of a non-contractible closed geodesic. We call this minimum the *homotopical systole* and denote it by sys_π. In particular the systole is well defined on such surfaces and satisfies $\operatorname{sys} \leq \operatorname{sys}_\pi$. This is a central result in systolic geometry that the homotopical systole can be uniformly bounded by the square root of the area. More precisely, every closed Riemannian surface (M, g) of genus $\gamma \geq 1$ satisfies the following systolic inequality

$$\operatorname{sys}_\pi(M, g) \leq C\,\frac{\log(\gamma+1)}{\sqrt{\gamma}}\,\sqrt{\operatorname{area}(M, g)} \tag{1.2}$$

where C is a universal constant, *cf.* [Gro83], [Bal04] and [KatzSab05] for three different proofs. In this theorem, the homotopical systole can be replaced by the *homological systole*— the shortest length of a homologically non-trivial loop—, see [Gro96]. The dependence on the genus in inequality (1.2) is optimal for both homotopical and homological systoles, see [BusSar94].

The existence of a closed geodesic on a Riemannian closed surface (M, g), possibly a two-sphere, can also be proved through a minimax argument on a different space, namely the one-cycle space $\mathcal{Z}_1(M;\mathbb{Z})$. Loosely speaking, this space arising from geometric measure theory is made of multiple curves (unions of oriented loops) endowed with some special topology, *cf.* [6] for a precise definition. The use of the one-cycle space, rather than the loop space, introduces some flexibility and permits to define a minimax process on any closed Riemannian surface (M, g) using F. Almgren's isomorphism [Almg60] between the relative fundamental group $\pi_1(\mathcal{Z}_1(M;\mathbb{Z}), \{0\})$ and the second homology group $H_2(M;\mathbb{Z}) \simeq \mathbb{Z}$. More precisely, we define the *diastole over the one-cycle space* as

$$\text{dias}(M, g) := \inf_{(z_t)} \sup_{0 \leq t \leq 1} \text{mass}(\mathrm{z_t}) \tag{1.3}$$

where (z_t) runs over the families of one-cycles inducing a generator of $\pi_1(\mathcal{Z}_1(M;\mathbb{Z}), \{0\})$ and $\text{mass}(\mathrm{z_t})$ represents the mass (or length) of z_t. From a result of J. Pitts [Pit74, p. 468], [Pit81, Theorem 4.10] (see also [CalCao92]), this minimax principle gives rise to a union of closed geodesics (counted with multiplicity) of total length $\text{dias}(M, g)$. Hence,

$$\text{sys}(M, g) \leq \text{dias}(M, g).$$

This principle has been used in [CalCao92], [NabRot02], [Sab04], [Rot05] and [Rot06] in the study of closed geodesics on Riemannian two-spheres.

In the article [6] we show in collaboration with S. Sabourau that the length of the shortest closed geodesic obtained by such a minimax process on the one-cycle space is uniformly bounded by the square root of the area. More precisely, we obtain the following diastolic inequality.

Theorem 1.1. *There exists a positive constant $C \leq 10^8$ such that every closed Riemannian surface (M, g) of genus $\gamma \geq 0$ satisfies*

$$\text{dias}(M, g) \leq C\,\sqrt{\gamma + 1}\,\sqrt{\text{area}(M, g)}. \tag{1.4}$$

Since the minimax principle (1.3) gives rise to a union of closed geodesics of length $\text{dias}(M, g)$, Theorem 1.1 yields a construction of short closed geodesics on surfaces through Morse theory over the one-cycle space.

The dependence on the genus in inequality (1.4) is optimal: the closed hyperbolic surfaces of arbitrarily large genus with Cheeger constant bounded away from zero constructed in [Bro86] provide examples of surfaces with area $\simeq \gamma$ and dias $\gtrsim \gamma$, see [6, Remark 7.3]. This dependence of the inequality (1.4) on the genus should be compared with the one in (1.2).

When the metric is bumpy, the diastole corresponds by definition to the length of a one-cycle of index one. Recall that the index of a one-cycle z of mass κ is defined as

$$\text{ind}_{\mathcal{Z}}(z) = \min\{i \in \mathbb{N} \mid \pi_i(\mathcal{Z}_1^\kappa(M, \mathbb{Z}) \cup \{\gamma\}, \mathcal{Z}_1^\kappa(M, \mathbb{Z})) \text{ is not trivial}\}$$

where $\mathcal{Z}_1^\kappa(M, \mathbb{Z}) = \{z \in \mathcal{Z}_1(M, \mathbb{Z}) \mid \text{mass}(\mathrm{z}) < \kappa\}$. It is important to remark that the relation between the filling radius (see [Gro83]) of a bumpy Riemannian two-sphere and the length of its shortest one-cycle of index one established in [Sab04] cannot be extended to the diastole. Indeed, from [Sab04, Theorem 1.6], there exists a sequence g_n of Riemannian metrics on the two-sphere such that

$$\lim_{n \to +\infty} \frac{\text{FillRad}(S^2, g_n)}{\text{dias}(S^2, g_n)} = 0.$$

This result illustrates the difference of nature between the length of the shortest closed geodesic or of the shortest one-cycle of index one, which can be bounded by the filling radius on the two-sphere, and the diastole. It also shows that the proof of Theorem 1.1 requires different techniques. The proof relies on an inequality between the area and the Cheeger constant, and a cut-and-paste argument on one-cycles.

1.2. **Pants decompositions.** Start with a hyperbolic surface S of genus γ with n cusps and recall that its area is equal to $2\pi(2\gamma-2+n)$. A *pants decomposition* is a maximal collection of disjoint simple closed geodesics. Such a collection is necessarily made of exactly $3\gamma-3+n$ loops whose complementary region is a disjoint union of $2\gamma-2+n$ surfaces of topological type $(0,3)$ (so-called pants). For a given pants decomposition $\mathcal{P}$ of S, we define its length as

$$\text{length}(\mathcal{P}) = \max_{\alpha\in\mathcal{P}} \text{length}(\alpha).$$

L. Bers [Bers74, Bers85] showed that there always exists a pants decomposition whose length is bounded from above by some universal constant which only depends on the topology of the surface. This result was quantified in the closed case by P. Buser [Bus81, Bus92], and P. Buser and M. Sëppala [BusSep92] who showed that the optimal constant —called *Bers' constant*— behaves at least like $\sim\sqrt{\gamma}$ and at most like $\sim\gamma$. In the punctured case P. Buser proved [Bus92] that Bers' constant grows at most linearly in the number of cusps. In any case the correct behavior remains unknown, but P. Buser conjectured the following.

Conjecture 1.1. *Bers' constants for surfaces of genus γ with n cusps behave roughly like $\sqrt{\gamma+n}$.*

In the sequel we will denote Bers' constant in signature (γ,n) by $\mathcal{B}_{\gamma,n}$. Before explaining our results in the hyperbolic case, we explain how to generalize pants decomposition to the Riemannian context. Given a Riemannian surface (M,g) of genus γ with n marked points, a pants decomposition is a collection of $3\gamma-3+n$ disjoint simple loops which cut the surface into $2\gamma-2+n$ pairs of pants. Here a pair of pants is either a three holed sphere, a cylinder with one marked point or a disk with two marked points. The length of such a pants decomposition is defined as the maximal length of its loops. As a byproduct of the proof of Theorem 1.1 we derive the following result, see [4, Proposition 6.3].

Corollary 1.1. *Let (M,g) be a closed Riemannian surface of genus γ with n marked points. Then (M,g) admits a pants decomposition with respect to the marked points of length at most*

$$C\sqrt{\gamma+1}\sqrt{\text{area}(M,g)}$$

for an explicit universal constant C.

This corollary implies previously known linear upperbounds by P. Buser and M. Sëppala for genus growth. It also implies the square root upperbound conjectured by P. Buser for hyperbolic punctured spheres. Indeed given a hyperbolic sphere S with n punctured points, cut a small punctured disk of area ε around each punctured point and define a new surface $\tilde{S}$ by gluing along each boundary component a round hemisphere with a marked point at its center. As the Riemannian surface $\tilde{S}$ with n marked points (the centers of the round hemispheres) admits a pants decomposition of length at most

$$C\sqrt{\text{area}(\tilde{S})},$$

this implies that

$$\mathcal{B}_{0,n} \leq C\sqrt{2\pi(n-2)}$$

by letting $\varepsilon\to 0$ as $\text{area}(S)=2\pi(n-2)$.

But the constant involved in this last inequality is very bad (more than 10^7), the techniques of [6] being not adapted to the study of pants decomposition. In collaboration with H. Parlier, we prove in [3] Buser's conjecture in the case of punctured spheres by showing that it fundamentally relies on a classical result: the so-called Besicovitch lemma. Then we prove Buser's conjecture in the closed case for hyperelliptic surfaces. More precisely, our results are the following.

Theorem 1.2. *1) Any hyperbolic sphere with $n \geq 4$ cusps admits a pants decomposition of length at most* $30\sqrt{2\pi(n-2)}$.

2) For any $n > 6$, there exists a hyperbolic sphere with n punctured points such that any pants decomposition has length at least

$$8\,\mathrm{arcsinh}\,1\,(\sqrt{\frac{n-4}{2}}-1).$$

3) Any hyperelliptic hyperbolic surface of genus $\gamma \geq 2$ admits a pants decomposition of length at most $51\sqrt{4\pi(\gamma-1)}$.

4) For any $\gamma \geq 2$, there exists a hyperelliptic hyperbolic surface of genus γ such that any pants decomposition has length at least

$$4\mathrm{arcsinh}1(\sqrt{\frac{\gamma-3}{2}}-1).$$

In particular, we derive from this theorem the optimal dependance of Bers' constant $\mathcal{B}_{0,n}$ in the number of punctured points:

$$4\sqrt{n-2} \lesssim \mathcal{B}_{0,n} \leq 76\sqrt{n-2}.$$

The proof of point *1)* in Theorem 1.2 relies both on an induction argument and Besicovitch lemma, and should be compared to the proof of Theorem 1.1. The proof of point *2)* is the explicit construction of the asymptotically optimal two-spheres. This construction is inspired by the example of P. Buser called the "hairy torus"—a hairy torus with hair tips pairwise glued together whose genus is γ and whose length of pants decompositions asymptotically grows at least like $\sim \sqrt{\gamma}$—. Point *3)* is obtained as follows. Our upper estimate for length of short pants decomposition of punctured spheres admits a version for two-spheres with conical points of angle π. Quotient of hyperelliptic surfaces by their hyperelliptic involution being such two-spheres, we pull back short pants decomposition on the quotient sphere to pants decomposition of controlled length on the initial surface. Lastly point *4)* is proved by adapting the examples of point *2)* to quotient of hyperelliptic surfaces and using the strategy of point *3)*.

Given a pants decomposition on a closed marked Riemannian surface or a punctured hyperbolic surface, we can also ask for bounds on the *total length* of the pants decomposition defined as the sum of the length of the curves involved in the pants decomposition. Our main theorem in this direction states that for any fixed genus γ, one can control the growth rate of total length of some pants decomposition of a surface of area $\sim \gamma + n$ by a factor which grows like $n \log n$, where n is the number of marked or punctured points. This result appeared in [4] written in collaboration with H. Parlier and S. Sabourau. More precisely, we have the following statement.

Theorem 1.3. *Fix $n \geq 1$ and $\gamma \geq 0$. Let (M, g) be a hyperbolic surface of genus γ with n cusps or a closed Riemannian surface of genus γ with n marked points whose area is normalized to* $2\pi(2\gamma + n - 2)$.

Then (M, g) admits a pants decomposition whose total length is bounded from above by

$$C_\gamma\, n \log(n+1),$$

where C_γ is an explicit genus dependent constant.

This estimate is sharp except possibly for the $\log(n+1)$ term. Indeed, the total lengths of the pants decompositions of hyperbolic surfaces of genus γ with n cusps and no closed geodesics of length less than $\frac{1}{100}$ are at least $C'_\gamma\, n$ for some positive constant C'_γ depending only on the genus.

As a corollary to the above we show that a hyperelliptic surface of genus γ admits a pants decomposition of total length at most $\sim \gamma \log \gamma$. This is in strong contrast with the general case according to a result of L. Guth, H. Parlier and R. Young [GutParY12]: "random" hyperbolic surfaces have all their pants decompositions of total length at least $\sim \gamma^{7/6-\varepsilon}$ for any $\varepsilon > 0$.

Theorem 1.3 is proved by considering naturally embedded graphs which capture part of the topology and geometry of the surface, and studying these graphs carefully. This method also permits us to generalize estimate (1.2) to almost γ homologically independent curves. This is the purpose of the next section.

1.3. **Homologically independent loops.** The homological systole of a closed Riemannian surface of genus γ with normalized area $4\pi(\gamma-1)$ is at most $\sim \log \gamma$. This result is equivalent to the version of inequality (1.2) for the homogical systole due to M. Gromov, see [Gro96, 2.C]. The dependence in the genus is optimal: there exist families of hyperbolic surfaces, one in each genus, whose homological systoles grow like $\sim \log \gamma$. The first of these were constructed by P. Buser and P. Sarnak in their seminal article [BusSar94], and there have been other constructions since by R. Brooks [Bro99] and M. Katz, M. Schaps and U. Vishne [KatzSchVish07]. By showing that the shortest homologically nontrivial loop on a hyperbolic surface lies in a "thick" embedded cylinder, P. Buser and P. Sarnak also derived new bounds on the minimal norm of nonzero period lattice vectors of Riemann surfaces. This result paved the way for a geometric approach of the Schottky problem which consists in characterizing Jacobians (or period lattices of Riemann surfaces) among abelian varieties.

In [BusSep02, BusSep03], P. Buser and M. Seppälä studied bounds on the lengths of curves in a homology basis for closed hyperbolic surfaces. Note however that without a lower bound on the homological systole, the $\gamma+1$ shortest homologically independent loop cannot be bounded by any function of the genus. Indeed, consider a hyperbolic surface with γ very short homologically independent (and thus disjoint) loops. Every loop homologically independent from these short curves must cross one of them, and via the collar lemma, can be made arbitrarily large by pinching our initial γ curves.

On the other hand, without assuming any lower bound on the homological systole, M. Gromov [Gro83, 1.2.D'] proved that on every closed Riemannian surface of genus γ with area normalized to $4\pi(\gamma-1)$, the length of the γ shortest homologically independent loops is at most $\sim \sqrt{\gamma}$. Furthermore, Buser's so-called hairy torus example [Bus81, Bus92] shows that this bound is optimal, even for hyperbolic surfaces.

In the article [4], we obtain with H. Parlier and S. Sabourau new bounds on the lengths of short homology basis for closed Riemannian surfaces with homological systole bounded from below.

Theorem 1.4. *Let (M,g) be a closed Riemannian surface of genus γ with homological systole at least ℓ and area equal to $4\pi(\gamma-1)$. Then there exist 2γ loops $\alpha_1,\dots,\alpha_{2\gamma}$ on M which induce a basis of $H_1(M;\mathbb{Z})$ such that*

$$\text{length}(\alpha_k) \leq C_0 \,\frac{\log(2\gamma-k+2)}{2\gamma-k+1}\,\gamma, \tag{1.5}$$

where $C_0 = \frac{2^{16}}{\min\{1,\ell\}}$.
In particular:

1. *the lengths of the α_i are bounded by $C_0\,\gamma$;*
2. *the median length of the α_i is bounded by $C_0 \log(\gamma+1)$.*

The linear upper bound in the genus of item (1) already appeared in [BusSep03] for hyperbolic surfaces, where the authors obtained a similar bound for the length of so-called canonical homology basis. They also constructed a genus γ hyperbolic surface all of whose homology bases have a loop of length at least $C\,\gamma$ for some positive constant C. This shows that the linear upper bound in (1) is roughly optimal. However, the general bound (1.5) on the length of the loops of a short homology basis, and in particular the item (2), cannot be derived from the arguments of [BusSep03] even in the hyperbolic case. The bound obtained in (2) is also roughly optimal. Indeed, the Buser-Sarnak surfaces [BusSar94] have their homological systole greater or equal to $\frac{4}{3}\log\gamma$ minus a constant.

A natural question is to find out for how many homologically independent curves does Gromov's $\log\gamma$ bound hold. In [4], we show using Theorem 1.4 that on every closed Riemannian surface of genus γ with normalized area there exist almost γ homologically independent loops of lengths at most $\sim\log\gamma$. More precisely, we prove the following.

Theorem 1.5. *Let $\eta:\mathbb{N}\to\mathbb{N}$ be a function such that*

$$\lambda := \sup_{\gamma}\frac{\eta(\gamma)}{\gamma} < 1.$$

Then there exists a constant C_λ such that for every closed Riemannian surface (M,g) of genus γ there are at least $\eta(\gamma)$ homologically independent loops $\alpha_1,\dots,\alpha_{\eta(\gamma)}$ which satisfy

$$\mathrm{length}(\alpha_i) \leq C_\lambda\,\frac{\log(\gamma+1)}{\sqrt{\gamma}}\,\sqrt{\mathrm{area}(M,g)}$$

for every $i\in\{1,\dots,\eta(\gamma)\}$.

Typically, this result applies to $\eta(\gamma)=[\lambda\gamma]$ where $\lambda\in(0,1)$.

Thus, the previous theorem generalizes Gromov's log bound on the homological systole to the lengths of almost γ homologically independent loops. Note that its proof differs from other systolic inequality proofs. Specifically, it directly yields a $\log\gamma$ bound on the homological systole without considering the homotopical systole (that is, the shortest length of a homotopically nontrivial loop). Initially, M. Gromov obtained his bound from the bound (1.2) on the homotopical systole using surgery, *cf.* [Gro96, 2.C]. However the original proof of the $\log\gamma$ bound on the homotopical systole, *cf.* [Gro83, 6.4.D'] and [Gro96], as well as the alternative proofs available, *cf.* [Bal04, KatzSab05], do not directly apply to the homological systole.

One can ask how far from being optimal our result on the number of short (homologically independent) loops is. Of course, in light of the Buser-Sarnak examples, one can not hope to do (roughly) better than a logarithmic bound on their lengths, but the question on the number of such curves remains. Now, because of Buser's hairy torus example, we know that the γ shortest homologically independent loops of a hyperbolic surface of genus γ can grow like $\sim\sqrt{\gamma}$ and that the result of Theorem 1.5 cannot be extended to $\eta(\gamma)=\gamma$. Still, one can ask for $\gamma-1$ homologically independent loops of lengths at most $\sim\log\gamma$, or more generally for any number of homologically independent loops of lengths at most $\sim\log\gamma$ which grows asymptotically like γ. Note that the surface constructed from Buser's hairy torus does not provide a counterexample in any of these cases.

Our next theorem shows this is impossible, which proves that the result of Theorem 1.5 on the number of homologically independent loops whose lengths satisfy a $\log\gamma$ bound is optimal. Before stating this theorem, it is convenient to introduce the following definition.

Definition 1.1. *Given $k\in\mathbb{N}^*$, the k-th* homological systole *of a closed Riemannian manifold (M,g), denoted by $\mathrm{sys}_k(M,g)$, is defined as the smallest real $L\geq 0$ such that there exist k homologically independent loops on M of length at most L.*

With this definition, under the assumption of Theorem 1.5 every closed Riemannian surface of genus γ with area $4\pi(\gamma-1)$ satisfies

$$\mathrm{sys}_{\eta(\gamma)}(M,g) \leq C_\lambda \log(\gamma+1)$$

for some constant C_λ depending only on λ. Furthermore, still under the assumption of Theorem 1.5, Gromov's sharp estimate, *cf.* [Gro83, 1.2.D'], with this notation becomes

$$\mathrm{sys}_\gamma(M,g) \leq C\sqrt{\gamma}$$

where C is a universal constant.

We can now state the second main result of [4].

Theorem 1.6. *Let $\eta : \mathbb{N} \to \mathbb{N}$ be a function such that*

$$\lim_{\gamma\to\infty} \frac{\eta(\gamma)}{\gamma} = 1.$$

Then there exists a sequence of genus γ_k hyperbolic surfaces S_{γ_k} with γ_k tending to infinity such that

$$\lim_{k\to\infty} \frac{\mathrm{sys}_{\eta(\gamma_k)}(S_{\gamma_k})}{\log \gamma_k} = \infty.$$

We now present an application of theorem 1.5 to the geometry of Jacobians of Riemann surfaces, extending the work [BusSar94] of P. Buser and P. Sarnak.

Consider a closed Riemann surface S of genus γ. We define the L^2-norm $|.|_{L^2}$, simply noted $|.|$, on $H^1(S;\mathbb{R}) \simeq \mathbb{R}^{2\gamma}$ by setting

$$|\Omega|^2 = \inf_{\omega\in\Omega} \int_S \omega \wedge *\omega \tag{1.6}$$

where $*$ is the Hodge star operator and the infimum is taken over all the closed one-forms ω on S representing the cohomology class Ω. The infimum in (1.6) is attained by the unique closed harmonic one-form in the cohomology class Ω. The space $H^1(S;\mathbb{Z})$ of the closed one-forms on S with integral periods (that is, whose integrals over the cycles of M are integers) is a lattice of $H^1(S;\mathbb{R})$. The Jacobian J of S is a principally polarized abelian variety isometric to the flat torus

$$\mathbb{T}^{2\gamma} \simeq H^1(S;\mathbb{R})/H^1(S;\mathbb{Z})$$

endowed with the metric induced by $|.|$.

In their geometric approach of the Schottky problem, P. Buser and P. Sarnak [BusSar94] also proved that the homological systole of the Jacobian of a Riemann surface S of genus γ is at most $\sim \sqrt{\log\gamma}$ and this bound is optimal. In other words, there is a nonzero lattice vector in $H^1(S;\mathbb{Z})$ whose L^2-norm satisfies a $\sqrt{\log\gamma}$ upper bound. We extend their result by showing that there exist almost γ linearly independent lattice vectors whose norms satisfy a similar upper bound. More precisely, we have the following.

Corollary 1.2. *Let $\eta : \mathbb{N} \to \mathbb{N}$ be a function such that*

$$\lambda := \sup_\gamma \frac{\eta(\gamma)}{\gamma} < 1.$$

Then there exists a constant C_λ such that for every closed Riemann surface S of genus γ there are at least $\eta(\gamma)$ linearly independent lattice vectors $\Omega_1, \ldots, \Omega_{\eta(\gamma)} \in H^1(S;\mathbb{Z})$ which satisfy

$$|\Omega_i|^2_{L^2} \leq C_\lambda \log(\gamma+1) \tag{1.7}$$

for every $i \in \{1, \ldots, \eta(\gamma)\}$.

The only extension of the Buser-Sarnak estimate we are aware of is due to B. Muetzel [Mue10], who recently proved a similar result with $\eta(\gamma) = 2$. Contrary to Theorem 1.5, we do not know whether the result of Corollary 1.2 is sharp regarding the number of independent lattice vectors of norm at most $\sim \sqrt{\log \gamma}$.

To prove Theorem 1.4 we consider naturally embedded graphs which capture a part of the topology and geometry of the surface, and recursively use on these graphs a systolic inequality due to B. Bollobás, E. Szemerédi and A. Thomason [BolTho97, BolSze02]. Then, we derive Theorem 1.5 in the absence of a lower bound on the homological systole. In the hyperbolic case (restricting ourselves to hyperbolic metrics in our constructions), we further obtain a crucial property for the proof of Corollary 1.2: the loops given by Theorem 1.5 have embedded collars of uniform width. To prove Theorem 1.6, we adapt known constructions of surfaces with large homological systole to obtain closed hyperbolic surfaces of large genus which asymptotically approach the limit case.

2. Local systolic geometry

In this second part of our report we present local systolic properties of some special metrics. All these metrics share the same property: they have a lot of closed geodesics whose length is precisely their systole. These results are included in the articles [1], [5], [7] and [8], the article [1] being written in collaboration with J.C. Álvarez Paiva while the article [7] was a collaboration with C. Croke and M. Katz. In a first part we consider the two-sphere endowed with two remarkable metrics: a singular metric with three conical singularities which was conjectured by E. Calabi to realize the minimum value of the ratio area/sys^2 over smooth metrics, and the round metric. Then we enlarge our study using contact geometry and give strong local systolic properties for any Zoll Finsler smooth metric on a closed manifold. In the sequel, otherwise stated, all Riemannian or Finsler metrics are supposed to be smooth.

2.1. **Local systolic and diastolic geometry of the two-sphere.** Given a smooth Riemannian metric g on the two-sphere, recall that the systole is defined as the shortest length of a nontrivial closed geodesic and is denoted by $\mathrm{sys}(S^2, g)$. According to Croke's inequality (1.1) the area is bounded from below by the systole as follows:

$$\mathrm{area}(S^2, g) \geq 1/31^2 \cdot \mathrm{sys}(S^2, g)^2.$$

The constant in the inequality was successively improved by A. Nabutovsky and R. Rotman [NabRot02], S. Sabourau [Sab04] and R. Rotman [Rot06]. The best known constant (due to R. Rotman) is 1/32 but the optimal constant remains unknown. Surprisingly, the round sphere is not a global minimum of the *systolic area*—the ratio area/sys^2—. Indeed E. Calabi, see [Cro88], has remarked that the singular metric g_c on S^2 defined by taking two identical equilateral triangles glued along their boundary has systolic area $1/(2\sqrt{3})$. While g_c is nor smooth or (strictly) convex, the surface (S^2, g_c) can be thickened to yield convex smooth two-spheres for which the systolic area is strictly smaller than $1/\pi$, the value which corresponds to the systolic area of the round metric g_0. This singular metric g_c is conjectured to achieve the global minimum of the systolic area for S^2:

Conjecture 2.1 (Optimal systolic inequality for the two-sphere)**.** *For any smooth Riemannian metric g on the two-sphere,*

$$\mathrm{area}(S^2, g) \geq \frac{1}{2\sqrt{3}} \cdot \mathrm{sys}(S^2, g)^2.$$

In section 1.1, we presented a diastolic inequality obtained in [6] with S. Sabourau that asserts that for any smooth Riemannian two-sphere (S^2, g)

$$\mathrm{area}(S^2, g) \geq C \cdot \mathrm{dias}(S^2, g)^2$$

where C is some positive constant. Recall that the diastole denoted by $\mathrm{dias}(S^2, g)$ is defined as the value obtained by a minimax process over the space of one-cycles and satisfies

$$\mathrm{dias}(S^2, g) \geq \mathrm{sys}(S^2, g).$$

Furthermore the two quantities coincide for smooth strictly convex Riemannian metrics by [CalCao92]. The *diastolic area*—the ratio area/dias^2—of the singular metric g_c equals to $1/(2\sqrt{3})$ and the following conjecture appears rather natural.

Conjecture 2.2 (Optimal diastolic inequality for the two-sphere)**.** *For any Riemannian two-sphere (S^2, g),*

$$\mathrm{area}(S^2, g) \geq \frac{1}{2\sqrt{3}} \cdot \mathrm{dias}(S^2, g)^2.$$

Observe that this conjecture is stronger than conjecture 2.1. By Pál's Theorem [Pál21], conjecture 2.2 holds for the special set of singular metrics obtained by gluing two copies of any convex disk of the plane along their boundary.

In this section we focus on local properties of systolic and diastolic area near the singular metric g_c and near the round metric g_0. By the uniformization theorem the two-sphere admits only one conformal structure up to diffeomorphism. So we reduce our study to the local behaviour of systolic and diastolic area over the space of metrics conformal respectively to g_c or g_0. Furthermore, these functionals being scale invariant, we use this additional symmetry to further reduce the space of metrics studied in the neighbourhood of g_0, see below. We already point out that we will obtained another type of local result for the round metric—and more generally for Zoll Finsler manifolds—by using contact geometry and perturbation theory in section 2.3.

We begin by presenting our results in the neighbourhood of the singular metric. Denote by $\mathcal{M}_c$ the space of Riemannian metrics of class C^1 with three conical singularities of angle $2\pi/3$. We remark in [5, Proposition 2.2] that the infimum of the diastolic area over $\mathcal{M}_c$ coincides with the infimum of the diastolic area over the space of smooth metrics. So $\mathcal{M}_c$ is relevant for our local study. The natural topology on $\mathcal{M}_c$ is the C^1-*topology*, defined as the topology induced by the C^1 compact-open topology on $C^1(S^2, \mathbb{R}_+^*)$ (see [Hir76, p. 34]). In [5] we prove the following local version of conjecture 2.2.

Theorem 2.1 (Local minimality of the singular metric for the C^1-topology)**.** *There exists an open neighborhood $\mathcal{O}$ of g_c in $\mathcal{M}_c$ with respect to the C^1-topology such that for all $g \in \mathcal{O}$,*

$$\operatorname{area}(S^2, g) \geq \frac{1}{2\sqrt{3}} \operatorname{dias}(S^2, g)^2$$

with equality if and only if g is isometric to some multiple of g_c.

As dias $\geq$ sys, Theorem 2.1 remains valid with the systole replacing the diastole. Our proof is based on the study of a degree 3 ramified cover of S^2 by the two-torus $\mathbb{T}^2$ and an optimal systolic inequality on the torus due to C. Loewner. In [Sab10], S. Sabourau found an alternative proof of our theorem which does not make use of the uniformization theorem, but is based on the study of the same ramified cover, and carries over to metrics which are closed enough for the Lipschitz distance (in particular thus authorizing small variations of the angles of the three conical points).

We now present our results in the neighbourhood of the round metric. Denote by $\mathcal{M}_0$ the space of smooth Riemannian metrics of the two-sphere. The space $\mathcal{D}$ of smooth diffeomorphism naturally acts on $\mathcal{M}_0$ and both diastolic and systolic area are invariant under this action, as well as under the action of $\mathbb{R}_+^*$ on $\mathcal{M}_0$ by homothety : for any metric $g \in \mathcal{M}_0$, any diffeomorphism $\phi \in \mathcal{D}$ and any scalar $\lambda > 0$, the metric $\lambda \cdot \phi^* g$ has the same diastolic (respectively systolic) area as g. Denote by

$$\mathcal{T}_0 := \left\{ f^2 \cdot g_0 \mid f \in C^\infty(S^2, \mathbb{R}_+^*) \text{ with } \int_{S^2} f dv_{g_0} = 4\pi \right\}$$

the space of normalized Riemannian metrics smooth conformal to g_0. (Here dv_{g_0} denotes the Riemannian volume associated to g_0 which satisfies $\int_{S^2} dv_{g_0} = 4\pi$.) The space $\mathcal{T}_0$ parametrizes the quotient of $\mathcal{M}_0$ under the action of $\mathcal{D}$ and $\mathbb{R}_+^*$, any smooth metric being isometric to some multiple of a metric in $\mathcal{T}_0$ thanks to the uniformization theorem. It is important to remind that diastole and systole coincide for smooth strictly convex metrics according to [CalCao92]. Thus near the round metric it is equivalent to study systolic or diastolic area. In order to state our next result, we need the following definition.

Definition 2.1 (Natural paths between metrics)**.** *Let g and h be two Riemannian metrics on some manifold. The natural path $\{g_t\}_{t\in[0,1]}$ of Riemannian metrics from g to h is defined by the following equation on the associated distance functions :*

$$d_{g_t} = (1-t)d_g + td_h$$

for $t \in [0,1]$.

A natural path between two metrics is thus the linearly parametrized segment between the corresponding distance functions. By a result of V. Guillemin [Gui76], there exist plenty of smooth deformations of the round metric by smooth (non-isometric) Zoll metrics—metrics all of whose geodesics are closed and of the same length—. As Zoll metrics all have the same diastolic area, the round metric can not be a strict local minimum of diastolic area. Somewhat surprising, a consequence of our results in [8] is that the round metric is a strict local minimum of diastolic area along natural paths in $\mathcal{T}_0$. More precisely, we have the following theorem.

Theorem 2.2 (Local minimality of the round metric along natural paths)**.** *Let g be a metric in $\mathcal{T}_0$ different from g_0 and denote by $\{g_t\}$ the natural path from g_0 to g. Then there exists a positive ε such that for $0 < t < \varepsilon$*

$$\text{area}(S^2, g_t) > \frac{1}{\pi}\,\text{dias}(S^2, g_t)^2.$$

This statement strongly suggests that the round metric is a local minimum of diastolic (and thus systolic) area. However recall that even in the finite dimensional case we can construct functions with a local minimum along natural paths which is not a local minimum.

We now briefly explain how to derive Theorem 2.2 from the main result of [8].

Proof. Recall that (see [8, Proposition 4]) *if $\Psi : S^2 \to \mathbb{R}$ is a non-trivial smooth function such that*

$$\int_{S^2} \Psi dv_{g_0} = 0,$$

then there exists a positive ε such that for $0 < |t| < \varepsilon$

$$\text{area}(S^2, (1+t\Psi)^2 \cdot g_0) > \frac{1}{\pi}\,\text{dias}(S^2, (1+t\Psi)^2 \cdot g_0)^2.$$

Now for any $g = f^2 \cdot g_0 \in \mathcal{T}_0$ the natural path from g_0 to g writes as

$$g_t = (1 + t(f-1))^2 \cdot g_0$$

with $\int_{S^2}(f-1)dv_{g_0} = 0$, and Theorem 2.2 directly follows. □

In [8] we derive the criticality of the metric g_0 for diastolic area. But we will present this result in a more general framework in section 2.3. In particular, we will see that in fact *any* smooth Zoll Finsler metric is a critical point of systolic area.

Both Theorems 2.1 and 2.2 are proved using the same strategy. For the variations of the metric in concern, the area could only increase while the average of the lengths of closed geodesics realizing the diastole is constant. So at least one of these lengths is not greater than the initial diastole and the local study of this closed curve permits to construct a short family of one-cycles sweeping out the deformed two-sphere.

2.2. **Systole and diameter of Zoll metrics on the two-sphere.** In this section we are interested in comparing the systole of a Riemannian two-sphere with another Riemannian invariant, namely the diameter. C. Croke proved in [Cro88] that any Riemannian two-sphere (S^2, g) satisfies the inequality

$$\mathrm{sys}(S^2, g) \leq 9 \cdot \mathrm{diam}(S^2, g).$$

The constant in the inequality was successively improved by M. Maeda [Mae94], A. Nabutovsky and R. Rotman [NabRot02], and S. Sabourau [Sab04]. The best known constant is 4, see [NabRot02] and [Sab04], and it has been strongly believed that the optimal constant was 2 meaning that the round metric is optimal for the relationship between these two invariants, see [NabRot02, Introduction].

In [7] we construct in collaboration with C. Croke and M. Katz counterexamples to this conjectured inequality. More precisely, we prove the following.

Theorem 2.3. *There exist smooth variations $\{g_t\}$ of the round metric g_0 by smooth Zoll metrics such that*

$$\mathrm{sys}(S^2, g_t) > 2 \cdot \mathrm{diam}(S^2, g_t)$$

for sufficiently small $t > 0$.

So the round metric is not optimal for the ratio sys/diam. Such families of Zoll metrics with sys $>$ $2 \cdot$ diam are obtained via the theorem of V. Guillemin's [Gui76] asserting that for any smooth odd function on the two-sphere there exists a smooth conformal deformation of the round metric by Zoll metrics whose first derivative is precisely the prescribed odd function. By carefully choosing the odd function, we are able to ensure that the diameter strictly decreases while the systole remains unchanged. It is important to remark that, as Guillemin's theorem is obtained by applying an implicit function theorem, we have no idea of how far from 2 is the optimal constant in the systole/diameter inequality.

2.3. **Local systolic geometry of Zoll Finsler manifolds.** In this section we present the results obtained in collaboration with J.C. Álvarez Paiva in [1]. We first introduce contact geometry as a natural setting for the study of systolic inequalities and then explain how we deduce new insights in systolic geometry by applying basic tools in perturbation theory.

A *contact manifold* is a pair (X, α) consisting of a $(2n+1)$-dimensional manifold together with a smooth 1-form α such that the top order form $\alpha \wedge d\alpha^n$ never vanishes. In the sequel we suppose our contact manifolds are closed and oriented in such a way that $\alpha \wedge d\alpha^n > 0$. The kernel of α defines a field of hyperplanes in the tangent space of X (a vector sub-bundle of TX of co-dimension one) that is maximally non-integrable called the *contact structure* associated to the contact form α. Note that if (X, α) is a contact manifold and $\rho : X \to \mathbb{R}$ is a smooth function that never vanishes, the form $\rho\alpha$ is also a contact form which defines the same contact structure as α. Contact manifolds come with a natural volume:

$$\mathrm{vol}(X, \alpha) := \int_X \alpha \wedge d\alpha^n \,.$$

They also carry a natural vector field, the *Reeb vector field* R_α, defined by the equations $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. The flow of the vector field R_α (remember our contact manifolds are all closed) is called the *Reeb flow* and its orbits are the *Reeb orbits.*

Definition 2.2. *The systole of a contact manifold (X, α), which we denote by* $\mathrm{sys}(X, \alpha)$*, is the smallest period of any of its periodic Reeb orbits. We define the systolic volume of a contact manifold (X, α) of dimension $2n + 1$ as the ratio*

$$\mathfrak{S}(X, \alpha) = \frac{\mathrm{vol}(X, \alpha)}{\mathrm{sys}(X, \alpha)^{n+1}} \,.$$

Remark that in this definition we are implicitly assuming the existence of periodic Reeb orbits on closed contact manifolds. We can bypass this thorny issue by setting $\mathfrak{S}(X,\alpha)=0$ if there are no periodic Reeb orbits, but Weinstein's conjecture is that they always exist. More importantly, their existence has been proved for all the contact manifolds that appear in our results, see [1].

From the viewpoint of perturbation theory, one advantage of working in the contact setting is that we may assume that every smooth deformation of a contact manifold (X,α_0) is of the form $\rho_s\alpha_0$, where ρ_s is a smooth function on M depending smoothly on the parameter. More precisely, *Gray's stability theorem* (see Theorem 2.2.2 in [Gei08]) states that given a smooth deformation α_s (s ranging over some compact interval), there exists an isotopy Φ_s such that $\Phi_s^*\alpha_s=\rho_s\alpha_0$. In other words, we may assume that the contact structure stays fixed along the deformation. Our main results about local systolic geometry of contact manifolds concern the following generalization of Zoll manifolds.

Definition 2.3. *A contact manifold (X,α) is said to be regular if its Reeb flow is periodic and all the Reeb orbits have the same prime period* $\mathrm{sys}(X,\alpha)$.

Our first main result in [1] states that the critical points of the systolic volume are precisely the regular contact manifolds.

Theorem 2.4. *A contact manifold (X,α) is regular if and only if for every smooth isosystolic deformation the derivative of the function $s\mapsto \mathrm{vol}(X,\alpha_s)$ vanishes at $s=0$.*

In studying systolic volume by perturbation techniques, we face the problem that it is not a differentiable function. In Theorem 2.4 we bypassed this difficulty by considering smooth isosystolic deformations—smooth deformations along which the systole remains constant—. However, one of the key features of the work [1] is that we are able to work with arbitrary smooth deformations, which we normalize to be volume-preserving. In order to state the next result we need the following.

Definition 2.4. *A smooth deformation α_s of a contact form α_0 is said to be trivial if there exist a smooth real-valued function $\lambda(s)$ and an isotopy Φ_s such that $\alpha_s=\lambda(s)\Phi_s^*\alpha_0$.*

A smooth deformation α_s is said to be formally trivial if for every $n\in\mathbb{N}$ there exists a trivial deformation $\alpha_s^{(n)}$ that has n-order contact with α_s at $s=0$.

Observe that systolic volume is constant along trivial deformations. Our second main result in [1] is the following description of the local behaviour of systolic volume near regular manifolds.

Theorem 2.5. *Let (X,α_s) be a smooth deformation of a regular contact manifold (X,α_0). If the deformation is not formally trivial, then the function $s\mapsto\mathfrak{S}(X,\alpha_s)$ attains a strict local minimum at $s=0$. If, on the other hand, the deformation is formally trivial, then*

$$\mathfrak{S}(X,\alpha_s)=\mathfrak{S}(X,\alpha_0)+\mathcal{O}(|s|^k)\ \textit{for all}\ k>0.$$

The main ingredient in proving Theorems 2.4 and 2.5 is the following. [1, Theorem 4.1] implies that if a deformation of a regular contact form α_0 is of the form $\alpha_s=\Phi_s^*\rho_s\alpha_0$, where Φ_s is an isotopy and ρ_s is a smooth one-parameter family of smooth *integrals of motion*—functions that are invariant under the Reeb flow of α_0—, then the systolic volume of α_s attains a minimum at $s=0$. Given the large number of isotopies and integrals of motion for a periodic Reeb flow, we could hope that every deformation of α_0 is of this form and thus prove the local systolic minimality of regular contact manifolds. Of course, this idea does not work. However, the theory of normal forms—which we adapt to contact geometry—tells us that it *almost* works.

We shall now explain the consequences for classical systolic geometry. It is well known that geodesic flows of Riemannian and Finsler metrics are Reeb flows (see, for example, Theorem 1.5.2 in [Gei08]). The precise setup is as follows: through the Legendre transform, a (not necessarily reversible) Finsler metric F on a manifold M gives rise to a Hamiltonian H defined in the cotangent bundle. The restriction α of the canonical one-form to the unit cotangent bundle $S^*_H M$ (i.e., the set of covectors where $H = 1$) is a contact form and its Reeb flow is the geodesic flow of the metric. A periodic Reeb orbit in $(S^*_H M, \alpha)$ projects down to a closed geodesic on M whose length equals the period (and the action) of the orbit. In particular, $\mathrm{sys}(S^*_H M, \alpha) = \mathrm{sys}(M, F)$. If the metric F is Riemannian and the manifold has dimension m, the Riemannian volume of (M, F) and the contact volume of $(S^*_H M, \alpha)$ are related by the equality

$$\mathrm{vol}(S^*_H M, \alpha) = m! b_m \mathrm{vol}(M, F) \,,$$

where b_m is the volume of the m-dimensional Euclidean unit ball. When the metric is Finsler, the *Holmes-Thompson* volume is defined by the preceding equality (see [Tho96] and [AlvTho04] for a detailed discussion of this definition). Once we remark that the Finsler metric F is Zoll—all geodesics are closed and of the same length—if and only if the restriction of the canonical one-form to the cotangent bundle $S^*_H M$ is regular, we deduce from Theorems 2.4 and 2.5 the following results.

Theorem 2.6. *A closed Finsler manifold (M, F_0) is Zoll if and only if for every smooth isosystolic deformation F_s the derivative of the function $s \mapsto \mathrm{vol}(M, F_s)$ vanishes at $s = 0$.*

Theorem 2.7. *Let (M, F_s) be a smooth volume-preserving Finsler deformation of a Zoll manifold (M, F_0). If the deformation is formally trivial—that is, if for every $n \in \mathbb{N}$ there exists a deformation $F_s^{(n)}$ by Zoll Finsler metrics that has n-order contact with F_s at $s = 0$—, then*

$$\mathrm{sys}(M, F_s) = \mathrm{sys}(M, F_0) + \mathcal{O}(|s|^k) \text{ for all } k > 0.$$

If, on the other hand, the deformation is not formally trivial, then the function $s \mapsto \mathrm{sys}(M, F_s)$ attains a strict local maximum at $s = 0$.

Our methods work better for Finsler (eventually non-reversible) metrics than for Riemannian metrics because only the former are stable under small contact perturbations.

One of the major open problems in systolic geometry is to determine whether the canonical Riemannian metric in $\mathbb{R}P^m$ ($m > 2$) is a minimum of the systolic volume. Specializing Theorem 2.7 to the Riemannian setting and using the solution of the infinitesimal Blaschke conjecture by R. Michel [Mich73] and C. Tsukamoto [Tsu81], we obtain in [1] the following result.

Theorem 2.8. *Let g_s be a smooth volume-preserving deformation of the canonical metric on one of the projective spaces $\mathbb{R}P^m$, $\mathbb{C}P^m$, $\mathbb{H}P^m$ or CaP^2 where $m \geq 2$. If at $s = 0$ the deformation g_s is not tangent to all orders to trivial deformations (i.e., to deformations of the form $\phi_s^* g_0$ for some isotopy ϕ_s), then the infimum of the lengths of periodic geodesics of the metric g_s attains π as a strict local maximum at $s = 0$.*

Spheres admit non-trivial Zoll deformations so the situation is more delicate. However, on the two-sphere the result takes a particularly simple form.

Theorem 2.9. *Let g_0 be the canonical metric on the two-sphere and let $g_s = e^{\rho_s} g_0$ be any smooth volume preserving deformation. If $d\rho_s/ds\,|_{s=0}$ is not odd, then the length of the shortest periodic geodesic of (S^2, g_s) attains 2π as a strict local maximum at $s = 0$.*

This result is sharp: the main theorem of [Gui76] states that if $\dot\rho$ is odd, then there exists a smooth deformation $e^{\rho_s} g_0$ by Zoll metrics satisfying $d\rho_s/ds\,|_{s=0} = \dot\rho$. The length of the shortest periodic geodesic is then constantly equal to 2π along the deformation.

3. Systolic geometry of homology classes and groups

Using systolic geometry one can construct invariants of homology classes and groups. In this last part of our report we present some properties about these invariants. The results presented here are part of the article [2] written in collaboration with I. Babenko and the article [4] written in collaboration with H. Parlier and S. Sabourau.

3.1. **Systolic volume of homology classes.** Let G be a finitely presentable group, and $a \in H_m(G, \mathbb{Z})$ a non-trivial homology class of dimension $m \geq 1$. We consider the various ways this class can be realized by a pseudomanifold endowed with a polyhedral metric. For such realizations, the two main geometrical ingredients are the volume of the pseudomanifold and the length of loops representing non-trivial elements of G. The systolic volume turns out to be the simplest natural way to compare these geometrical quantities in order to form an invariant and is defined as follows. A *geometric cycle* (X, f) representing a is a pair (X, f) consisting of an orientable pseudomanifold X of dimension m and a continuous map $f : X \to K(G, 1)$ such that $f_*[X] = a$ where $[X]$ denotes the fundamental class of X and $K(G, 1)$ the Eilenberg-MacLane space. The representation is said to be *normal* if in addition the induced map $f_\sharp : \pi_1(X) \to G$ is an epimorphism. Given a geometric cycle (X, f), we can consider for any polyhedral metric g on X (see [Bab06]) the *relative homotopic systole* denoted by $\text{sys}_f(X, g)$ and defined as the least length of a loop γ of X whose image under f is not contractible. The *systolic constant* of the geometric cycle (X, f) is then the value

$$\mathfrak{S}_f(X) := \inf_g \frac{\text{vol}(X, g)}{\text{sys}_f(X, g)^m},$$

where the infimum is taken over all polyhedral metrics g on X and $\text{vol}(X, g)$ denotes the m-dimensional volume of X. In the case where $f : X \to K(\pi_1(X), 1)$ is the classifying map (induced by an isomorphism between the fundamental groups), we simply denote by $\mathfrak{S}(X)$ the systolic constant of the pair (X, f). From [Gro83, Section 6], we have for any $m \geq 1$ that

$$\sigma_m := \inf_{(X,f)} \mathfrak{S}_f(X) > 0,$$

the infimum being taken over all geometric cycles (X, f) representing a non trivial homology class of dimension m. The following notion was introduced by M. Gromov in [Gro83, Section 6]:

Definition 3.1. *The systolic volume of the pair (G, a) is defined as the number*

$$\mathfrak{S}(G, a) := \inf_{(X,f)} \mathfrak{S}_f(X),$$

where the infimum is taken over all geometric cycles (X, f) representing the class a.

Any integer class is representable by a geometric cycle, see Theorem 3.1 below. The systolic volume of (G, a) is thus well defined and satisfies $\mathfrak{S}(G, a) \geq \sigma_m$. But it is not clear if the infimum value $\mathfrak{S}(G, a)$ is actually a minimum and what is the structure of a geometric cycle that might achieve it. In the case where the homology class a is representable by a manifold, we know that the systolic volume coincides with the systolic constant of any normal representation of a by a manifold, see [Bab06, Bab08, Bru08]. A manifold is an example of *admissible* pseudomanifold, that is a special type of pseudomanifolds for which any element of the fundamental group can be represented by a curve not going through the singular locus of X. In the article [2] written in collaboration with I. Babenko, we first prove the following result.

Theorem 3.1. *Let G be a finitely presentable group and $a \in H_m(G, \mathbb{Z})$ a homology class of dimension $m \geq 3$. For any normal representation of a by an admissible geometric cycle (X, f),*

$$\mathfrak{S}(G, a) = \mathfrak{S}_f(X).$$

Furthermore, there always exists a normal representation of a by an admissible geometric cycle.

Here an admissible geometric cycle (X, f) stands for a geometric cycle whose pseudomanifold X is admissible. Thus the infimum in the definition of systolic volume of a homology class is a minimum and the systolic constant of an admissible orientable pseudomanifold X depends only on the image of its fundamental class $f_*[X] \in H_m(\pi_1(X), \mathbb{Z})$ as in the case of orientable manifolds. We will see in the sequel an example showing that the condition of normalization (that is, $f_\sharp$ is an epimorphism between fundamental groups) can not be relaxed in our theorem. Loosely speaking, this theorem is proved as follows. Any geometric cycle representing the class $a \in H_m(G, \mathbb{Z})$ can be used as starting point for the construction of the Eilenberg-MacLane space $K(G, 1)$. This allows to compare its systolic volume with the systolic volume of a fixed admissible geometric cycle normally representing the homology class. The comparison is done using techniques initiated by I. Babenko in [Bab06]. The existence of such normal representations by admissible geometric cycles is deduced from a construction of singular manifolds representing homology classes due to N. Baas in [Baas73].

In order to understand the systolic volume invariant, one can ask for its distribution along the real line. In [2] we show two new phenomena. First, the systolic volume function does not avoid arbitrarily large intervals.

Proposition 3.1. *Let $m \geq 3$. For any interval $I \subset \mathbb{R}^+$ of length at least σ_m, there exists a pair (G, a) consisting of a finitely presentable group and a homology class of dimension m such that $\mathfrak{S}(G, a) \in I$.*

Secondly, there is no finiteness result in great generality for systolic volume in dimension $m \geq 3$. In order to give this statement content, we introduce the following definition. A class $a \in H_m(G, \mathbb{Z})$ is said *reducible* if there exists a proper subgroup $H \subset G$ and a class $b \in H_m(H, \mathbb{Z})$ such that $i_*(b) = a$ where i denotes the canonical inclusion. Otherwise the class will be said *irreducible*.

Theorem 3.2. *For any dimension $m \geq 3$ there exists an infinite sequence of finitely presentable pairwise distinct groups $\{G_i\}$ for which at least one irreducible class $a_i \in H_m(G_i, \mathbb{Z})$ satisfies $\mathfrak{S}(G_i, a_i) \leq 1$*

This theorem is proved by explicitely constructing the pairs (G_i, a_i) as reductions modulo prime numbers of the pair $(\mathbb{Z}^m, [\mathbb{T}^m])$. This implies the following unexpected result using surgery on representations by manifolds for dimensions at least 4.

Corollary 3.1. *For any dimension $m \geq 4$, there exists an infinite number of irreducible orientable manifolds M of dimension m with pairewise non-isomorphic fundamental groups such that $\mathfrak{S}(M) \leq 1$.*

So we have to introduce topological or algebraic restrictions in order to derive finiteness results. For instance, given a finitely presentable group G, a homology class $a \in H_m(G, \mathbb{Z})$ and a positive number T, the number of integer multiple classes ka whose systolic volume is less than T is at most $T. \ln T$ (up to some multiplicative constant). More precisely we show in [2, section 5] the following.

Theorem 3.3. *Let G be a finitely presentable group and $a \in H_m(G, \mathbb{Z})$ where $m \geq 3$. There exists a positive number $C(G, a)$ depending only on the pair (G, a) such that*

$$\mathfrak{S}(G, ka) \leq C(G, a) \cdot \frac{k}{\ln(1 + k)}$$

for any integer $k \geq 1$. In particular,

$$\lim_{k \to \infty} \frac{\mathfrak{S}(G, ka)}{k} = 0.$$

The proof of this result relies on the behaviour of systolic volume of geometric cycles under the operation of connected sum, and is related to a previous collaboration with I. Babenko, see

[BabBal05]. It shows that systolic volume of multiples of a class is a sublinear function. For classes a whose simplicial volume is not zero, we know after Gromov [Gro83] that there exists a positive number $C'(G,a)$ depending only on the pair (G,a) such that

$$\mathfrak{S}(G,ka) \geq C'(G,a) \cdot \frac{k}{(\ln(1+k))^m}. \tag{3.1}$$

Moreover, for fundamental groups π_l of orientable surfaces Σ_l of genus $l \geq 1$ and their corresponding fundamental classes $[\Sigma_l]$, we know by [Gro83] and [BusSar94] that

$$\mathfrak{S}(\pi_l, k[\Sigma_l]) \sim \frac{k}{(\ln(1+k))^2}.$$

where $f \sim g$ means that there exists some positive constants c and C such that $c.f \leq g \leq C.f$. This naturally leads to the following:

Conjecture 3.1. *Let G be a finitely presentable group and $a \in H_m(G,\mathbb{Z})$ a class of non-zero simplicial volume where $m \geq 3$. Then*

$$\mathfrak{S}(G,ka) \sim \frac{k}{(\ln(1+k))^m}.$$

The dependence of the systolic volume on torsion is another natural question. In [Gro96] Gromov mentions that it may be possible to use the torsion of $H_*(\pi_1(M),\mathbb{Z})$ to bound from below the systolic volume of a manifold M. Given a finitely presentable group G and a homology class a of dimension m, we define the 1-*torsion of the class a* as the integer

$$t_1(a) := \min_{(X,f)} |\operatorname{Tors} H_1(X,\mathbb{Z})|,$$

where the minimum is taken over the set of geometric cycles (X,f) representing the class a and $|\operatorname{Tors} H_1(X,\mathbb{Z})|$ denotes the number of torsions elements in the first homology group of X. We now state the main result of [2, section 6] :

Theorem 3.4. *Let G be a finitely presentable group and $a \in H_m(G,\mathbb{Z})$ where $m \geq 2$. Then*

$$\mathfrak{S}(G,a) \geq C_m \frac{\ln t_1(a)}{\exp(C'_m\sqrt{\ln(\ln t_1(a))}},$$

where C_m and C'_m are two positive numbers depending on m.

In particular for any $\varepsilon > 0$

$$\mathfrak{S}(G,a) \geq (\ln t_1(a))^{1-\varepsilon}$$

if $t_1(a)$ is large enough.

This result is optimal in the following sense: for any dimension m, there exists a sequence of groups G_n and homology classes $a_n \in H_m(G_n,\mathbb{Z})$ such that

$$\lim_{n\to\infty} \frac{\mathfrak{S}(G_n,a_n)}{\ln t_1(a_n)} = 0.$$

In general the 1-torsion of a class is difficult to compute. In the case of $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, we can bound from below the 1-torsion of any generator by the number n. In particular, the fundamental classes of lens spaces $L_m(n)$ realize exactly the generators of the group $H_{2m+1}(\mathbb{Z}_n,\mathbb{Z})$ and we obtain that

$$\mathfrak{S}(L_m(n)) \geq (\ln n)^{1-\varepsilon}$$

for any $\varepsilon > 0$ if n is large enough.

Theorem 3.4 is proved by first bounding from below the minimal number of 2-simplices of any geometric cycle (X,f) representing a by its 1-torsion, and then using an estimate due to M. Gromov of the systolic volume $\mathfrak{S}(G,a)$ by the *simplicial height* of a—the minimal number of simplices (of all dimensions) of a geometric cycle representing the class a—, see [Gro83, 6.4.C"] and [Gro96, 3.C.3]. Theorem 3.4 allows us to derive the following result.

Theorem 3.5. *There exists two positive constants a and b such that, for any manifold M of dimension 3 with finite fundamental group,*

$$\mathfrak{S}(M) \geq a \frac{\ln |\pi_1(M)|}{\exp(b\sqrt{\ln(\ln |\pi_1(M)|)})},$$

where $|\pi_1(M)|$ denotes the cardinal of $\pi_1(M)$.

Now we present the particular case of the Heisenberg group of dimension 3. We obtain a new illustration of the possible behaviour of the systolic volume of cyclic coverings. The Heisenberg group $\mathcal{H}$ of dimension 3 is the group of triangular matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

The subset $\mathcal{H}(\mathbb{Z})$ of $\mathcal{H}$ composed of matrices with integer coefficients (*i.e.* matrices for which $x, y, z \in \mathbb{Z}$) is a lattice, and we will denote by $M_{\mathcal{H}} = \mathcal{H}/\mathcal{H}(\mathbb{Z})$ the corresponding quotient space. First of all, we obtain the following explicit upper bound for the systolic volume of multiples of the fundamental class of $M_{\mathcal{H}}$, see [2].

Theorem 3.6. *Let $a = [M_{\mathcal{H}}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ be the fundamental class of $M_{\mathcal{H}}$. Then*

$$\mathfrak{S}(\mathcal{H}(\mathbb{Z}), ka) \leq 19 \cdot \mathfrak{S}(\mathcal{H}(\mathbb{Z}), a)$$

for any integer $k \geq 1$.

The constant appearing here is the one involved in the resolution of the classical Waring problem (see [BDD86]): any integer number decomposes into a sum of at most 19 fourth powers. The idea of using the solution of the Waring problem in order to bound from above the function $\mathfrak{S}(G, ka)$ when $(G, a) = (\mathcal{H}(\mathbb{Z}), [M_{\mathcal{H}}])$ carries over to any pair (G, a) where G is a nilpotent graded group without torsion and a denotes the fundamental class of the corresponding nilmanifold, see [2, Theorem 7.2].

Now consider the sequence of lattices $\{\mathcal{H}_n(\mathbb{Z})\}_{n=1}^{\infty}$ of $\mathcal{H}$, where $\mathcal{H}_n(\mathbb{Z})$ denotes the subset of matrices whose integer coefficients satisfy $x \in n\mathbb{Z}$ and $y, z \in \mathbb{Z}$. Denote by $M_{\mathcal{H}_n} = \mathcal{H}/\mathcal{H}_n(\mathbb{Z})$ the corresponding nilmanifolds. The manifold $M_{\mathcal{H}_n}$ is a cyclic covering with n sheets of $M_{\mathcal{H}}$, and the techniques involved in the proof of Theorem 3.3 implies that

$$\mathfrak{S}(M_{\mathcal{H}_n}) \leq C \cdot \frac{n}{\ln(1+n)}.$$

The fact that the function $\mathfrak{S}(M_{\mathcal{H}_n})$ goes to infinity is a consequence of Theorem 3.4.

Corollary 3.2. *The function $\mathfrak{S}(M_{\mathcal{H}_n})$ satisfies the following inequality:*

$$\mathfrak{S}(M_{\mathcal{H}_n}) \geq a \frac{\ln n}{\exp(b\sqrt{\ln(\ln n)})},$$

where a an b are two positive constants. In particular,

$$\lim_{n \to +\infty} \mathfrak{S}(M_{\mathcal{H}_n}) = +\infty.$$

Note that in this case $\|M_{\mathcal{H}_n}\|_\Delta = 0$ and the lower bound 3.1 does not apply. For any integer n the manifold $M_{\mathcal{H}_n}$ gives a non-normal realization of the class $n[M_{\mathcal{H}}]$. So normalization condition in Theorem 3.1 cannot be relaxed.

3.2. **Systolic area of groups.** Let G be a finitely presentable group and consider the various ways this group can be realized as the fundamental group of a finite simplicial complexe of dimension 2. Using the systolic geometry of these realizations, one defines an invariant of the group called *systolic area* and introduced by M. Gromov in [Gro96, 3.C.8].

Definition 3.2. *The systolic area of G is defined as*

$$\mathfrak{S}(G) = \inf_X \mathfrak{S}(X),$$

where the infimum is taken over all finite 2-complexes X with fundamental group isomorphic to G.

If the group is free, its systolic area is zero. In converse any group which is not free has positive systolic area by [Gro83, 6.7.A]. The best known lower bound is due to Y. Rudyak and S. Sabourau in [RudSab08] who proved that

$$\mathfrak{S}(G) \geq \frac{\pi}{16}$$

for such groups. In the same article they also prove a finiteness result for systolic area of groups.

In [4] we prove the analog of Gromov's $\log \gamma$ bound (1.2) for systolic area of groups. Recall that the first Betti number of a group G is defined as the dimension of its first real homology group

$$H_1(G, \mathbb{R}) := H_1(K(G,1), \mathbb{R}),$$

where $K(G,1)$ denotes the Eilenberg-MacLane space associated to G. Using the techniques explained in section 1.3 we proved with H. Parlier and S. Sabourau in [4] the following result.

Theorem 3.7. *Let G be a finitely presentable nontrivial group with no free factor isomorphic to $\mathbb{Z}$. Then*

$$\mathfrak{S}(G) \geq C \, \frac{b_1(G)+1}{(\log(b_1(G)+2))^2}$$

for some positive universal constant C.

Consider the free product $G_n = F_n * G$, where F_n is the free group with n generators and G is a finitely presentable nontrivial group. The first Betti number of G_n goes to infinity with n, while its systolic area remains bounded by the systolic area of G. This example shows that a restriction on the free factors is needed in the previous theorem. The order of the bound in the previous theorem is asymptotically optimal, see [4, example 7.4].

We can also bound from below the systolic area of such groups by the number of torsion elements in their first homology group. Indeed, we prove with I. Babenko in [2] the following result.

Theorem 3.8. *Let G be a finitely presentable group with no free factor isomorphic to $\mathbb{Z}$. Then*

$$\mathfrak{S}(G) \geq C \frac{\ln |\, Tors\, H_1(G,\mathbb{Z})|}{\exp(C' \sqrt{\ln(\ln |\, Tors\, H_1(G,\mathbb{Z})|)})},$$

where C and C' are two positive numbers.

In particular for any $\varepsilon > 0$

$$\mathfrak{S}(G) \geq (\ln\, |\, Tors\, H_1(G,\mathbb{Z})|)^{1-\varepsilon}$$

if $|\, Tors\, H_1(G,\mathbb{Z})|$ is large enough.

This result should be compared with Theorem 3.4.

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