CHARACTERISTIC IDEALS AND SELMER GROUPS

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Abstract. Let \( A \) be an abelian variety defined over a global field \( F \) of positive characteristic \( p \) and let \( \mathcal{F}/F \) be a \( \mathbb{Z}_p^n \)-extension, unramified outside a finite set of places of \( F \). Assuming that all ramified places are totally ramified, we define a pro-characteristic ideal associated to the Pontrjagin dual of the \( p \)-primary Selmer group of \( A \). To do this we first show the relation between the characteristic ideals of duals of Selmer groups for a \( \mathbb{Z}_d \)-extension \( \mathcal{F}_d/F \) and for any \( \mathbb{Z}_d \)-extension contained in \( \mathcal{F}_d \), and then use a limit process. Finally, we give an application to an Iwasawa Main Conjecture for the non-noetherian commutative Iwasawa algebra \( \mathbb{Z}_p[[\text{Gal}(\mathcal{F}/F)]] \) in the case \( A \) is a constant abelian variety.

1. Introduction

Let \( F \) be a global function field of characteristic \( p \) and \( \mathcal{F}/F \) a \( \mathbb{Z}_p^n \)-extension unramified outside a finite set of places. We take an abelian variety \( A \) defined over \( F \) and let \( S_A \) be a finite set of places of \( F \) containing exactly the primes of bad reduction for \( A \) and those which ramify in \( \mathcal{F}/F \). For any extension \( v \) of some place of \( F \) to the algebraic closure \( \overline{F} \) and for any finite extension \( E/F \), we denote by \( \overline{E}_v \) the completion of \( E \) with respect to \( v \) and, if \( L/F \) is infinite, we put \( L_v := \bigcup \overline{E}_v \), where the union is taken over all finite subextensions of \( L \). We define the \( p \)-part of the Selmer group of \( A \) over \( E \) as

\[
Sel(E) := Sel_A(E)_p := \text{Ker} \left\{ \prod_v H^1_{fl}(X_{E_v}, A[p^{\infty}]) \to \prod_v H^1_{fl}(X_{E_v}, A)[p^{\infty}] \right\}
\]

(where \( H^1_{fl} \) denotes flat cohomology, \( X_E := \text{Spec}(E) \) and the map is the product of the natural restrictions at all places \( v \) of \( E \)). For infinite algebraic extensions we define the Selmer groups by taking direct limits on all the finite subextensions. For any algebraic extension \( K/F \), let \( S(K) \) denote the Pontrjagin dual of \( Sel(K) \) (other Pontrjagin duals will be indicated by the symbol \( \check{\_} \)).

For any infinite Galois extension \( L/F \), let \( \Lambda(L) := \mathbb{Z}_p[[\text{Gal}(L/F)]] \) be the associated Iwasawa algebra: we recall that, if \( \text{Gal}(L/F) \simeq \mathbb{Z}_d \), then \( \Lambda(L) \simeq \mathbb{Z}_p[[t_1, \ldots, t_d]] \) is a Krull domain. It is well known that \( S(L) \) is a \( \Lambda(L) \)-module and its structure has been described in several recent papers (see, e.g., [13] for \( \text{Gal}(L/F) \simeq \mathbb{Z}_d \) and [4] for the non abelian case). When \( S(L) \) is a finitely generated module over a noetherian abelian Iwasawa algebra, it is possible to associate to \( S(L) \) a characteristic ideal which is a key ingredient in Iwasawa Main Conjectures. We are interested in the definition of the analogue of a characteristic ideal in \( \Lambda(F) \) for \( S(F) \) (a similar result providing a pro-characteristic ideal for the Iwasawa module of class groups is described in [3]).

If \( R \) is a noetherian Krull domain and \( M \) a finitely generated torsion \( R \)-module, the structure theorem for \( M \) provides an exact sequence

\[
0 \to P \to M \to \bigoplus_{i=1}^n R/p_i^{e_i}R \to Q \to 0
\]

2010 Mathematics Subject Classification. 11R23; 11G35.

Key words and phrases. Characteristic ideals; Iwasawa theory; Selmer groups.

F. Bars supported by MTM2013-40680-P.
where the $p_i$'s are height 1 prime ideals of $R$ and $P$ and $Q$ are pseudo-null $R$-modules (i.e., torsion modules with annihilator of height at least 2). With this sequence one defines the characteristic ideal of $M$ as

$$Ch_R(M) := \prod_{i=1}^{n} p_i^{e_i}$$

(if $M$ is not torsion, we put $Ch_R(M) = 0$, moreover note that $M$ is pseudo-null if and only if $Ch_R(M) = (1)$). In commutative Iwasawa theory characteristic ideals provide the algebraic counterpart for the $p$-adic $L$-functions associated to Iwasawa modules (such as duals of Selmer groups).

We fix a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i \in \mathbb{N}}$ for $\Gamma$ and, for any $d \geq 0$, we let $\mathcal{F}_d \subset F$ be the fixed field of $\{\gamma_i\}_{i > d}$. Then we have $\Lambda(\mathcal{F}) = \varprojlim \Lambda(\mathcal{F}_d)$ and $S(\mathcal{F}) = \varprojlim S(\mathcal{F}_d)$. Note that the filtration $\{\mathcal{F}_d\}$ of $\mathcal{F}$ is uniquely determined once the $\gamma_i$ have been fixed, but we allow complete freedom in their initial choice. Put $t_i := \gamma_i - 1$: the subring $\mathbb{Z}_p[[t_1, \ldots, t_d]]$ of $\Lambda(\mathcal{F})$ is isomorphic to $\Lambda(\mathcal{F}_d)$ and, by a slight abuse of notation, the two shall be identified in this paper. In particular, for any $d \geq 1$ we have $\Lambda(\mathcal{F}_d) = \Lambda(\mathcal{F}_{d-1})[t_d]$. Let $\pi^d_{d-1}: \Lambda(\mathcal{F}_d) \to \Lambda(\mathcal{F}_{d-1})$ be the canonical projection, denote its kernel by $I^d_{d-1} = (t_d)$ and put $I^d_d := \text{Gal}(\mathcal{F}_d/\mathcal{F}_{d-1})$.

Our goal is to define an ideal attached to $S(\mathcal{F})$ in the non-noetherian Iwasawa algebra $\Lambda(\mathcal{F})$: we will do this via a limit of the characteristic ideals $Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))$. Thus we need to study the relation between $\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d)))$ and $Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}))$. A general technique to deal with this type of descent and ensure that the limit does not depend on the filtration has been described in [3, Theorem 2.13]. That theorem is based on a generalization of some results of [10, Section 3] (which directly apply to our algebras $\Lambda(\mathcal{F}_d)$, even without the generalization to Krull domains provided in [3]) and can be applied to the $\mathcal{L}(\mathcal{F})$-module $S(\mathcal{F})$. In our setting [3, Theorem 2.13] reads as follows

**Theorem 1.1.** If, for every $d \geq 0$,

1. the $t_d$-torsion submodule of $S(\mathcal{F}_d)$ is a pseudo-null $\Lambda(\mathcal{F}_{d-1})$-module, i.e.,

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/t_d) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/I^d_{d-1}) = (1)$$

2. $Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/t_d) \subseteq Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/I^d_{d-1})$,

then the ideals $Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))$ form a projective system (with respect to the maps $\pi^d_{d-1}$).

In Section 2 we show that if $S(\mathcal{F}_d)$ is $\Lambda(\mathcal{F}_d)$-torsion, then $S(\mathcal{F}_d)$ is $\Lambda(\mathcal{F}_d)$-torsion for all $d \geq e$ and use [3, Proposition 2.10] to provide a general relation

$$(1.2) \quad Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/I^d_{d-1}) \cdot \pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}) \cdot J_d)$$

(see (2.9) where the extra factor $J_d$ is more explicit). Then we move to the totally ramified setting, i.e., extensions in which all ramified primes are assumed to be totally ramified (an example are the extensions obtained from $F$ by adding the $\mathfrak{a}^n$-torsion points of a normalized rank 1 Drinfeld module over $F$). In this setting, adapting some techniques and results of K.-S. Tan ([14]), we check the hypotheses of Theorem 1.1 using equation (1.2), and obtain (see Corollary 3.5 and Definition 3.6)

**Theorem 1.2.** Assume all ramified primes in $\mathcal{F}/F$ are totally ramified. Then, for $d \gg 0$, 

$$\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}))$$

and the pro-characteristic ideal

$$\tilde{Ch}_{\Lambda(\mathcal{F})}(S(\mathcal{F})) := \varprojlim_d Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d)) \subseteq \mathcal{L}(\mathcal{F})$$

is well defined.
As an application, we use a deep result of Lai - Longhi - Tan - Trihan [8] to prove an Iwasawa Main Conjecture for constant abelian varieties in our non-noetherian setting (see Theorem 3.7).

2. General $\mathbb{Z}_p$-descent for Selmer groups

To be able to define characteristic ideals we need the following

**Theorem 2.1.** (Tan) Assume that $A$ has good ordinary or split multiplicative reduction at all places of the finite set $S_A$. Then, for any $d$ and any $\mathbb{Z}_p^d$-extension $\mathcal{L}/F$ contained in $\mathcal{F}$, the group $S(\mathcal{L})$ is a finitely generated $\Lambda(\mathcal{L})$-module.

**Proof.** In this form the theorem is due to Tan ([13, Theorem 5]). See also [2, Section 2] and the references there.

If there is a place $v$ ramified in $\mathcal{L}/F$ and of supersingular reduction for $A$, then the module $S(\mathcal{L})$ is not finitely generated over $\Lambda(\mathcal{L})$ by [14, Proposition 1.1 and Theorem 3.10]. In order to obtain a nontrivial relation between the characteristic ideals, we need no ramified supersingular primes and something more than just Theorem 2.1, so we make the following

**Assumptions 2.2.**
1. All places ramified in $F/F_e$ are of ordinary reduction.
2. There exists an $e > 0$ such that $S(F_e)$ is a torsion $\Lambda(F_e)$-module.

**Remarks 2.3.**
1. Hypothesis 2 is satisfied in many cases: for example when $F_e$ contains the arithmetic $\mathbb{Z}_p$-extension of $F$ (proof in [14, Theorem 2], extending [11, Theorem 1.7]) or when $\text{Sel}(F)$ is finite and $A$ has good ordinary reduction at all places which ramify in $F_e/F$ (easy consequence of [13, Theorem 4]).
2. Our goal is an equation relating $\pi_{d-1}^{\text{Ch}}(\Lambda(\mathcal{F}_d))$ and the characteristic ideal of $S(\mathcal{F}_{d-1})$. If the above assumption 2 is not satisfied for any $e$, then all characteristic ideals are 0 and there is nothing to prove.

In this section we also assume that none of the ramified prime has trivial decomposition group in $\text{Gal}(\mathcal{F}_1/F)$. In Section 3 we shall work in extensions in which ramified places are totally ramified, so this assumption will be automatically verified. Anyway this is not restrictive in general because of the following

**Lemma 2.4.** If $d \geq 2$, one can always find a $\mathbb{Z}_p$-subextension $\mathcal{F}_1/F$ of $\mathcal{F}_d/F$ in which none of the ramified places splits completely.

**Proof.** See [3, Lemma 3.1] □

Consider the diagram

\[
\begin{array}{c}
\text{Sel}(\mathcal{F}_{d-1}) \xrightarrow{\alpha_{d-1}} H^1_{\text{fl}}(X_{d-1}, A[p^\infty]) \xrightarrow{b_{d-1}} \mathcal{G}(X_{d-1}) \\
\downarrow c_{d-1} \downarrow \downarrow c_{d-1} \\
\text{Sel}(\mathcal{F}_d) & \xrightarrow{\Gamma_{d-1}} & H^1_{\text{fl}}(X_d, A[p^\infty])^{\Gamma_{d-1}} & \xrightarrow{\mathcal{G}(X_d)^{\Gamma_{d-1}}} \\
& & \mathcal{G}(X_d)^{\Gamma_{d-1}} &
\end{array}
\]

where $X_d := \text{Spec}(\mathcal{F}_d)$ and $\mathcal{G}(X_d)$ is the image of the product of the restriction maps

$H^1_{\text{fl}}(X_d, A[p^\infty]) \rightarrow \prod_w H^1_{\text{fl}}(X_{d,w}, A[p^\infty]),$

with $w$ running over all places of $\mathcal{F}_d$.

**Lemma 2.5.** For any $d \geq 2$, the Pontrjagin dual of $\text{Ker} c_{d-1}^{\Gamma_{d-1}}$ is a finitely generated torsion $\Lambda(\mathcal{F}_{d-1})$-module.
Proof. For any place $v$ of $F$ we fix an extension to $\mathcal{F}$, which by a slight abuse of notation we still denote by $v$, so that the set of places of $\mathcal{F}_d$ above $v$ will be the Galois orbit $\text{Gal}(\mathcal{F}_d/F)\cdot v$. For any field $L$ let $\mathcal{P}_L$ be the set of places of $L$. We have an obvious injection

$$
(2.2) \quad \text{Ker } \mathcal{C}_d^{d-1} \hookrightarrow \prod_{u \in \mathcal{P}_{\mathcal{F}_d}} \text{Ker } \left\{ H^1_f(X_{d-1,u}, A)[p^\infty] \rightarrow \prod_{w|u} H^1_f(X_{d,w}, A)[p^\infty] \right\}
$$

(the map is the product of the natural restrictions $r_w$). By the Hochschild-Serre spectral sequence, we get

$$
(2.3) \quad \text{Ker } r_w \simeq H^1(\Gamma^d_{d-1,w}, A(\mathcal{F}_{d,w}))[p^\infty]
$$

where $\Gamma^d_{d-1,w}$ is the decomposition group of $w$ in $\Gamma^d_{d-1}$. Those kernels really depend only on the place $u$ of $\mathcal{F}_d$ lying below $w$ (for any $w_1, w_2$ dividing $u$ we obviously have $\text{Ker } r_{w_1} \simeq \text{Ker } r_{w_2}$). Hence for any $v$ of $F$ and any $u \in \mathcal{P}_{\mathcal{F}_d}$ dividing $u$, we fix a $u(w)$ of $\mathcal{F}_d$ over $u$ and define

$$
\mathcal{H}_v(\mathcal{F}_d) := \prod_{u \in \text{Gal}(\mathcal{F}_d/F)\cdot v} H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty].
$$

Equation (2.2) now reads as

$$
(2.4) \quad \text{Ker } \mathcal{C}_d^{d-1} \hookrightarrow \bigoplus_{v \in \mathcal{P}_F} \mathcal{H}_v(\mathcal{F}_d).
$$

Obviously $\mathcal{H}_v(\mathcal{F}_d) = 0$ for all primes which totally split in $\mathcal{F}_d/\mathcal{F}_{d-1}$ and, from now on, we only consider places that $\Gamma^d_{d-1,w(u)} \neq 0$.

Let $\Lambda(\mathcal{F}_{d,v}) := \mathbb{Z}_p[[\text{Gal}(\mathcal{F}_{d,v}/F_v)]]$ be the Iwasawa algebra associated to the decomposition group of $v$ and note that each $\text{Ker } r_w$ is a $\Lambda(\mathcal{F}_{d-1,v})$-module. Moreover, we get an action of $\text{Gal}(\mathcal{F}_{d-1}/F)$ on $\mathcal{H}_v(\mathcal{F}_d)$ by permutation of the primes $u$ and an isomorphism

$$
(2.5) \quad \mathcal{H}_v(\mathcal{F}_d) \simeq \Lambda(\mathcal{F}_{d-1}) \otimes_{\Lambda(\mathcal{F}_{d-1,v})} H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]
$$

(more details can be found in [14, Lemma 3.2], note that $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]$ is finitely generated over $\Lambda(\mathcal{F}_{d-1,v})$).

First assume that the place $v$ is unramified in $\mathcal{F}_d/F$ (hence inert in $\mathcal{F}_d/\mathcal{F}_{d-1}$). Then $\mathcal{F}_{d-1,v} = F_v \neq \mathcal{F}_{d,v}$ and one has, by [9, Proposition I.3.8],

$$
H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)})) \simeq H^1(\Gamma^d_{d-1,w(u)}, \pi_0(\mathcal{A}_{0,v})),
$$

where $\mathcal{A}_{0,v}$ is the closed fiber of the Néron model of $A$ over $F_v$ and $\pi_0(\mathcal{A}_{0,v})$ is its set of connected components. It follows that $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]$ is trivial when $v$ does not lie above $S_A$ and that it is finite of order bounded by (the $p$-part of) $|\pi_0(\mathcal{A}_{0,v})|$ for the unramified places of bad reduction. Hence (2.4) reduces to

$$
(2.6) \quad \text{Ker } \mathcal{C}_d^{d-1} \subseteq \bigoplus_{v \in S_A(d)} \mathcal{H}_v(\mathcal{F}_d)
$$

(where $S_A(d)$ is the set of primes in $S_A$ which are not totally split in $\mathcal{F}_d/\mathcal{F}_{d-1}$) and, by (2.5), $\mathcal{H}_v(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_{d-1})$-module for unramified $v$.

For the ramified case the exact sequence

$$
A(\mathcal{F}_{d,w(u)})[p] \longrightarrow A(\mathcal{F}_{d,w(u)}) \longrightarrow pA(\mathcal{F}_{d,w(u)})
$$

yields a surjection

$$
H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p] \longrightarrow H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p].
$$
The first module is obviously finite, so \( H^1(T_{d-1}, A(F_d))(p) \) is finite as well: this implies that \( H^1(T_{d-1}, A(F_d))(p) \) has finite \( p \)-rank. As a finitely generated \( \mathbb{Z}_p \)-module, \( H^1(T_{d-1}, A(F_d)) \) is a \( p \)-torsion module, \( H^1(T_{d-1}, A(F_d)) \) must be \( p \)-\( \mathbb{Z}_p \)-torsion for any \( d \geq 2 \) (because of our choice of \( F \) and (2.5) shows once again that \( H_0(F_d) \) is finitely generated and torsion over \( \Lambda(F_{d-1}) \).

\[ \square \]

**Remark 2.6.** One can go deeper in the details and compute those kernels according to the reduction of \( A \) at \( v \) and the behaviour of \( v \) in \( F_d/F \). We will do this in Section 3 but only for the particular case of a totally ramified extension (with the statement of a Main Conjecture as a final goal). See [14] for a more general analysis.

The following proposition provides a crucial step towards equation (1.2) (in particular it also takes care of hypothesis 2 of Theorem 1.1).

**Proposition 2.7.** Let \( e \) be as in Assumption 2.2.2. For any \( d > e \), the module \( S(F_d)/I_{d-1}^{2} \) is a finitely generated torsion \( \Lambda(F_{d-1}) \)-module and \( S(F_d) \) is a finitely generated torsion \( \Lambda(F_{d-1}) \)-module. Moreover, if \( d > \max\{2, e\} \),

\[ C_{\Lambda(F_{d-1})}(S(F_d)/I_{d-1}^{2}) = C_{\Lambda(F_{d-1})}(S(F_d)) \cdot C_{\Lambda(F_{d-1})}((Coker a_{e+1}^{d})^\vee). \]

**Proof.** It suffices to prove the first statement for \( d = e + 1 \), then a standard argument (detailed, e.g., in [7, page 207]) shows that \( S(F_{e+1}) \) is \( \Lambda(F_{e+1}) \)-torsion and we can iterate the process. From diagram (2.1) one gets a sequence

\[ (2.7) \quad (Coker a_{e+1}^{d})^\vee \longrightarrow (Sel(F_{e+1})I_{e+1}^{2})^\vee \longrightarrow S(F_{e}) \longrightarrow (Ker a_{e+1}^{d})^\vee. \]

By the Hochschild-Serre spectral sequence, it follows

\[ \text{Coker } b_{e+1}^{d} \longrightarrow H^2(\Gamma_{e+1}^{d}, A[p](F_{e+1})) = 0 \]

(because \( \Gamma_{e+1}^{d} \) has \( p \)-cohomological dimension 1). Therefore there is a surjective map

\[ \text{Ker } c_{e+1}^{d} \longrightarrow \text{Coker } a_{e+1}^{d} \]

and, by Lemma 2.5, \( (Ker a_{e+1}^{d})^\vee \) is \( \Lambda(F_{e}) \)-torsion. Hence Assumption 2.2.2 and sequence (2.7) yield that

\[ (Sel(F_{e+1})I_{e+1}^{2})^\vee \simeq S(F_{e+1})/I_{e+1}^{2} \]

is \( \Lambda(F_{e}) \)-torsion. To conclude note that (for any \( d \)) the duals of

\[ \text{Ker } a_{d-1}^{d} \hookrightarrow \text{Ker } b_{d-1}^{d} \simeq H^1(\Gamma_{d-1}^{d}, A[p](F_d)) \simeq A[p](F_d)/I_{d-1}^{2} \]

are finitely generated \( \mathbb{Z}_p \)-modules (hence pseudo-null over \( \Lambda(F_{d-1}) \) for any \( d \geq 3 \)). Taking characteristic ideals in the sequence (2.7), for large enough \( d \), one finds

\[ C_{\Lambda(F_{d-1})}(S(F_d)/I_{d-1}^{2}) = C_{\Lambda(F_{d-1})}(S(F_d)) \cdot C_{\Lambda(F_{d-1})}((Coker a_{d-1}^{d})^\vee). \]

\[ \square \]

**Remark 2.8.** In [11, Theorem 1.7], the authors prove that \( S(F^{(p)}) \) is a finitely generated torsion \( \mathbb{Z}_p[[\text{Gal}(F^{(p)}/F)]] \)-module (where \( F^{(p)} \) is the arithmetic \( \mathbb{Z}_p \)-extension of \( F \)). The first part of the proof above provides a more direct approach to the generalization of this result given in [14, Theorem 2].

Whenever \( S(F_d) \) is a finitely generated torsion \( \Lambda(F_d) \)-module, [3, Proposition 2.10] yields

\[ (2.8) \quad C_{\Lambda(F_{d-1})}(S(F_d)) \cdot \pi_{d-1}^{d}(C_{\Lambda(F_d)}(S(F_d))) = C_{\Lambda(F_{d-1})}(S(F_d)) \cdot J_{d}. \]

If \( d > \max\{2, e\} \), equation (2.8) turns into

\[ (2.9) \quad C_{\Lambda(F_{d-1})}(S(F_d)) \cdot \pi_{d-1}^{d}(C_{\Lambda(F_d)}(S(F_d))) = C_{\Lambda(F_{d-1})}(S(F_{d-1})) \cdot J_{d}, \]
where \( J_d := Ch_{\Lambda(F_{d-1})}((\text{Coker } a_{d-1}^d)^\vee) \).

Therefore, whenever we can prove that \( S(F_0)^{\Gamma_0}_{d-1} \) is a pseudo-null \( \Lambda(F_{d-1}) \)-module (i.e., hypothesis 1 of Theorem 1.1), we immediately get

\[
\pi_d^{d-1}(Ch_{\Lambda(F_d)}(S(F_d))) \subseteq Ch_{\Lambda(F_{d-1})}(S(F_{d-1})
\]

and Theorem 1.1 will provide the definition of the pro-characteristic ideal for \( S(F) \) in \( \Lambda(F) \) we were looking for.

3. \( \mathbb{Z}_p \)-descent for totally ramified extensions

The main examples we have in mind are extensions satisfying the following

**Assumption 3.1.** The (finitely many) ramified places of \( F/F \) are totally ramified.

In what follows an extension satisfying this assumption will be called a **totally ramified extension**. A prototypical example is the \( \omega \)-cyclic extension of \( \mathbb{F}_q(T) \) generated by the \( \omega \)-torsion of the Carlitz module (\( \omega \) an ideal of \( \mathbb{F}_q[T] \), see, e.g., [12, Chapter 12]). As usual in Iwasawa theory over number fields, most of the proofs will work (or can be adapted) simply assuming that ramified primes are totally ramified in \( F/F_\epsilon \) for some \( \epsilon > 0 \), but, in the function field setting, one would need some extra hypothesis on the behaviour of these places in \( F/F_\epsilon \) (as we have seen with Lemma 2.4, note that in totally ramified extensions any \( \mathbb{Z}_p \)-subextension can play the role of \( F_1 \)).

A relevant example for the last case is the composition of a \( \omega \)-cyclic extension and of the arithmetic \( \mathbb{Z}_p \)-extension of \( \mathbb{F}_q(T) \) (with the second one playing the role of \( F_1 \)). Note that Assumption 2.2.2 is verified in this case with \( \epsilon = 1 \), thanks to [11, Theorem 1.7], hence our next results hold for all these extensions as well.

Let \( v \in S_A \) be unramified in \( F/F \), then it is either totally split or it is inert in just one \( \mathbb{Z}_p \)-extension \( F_{d(v)}/F_{d(v)-1} \) and totally split in all the others. Since \( |S_A| \) is finite we can fix an index \( d_0 \) such that all unramified places of \( S_A \) are totally split in \( F/F_{d_0} \).

**Theorem 3.2.** Assume \( F/F \) is a totally ramified extension, then, for any \( d \geq d_0 + 1 \), we have

\[
Ch_{\Lambda(F_{d-1})}((\text{Coker } a_{d-1}^d)^\vee) = (1).
\]

**Proof.** The proof of Proposition 2.7 shows that the \( \Lambda(F_{d-1}) \)-modules (Coker \( a_{d-1}^d \)) and (Ker \( c_{d-1}^d \)) are pseudo-isomorphic for \( d \geq 3 \). Moreover, by the proof of Lemma 2.5 (recall, in particular, equation (2.6)), we know that (Ker \( c_{d-1}^d \)) is a quotient of \( \bigoplus_{v \in S(d)} \mathcal{H}_v(F_\epsilon)^\vee \). Hence we only consider the contributions of the places of \( S_A \) which are not totally split in \( F/F \). By equation (2.5), we have (for a fixed \( w \) dividing \( v \))

\[
Ch_{\Lambda(F_{d-1})}((\mathcal{H}_v(F_\epsilon))^\vee) = \Lambda(F_{d-1})Ch_{\Lambda(F_{d-1},w)}(H^1(T_{d-1,w}^d, A(F_{d,w}))[p^\infty]^\vee).
\]

We also saw that, for a ramified prime \( v \), \( \mathcal{H}_v(F_\epsilon)^\vee \) (which is \( H^1(T_{d-1,w}^d, A(F_{d,w}))[p^\infty]^\vee \), because \( v \) is totally ramified) is finitely generated over \( \mathbb{Z}_p \), hence pseudo-null over \( \Lambda(F_{d-1,v}) = \Lambda(F_{d-1}) \) for \( d \geq 3 \).

We are left with the unramified (not totally split) primes in \( S_A \). Assume \( v \) is inert in an extension \( F_r/F_{r-1} \) (\( r \leq d_0 \) by definition), then

\[
\Lambda(F_{r-1,v}) \simeq \mathbb{Z}_p \quad \text{and} \quad \Lambda(F_{d,v}) \simeq \mathbb{Z}_p[[t_r]] \quad \text{for any } d \geq r.
\]

Since (again by Lemma 2.5) \( H^1(T_{r-1,w}^d, A(F_{r,w}))[p^\infty] \) is finite and \( H^1(T_{d-1,w}^d, A(F_{d,w}))[p^\infty] = 0 \) for any \( d \geq r \), we have

\[
Ch_{\Lambda(F_{r-1,v})}(H^1(T_{r-1,w}^d, A(F_{r,w}))[p^\infty]^\vee) = (p^{k(v)}).
\]
for some \( \nu(v) \) depending on \( \pi_0(\mathcal{A}_{0,v}) \), and

\[
Ch_{\Lambda}(\mathcal{F}_{d-1,v})(H^1(\Gamma^d_{d-1,v}, \mathcal{A}(\mathcal{F}_{d,v}))|p^\infty)|^v = (1) \text{ for any } d \geq d_0 + 1 \geq r + 1.
\]

These local informations and (3.1) yield the theorem.

\( \square \)

Now we deal with the other extra term of equation (2.9), i.e., \( Ch_{\Lambda}(\mathcal{F}_{d-1}) (\mathcal{S}(\mathcal{F}_d)|^{\Gamma^d_{d-1}}) \). Note first that, taking duals

\[
(\mathcal{S}(\mathcal{F}_d)|^{\Gamma^d_{d-1}})^\vee \simeq \mathcal{S}(\mathcal{F}_d)^\vee / (\gamma_d - 1) = Sel(\mathcal{F}_d)/(\gamma_d - 1),
\]

so we work on the last module.

From now on we put \( \gamma := \gamma_d \) and we shall need the following (see also [14, Proposition 4.4])

**Lemma 3.3.** We have

\[
H^1_{fl}(\mathcal{X}_d, \mathcal{A}[p^m]) = (\gamma - 1)H^1_{fl}(\mathcal{X}_d, \mathcal{A}[p^\infty]).
\]

**Proof.** Since

\[
H^1_{fl}(\mathcal{X}_d, \mathcal{A}[p^\infty]) = \varprojlim_{K \subset \mathcal{F}_d, [K:F] < \infty} \varprojlim_m H^1_{fl}(X_K, \mathcal{A}[p^m]),
\]

an element \( \alpha \in H^1_{fl}(\mathcal{X}_d, \mathcal{A}[p^\infty]) \) belongs to some \( H^1_{fl}(X_K, \mathcal{A}[p^m]) \). Now let \( \gamma_{\mathcal{F}_{d}(K)} \) be the largest power of \( \gamma \) which acts trivially on \( K \), and define a \( \mathbb{Z}_p \)-extension \( K_\infty \) with \( \text{Gal}(K_\infty/K) = \langle \gamma_{\mathcal{F}_{d}(K)} \rangle \) and layers \( K_n \). Take \( t \geq m \), consider the restrictions

\[
H^1_{fl}(X_K, \mathcal{A}[p^m]) \rightarrow H^1_{fl}(X_{K_t}, \mathcal{A}[p^m]) \rightarrow H^1_{fl}(X_{K_\infty}, \mathcal{A}[p^m])
\]

and denote by \( x_t \) the image of \( x \). Now \( x_t \) is fixed by \( \text{Gal}(K_t/K) \) and \( p^m x_t = 0 \), so \( x_t \) is in the kernel of the norm \( N^{K_t}_K \), i.e., \( x_t \) belongs to the (Galois) cohomology group

\[
H^1(K_t/K, H^1_{fl}(X_{K_\infty}, \mathcal{A}[p^m])) \rightarrow H^1(K_\infty/K, H^1_{fl}(X_{K_\infty}, \mathcal{A}[p^m])).
\]

Let \( \text{Ker}_{m}^2 \) be the kernel of the restriction map \( H^2_{fl}(X_K, \mathcal{A}[p^m]) \rightarrow H^2_{fl}(X_{K_\infty}, \mathcal{A}[p^m]) \), then, from the Hochschild-Serre spectral sequence, we have

\[
(3.2) \quad \text{Ker}_{m}^2 \rightarrow H^1(K_\infty/K, H^1_{fl}(X_{K_\infty}, \mathcal{A}[p^m])) \rightarrow H^3(K_\infty/K, A[\mathcal{A}[p^m]]) = 0
\]

(because the \( p \)-cohomological dimension of \( \mathbb{Z}_p \) is 1). To get rid of \( \text{Ker}_{m}^2 \) note that, by [6, Lemma 3.3], \( H^1_{fl}(X_K, \mathcal{A}) = 0 \). Hence, the cohomology sequence arising from

\[
\begin{array}{ccc}
A[p^m] & \longrightarrow & A \longrightarrow A[p^m] \\
\end{array}
\]

yields an isomorphism \( H^1_{fl}(X_K, \mathcal{A}[p^m]) \simeq H^1_{fl}(X_K, \mathcal{A}[p^m]) \). Consider the commutative diagram (with \( m_2 \geq m_1 \))

\[
\begin{array}{ccc}
H^1_{fl}(X_K, \mathcal{A})/p^{m_1} & \longrightarrow & H^2_{fl}(X_K, \mathcal{A}[p^{m_1}]) \\
\downarrow & & \downarrow \\
H^1_{fl}(X_K, \mathcal{A})/p^{m_2 - m_1} & \longrightarrow & H^2_{fl}(X_K, \mathcal{A}[p^{m_2}]) \\
\end{array}
\]

An element of \( H^1_{fl}(X_K, \mathcal{A})/p^{m_1} \) of order \( p^r \) goes to zero via the vertical map on the left as soon as \( m_2 \geq m_1 + r \), hence the direct limit provides \( \varinjlim_{m} H^1_{fl}(X_K, \mathcal{A})/p^{m} = 0 \) and, eventually,

\[
\varinjlim_{m} \text{Ker}_{m}^2 = 0 \text{ as well. By (3.2)}
\]

\[
0 = \varinjlim_{m} H^1(K_\infty/K, H^1_{fl}(X_{K_\infty}, \mathcal{A}[p^m])) = H^1(K_\infty/K, H^1_{fl}(X_{K_\infty}, \mathcal{A}[p^\infty])),
\]
which yields
\[ H^1_f(X_{K^\infty}, A[p^\infty]) = (\gamma p^{(K)} - 1)H^1_f(X_{K^\infty}, A[p^\infty]) = (\gamma - 1)H^1_f(X_{K^\infty}, A[p^\infty]). \]

We get the claim by taking the direct limit on the finite subextensions \( K \).

\[ \textbf{Theorem 3.4.} \text{ For any } d \geq 3 \text{ we have } Ch_A(F_{d-1})(S(F_d)^{\Gamma_d}_{d-1}) = (1). \]

\[ \textbf{Proof.} \text{ Consider the following diagram} \]
\[ (3.3) \]
\[ \xymatrix{ Sel(F_d) \ar[r]^{\phi} & H^1_f(X_d, A[p^\infty]) \ar[d]^{\gamma - 1} & H^1(X_d, A) \ar[d]^{\gamma - 1} & Coker(\phi) \ar[d]^{\gamma - 1} \\
Sel(F_d) \ar[r]^{\phi} & H^1_f(X_d, A[p^\infty]) \ar[d]^{\gamma - 1} & H^1(X_d, A) \ar[d]^{\gamma - 1} & Coker(\phi) \ar[d]^{\gamma - 1} }
\]

(where \( H^i(X_d, A) := \prod_w H^i_f(X_{d, w}, A[p^\infty]) \) and the surjectivity of the second vertical arrow comes from the previous lemma). Inserting \( G(F_d) := \text{Im}(\phi) \), we get two diagrams
\[ (3.4) \]
\[ \xymatrix{ Sel(F_d) \ar[r]^{\phi} & H^1_f(X_d, A[p^\infty]) \ar[d]^{\gamma - 1} & H^1(X_d, A) \ar[d]^{\gamma - 1} & Coker(\phi) \ar[d]^{\gamma - 1} \\
Sel(F_d) \ar[r]^{\phi} & H^1_f(X_d, A[p^\infty]) \ar[d]^{\gamma - 1} & H^1(X_d, A) \ar[d]^{\gamma - 1} & Coker(\phi) \ar[d]^{\gamma - 1} }
\]

From the snake lemma sequence of the first one, we obtain the isomorphism
\[ (3.5) \]
\[ G(F_d)^{\Gamma_d}_{d-1}/\text{Im}(\phi^{\Gamma_d}_{d-1}) \cong Sel(F_d)/(\gamma - 1) \]

(where \( \phi^{\Gamma_d}_{d-1} \) is the restriction of \( \phi \) to \( H^1_f(X_d, A[p^\infty])^{\Gamma_d}_{d-1} \)). The snake lemma sequence of the second diagram (its “upper” row) yields an isomorphism
\[ (3.6) \]
\[ H^1(X_d, A)^{\Gamma_d}_{d-1}/G(F_d)^{\Gamma_d}_{d-1} \cong Coker(\phi)^{\Gamma_d}_{d-1}. \]

The injection \( G(F_d)^{\Gamma_d}_{d-1} \hookrightarrow H^1(X_d, A)^{\Gamma_d}_{d-1} \) induces an exact sequence
\[ (3.7) \]
\[ G(F_d)^{\Gamma_d}_{d-1}/\text{Im}(\phi^{\Gamma_d}_{d-1}) \hookrightarrow H^1(X_d, A)^{\Gamma_d}_{d-1}/\text{Im}(\phi^{\Gamma_d}_{d-1}) \twoheadrightarrow H^1(X_d, A)^{\Gamma_d}_{d-1}/G(F_d)^{\Gamma_d}_{d-1} \]

(with a little abuse of notation we are considering \( \text{Im}(\phi^{\Gamma_d}_{d-1}) \) as a submodule of \( H^1(X_d, A)^{\Gamma_d}_{d-1} \) via the natural injection above) which, by (3.5) and (3.6), yields the sequence
\[ (3.8) \]
\[ Sel(F_d)/(\gamma - 1) \hookrightarrow Coker(\phi^{\Gamma_d}_{d-1}) \twoheadrightarrow Coker(\phi)^{\Gamma_d}_{d-1}. \]

Now consider the following diagram
\[ (3.9) \]
\[ \xymatrix{ H^1(\Gamma_{d-1}^d, A[p^\infty]) \ar[r]^{\phi^{d}_{d-1}} & H^1_f(X_{d-1}, A[p^\infty]) \ar[d]^{\phi^{d}_{d-1}} & H^1_f(X_d, A[p^\infty])^{\Gamma_d}_{d-1} \ar[d]^{\phi^{d}_{d-1}} & 0 \\
H^1(\Gamma_{d-1}^d, A) \ar[r] & H^1(X_{d-1}, A) \ar[d]^{\phi^{d}_{d-1}} & H^1(X_d, A)^{\Gamma_d}_{d-1} \ar[d]^{\phi^{d}_{d-1}} & H^2(\Gamma_{d-1}^d, A)}
\]

where:
- the vertical maps are all induced by the product of restrictions \( \phi \);
- the horizontal lines are just the Hochschild-Serre sequences for global and local cohomology;
- the 0 in the upper right corner comes from \( H^2(\Gamma_{d-1}^d, A[p^\infty]) = 0 \);
- the surjectivity on the lower right corner comes from \( H^2(X_{d-1}, A) = 0 \), which is a direct consequence of [9, Theorem III.7.8].
This yields a sequence (from the snake lemma)

\[ Coker(\phi_{d-1}) \to Coker(\phi_{d-1}^d) \to H^2(\Gamma_{d-1}^d, A) = \prod_{w} H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p^{\infty}] \, . \]

**The module** \( Coker(\phi_{d-1}) \). The Kummer map induces a surjection \( H^1(X_{d-1}, A[p^{\infty}]) \to H^1(X_{d-1}, A)[p^{\infty}] \) which fits in the diagram

\[ \begin{array}{ccc}
H^1(X_{d-1}, A[p^{\infty}]) & \xrightarrow{\phi_{d-1}} & H^1(X_{d-1}, A) \\
\downarrow & & \downarrow \\
H^1(X_{d-1}, A)[p^{\infty}] & \rightarrow & H^1(X_{d-1}, A)[p^{\infty}]
\end{array} \]

This induces natural surjective maps \( Im(\phi_{d-1}) \to Im(\lambda_{d-1}) \) and, eventually, \( Coker(\lambda_{d-1}) \to Coker(\phi_{d-1}) \). For any finite extension \( K/F \) we have a similar map

\[ \lambda_K : H^1(X_K, A)[p^{\infty}] \to H^1(X_K, A) \]

whose cokernel verifies

\[ Coker(\lambda_{d-1}) \cong T_p(Sel_A(K)) \]

(by [5, Main Theorem]), where \( A^t \) is the dual abelian variety of \( A \) and \( T_p \) denotes the \( p \)-adic Tate module. Moreover there is an embedding of \( T_p(Sel_A(K)) \) into the \( p \)-adic completion of \( H^0(K, A^t) \) (recall that, by Tate local duality, \( A^t(K_v) = H^0(K_v, A^t) \) is the Pontrjagin dual of \( H^1(K_v, A) \), see [9, Theorem III.7.8]). Taking limits on all the finite subextensions of \( F_{d-1} \) we find similar relations

\[ Coker(\lambda_{d-1}) \cong T_p(Sel_A(F_{d-1})) \]

Hence \( Coker(\lambda_{d-1}) \) embeds into a finitely generated \( \mathbb{Z}_p \)-module, i.e., it is \( \Lambda(F_{d-1}) \) pseudo-null for any \( d \geq 3 \).

**The modules** \( H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p^{\infty}] \). If the prime splits completely in \( \mathcal{F}_d/F_{d-1} \), then obviously \( H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p^{\infty}] = 0 \). If the place is ramified or inert, then \( \Gamma_{d-1}^d \cong \mathbb{Z}_p \).

Consider the exact sequence

\[ A(\mathcal{F}_{d,w})[p] \xrightarrow{\phi} A(\mathcal{F}_{d,w}) \xrightarrow{\phi} pA(\mathcal{F}_{d,w}) \]

which yields a surjection

\[ H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p] \to H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p] \, . \]

The module on the left is trivial because \( cd_p(\mathbb{Z}_p) = 1 \), hence \( H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p] = 0 \) and this yields \( H^2(\Gamma_{d-1}^d, A(\mathcal{F}_{d,w}))[p^{\infty}] = 0 \).

The sequence (3.8) implies that \( Coker(\phi_{d-1}^d) \) is \( \Lambda(F_{d-1}) \) pseudo-null for \( d \geq 3 \) and, by (3.7), we get \( Sel(\mathcal{F}_d)/(\gamma - 1) \) is pseudo-null as well. Therefore

\[ Ch_{\Lambda(F_{d-1})}(S(\mathcal{F}_d)^{\Gamma_{d-1}^d}) = Ch_{\Lambda(F_{d-1})}((Sel(\mathcal{F}_d)/(\gamma - 1))^\vee) = (1) \, . \]

\[ \square \]

**Corollary 3.5.** Assume \( \mathcal{F}/F \) is a totally ramified extension, then, for any \( d \geq 0 \) and any \( \mathbb{Z}_p \)-subextension \( \mathcal{F}_d/F_{d-1} \), one has

\[ \pi_{d-1}(Ch_{\Lambda(F_d)}(S(\mathcal{F}_d))) = Ch_{\Lambda(F_{d-1})}(S(\mathcal{F}_{d-1})) \, . \]

The modules \( S(\mathcal{F}_d) \) verify the hypotheses of Theorem 1.1 (because of Proposition 2.7 and Theorem 3.4), so we can define
Definition 3.6. The pro-characteristic ideal of $S(F)$ is

$$\widehat{Ch}_{A(F)}(S(F)) := \lim_{\to} Ch_{A(F_d)}(S(F_d)) \subseteq \Lambda .$$

We remark that Definition 3.6 only depends on the extension $\bar{F} / F$ and not on the filtration of $\mathbb{Z}_p^d$-extension we choose inside it. Indeed with two different filtrations $\{ F_d \}$ and $\{ F'_d \}$ we can define a third one by putting

$$F'_0 := F \quad \text{and} \quad F'_n = F_n F'_n \quad \forall \ n \geq 1 .$$

By Corollary 3.5, the limits of the characteristic ideals of the filtrations we started with coincide with the limit on the filtration $\{ F'_n \}$ (see [3, Remark 3.11] for an analogous statement for characteristic ideals of class groups).

This pro-characteristic ideal could play a role in the Iwasawa Main Conjecture (IMC) for a totally ramified extension of $F$ as the algebraic counterpart of a $p$-adic $L$-function associated to $A$ and $\bar{F}$ (see [1, Section 5] or [2, Section 3] for similar statements but with Fitting ideals). Anyway, at present, the problem of formulating a (conjectural) description of this ideal in terms of a natural $p$-adic $L$-functions (i.e., a general non-noetherian Iwasawa Main Conjecture) is still wide open. However, we can say something if $A$ is already defined over the constant field of $F$.

Theorem 3.7. [Non-noetherian IMC for constant abelian varieties] Assume $A/F$ is a constant abelian variety and let $\bar{F} / F$ be a totally ramified extension as above. Then there exists an element $\theta_{A,F}$ interpolating the classical $L$-function $L(A, \chi, 1)$ (where $\chi$ varies among characters of $\text{Gal}(\bar{F}/F)$) such that one has an equality of ideals in $\Lambda(F)$

$$\widehat{Ch}_{A(F)}(S(F)) = (\theta_{A,F}) .$$

Proof. This is a simple consequence of [8, Theorem 1.3]. Namely, the element $\theta_{A,L}$ is defined in [8, Section 7.2.1] for any abelian extension $L/F$ unramified outside a finite set of places. It satisfies $\pi_{d-1}^d(\theta_{A,F_d}) = \theta_{A,F_d-1}$ by construction and the interpolation formula (too complicated to report it here) is proved in [8, Theorem 7.3.1]. Since $A$ has good reduction everywhere, our results apply here and both sides of (3.10) are defined. Finally [8, Theorem 1.3] proves that $Ch_{A(F_d)}(S(F_d)) = (\theta_{A,F_d})$ for all $d$ and (3.10) follows by just taking a limit. $\square$

References


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