

# Infinitely many cubic points on $X_0(N)/\langle w_d \rangle$ with $N$ square-free

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## Abstract

We determine all modular curves  $X_0(N)/\langle w_d \rangle$  that admit infinitely many cubic points over the rational field  $\mathbb{Q}$ , when  $N$  is square-free.

## 1 Introduction

A non-singular smooth curve  $C$  over a number field  $K$  of genus  $g_C > 1$  always has a finite set of  $K$ -rational points  $C(K)$  by a celebrated result of Faltings (here we fix once and for all  $\overline{K}$ , an algebraic closure of  $K$ ). We consider the set of all points of degree at most  $n$  for  $C$  by  $\Gamma_n(C, K) = \cup_{[L:K] \leq n} C(L)$  and exact degree  $n$  by  $\Gamma'_n(C, K) = \cup_{[L:K]=n} C(L)$ , where  $L \subseteq \overline{K}$  runs over the finite extensions of  $K$ . A point  $P \in C$  is said to be a point of degree  $n$  over  $K$  if  $[K(P) : K] = n$ .

The set  $\Gamma_n(C, M)$  is infinite for a certain  $M/K$  finite extension if  $C$  admits a degree at most  $n$  map, all defined over  $M$ , to a projective line or an elliptic curve with positive  $M$ -rank. The converse is true for  $n = 2$  [HaSi91],  $n = 3$  [AbHa91] and  $n = 4$  under certain restrictions [AbHa91][DeFa93]. If we fix the number field  $M$  in the above results (i.e. an arithmetic statement for  $\Gamma_n(C, M)$  with  $M$  fix), we need a precise understanding over  $M$  of the set  $W_n(C) = \{v \in Pic^n(C) | h^0(C, \mathcal{L}_v) > 0\}$  where  $Pic^n$  is the usual  $n$ -Picard group and  $\mathcal{L}_v$  the line bundle of degree  $n$  on  $C$  associated to  $v$ . If  $W_n(C)$  contains no translates of abelian subvarieties with positive  $M$ -rank of  $Pic^n(C)$  then  $\Gamma'_n(C, M)$  is finite, (under the assumption that  $C$  admits no maps of degree at most  $n$  to a projective line over  $M$ ).

For  $n = 2$  the arithmetic statement for  $\Gamma_n(C, K)$  follows from [AbHa91], (for a sketch of the proof and the precise statement see [Ba, Theorem 2.14]).

For  $n = 3$ , Daeyeol Jeon introduced an arithmetic statement and its proof in [Jeo21] following [AbHa91] and [DeFa93]. In particular, if  $g_C \geq 3$  and  $C$  has no degree 3 or 2 map to a projective line and no degree 2 map to an elliptic curve over  $\overline{K}$  then the set of exact cubic points of  $C$  over  $K$ ,  $\Gamma'_3(C, K)$ , is an infinite set if and only if  $C$  admits a degree three map to an elliptic curve over  $K$  with positive  $K$ -rank.

Observe that if  $g_C \leq 1$  (with  $C(K) \neq \emptyset$  for  $g_C = 1$ ), then  $C$  has a degree three map over  $K$  to the projective line, thus  $\Gamma'_3(C, K)$  is always an infinite set. Thus for curves  $C$  with  $C(K) \neq \emptyset$

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we restrict to  $g_C \geq 2$  in order to study the finiteness of the set  $\Gamma'_3(C, K)$ . We now introduce a few more notations.

For any  $N \in \mathbb{N}$ , let  $X_0(N)$  denote the modular curve corresponding to the group  $\Gamma_0(N)$ . The modular curve  $X_0(N)$  is the coarse moduli space over  $\mathbb{Q}$  of the isomorphism classes of the generalized elliptic curves  $E$  with a cyclic subgroup  $C$  of order  $N$ . For any  $d|N$ , the  $d$ -th (partial) Atkin-Lehner operator  $w_d$  defined by the action of the matrix  $\begin{pmatrix} dx & y \\ Nu & dv \end{pmatrix}$  such that  $d^2xv - Nuy = d$ . Let  $N = d \cdot N'$  where  $(d, N') = 1$ . Then  $w_d$  defines an involution on  $X_0(N)$  given by

$$(E, C) \rightarrow (E/A_{N'}, (E_{N'} + A)/A_{N'}),$$

where  $E_{N'}$  denotes the kernel of the multiplication by  $N'$  and  $A_{N'} := \ker([N'] : A \rightarrow A)$ . Let  $X_0^{+d}(N) := X_0(N)/\langle w_d \rangle$  denote the quotient curve. Thus there is a  $\mathbb{Q}$ -rational degree 2 mapping  $X_0(N) \rightarrow X_0^{+d}(N)$ . In [BaDa22], the authors computed all the values of  $N$  for which the curve  $X_0(N)/w_N$  has infinitely many cubic points over  $\mathbb{Q}$ . In this article, we determine all the values of the pair  $(N, d)$  such that the curve  $X_0^{+d}(N) := X_0(N)/w_d$  has infinitely many cubic points over the rational field  $\mathbb{Q}$  when  $N$  is a square free integer and  $1 < d < N$ . Unless otherwise stated explicitly, throughout the article we always assume that  $N$  is a square-free positive integer. Since the the modular curves  $X_0^{+d}(N)$  always has a  $\mathbb{Q}$ -rational points (cusp), we restrict ourselves to the cases where  $g_{X_0^{+d}(N)} := \text{genus}(X_0^{+d}(N)) \geq 2$ . The main result of this article is the following:

**Theorem 1.1.** *Suppose  $g_{X_0^{+d}(N)} \geq 2$ . Then  $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$  is infinite if and only if  $g_{X_0^{+d}(N)} = 2$  or  $(N, d)$  is in the following list:*

$g_{X_0^{+d}(N)}$	$(N, w_d)$
3	$(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (77, w_7), (82, w_{41}), (91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47}),$
4	$(66, w_{33}), (74, w_{37}), (86, w_{43})$
5	$(106, w_{53}), (158, w_{79})$
6	$(122, w_{61}), (166, w_{83})$
7	$(130, w_{65})$
8	$(178, w_{89})$
9	$(202, w_{101}), (262, w_{131})$

One of the key idea to decide the set  $\Gamma'_3(X, \mathbb{Q})$  is finite or infinite is to check whether there is a  $\mathbb{Q}$ -rational degree 3 mapping between  $X$  and an elliptic curve  $E$  with positive  $\mathbb{Q}$ -rank. Using the existing ideas we can not directly compute the values of  $N$  and  $d$  such that the set  $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$  is infinite. For example, with the existing ideas we can not determine whether there is a degree 3 mapping  $X_0(Np)/\langle w_p \rangle \rightarrow E$ , where  $p$  is a prime,  $p \nmid N$  and  $\text{Cond}(E) = p$ . But, in this article, based on the ideas of M.Derick and P.Orlić from [DeOr24], we develop a criterion to check whether there is a  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(Np)/\langle w_p \rangle \rightarrow E$ , where  $p$  is a prime,  $p \nmid N$  and  $\text{Cond}(E) = p$ .

The MAGMA codes for computing the models of  $X_0^{+d}(N)$  can be found at <https://github.com/Tarunda1> and the MAGMA codes for computing the points of  $X_0^{+d}(N)$  over finite fields can be found at <https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves>.

## 2 General considerations

We recall the notion of gonality of a curve. Given a complete curve  $C$  over  $K$ , the gonality of  $C$  is defined as follows:

$$\text{Gon}(C) := \min\{\deg(\varphi) \mid \varphi : C \rightarrow \mathbb{P}^1 \text{ defined over } \overline{K}\}.$$

The following theorem plays a crucial role in deciding whether the set  $\Gamma'_3(C, K)$  is infinite or not.

**Lemma 2.1.** *[Jeo21, Lemma 1.2] (see also [BaDa22, Lemma 2]) Suppose  $C$  has  $\text{Gon}(C) \geq 4$ ,  $P \in C(K)$  and does not have a degree  $\leq 2$  map to an elliptic curve. If  $\Gamma'_3(C, K)$  is an infinite set then  $C$  admits a  $K$ -rational map of degree 3 to an elliptic curve with positive  $K$ -rank.*

The following results are well known:

**Theorem 2.2.** *(i). ([FuHa99]) The values of  $(N, d)$  such that  $X_0(N)/\langle w_d \rangle$  is hyperelliptic are given in Table 5.*

*(ii). ([HaSh99a]) The values of  $(N, d)$  such that  $\text{Gon}(X_0(N)/\langle w_d \rangle) = 3$  are given in Table 6.*

*(iii). ([BaGoKa20]) The values of  $(N, d)$  such that  $X_0(N)/\langle w_d \rangle$  is bielliptic are given in Table 7.*

We say that a triple  $(N, d, E)$ , where  $N$  is a natural number,  $d \mid N$  and  $E$  is an elliptic curve over  $\mathbb{Q}$  with positive  $\mathbb{Q}$ -rank, is admissible if there is a  $\mathbb{Q}$ -rational degree 3 mapping  $X_0^{+d}(N) \rightarrow E$ .

Throughout this section we consider the values of  $N, d$  such that  $\text{Gon}(X_0^{+d}(N)) \geq 4$  and  $X_0^{+d}(N)$  has no degree  $\leq 2$  map to an elliptic curve. By Lemma 2.1, the set  $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$  is infinite if and only if the triple  $(N, d, E)$  is admissible for some elliptic curve  $E$  with positive  $\mathbb{Q}$ -rank.

The following lemma gives a criterion to rule out the triples which are not admissible (cf. [BaDa22, Lemma 4]).

**Lemma 2.3.** *If  $(N, d, E)$  is admissible, then:*

1.  *$E$  has conductor  $M$  with  $M \mid N$  and for any prime  $p \nmid N$  we have  $|\overline{X}_0^{+d}(N)(\mathbb{F}_{p^n})| \leq 3|\overline{E}(\mathbb{F}_{p^n})|$  and  $|\overline{X}_0(N)(\mathbb{F}_{p^n})| \leq 6|\overline{E}(\mathbb{F}_{p^n})|$ ,  $\forall n \in \mathbb{N}$ .*
2. *if conductor of  $E$  is  $N$ , then the degree of the strong Weil parametrization of  $E$  divides 6.*
3. *for any prime  $p \nmid N$  we have  $\frac{p-1}{12}\psi(N) + 2^{\omega(N)} \leq 6(p+1)^2$ , where  $\omega(N)$  is the number of prime divisors of  $N$  and  $\psi = N \prod_{q \mid N, q \text{ prime}} (1 + 1/q)$  is the  $\psi$ -Dedekind function,*
4. *for any Atkin-Lehner involution  $w_r$  of  $X_0(N)$  with  $r \neq d$ , we have  $g_{X_0^{+d}(N)} \leq 3 + 2 \cdot g_{X_0^{+d}(N)/w_r} + 2$ .*

**Corollary 2.4.** *For  $N > 623$ , the triple  $(N, d, E)$  is not admissible.*

*Proof.* The proof is similar to that of [BaDa22, Corollary 5]. □

We recall the following result (cf. [BaDa22, Lemma 6])

**Lemma 2.5.** *Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$  and  $\varphi : X_0(N) \rightarrow E$  be the strong Weil parametrization of degree  $k$  defined over  $\mathbb{Q}$ . Suppose that  $w_N$  acts as  $+1$  on the modular form  $f_E$  associated to  $E$ . Then  $\varphi$  factors through  $X_0^+(N)$  (and  $k$  is even).*

We now make a minor improvement to [BaDa22, Lemma 7].

**Lemma 2.6.** *Consider a degree  $k$  map  $\varphi : X \rightarrow E$  defined over  $\mathbb{Q}$ , where  $X$  is a quotient modular curve  $X_0(N)/W_N$  with  $W_N$  a proper subgroup of  $B(N)$  ( $B(N)$  is the subgroup of  $\text{Aut}(X_0(N))$  generated by all Atkin-Lehner involutions). Assume that  $\text{cond}(E) = M$  ( $M|N$ ). Let  $d \in \mathbb{N}$  with  $d|M$ ,  $(d, N/d) = 1$  and  $w_d \notin W_N$ , such that  $w_d$  acts as  $+1$  on the modular form  $f_E$  associated to  $E$ . Then,*

1. *if  $E$  has no non-trivial 2-torsion over  $\mathbb{Q}$ , then  $\varphi$  factors through  $X/\langle w_d \rangle$  and  $k$  is even.*
2. *if  $w_d$  has a fixed point on  $X$ , then  $\varphi$  factors through  $X/\langle w_d \rangle$  and  $k$  is even.*
3. *if  $E$  has non-trivial 2-torsion over  $\mathbb{Q}$  and  $k$  is odd, then we obtain a degree  $k$  map  $\varphi' : X/\langle w_d \rangle \rightarrow E'$ , by taking  $w_d$ -invariant to  $\varphi$ , where  $E'$  is an elliptic curve isogenous to  $E$ .*

*Proof.* The proofs of (1) and (3) can be found in [BaDa22, Lemma 7]. Moreover, from the proof of [BaDa22, Lemma 7], we get  $\varphi \circ w_d = \varphi + P$ , where  $P$  is a 2-torsion point of  $E(\mathbb{C})$ . Thus it is easy to see that if  $w_d$  has a fixed point on  $X$ , then  $P$  is the trivial zero of  $E$  and  $\varphi$  factors through  $X/\langle w_d \rangle$ .  $\square$

As an immediate consequence of Lemma 2.6(2), we obtain the following result:

**Corollary 2.7.** *The following triples are not admissible*

(130, 2, 65a1), (130, 5, 65a1), (130, 10, 65a1), (130, 13, 65a1), (130, 26, 65a1), (185, 5, 185c1)  
(185, 37, 185c1), (195, 3, 65a1), (195, 5, 65a1), (195, 13, 65a1), (195, 39, 65a1).

*Proof.* Consider the triple (130, 5, 65a1). Suppose  $f : X_0^{+5}(130) \rightarrow 65a1$  is a  $\mathbb{Q}$ -rational degree 3 mapping. By Riemann-Hurwitz theorem, we see that  $w_{13}$  has fixed points on  $X_0^{+5}(130)$ . By Lemma 2.6 (2),  $f$  must factor through  $X_0(130)/\langle w_5, w_{13} \rangle$ . Which is not possible since  $\deg(f)$  is odd. Hence the triple (130, 5, 65a1) is not admissible. A similar argument will work for the other cases (for the cases (195, 3, 65a1) and (195, 39, 65a1) apply Lemma 2.6(2) with the involutions  $w_{13}$  and  $w_5$  respectively).  $\square$

After applying Lemma 2.3, Lemma 2.6(1) and Corollary 2.7, we can conclude that if  $\text{Gon}(X_0^{+d}(N)) \geq 4$  and  $X_0^{+d}(N)$  has no degree  $\leq 2$  map to an elliptic curve, then the triple  $(N, d, E)$  is not admissible possibly only for the following values of  $N, d$  and  $E$ :

N	$d$	E	N	$d$	E	N	$d$	E
106	53	53a1	182	91	91b1	249	83	83a1
122	61	61a1	183	61	61a1	262	131	131a1
129	43	43a1	195	65	65a1	267	89	89a1
130	65	65a1	202	101	101a1	273	91	91b1
158	79	79a1	215	43	43a1	303	101	101a1
166	83	83a1	222	37	37a1	305	61	61a1
178	89	89a1	237	79	79a1	395	79	79a1

To handle the remaining triples  $(N, d, E)$  mentioned in the last tables, we study the degeneracy maps between quotient modular curves.

### 3 Degeneracy maps:

Let  $M$  be a positive integer such that  $M|N$ . For every positive divisor  $d$  of  $\frac{N}{M}$ , we have  $\iota_d \Gamma_0(N) \iota_d^{-1} \subset \Gamma_0(M)$ , where  $\iota_d := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ . Hence  $\iota_d$  induces a degeneracy map

$$\iota_{d,N,M} : X_0(N) \rightarrow X_0(M),$$

where  $\iota_{d,N,M}$  acts on  $\tau \in \mathbb{H}^*$  in the extended upper half plane as  $\iota_{d,N,M}(\tau) = \iota_d \cdot \tau = d\tau$ . For any positive integer  $r$  with  $r||M$ , let  $w_r^{(N)}$  denote the  $r$ -th (partial) Atkin-Lehner operator acting on  $X_0(N)$  and  $w_r^{(M)}$  denote the  $r$ -th (partial) Atkin-Lehner operator acting on  $X_0(M)$ . For any positive divisor  $d$  of  $\frac{N}{M}$  with  $(d, r) = 1$ , a simple calculation shows that  $\iota_d w_r^{(N)} \iota_d^{-1} = w_r^{(M)}$ . Thus  $\iota_{d,N,M}$  induces a mapping

$$\iota_{d,N,M,r} : X_0(N)/\langle w_r^{(N)} \rangle \rightarrow X_0(M)/\langle w_r^{(M)} \rangle.$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} X_0(N) & \xrightarrow{\iota_{d,N,M}} & X_0(M) \\ w_r^{(N)} \downarrow & & \downarrow w_r^{(M)} \\ X_0(N)/\langle w_r^{(N)} \rangle & \xrightarrow{\iota_{d,N,M,r}} & X_0(M)/\langle w_r^{(M)} \rangle. \end{array} \quad (3.1)$$

By abuse of notations we write  $w_r$  for  $w_r^{(N)}$  and  $w_r^{(M)}$ . As an immediate consequence we obtain:

**Corollary 3.1.** *The modular curves  $X_0^{+53}(106)$ ,  $X_0^{+61}(122)$ ,  $X_0^{+65}(130)$ ,  $X_0^{+79}(158)$ ,  $X_0^{+83}(166)$ ,  $X_0^{+89}(178)$ ,  $X_0^{+101}(202)$ ,  $X_0^{+131}(262)$  have infinitely many cubic points over  $\mathbb{Q}$ .*

*Proof.* By the previous discussions, there are natural  $\mathbb{Q}$ -rational degree 3 mappings

$$\begin{aligned} X_0(106)/w_{53} &\rightarrow X_0(53)/w_{53} \cong 53a1 \\ X_0(122)/w_{61} &\rightarrow X_0(61)/w_{61} \cong 61a1 \\ X_0(130)/w_{65} &\rightarrow X_0(65)/w_{65} \cong 65a1 \\ X_0(158)/w_{79} &\rightarrow X_0(79)/w_{79} \cong 79a1 \\ X_0(166)/w_{83} &\rightarrow X_0(83)/w_{83} \cong 83a1 \\ X_0(178)/w_{89} &\rightarrow X_0(89)/w_{89} \cong 89a1 \\ X_0(202)/w_{101} &\rightarrow X_0(101)/w_{101} \cong 101a1 \\ X_0(262)/w_{131} &\rightarrow X_0(131)/w_{131} \cong 131a1. \end{aligned}$$

Since all the elliptic curves  $53a1, 61a1, 65a1, 79a1, 83a1, 89a1, 101a1$  and  $131a1$  have positive  $\mathbb{Q}$ -ranks, we conclude that all the modular curves mentioned in the statement of this corollary have infinitely many cubic points over  $\mathbb{Q}$ .  $\square$

We now introduce some notations. For any curves  $C, C'$  over a field  $k$ , and any morphism  $f : C \rightarrow C'$  we define the following mappings

$$f_* : J(C) \rightarrow J(C') \text{ defined by } f_*([\sum n_i P_i]) = [\sum n_i f(P_i)], \text{ and}$$

$$f^* : J(C) \rightarrow J(C') \text{ defined by } f^*([\sum n_i P_i]) = [\sum n_i f^{-1}(P_i)],$$

where  $J(C)$  (resp.,  $J(C')$ ) denote the Jacobian of  $C$  (resp.,  $C'$ ). It is easy to check that  $f_* \circ f^* = [\deg(f)]$ .

For any positive integer  $M$ , we use the notation  $[E, r]_M$  to denote a pair  $(E, r)$  where  $E$  is an elliptic curve of conductor  $M$  and  $r \parallel M$  is a positive integer such that  $w_r$  acts as  $+1$  on  $E$ , furthermore if  $r < M$ , then at least one of the following is true:

1.  $E$  has no non-trivial 2-torsion point,
2.  $w_r$  has a fixed point on  $X_0(M)$ .

For a pair  $[E, r]_M$ , if  $f : X_0(M) \rightarrow E$  is the modular parametrization, then by Lemma 2.5 and Lemma 2.6,  $f$  induces a mapping

$$f_r : X_0(M)/\langle w_r \rangle \rightarrow E, \tag{3.2}$$

and the following diagram commutes:

$$\begin{array}{ccc} X_0(M) & \xrightarrow{f} & E \\ w_r \downarrow & \nearrow f_r & \\ X_0(M)/\langle w_r \rangle & & \end{array} \tag{3.3}$$

Observe that if  $E'$  is any elliptic curve isogenous to  $E$  and  $f' : X_0(M)/\langle w_r \rangle \rightarrow E'$  be a mapping, then  $f'$  factors through  $f_r$ . Also, for any positive divisor  $d$  of  $\frac{N}{M}$  with  $(d, r) = 1$ , we have the following commutative diagram:

$$\begin{array}{ccccc} X_0(N) & \xrightarrow{\iota_{d,N,M}} & X_0(M) & \xrightarrow{f} & E \\ w_r^{(N)} \downarrow & & w_r^{(M)} \downarrow & \nearrow f_r & \\ X_0(N)/\langle w_r \rangle & \xrightarrow{\iota_{d,N,M,r}} & X_0(M)/\langle w_r \rangle & & \end{array} \tag{3.4}$$

Let  $J_0(N)^{w_r}$  (resp.,  $J_0(M)^{w_r}$ ) denote the Jacobian of  $X_0(N)/\langle w_r \rangle$  (resp.,  $X_0(M)/\langle w_r \rangle$ ) and  $n$  be the number of divisors of  $\frac{N}{M}$ . Considering the pushforward maps of (3.4), we obtain the commutative diagram:

$$\begin{array}{ccccc} J_0(N) & \xrightarrow{\iota_{d,N,M,*}} & J_0(M) & \xrightarrow{f_*} & E \\ w_{r,*}^{(N)} \downarrow & & w_{r,*}^{(M)} \downarrow & \nearrow f_{r,*} & \\ J_0(N)^{w_r} & \xrightarrow{\iota_{d,N,M,r,*}} & J_0(M)^{w_r} & & \end{array} \tag{3.5}$$

Note that there is a natural mapping  $\delta_{N,M,r} : J_0(N)^{w_r} \rightarrow (J_0(M)^{w_r})^n$  defined by

$$\delta_{N,M,r} := (\iota_{1,N,M,r,*}, \dots, \iota_{\frac{N}{M},N,M,r,*}).$$

We recall the notion of optimal  $B$ -isogenous quotient (cf. [DeOr24, Definition 3.2]).

**Definition 3.2.** Let  $A, B$  be abelian varieties over a field  $k$  with  $B$  simple. An abelian variety  $A'$  together with a quotient map  $\pi : A \rightarrow A'$  is an optimal  $B$ -isogenous quotient if  $A'$  is isogenous to  $B^n$  for some integer  $n$  and every morphism  $A \rightarrow B'$  with  $B'$  isogenous to  $B^m$  for some integer  $m$  uniquely factors through  $\pi$ .

**Proposition 3.3.** *Suppose  $N < 408$  is a positive integer. For a positive divisor  $M$  of  $N$  consider a pair  $[E, r]_M$ , where  $E$  is a strong Weil curve over  $\mathbb{Q}$  of positive rank and conductor  $M$ . Let  $f_r : X_0(M)/\langle w_r \rangle \rightarrow E$  be the mapping induced by the modular parametrization  $f : X_0(M) \rightarrow E$ . Consider the mapping  $\xi_{E,N,r} := (f_{r,*})^n \circ \delta_{N,M,r} : J_0(N)^{w_r} \rightarrow E^n$ , where  $n$  denotes the number of divisors of  $\frac{N}{M}$ . Then  $\xi_{E,N,r}^\vee : E^n \rightarrow J_0(N)^{w_r}$  has a trivial kernel (where  $\xi_{E,N,r}^\vee$  denotes the dual mapping of  $\xi_{E,N,r}$ ). Hence  $\xi_{E,N,r}^\vee : E^n \rightarrow J_0(N)^{w_r}$  is a maximal  $E$ -isogenous abelian subvariety and  $\xi_{E,N,r} : J_0(N)^{w_r} \rightarrow E^n$  is an optimal  $E$ -isogenous quotient of  $J_0(N)^{w_r}$ .*

*Proof.* Consider the mapping  $\xi_{E,N} := (f_*)^n \circ (\iota_{1,N,M,*}, \dots, \iota_{\frac{N}{M},N,M,*}) : J_0(N) \rightarrow E^n$  (cf. [DeOr24, Definition 3.8]). From the commutative diagram (3.5) we get

$$\xi_{E,N} = \xi_{E,N,r} \circ w_{r,*}^{(N)}. \quad (3.6)$$

Hence  $\xi_{E,N}^\vee = (w_{r,*}^{(N)})^\vee \circ \xi_{E,N,r}^\vee$ . Since  $\xi_{E,N}^\vee$  has trivial kernel (cf. [DeOr24, Proposition 3.9]), we conclude that  $\xi_{E,N,r}^\vee$  has trivial kernel. The remaining statements follows from [DeOr24, Proposition 3.9] and the commutative diagram (3.5).  $\square$

Let  $E$  be a strong Weil curve of conductor  $M$  ( $M|N$ ) and odd  $\mathbb{Q}$ -rank, with  $w_M$  acts as  $+1$  on  $E$ , and  $f_M : X_0(M)/\langle w_M \rangle \rightarrow E$  be the mapping induced by the modular parametrization of  $E$ . Since  $\xi_{E,r,N} : J_0(N)^{w_M} \rightarrow E^n$  is an  $E$ -isogenous optimal quotient when  $N < 408$ , every mapping  $J_0(N)^{w_M} \rightarrow E$  uniquely factors through  $E^n$ . Therefore we get that the maps  $f_M \circ d_i$ , where  $d_i$  runs over the degeneracy maps  $X_0(N)/\langle w_M \rangle \rightarrow X_0(M)/\langle w_M \rangle$ , form a basis for  $\text{Hom}_{\mathbb{Q}}(J_0(N)^{w_M}, E) \cong \text{Hom}_{\mathbb{Q}}(E^n, E) \cong \mathbb{Z}^n$ .

### 3.1 Degree Pairing

We recall the definition of degree pairing from [DeOr24].

**Definition 3.4.** [DeOr24, Definition 2.2] Let  $C, E$  be curves over a field  $k$ , where  $E$  is an elliptic curve. The degree pairing is defined on  $\text{Hom}(C, E)$  as

$$\begin{aligned} \langle -, - \rangle : \text{Hom}(C, E) \times \text{Hom}(C, E) &\rightarrow \text{End}(\text{Jac}(E)) \\ f, g &\rightarrow f_* \circ g^*. \end{aligned}$$

Note that with this notation we have  $\langle f, f \rangle = [\deg(f)]$  for  $f \in \text{Hom}(C, E)$ . If  $P \in C(k)$ , then we can define the degree pairing on  $\text{Hom}(J(C), E)$  as follows

$$\begin{aligned} \langle -, - \rangle : \text{Hom}(J(C), E) \times \text{Hom}(J(C), E) &\rightarrow \text{End}(J(E)) \\ f, g &\rightarrow (f \circ f_P)_* \circ (g \circ g_P)^*, \end{aligned}$$

where the mapping  $f_P$  (similar for  $g_P$ ) is defined as

$$\begin{aligned} f_P : C &\rightarrow J(C) \\ x &\rightarrow [x - P]. \end{aligned}$$

**Proposition 3.5.** *Assume that  $N$  is a square free positive integer and  $M|N$ . Let  $E$  be an elliptic curve of conductor  $M$  with newform  $\sum_{n=1}^{\infty} a_n q^n$  such that  $w_M^{(M)}$  acts as  $+1$  on  $E$ . Suppose that  $f_M : X_0(M)/\langle w_M^{(M)} \rangle \rightarrow E$  be the mapping induced by the modular parametrization  $f : X_0(M) \rightarrow E$ . For any positive divisors  $d_1, d_2$  of  $\frac{N}{M}$  we define  $a := a_{\frac{d_1 d_2}{\gcd(d_1, d_2)^2}}$ , then we have the following equality in  $\text{End}(\text{Jac}(E))$ :*

$$[2]\langle f_M \circ \iota_{d_1, N, M, M}, f_M \circ \iota_{d_2, N, M, M} \rangle = [2][\deg(f_M)] \left[ a \frac{\psi(N)}{\psi\left(\frac{M \cdot \text{lcm}(d_1, d_2)}{\gcd(d_1, d_2)}\right)} \right],$$

where  $\psi(s) = s \prod_{p|s} \left(1 + \frac{1}{p}\right)$ .

*Proof.* By the definition of the pairing, we have

$$\begin{aligned} \langle f_M \circ \iota_{d_1, N, M, M} \circ w_M^{(N)}, f_M \circ \iota_{d_2, N, M, M} \circ w_M^{(N)} \rangle &= (f_M \circ \iota_{d_1, N, M, M} \circ w_M^{(N)})_* \circ (f_M \circ \iota_{d_2, N, M, M} \circ w_M^{(N)})^* \\ &= (f_M)_* \circ (\iota_{d_1, N, M, M})_* \circ (w_M^{(N)})_* \circ (w_M^{(N)})^* \circ (\iota_{d_2, N, M, M})^* \circ (f_M)^* \\ &= (f_M)_* \circ (\iota_{d_1, N, M, M})_* \circ [\deg(w_M^{(N)})] \circ (\iota_{d_2, N, M, M})^* \circ (f_M)^* \\ &= (f_M)_* \circ (\iota_{d_1, N, M, M})_* \circ [2] \circ (\iota_{d_2, N, M, M})^* \circ (f_M)^* \\ &= [2] \circ (f_M)_* \circ (\iota_{d_1, N, M, M})_* \circ (\iota_{d_2, N, M, M})^* \circ (f_M)^* \\ &= [2] \circ \langle f_M \circ \iota_{d_1, N, M, M}, f_M \circ \iota_{d_2, N, M, M} \rangle \end{aligned}$$

On the other hand, from the diagram (3.4) we have the following equalities inside  $\text{End}(E)$ :

$$\begin{aligned} \langle f_M \circ \iota_{d_1, N, M, M} \circ w_M^{(N)}, f_M \circ \iota_{d_2, N, M, M} \circ w_M^{(N)} \rangle &= \langle f_M \circ w_M^{(M)} \circ \iota_{d_1, N, M}, f_M \circ w_M^{(M)} \circ \iota_{d_2, N, M} \rangle \\ &= \langle f \circ \iota_{d_1, N, M}, f \circ \iota_{d_2, N, M} \rangle. \end{aligned}$$

By [DeOr24, Theorem 2.13], we have

$$\langle f \circ \iota_{d_1, N, M}, f \circ \iota_{d_2, N, M} \rangle = \left[ a \frac{\psi(N)}{\psi\left(\frac{M \cdot \text{lcm}(d_1, d_2)}{\gcd(d_1, d_2)}\right)} \deg(f) \right] = \left[ a \frac{\psi(N)}{\psi\left(\frac{M \cdot \text{lcm}(d_1, d_2)}{\gcd(d_1, d_2)}\right)} \deg(f_M) \right] [2]$$

Therefore we conclude that

$$[2] \circ \langle f_M \circ \iota_{d_1, N, M, M}, f_M \circ \iota_{d_2, N, M, M} \rangle = \langle f \circ \iota_{d_1, N, M}, f \circ \iota_{d_2, N, M} \rangle = [2][\deg(f_M)] \left[ a \frac{\psi(N)}{\psi\left(\frac{M \cdot \text{lcm}(d_1, d_2)}{\gcd(d_1, d_2)}\right)} \right].$$

This completes the proof.  $\square$

We now explain via some examples how to use the above recipe to study the triples  $(N, d, E)$ .

## 4 The remaining cases from §2

In this section we give complete explanation the cases  $(222, 37, 37a1)$  and  $(195, 65, 65a1)$ . The arguments for the other cases are similar to these.



#### 4.1 The curve $X_0(222)/\langle w_{37} \rangle$ :

We want to check whether there is a  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(222)/\langle w_{37} \rangle \rightarrow 37a1$ . The modular parametrization  $f : X_0(37) \rightarrow 37a1$  has degree 2 and  $w_{37}$  acts as +1 on  $37a1$ . By Lemma 2.6,  $f$  factors through  $X_0(37)/\langle w_{37} \rangle$  and the induced map  $f_{37} : X_0(37)/\langle w_{37} \rangle \rightarrow 37a1$  has degree 1 (note that  $X_0(37)/\langle w_{37} \rangle \cong 37a1$ ). The degeneracy maps are  $\iota_{1,222,37,37}, \iota_{2,222,37,37}, \iota_{3,222,37,37}$  and  $\iota_{6,222,37,37}$  (note that all these maps has degree 12). Moreover, by Proposition 3.5, we have

$$\begin{aligned} [2]\langle f_{37} \circ \iota_{1,222,37,37}, f_{37} \circ \iota_{2,222,37,37} \rangle &= [2][a_2.\psi(3).deg(f_{37})] = [2][-2.4.1] = [-16], \\ [2]\langle f_{37} \circ \iota_{1,222,37,37}, f_{37} \circ \iota_{3,222,37,37} \rangle &= [2][a_3.\psi(2).deg(f_{37})] = [2][-3.3.1] = [-18], \\ [2]\langle f_{37} \circ \iota_{1,222,37,37}, f_{37} \circ \iota_{6,222,37,37} \rangle &= [2][a_6.\psi(1).deg(f_{37})] = [2][6.1.1] = [12], \\ [2]\langle f_{37} \circ \iota_{2,222,37,37}, f_{37} \circ \iota_{3,222,37,37} \rangle &= [2][a_6.\psi(1).deg(f_{37})] = [2][6.1.1] = [12], \\ [2]\langle f_{37} \circ \iota_{2,222,37,37}, f_{37} \circ \iota_{6,222,37,37} \rangle &= [2][a_3.\psi(2).deg(f_{37})] = [2][-3.3.1] = [-18], \\ [2]\langle f_{37} \circ \iota_{3,222,37,37}, f_{37} \circ \iota_{6,222,37,37} \rangle &= [2][a_2.\psi(3).deg(f_{37})] = [2][-2.4.1] = [-16]. \end{aligned}$$

The maps  $f_{37} \circ \iota_{1,222,37,37}, f_{37} \circ \iota_{2,222,37,37}, f_{37} \circ \iota_{3,222,37,37}$  and  $f_{37} \circ \iota_{6,222,37,37}$  form a basis for  $\text{Hom}_{\mathbb{Q}}(J_0(222)^{w_{37}}, 37a1)$ . If  $\varphi : X_0(222)/\langle w_{37} \rangle \rightarrow 37a1$  is a  $\mathbb{Q}$ -rational mapping, then we can write

$$\varphi = x_1(f_{37} \circ \iota_{1,222,37,37}) + x_2(f_{37} \circ \iota_{2,222,37,37}) + x_3(f_{37} \circ \iota_{3,222,37,37}) + x_4(f_{37} \circ \iota_{6,222,37,37}), \quad (4.1)$$

for some  $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ . From (4.1) we have

$$\begin{aligned} [\deg \varphi] &= [12x_1^2] + [12x_2^2] + [12x_3^2] + [12x_4^2] + [-16x_1x_2] + [-18x_1x_3] + [12x_1x_4] + [12x_2x_3] \\ &\quad + [-18x_2x_4] + [-16x_3x_4], \end{aligned}$$

where “[ $a$ ]” denotes the multiplication by  $a$ -mapping on the elliptic curve  $E = 37a1$ . Hence we must have

$$\deg \varphi = 12x_1^2 + 12x_2^2 + 12x_3^2 + 12x_4^2 - 16x_1x_2 - 18x_1x_3 + 12x_1x_4 + 12x_2x_3 - 18x_2x_4 - 16x_3x_4. \quad (4.2)$$

This shows that  $\deg \varphi$  is of the form  $2g(x_1, x_2, x_3, x_4)$ , so it can not take the value 3. Consequently, the triple  $(222, 37, 37a1)$  is not admissible.

#### 4.2 The curve $X_0(195)/w_{65}$ :

We want to check whether there is a  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(195)/w_{65} \rightarrow 65a1$ . The modular parametrization  $f : X_0(65) \rightarrow 65a1$  has degree 2. By Lemma 2.6,  $f$  factors through  $X_0(65)/\langle w_{65} \rangle$  and the induced map  $f_{65} : X_0(65)/w_{65} \rightarrow 65a1$  has degree 1, since  $X_0(65)/w_{65} \cong 65a1$ . Both the degeneracy maps  $\iota_{1,195,65,65}$  and  $\iota_{3,195,65,65}$  has degree 4. Moreover, we have

$$[2]\langle f_{65} \circ \iota_{1,195,65,65}, f_{65} \circ \iota_{3,195,65,65} \rangle = [2][a_3.\frac{\psi(195)}{\psi(65.3)}.deg(f_{65})] = [2][-2.1.1] = [-4].$$

The maps  $f_{65} \circ \iota_{1,195,65,65}, f_{65} \circ \iota_{3,195,65,65}$  form a basis for  $\text{Hom}_{\mathbb{Q}}(J_0(195)^{w_{65}}, 65a1)$ . If  $\varphi : X_0(195)/w_{65} \rightarrow 65a1$  is  $\mathbb{Q}$ -rational mapping then we can write

$$\varphi = x_1(f_{65} \circ \iota_{1,195,65,65}) + x_2(f_{65} \circ \iota_{3,195,65,65}).$$

Thus we have

$$\begin{aligned} [\deg(\varphi)] &= [x_1^2 \deg(f_{65} \circ \iota_{1,195,65,65})] + [2x_1x_2 \langle f_{65} \circ \iota_{1,195,65,65}, f_{65} \circ \iota_{3,195,65,65} \rangle] + [x_2^2 \deg(f_{65} \circ \iota_{3,195,65,65})], \\ &= [4x_1^2] + [x_1x_2][-4] + [4x_2^2], \end{aligned}$$

where “[ $a$ ]” denotes the multiplication by  $a$ -mapping on the elliptic curve  $E = 65a1$ .

Hence we must have

$$\deg(\varphi) = 4x_1^2 - 4x_1x_2 + 4x_2^2. \quad (4.3)$$

From (4.3), we see that  $\deg(\varphi)$  can not take the value 3. Consequently,  $\text{Hom}_{\mathbb{Q}}(J_0(195)^{w_{65}}, 65a1)$  has no element of order 3. Thus there is no  $\mathbb{Q}$ -rational degree 3-mapping  $X_0(195)/w_{65} \rightarrow 65a1$ .

The quadratic forms for the other remaining cases are given in the following table. It is clear from the following table that the triples  $(N, d, E)$  appearing in the following table are not admissible. Consequently, the curves  $X_0^{+d}(N)$  has finitely many cubic points over  $\mathbb{Q}$  for  $N, d$  appearing in the following table.

N	$d$	E	$g(X_0^{+d}(N))$	Quadratic Form
129	43	43a1	7	$4x_1^2 - 4x_1x_2 + 4x_2^2$
182	91	91b1	10	$6x_1^2 + 6x_2^2$
183	61	61a1	10	$4x_1^2 - 4x_1x_2 + 4x_2^2$
195	65	65a1	9	$4x_1^2 - 4x_1x_2 + 4x_2^2$
215	43	43a1	11	$6x_1^2 - 8x_1x_2 + 6x_2^2$
222	37	37a1	18	$2 \cdot g(x_1, x_2, x_3, x_4)$
237	79	79a1	13	$4x_1^2 - 2x_1x_2 + 4x_2^2$
249	83	83a1	8	$4x_1^2 - 2x_1x_2 + 4x_2^2$
267	89	89a1	9	$4x_1^2 - 2x_1x_2 + 4x_2^2$
273	91	91b1	17	$8x_1^2 - 8x_1x_2 + 8x_2^2$
303	101	101a1	10	$4x_1^2 - 4x_1x_2 + 4x_2^2$
305	61	61a1	12	$6x_1^2 - 6x_1x_2 + 6x_2^2$
395	79	79a1	15	$6x_1^2 - 6x_1x_2 + 6x_2^2$

Table 2: Quadratic forms for remaining cases

## 5 Hyperelliptic cases

In this section we deal with the pairs  $(N, w_d)$  such that  $X_0(N)/\langle w_d \rangle$  is hyperelliptic. We first consider the hyperelliptic curves of genus 2.

**Theorem 5.1.** *The curve  $X_0(N)/\langle w_d \rangle$  has infinitely many cubic points over  $\mathbb{Q}$  for the following pairs of  $(N, w_d)$ :*

$$\begin{aligned} &(30, w_2), (30, w_3), (30, w_{10}), (33, w_3), (35, w_7), (39, w_{13}), (42, w_3), (42, w_6), (42, w_{21}), (57, w_3), \\ &(58, w_{29}), (66, w_{11}), (70, w_{35}), (78, w_{39}), (142, w_{71}). \end{aligned}$$

*Proof.* Using the MAGMA code “`#Points(SimplifiedModel(X0NQuotient(N, [d])):Bound:=10)`”, we see that for the above mentioned pairs of  $(N, w_d)$ , the curve  $X_0(N)/\langle w_d \rangle$  has at least three rational points over  $\mathbb{Q}$ . From [JKS04, Lemma 2.1], we conclude that for each of these cases there is a  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(N)/\langle w_d \rangle \rightarrow \mathbb{P}^1$ . Consequently,  $X_0(N)/\langle w_d \rangle$  has infinitely many cubic points over  $\mathbb{Q}$ .  $\square$

The remaining genus two cases are  $X_0(38)/\langle w_2 \rangle$  and  $X_0(87)/\langle w_{29} \rangle$ .

**Theorem 5.2.** *Both the sets  $\Gamma'_3(X_0(38)/\langle w_2 \rangle, \mathbb{Q})$  and  $\Gamma'_3(X_0(87)/\langle w_{29} \rangle, \mathbb{Q})$  are infinite.*

*Proof.* An affine model of  $X_0(38)/\langle w_2 \rangle$  is given by

$$X_0(38)/\langle w_2 \rangle : y^2 = x^6 - 4x^5 - 6x^4 + 4x^3 - 19x^2 + 4x - 12. \quad (5.1)$$

It is easy to see that  $X_0(38)/\langle w_2 \rangle$  has two  $\mathbb{Q}$ -rational points  $(1 : 1 : 0)$ ,  $(1 : -1 : 0)$  and the hyperelliptic involution permutes these points. From [Jeo21, Lemma 2.2], we conclude that there is a  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(38)/\langle w_2 \rangle \rightarrow \mathbb{P}^1$ . Consequently the set  $\Gamma'_3(X_0(38)/\langle w_2 \rangle, \mathbb{Q})$  is infinite. A similar argument works for the curve  $X_0(87)/\langle w_{29} \rangle$ , which has an affine model  $y^2 = x^6 - 2x^4 - 6x^3 - 11x^2 - 6x - 3$ .  $\square$

Now consider the pairs  $(N, w_d)$  such that  $X_0(N)/\langle w_d \rangle$  is hyperelliptic and  $g(X_0(N)/\langle w_d \rangle) \geq 3$ . If the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is infinite, then  $W_3(X_0(N)/\langle w_d \rangle)$  must contain an elliptic curve with positive  $\mathbb{Q}$ -rank (cf [Jeo21, §2.3]).

Thus, by Cremona tables [Cre] we obtain (because there is no elliptic curve with positive  $\mathbb{Q}$ -rank for levels dividing  $N$ ):

**Theorem 5.3.** *The set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is finite for the following pairs of  $(N, w_d)$ :*

$$(46, w_2), (51, w_3), (55, w_5), (70, w_{14}), (78, w_{26}), (95, w_{19}), (62, w_2), (66, w_6), (69, w_3), (70, w_{10}), \\ (119, w_{17}), (87, w_3), (95, w_5), (78, w_6), (94, w_2), (119, w_7).$$

## 6 Trigonal cases

It is well known that, if  $C/\mathbb{Q}$  is a trigonal curve of genus 3 with a  $\mathbb{Q}$ -rational point, then the projection from the  $\mathbb{Q}$ -rational point defines a degree 3 mapping  $C \rightarrow \mathbb{P}^1$  over  $\mathbb{Q}$ . Consequently, in these cases the set  $\Gamma'_3(C, \mathbb{Q})$  is infinite.

Now consider the pairs  $(N, w_d)$  such that  $X_0(N)/\langle w_d \rangle$  is trigonal and  $g(X_0(N)/\langle w_d \rangle) = 4$ . A model of  $X_0(N)/\langle w_d \rangle$  can be constructed using Petri’s theorem. It is well known that a non hyperelliptic curve  $X$  (defined over  $\mathbb{Q}$ ) of genus 4 lies either on a ruled surface or on a quadratic cone (defined over either  $\mathbb{Q}$ , a quadratic field or a biquadratic field) (cf. [HaSh99, Page 131]). If the ruled surface or the quadratic cone is defined over  $\mathbb{Q}$ , then there is a degree 3 mapping  $X \rightarrow \mathbb{P}^1$  defined over  $\mathbb{Q}$ . For example, consider the curve  $X_0(70)/\langle w_5 \rangle$ . Choosing the following basis of weight 2 cusp forms  $S_2(\langle \Gamma_0(70), w_5 \rangle)$ ,

$$q + q^8 - 2q^9 - q^{10} + q^{11} + O(q^{12}) \\ q^2 + q^6 - 2q^8 - q^{10} + O(q^{12}) \\ q^3 - 3q^5 - 2q^6 - q^7 + 3q^8 - 2q^9 + 3q^{10} + 2q^{11} + O(q^{12}) \\ q^4 - q^5 - q^6 - q^7 + 2q^8 - q^9 + q^{10} + 3q^{11} + O(q^{12}),$$

and using **MAGMA**, a model of  $X_0(70)/\langle w_5 \rangle$  is given by

$$\begin{cases} x^2w + 4xyw - 11xw^2 - y^3 - 3y^2z + 8y^2w - 3yz^2 + 16yzw - 24yw^2 - z^3 + 7z^2w - 9zw^2 + 3w^3, \\ xz + 4xw - y^2 - 4yz + 9yw - 2z^2 + 3zw - w^2. \end{cases}$$

After a suitable coordinate change, the degree 2 homogeneous equation can be written as:

$$-2x^2 + y^2 - 2z^2 + 445w^2 = (y + \sqrt{2}x)(y - \sqrt{2}x) - (\sqrt{2}z + \sqrt{445}w)(\sqrt{2}z - \sqrt{445}w),$$

which is isomorphic to the ruled surface  $uv - st$  over  $\mathbb{Q}(\sqrt{2}, \sqrt{445})$ . Thus  $X_0(70)/\langle w_5 \rangle$  is not trigonal over  $\mathbb{Q}$ . The models of the quadratic surfaces for genus 4 curves are given in Table 8. Since the curves  $X_0(N)/\langle w_d \rangle$  always has a  $\mathbb{Q}$ -rational cusp, from the discussions above we conclude that

**Theorem 6.1.** *Suppose that  $g(X_0(N)/\langle w_d \rangle) \geq 3$ . Then  $X_0(N)/\langle w_d \rangle$  is trigonal over  $\mathbb{Q}$  if and only if  $(N, w_d)$  is in the following list.*

$g(X_0(N)/\langle w_d \rangle)$	$(N, w_d)$
3	$(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (77, w_7), (82, w_{41}),$ $(91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47}),$
4	$(66, w_{33}), (74, w_{37}), (86, w_{43})$

Consequently, for such pairs  $(N, w_d)$  the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is infinite.

Now consider the pairs  $(N, w_d)$  such that  $\text{Gon}(X_0(N)/\langle w_d \rangle) = 3$ , but there is no  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(N)/\langle w_d \rangle \rightarrow \mathbb{P}^1$ . A similar argument as in [Jeo21, p. 352] shows that if the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is infinite, then  $W_3(X_0(N)/\langle w_d \rangle)$  contains a translation of an elliptic curve  $E$  with positive  $\mathbb{Q}$ -rank.

**Theorem 6.2.** *The set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is finite for the following pairs of  $(N, w_d)$ :*

$$(66, w_2), (70, w_5), (74, w_2), (77, w_{11}), (82, w_2), (85, w_5), (85, w_{17}), (91, w_7), (93, w_3), (110, w_{55}),$$

$$(133, w_{19}), (145, w_{29}), (177, w_{59}).$$

*Proof.* For  $N = 66, 70, 85, 93, 110, 133, 177$ , there is no elliptic curve  $E$  of positive  $\mathbb{Q}$ -rank with  $\text{cond}(E) \mid N$ . Hence in these cases, the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is finite. In the remaining cases, the Jacobian decomposition of  $X_0(N)/\langle w_d \rangle$  are given by:

$$\begin{aligned} J_0(74)^{\langle w_2 \rangle} &\sim^{\mathbb{Q}} 37a1 \times 37b1 \times A_{f, \dim=2} \\ J_0(77)^{\langle w_{11} \rangle} &\sim^{\mathbb{Q}} 77a1 \times 77b1 \times A_{f, \dim=2} \\ J_0(82)^{\langle w_2 \rangle} &\sim^{\mathbb{Q}} 82a1 \times A_{f, \dim=3} \\ J_0(91)^{\langle w_7 \rangle} &\sim^{\mathbb{Q}} 91a1 \times A_{f, \dim=3} \\ J_0(145)^{\langle w_{29} \rangle} &\sim^{\mathbb{Q}} 145a1 \times A_{f, \dim=3}. \end{aligned}$$

Note that in these cases,  $X_0(N)/\langle w_d \rangle$  is bielliptic and there are elliptic curves of positive  $\mathbb{Q}$ -rank with  $\text{cond}(E) \mid N$ . By arguments in [Jeo21, Page 353], if there is no  $\mathbb{Q}$ -rational degree 3 mapping  $X_0(N)/\langle w_d \rangle \rightarrow E$  where  $E$  is an elliptic curve of positive  $\mathbb{Q}$ -rank and  $\text{cond}(E) \mid N$ , then  $W_3(X_0(N)/\langle w_d \rangle)$  has no translation of an elliptic curve with positive  $\mathbb{Q}$ -rank. Form

the Jacobian decomposition, we only need to check whether triples (74,2,37a1), (77,11,77a1), (82,2,82a1), (91,7,91a1), (145,29,145a1) are admissible or not.

Since  $w_{37}$  acts as +1 on 37a1 and 37a1 has no non-trivial 2-torsion over  $\mathbb{Q}$ , from Lemma 2.6 we conclude that the triple (74,2,37a1) is not admissible. In the remaining cases, if any of the triple  $(N, d, E)$  is admissible (consequently, there is a  $\mathbb{Q}$ -rational degree 6 mapping  $X_0(N) \rightarrow E$ ), then the degree of the strong Weil parametrization of  $E$  should divide 6. For all the curves 77a1, 82a1, 91a1 and 145a1 the degree of the strong Weil parametrization is 4. Thus none of the triples is admissible. The result follows.  $\square$

## 7 Bielliptic cases

We are now left to discuss the pairs  $(N, w_d)$  such that  $X_0(N)/\langle w_d \rangle$  is bielliptic and  $\text{Gon}(X_0(N)/\langle w_d \rangle) > 3$ . For the benefit of the readers, we recall such pairs  $(N, w_d)$ :

Table 4: Bielliptic remaining cases

$g(X_0(N)/\langle w_d \rangle)$	$(N, w_d)$
5	$(66, w_3), (66, w_{22}), (70, w_2), (70, w_7), (78, w_3), (86, w_2), (105, w_5), (105, w_{21}), (110, w_{11}), (111, w_3), (155, w_{31})$
6	$(78, w_2), (78, w_{13}), (111, w_{37}), (143, w_{13}), (145, w_5), (159, w_{53})$
7	$(105, w_3), (105, w_7), (105, w_{15}), (110, w_{10}), (118, w_2), (123, w_3), (143, w_{11})$
8	$(110, w_2), (110, w_5), (141, w_3), (155, w_5)$
9	$(142, w_2), (159, w_3)$ .

Following [Jeo21, Page 353], for the above pairs  $(N, w_d)$ , if the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is infinite, then  $W_3(X_0(N)/\langle w_d \rangle)$  contains a translation of an elliptic curve  $E$  with positive  $\mathbb{Q}$ -rank, equivalently the triple  $(N, d, E)$  is admissible.

**Theorem 7.1.** *For the pairs  $(N, w_d)$  in Table 4, the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is finite.*

*Proof.* For  $N = 66, 70, 78, 105, 110$ , there is no elliptic curve  $E$  of positive  $\mathbb{Q}$ -rank with  $\text{cond}(E) \mid N$ . Hence for such values of  $N$  and the corresponding values of  $d$  as in Table 4, the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is finite.

For  $N = 118, 123, 141, 142, 143, 145, 155$ , the only elliptic curves  $E$  with positive  $\mathbb{Q}$ -rank has conductor equal to  $N$ . For such  $N, E$  and the corresponding  $d$  as in Table 4, if the triple  $(N, d, E)$  is admissible, then the degree of the strong Weil parametrization of  $E$  should divide 6. From Cremona's table we see that for elliptic curves  $E$  with positive  $\mathbb{Q}$ -rank of conductors 118, 123, 141, 142, 143, 145 and 155, the degree of the strong Weil parametrization of  $E$  does not divide 6. Consequently, for  $N = 118, 123, 141, 142, 143, 145, 155$ , and the corresponding  $d$  as in Table 4, the set  $\Gamma'_3(X_0(N)/\langle w_d \rangle, \mathbb{Q})$  is finite.

Finally, we are left to check whether the triples (86, 2, 43a1), (111, 3, 37a1), (111, 37, 37a1) and (159, 3, 53a1) are admissible or not. Since the elliptic curves 43a1, 37a1 and 53a1 has no non-trivial 2-torsion points, by Lemma 2.6 (1), we conclude that the triples (86, 2, 43a1), (111, 3, 37a1) and (159, 3, 53a1) are not admissible. Consequently, the sets  $\Gamma'_3(X_0(86)/\langle w_2 \rangle, \mathbb{Q})$ ,  $\Gamma'_3(X_0(111)/\langle w_3 \rangle, \mathbb{Q})$  and  $\Gamma'_3(X_0(153)/\langle w_3 \rangle, \mathbb{Q})$  are finite.

A similar argument as in §4, shows that if  $\varphi : X_0(111)/\langle w_{37} \rangle \rightarrow 37a1$  is a  $\mathbb{Q}$ -rational mapping then we must have

$$\deg(\varphi) = 4x_1^2 - 6x_1x_2 + 4x_2^2, \text{ for some } x_1, x_2 \in \mathbb{Z}. \quad (7.1)$$

Thus  $\deg(\varphi)$  can not take the value 3. Consequently, the triple  $(111, 37, 37a1)$  is not admissible. This completes the proof.  $\square$

## A Appendix

Let  $N \leq 623$ . Suppose that  $\text{Gon}(X_0^{+d}(N)) \geq 4$  and  $X_0^{+d}(N)$  has no degree  $\leq 2$  map to an elliptic curve. After applying Lemma 2.3 (2 and 3), we are left to check the existence of  $\mathbb{Q}$ -rational degree 3 mapping  $X_0^{+d}(N) \rightarrow E$  where  $E$  is an elliptic curve with positive  $\mathbb{Q}$ -rank, only for the following values of  $N$ :

106, 114, 122, 129, 130, 154, 158, 159, 166, 174, 178, 182, 183, 185, 195, 202, 215, 222, 231, 237, 246, 249, 258, 259, 262, 265, 267, 273, 282, 285, 286, 301, 303, 305, 326, 371, 393, 395, 407, 415, 427, 445, 473, 481.

Table 5: Hyperelliptic curve  $X_0(N)/w_d$

$g(X_0(N)/\langle w_d \rangle)$	$(N, w_d)$
2	$(30, w_2), (30, w_3), (30, w_{10}), (33, w_3), (35, w_7), (38, w_2), (39, w_{13}), (42, w_3), (42, w_6), (42, w_{21}), (57, w_3), (58, w_{29}), (66, w_{11}), (70, w_{35}), (78, w_{39}), (87, w_{29}), (142, w_{71})$
3	$(46, w_2), (51, w_3), (55, w_5), (70, w_{14}), (78, w_{26}), (95, w_{19})$
4	$(62, w_2), (66, w_6), (69, w_3), (70, w_{10}), (119, w_{17})$
5	$(87, w_3), (95, w_5)$
6	$(78, w_6), (94, w_2), (119, w_7).$

Table 6:  $X_0(N)/\langle w_d \rangle$  with  $\text{Gon}(X_0(N)/\langle w_d \rangle) = 3$

$g(X_0(N)/\langle w_d \rangle)$	$(N, w_d)$
3	$(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (77, w_7), (82, w_{41}), (91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47}),$
4	$(66, w_2), (66, w_{33}), (70, w_5), (74, w_2), (74, w_{37}), (77, w_{11}), (82, w_2), (85, w_5), (85, w_{17}), (86, w_{43}), (91, w_7), (93, w_3), (110, w_{55}), (133, w_{19}), (145, w_{29}), (177, w_{59})$

Table 7: Bielliptic curve  $X_0(N)/\langle w_d \rangle$

$g(X_0(N)/\langle w_d \rangle)$	$(N, w_d)$
2	$(30, w_2), (30, w_3), (30, w_{10}), (42, w_3), (42, w_6), (42, w_{21}), (57, w_3), (58, w_{29}), (66, w_{11}), (70, w_{35}), (78, w_{39}), (142, w_{71})$



$X_0(82)/\langle w_2 \rangle$	$\begin{cases} x^2w + 4xyw - 12xw^2 - 8y^3 - 24y^2z + 24y^2w - 44yz^2 + 68yzw - 8yw^2 \\ \quad -40z^3 + 128z^2w - 108zw^2 + 31w^3, \\ xz + xw - 2y^2 - 4yz - 8z^2 + 9zw - 3w^2. \end{cases}$ <p>Diagonal form <math>6x^2 - 24y^2 - 8z^2 - 18w^2</math>, lies over a ruled surface over <math>\mathbb{Q}(i)</math>.</p>
$X_0(85)/\langle w_5 \rangle$	$\begin{cases} 72x^2w + 4xyw - 28xw^2 - 18y^3 + 25y^2z + 75y^2w - 81yz^2 - 58yzw \\ \quad -107yw^2 + 81z^3 + 252z^2w + 144zw^2 + 71w^3, \\ 18xz + 2xw - 9y^2 - yz + 15yw - 18z^2 - 27zw - 16w^2 \end{cases}$ <p>Diagonal form: <math>1886652x^2 - 11646y^2 - 18z^2 - 14623740w^2</math>, lies on a ruled surface over <math>\mathbb{Q}(\sqrt{2}, \sqrt{-5015})</math>.</p>
$X_0(85)/\langle w_{17} \rangle$	$\begin{cases} 18x^2w - 6xyw + 40xw^2 - 9y^3 + 3y^2z + 12y^2w - 15yz^2 - 32yzw \\ \quad -118yw^2 + 21z^3 + 80z^2w + 202zw^2 + 212w^3, \\ 3xz - 2xw - 3y^2 + yz - yw - z^2 + 13zw + 2w^2 \end{cases}$ <p>Diagonal form: <math>297x^2 - 11y^2 - z^2 + 29835w^2</math> lies on ruled surface over <math>\mathbb{Q}(\sqrt{3}, \sqrt{1105})</math>.</p>
$X_0(86)/\langle w_{43} \rangle$	$\begin{cases} x^2z - xy^2 + y^2z - 2y^2w + 5yz^2 + 3yzw + 4yw^2 + 4z^3 + 4z^2w + 2zw^2 - 2w^3, \\ xw - yz - z^2 - zw \end{cases}$ <p>Diagonal form: <math>-x^2 + 3y^2 - 3z^2 + w^2</math>, lies over a ruled surface over <math>\mathbb{Q}</math>.</p>
$X_0(91)/\langle w_7 \rangle$	$\begin{cases} 72x^2w - 60xyw + 52xw^2 - 18y^3 + 57y^2z - 93y^2w + 75yz^2 - 146yzw \\ \quad +25yw^2 + 48z^3 - 109z^2w + 81zw^2 - 8w^3, \\ 6xz - 10xw - 3y^2 + 5yz - 7yw + 4z^2 - 3zw - 4w^2 \end{cases}$ <p>Diagonal form: <math>-7884x^2 - 292y^2 + 4z^2 + 6588w^2</math>, lies on a ruled surface over <math>\mathbb{Q}(\sqrt{-3}, \sqrt{61})</math>.</p>
$X_0(93)/\langle w_3 \rangle$	$\begin{cases} 4500x^2w - 1050xyw + 25xw^2 - 180y^3 + 30y^2z + 270y^2w - 96yz^2 \\ \quad +319yzw - 274yw^2 + 30z^3 + 563z^2w - 412zw^2 + 282w^3, \\ 30xz - 35xw - 6y^2 + 7yz + 2yw - z^2 + 8zw - 6w^2 \end{cases}$ <p>Diagonal form: <math>-5400x^2 + 25y^2 - z^2 + 1171800w^2</math> lies on a ruled surface over <math>\mathbb{Q}(\sqrt{6}, \sqrt{217})</math>.</p>
$X_0(110)/\langle w_{55} \rangle$	$\begin{cases} x^2w - xyw + xw^2 - y^3 + y^2w + 3yzw + yw^2 + z^2w + zw^2 - w^3, \\ xz - xw - y^2 + yz + 2yw + 2zw - w^2 \end{cases}$ <p>Diagonal form: <math>-x^2 - 3y^2 + 3z^2 + 13w^2</math>, lies on a ruled surface over <math>\mathbb{Q}(\sqrt{13})</math></p>
$X_0(133)/\langle w_{19} \rangle$	$\begin{cases} 54x^2w - 9xyw - 18xw^2 - 6y^3 - 3y^2z + 3y^2w - 4yz^2 + 10yzw - 7yw^2 \\ \quad -4z^3 + 16z^2w - 16zw^2 + 12w^3, \\ 6xz - 3xw - 2y^2 + yz - yw + 2zw - 3w^2 \end{cases}$ <p>Diagonal form: <math>-6x^2 - 282y^2 + 3384z^2 - 504w^2</math>, lies on a ruled surface over <math>\mathbb{Q}(\sqrt{3}, \sqrt{-7})</math>.</p>
$X_0(145)/\langle w_{29} \rangle$	$\begin{cases} x^2w - 2xyw - xw^2 - y^3 + y^2z - 2y^2w + 5yz^2 - 4yzw + 2z^3 - 2z^2w, \\ xz - 2xw - y^2 + 2yz - yw + 2z^2 - 3zw + w^2 \end{cases}$ <p>Diagonal form: <math>-3x^2 - 6y^2 + 2z^2 + 21w^2</math>, lies on a ruled surface over <math>\mathbb{Q}(\sqrt{3}, \sqrt{7})</math>.</p>



$X_0(177)/\langle w_{59} \rangle$	$\begin{cases} x^2w - xw^2 - y^3 - y^2z - yz^2 + w^3, \\ xz - y^2 - yw - zw - w^2 \end{cases}$ Diagonal form: $-x^2 - y^2 + z^2 - 3w^2$ , lies on a ruled surface over $\mathbb{Q}(\sqrt{-3})$ .
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