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ON JANNSEN'S CONJECTURE FOR HECKE CHARACTERS OF IMAGINARY QUADRATIC FIELDS

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Abstract

We present a collection of results on a conjecture of Jannsen about the p-adic realizations associated to Hecke characters over an imaginary quadratic field K of class number 1.

The conjecture is easy to check for Galois groups purely of local type (Section 1). In Section 2 we define the p-adic realizations associated to Hecke characters over K. We prove the conjecture under a geometric regularity condition for the imaginary quadratic field K at p, which is related to the property that a global Galois group is purely of local type. Without this regularity assumption at p, we present a review of the known situations in the critical case (Section 3) and in the non-critical case (Section 4) for these realizations. We relate the conjecture to the non-vanishing of some concrete non-critical values of the associated p-adic L-function of the Hecke character.

Finally, in Section 5 we prove that the conjecture follows from a general conjecture on Iwasawa theory for almost all Tate twists.

1. The Jannsen conjecture on local type Galois groups

Jannsen's conjecture [**9**] predicts the vanishing of a second Galois cohomology group for the p-adic realization of almost all Tate twists of a pure Chow motive. It also specifies the Tate twists where this cohomology group could not vanish. Without this specification the conjecture is a generalization of the classical weak Leopoldt conjecture.

We refer to Jannsen's original paper [**9**] and Perrin-Riou's paper [**14**, Appendix B] for the relations with other conjectures and for general results.

Let F be a number field with algebraic closure $\overline{F}$. Let X be a smooth, projective variety of pure dimension d over F. Let p be a prime number,

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and S a finite set of places of F, containing all places above ∞ and p, and all primes where X has bad reduction. Let G_S be the Galois group over F of the maximal S-ramified (unramified outside S) extension of F, that we call F_S.

Conjecture 1.1 (Jannsen)**.** *If $\overline{X} = X \times_F \overline{F}$, then*

$$H^2(G_S, H^i_{et}(\overline{X}, \mathbb{Q}_p(n))) = 0 \text{ if } \begin{cases} a) & i+1 < n, \quad \text{or} \\ b) & i+1 > 2n. \end{cases}$$

This conjecture can be verified also from the étale cohomology with $\mathbb{Z}_p$ or $\mathbb{Q}_p/\mathbb{Z}_p$ coefficients.

Lemma 1.2 ([**9**, Lemma 1])**.** *The following statements are equivalent:*

1. $H^2(G_S, H^i_{et}(\overline{X}, \mathbb{Q}_p(n))) = 0$.
2. $H^2(G_S, H^i_{et}(\overline{X}, \mathbb{Z}_p(n)))$ *is finite.*
3. $H^2(G_S, H^i_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ *is finite.*
4. *(If $p \neq 2$ or F is totally imaginary)* $H^2(G_S, \tilde{H}^i_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) = 0$ *where* $\tilde{H}^i_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ *is the p-divisible part of the group* $H^i_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))$.

There is an analogous conjecture if one replaces G_S by $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$, the absolute Galois group of the local field $F_{\mathfrak{p}}$, where $F_{\mathfrak{p}}$ is the completion at $\mathfrak{p}$ of F.

Conjecture 1.3 (Jannsen)**.** $H^2(\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}}), H^i_{et}(\overline{X}, \mathbb{Q}_l(n))) = 0$ *if* $i+1 < n$ *or* $i+1 > 2n$.

Lemma 1.2 is also valid in this situation, i.e. replacing G_S by $\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})$.

Denote by $G_S(p)$ the Galois group $\mathrm{Gal}(F_S(p)/F)$, where $F_S(p)$ means the maximal p-extension of F inside F_S. Let S_p be the set of primes of F above p.

Let us denote by $\mathcal{G}_v$ the Galois group of the maximal p-extension of the local field F_v (with $v \in S$) over F_v; we write also $\mathcal{G}_v = \mathrm{Gal}(F_v(p)/F_v)$. We have the natural surjective map

$$i_{v,S} \colon \mathcal{G}_v \to G_v$$

where G_v denotes the p-part of the decomposition group at F_v of G_S. Let $\varphi_v \colon G_v \to G_S(p)$ be the natural inclusion.

Definition 1.4. The Galois group $G_S(p)$ is purely of local type iff the map $\varphi_v \circ i_{v,S}$ is an isomorphism.

It is easy to see that this definition is equivalent to the one given in [**18**] ([**2**, B.2.6]).

Lemma 1.5. *Let $M(j)$ be a p-primary divisible $\mathcal{G}_v = \mathrm{Gal}(F_v(p)/F_v)$-module of cofinite type. If $\mu_p \notin F_v$ then $H^2(\mathcal{G}_v, M(j)) = 0$, where μ_p denotes a primitive p-root of unity. If $\mu_p \in F_v$, then $H^2(\mathcal{G}_v, M(j)) = M(j-1)_{\mathcal{G}_v}$.*

Finally, let $M(j)$ be any p-primary divisible $\mathrm{Gal}(\overline{F}_v/F_v)$-module of cofinite type. Then $H^2(\mathrm{Gal}(\overline{F}_v/F_v), M(j)) = M(j-1)_{\mathrm{Gal}(\overline{F}_v/F_v)}$.

Proof: If $\mu_p \notin F_v$ then $\mathcal{G}_v$ is free, hence $H^2(\mathcal{G}_v, M(j)) = 0$. Otherwise it is a Poincaré group of dimension two with dualizing module $\mathbb{Q}_p/\mathbb{Z}_p(1)$. Using local Tate duality we get $H^2(\mathcal{G}_v, M(j)) = M(j-1)_{\mathcal{G}_v}$.

The last statement follows from local Tate duality. □

Lemma 1.6. *Let S' be a subset of S which contains S_p, the places of F above p. Suppose that the Galois group $\mathrm{Gal}(F_{S'}(p)/F)$ is purely of local type for a place $w \in S'$ of F. Furthermore let M be a p-primary divisible $\mathrm{Gal}(F_{S'}(p)/F)$-module of cofinite type such that $M(j-1)_{\mathrm{Gal}(\overline{F}_v/F_v)} = 0$ for all $v \in S \setminus (S' \setminus \{w\})$. Then*

$$H^2(G_S, M(j) = 0.$$

Proof: Let us consider part of Soulé's exact sequence recalled in [**9**, §3, (13)]:

$$H^2(G_{S'}, M(j)) \to H^2(G_S, M(j)) \to \oplus_{v \in S \setminus S'} M(j-1)_{\mathrm{Gal}(\overline{F}_v/F_v)} \to 0,$$

where the last term of this sequence is zero by hypothesis. Observe that

$$H^2(G_{S'}, M(j)) = H^2(G_{S'}(p), M(j))$$

by a result of Neumann [**13**, Theorem 1]. Since $G_{S'}(p)$ is purely of local type,

$$H^2(G_{S'}(p), M(j)) = H^2(\mathcal{G}_w, M(j)).$$

If $\mu_p \notin F_w$ then $H^2(\mathcal{G}_w, M(j)) = 0$. Otherwise $H^2(\mathcal{G}_w, M(j)) = M(j-1)_{\mathrm{Gal}(F_w(p)/F_w)}$. But, in our case, the action of $\mathrm{Gal}(\overline{F}_w/F_w)$ factors through the pro-p-quotient $\mathcal{G}_w$, and so $M(j-1)_{\mathrm{Gal}(F_w(p)/F_w)} = 0$ by hypothesis. □

Remark 1.7. If X has good reduction at $\mathfrak{p}$ and $\mathfrak{p} \nmid p$, the vanishing of the Galois group $H^2(F_{\mathfrak{p}}, H^i_{et}(\overline{X}, \mathbb{Q}_p(j)))$ with $i \neq 2(j-1)$ follows from Lemma 1.5 and weights arguments (see [**9**, Lemma 3]). As usual $H^i(F_v, -)$ means $H^i(\mathrm{Gal}(\overline{F_v}/F_v), -)$. This result can also be proved when X has potentially good reduction at $\mathfrak{p}$ [**9**, Lemma 12].

When $\mathfrak{p} \mid p$, Soulé proves the vanishing of this group for $j > i+1$ or $j < 1$ [**9**, Corollary 65, Lemma 11]. Moreover if $\mathfrak{p}$ is unramified in $F/\mathbb{Q}$ and X has good reduction at $\mathfrak{p}$ then $H^2(F_{\mathfrak{p}}, H^i_{et}(\overline{X}, \mathbb{Q}_p(j))) = 0$ if $i \neq 2(j-1)$ [**9**, Corollary 6]. In this situation if $i-2j \neq 2$ and $i-2j \neq 0$ we obtain:

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^1(F_{\mathfrak{p}}, H^i_{et}(\overline{X}, \mathbb{Q}_p(j))) &= -\chi(H^i_{et}(\overline{X}, \mathbb{Q}_p(j))) \\ &= [F_{\mathfrak{p}} : \mathbb{Q}_p] \dim_{\mathbb{Q}_p}(H^i_{et}(\overline{X}, \mathbb{Q}_p(j))),\end{aligned}$$

where χ is the Euler characteristic in local Galois cohomology. The last equality follows from [**12**, 7.3.8].

2. The geometric regularity condition

We keep once and for all the notations of Section 1. Let $\mathfrak{p}$ be a prime of F such that $\mathfrak{p} \mid p$ and the inertia group at $\mathfrak{p}$ acts trivially on M, or on $\overline{X}$.

Let E be a fixed elliptic curve defined over an imaginary quadratic field K, and with CM by $\mathcal{O}_K$, the ring of integers of K. This hypothesis implies that $cl(K) = 1$. Associated to this elliptic curve there is a Hecke character φ of the imaginary quadratic field K with conductor $\mathfrak{f}$, an ideal of $\mathcal{O}_K$ which coincides with the conductor of the elliptic curve E.

Consider the category of Chow motives $\mathcal{M}_{\mathbb{Q}}(K)$ over K with morphisms induced by graded correspondences in Chow theory tensored with $\mathbb{Q}$. Then, the motive of the elliptic curve E has a canonical decomposition $h(E)_{\mathbb{Q}} = h^0(E)_{\mathbb{Q}} \oplus h^1(E)_{\mathbb{Q}} \oplus h^2(E)_{\mathbb{Q}}$. The motive $h^1(E)_{\mathbb{Q}}$ has a multiplication by K [**5**]. For w a strictly positive integer, let us consider the motive $\otimes^w_{\mathbb{Q}} h^1(E)_{\mathbb{Q}}$, which has multiplication by $T_w := \otimes^w_{\mathbb{Q}} K$. Observe that T_w has a decomposition $\prod_\theta T_\theta$ as a product of fields T_θ, where θ runs through the $\mathrm{Aut}(\mathbb{C})$-orbits of $\Upsilon^w = \mathrm{Hom}(T_w, \mathbb{C})$, where $\Upsilon = \mathrm{Hom}(K, \mathbb{C})$. This decomposition defines some idempotents e_θ and gives also a decomposition of the motive and its realizations. Let us fix once and for all an immersion $\lambda\colon K \to \mathbb{C}$ as in [**5**, p. 135].

The L-function associated to the motive $e_\theta(\otimes^w h^1(E)_{\mathbb{Q}})$ corresponds to the L-function associated to the CM-character $\psi_\theta = e_\theta(\otimes^w \varphi)\colon \mathbb{A}^*_K \to K^*$ [**5**, §1.3.1] (for equivalent definitions of Hecke characters over an imaginary quadratic field, see [**7**, §2.2]). With respect to the fixed embedding λ, this CM-character corresponds to $\varphi^a \overline{\varphi}^b$, where $a, b \geq 0$ are integers such that $w = a+b$. The pair (a, b) is the infinite type of ψ_θ. We note that there are different θ with the same infinite type. Every θ gives two elements of Υ^w, one given by the infinite type $\vartheta \in \theta \cap \mathrm{Hom}_K(T_w, \mathbb{C})$ and the other one coming from the other embedding. When θ corresponds to $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \mathrm{Hom}_K(T_w, \mathbb{C})$ with type (a, b), we denote by $\overline{\theta}$ the one that corresponds to the element $\overline{\vartheta} = (\overline{\lambda_1}, \ldots, \overline{\lambda_w}) \in \mathrm{Hom}_K(T_w, \mathbb{C})$, where λ_i are $\mathbb{Q}$-immersions of K into $\mathbb{C}$ and the bar denotes complex conjugation. Observe that $\overline{\theta}$ has type (b, a).

Let us denote by $\otimes^w h^1(E)$ the integer Chow motive (similar to $\otimes^w h^1(E)_{\mathbb{Q}}$ but without tensoring by $\mathbb{Q}$ the correspondences). The Chow motive

$$M_\theta := e_\theta(\otimes^w h^1(E)),$$

is a Chow motive with coefficients in $\mathcal{O}_K[1/DK]$ where D_K is the discriminant of K.

The p-adic realization of the motive $M_\theta(n)$:

$$M_{\theta\mathbb{Q}_p}(n) := H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Q}_p(n)) \cong e_\theta(\otimes^w(T_pE \otimes \mathbb{Q}_p))(n - w),$$

is unramified outside any set S which contains the finite primes of K which divide $p\mathfrak{f}_\theta$, where $\mathfrak{f}_\theta$ is the conductor of ψ_θ. We have also a natural lattice associated to it, corresponds for $p \nmid D_K$ to $H^w_{ét}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(n)) = e_\theta(\otimes^w T_pE)(n - w)$ and we denote it by

$$M_{\theta\mathbb{Z}_p}(n) := H^w_{ét}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(n)).$$

Since $e_\theta(\otimes^w h^1(E)) \subseteq h^w(E^w)$ Jannsen's conjecture for $h^w(E^w)$ implies the following conjecture for Hecke characters.

Conjecture 2.1. *Let p be a prime such that E has good reduction at the primes over p in K. Let S be a set of primes of K which contains the primes of K over p and the primes of $\mathfrak{f}$. Then,*

$$H^2(\mathrm{Gal}(K_S/K), M_{\theta\mathbb{Q}_p}(n)) = 0,$$

for $n > w + 1$ or $w + 1 > 2n$.

Consider $\underline{F} \subseteq K(E[p])$ with $K \subseteq \underline{F}$. We suppose also $p > 3$. Denote by $\mathcal{F} = K(E[p])$. We know that $E \times_K \mathcal{F}$ has good reduction everywhere.

We introduce the notion of regularity for $\mathfrak{p}$ at $\underline{F}$, which turns out to be closely related to the condition that a global Galois group is purely of local type.

Let us assume in this section once for all that p splits in K with $p = \mathfrak{p}\mathfrak{p}^*$ where $\mathfrak{p}$, $\mathfrak{p}^*$ are different primes of K.

Definition 2.2. Suppose that $p > 3$, $(p) = \mathfrak{p}\mathfrak{p}^*$ in K, $\mathfrak{p} \neq \mathfrak{p}^*$ and E has good reduction at $\mathfrak{p}$ and $\mathfrak{p}^*$. Let $S_\mathfrak{p}$ be the set of primes of $\underline{F}$ that divide $\mathfrak{p}$. The prime $\mathfrak{p}$ is called regular for E and $\underline{F}$ if $\underline{F}_{S_\mathfrak{p}}(p)$ is a $\mathbb{Z}_p$-extension of $\underline{F}$. We say that p is regular for E and $\underline{F}$ if $\mathfrak{p}$ and $\mathfrak{p}^*$ are regular for E and $\underline{F}$.

Remark 2.3. The above regularity condition admits an analogue of Kummer's criterion as in the classical notion of regularity at p [**2**, B.1.2, 1.7, 2.16–17]. One of the equivalent notions of regularity is related to a critical value of the L-function associated to some concrete Hecke characters of the form $\varphi^a\overline{\varphi}^b$ [**17**] ([**2**, B.1.7]).

Proposition 2.4 (Wingberg [**17**])**.** *Let $\underline{F}$ be a number field between K and $\mathcal{F}$. The prime $\mathfrak{p}$ is regular for E and $\underline{F}$ if and only if* $\mathrm{Gal}(\underline{F}_{S_p}(p)/\underline{F})$ *is purely of local type with respect to $\mathfrak{p}^*$, where S_p denotes the places of $\underline{F}$ above p.*

Remark 2.5. Thus, for this notion of regular primes the Grunwald-Wang theorem holds: the maximal p-extension of $\underline{F}_{\mathfrak{p}^*}$ coincides with the completion at $\mathfrak{p}^*$ of the maximal p-extension of $\underline{F}_{S_p}$ (use [**2**, Corollary B.2.6] with Proposition 2.4).

Let us write a consequence of Proposition 2.4 (the case $w = 1$ is a result of Wingberg [**17**]).

Corollary 2.6. *Let p be a regular prime for E and $\mathcal{F} = K(E[p])$. Let $\underline{F}$ be an extension of K inside $\mathcal{F}$. Denote by S^* a finite set of primes of $\underline{F}$ containing the primes above p and those where $E \times_K \underline{F}$ has bad reduction. Then,*

$$H^2(\mathrm{Gal}(\underline{F}_{S^*}/\underline{F}), H^w(M_\theta, \mathbb{Q}_p/\mathbb{Z}_p(j+w))) = 0$$

for any integer $w = a + b$ with $w + 2(j-1) \neq 0$.

Proof: By Kummer theory, we have that

$$H^1_{et}(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(1)) = E[p^\infty] = E[\mathfrak{p}^\infty] \oplus E[(\mathfrak{p}^*)^\infty]$$

where $E[a^\infty] = \varinjlim_n E[a^n]$ is the inductive limit of the a^n-torsion points of the elliptic curve E. Thus $M(w+j) := H^w(M_\theta \otimes_K \overline{K}, \mathbb{Q}_p/\mathbb{Z}_p(w+j)) \cong e_\theta(\otimes^w E[p^\infty])(j)$. Since $K \subseteq \underline{F} \subseteq \mathcal{F}$, we need only to prove

$$H^2(\mathrm{Gal}(\mathcal{F}_{S^*}/\underline{F}), M(w+j)) = 0,$$

because $\underline{F} \subseteq \mathcal{F}$ is unramified outside S^* [**6**, II.1.8]. Since $\mathcal{F}/\underline{F}$ is prime to p it is enough to show

$$H^2(\mathrm{Gal}(\mathcal{F}_{S^*}/\mathcal{F}), M(w+j)) = 0.$$

Now, $M(w+j)$ is a $\mathrm{Gal}(\mathcal{F}_{S_p}(p)/\mathcal{F})$-module because the Galois action on $M(w+j)$ factors through $\mathrm{Gal}(\mathcal{F}(E[p^\infty])/\mathcal{F})$ and $E \times_K \mathcal{F}$ has good reduction [**6**, 1.9, (i), (ii)].

Using Lemma 1.6 it is enough to prove the vanishing of some coinvariant modules in local Galois groups.

Since $E \times_K \mathcal{F}$ has good reduction everywhere, in particular in the places of $S^* \setminus S_p$, the same is true for $E^w = E \times_K E \times_K \cdots \times_K E$ over the field $\mathcal{F}$. Then, by the proved Weil conjectures on $H^w(\overline{E}^w, \mathbb{Q}_p(n))$, the Frobenius at v over $\mathcal{F}$ does not act as identity in any subspace of the above cohomology group for $w - 2n \neq 0$. By local Tate duality $M(w+j-1)_{\mathrm{Gal}(\overline{\mathcal{F}_v}/\mathcal{F}_v)}$ is dual to $M_{\theta\mathbb{Z}_p}^{\mathrm{Gal}(\overline{\mathcal{F}_v}/\mathcal{F}_v)}$ and this module vanishes for $w - 2(w+j-1) \neq 0$.

By hypothesis $\mathfrak{p}^*$ is regular in $\mathcal{F}$, hence (Theorem 2.4)

$$\mathrm{Gal}(\mathcal{F}_{S_p}(p)/\mathcal{F}) \cong \mathrm{Gal}(\mathcal{F}_{\mathfrak{p}}(p)/\mathcal{F}_{\mathfrak{p}}).$$

Thus, it remains only to show that

$$M(w+j-1)_{\mathrm{Gal}(\mathcal{F}_{\mathfrak{p}}(p)/\mathcal{F}_{\mathfrak{p}})} = M(w+j-1)_{\mathrm{Gal}(\overline{\mathcal{F}_{\mathfrak{p}}}/\mathcal{F}_{\mathfrak{p}})} = 0.$$

The $\mathrm{Gal}(\overline{\mathcal{F}_{\mathfrak{p}}}/\mathcal{F}_{\mathfrak{p}})$-action on $M(w+j-1)$ factors though the pro-p-quotient by the regularity and the condition $\mu_p \subseteq \mathcal{F}_{\mathfrak{p}}$. By [**7**, proof of Lemma 3.2.5] these coinvariants are always zero unless $a-(j-1+w)=0$ and $b-(j-1+w)=0$. If both are zero, we have $0 = a-(j-1+w)+b-(j-1+w) = w-2j+2-2w = -(w+2j-2)$ which is impossible by our hypothesis. □

Remark 2.7. The above result, and Corollaries 2.8 and 2.10 below, are true assuming only that S^* is a finite set of places which contains the primes above p and the primes of $\underline{F}$ such that the inertia acts nontrivially in $M_\theta \times_K \underline{F}$ (use standard arguments like in the proof of [**11**, 2.2.16]).

Corollary 2.8 (Jannsen's conjecture under regularity)**.** *Let p be a regular prime for E and $\mathcal{F} = K(E[p])$. Let $\underline{F}$ be an extension of K inside $\mathcal{F}$. Let S^* be a finite set of primes containing the primes of $\underline{F}$ that are over p and the primes where the elliptic curve $E \times_K \underline{F}$ has bad reduction. Let M_θ be $e_\theta(\otimes^w h^1(E))$. Then*

$$H^2(\mathrm{Gal}(\underline{F}_{S^*}/\underline{F}), M_{\theta\mathbb{Q}_p}(n)) = 0$$

if $w + 2(n-w-1) \neq 0$.

Proof: Is a direct consequence of the above Corollary 2.6 and Lemma 1.2. □

Corollary 2.9. *Let p be a regular prime for E and $\mathcal{F} = K(E[p])$. Let $\underline{F}$ be an extension of K inside $\mathcal{F}$. Let S^* be a finite set of primes of $\underline{F}$ containing the primes above p and the primes where the elliptic curve $E \times_K \underline{F}$ has bad reduction. Fix an integer $w \geq 1$. Then*

$$H^2(\mathrm{Gal}(\underline{F}_{S^*}/\underline{F}), (\otimes^w h^1(E)(1))_{et}(j)) = 0$$

for any integer j such that $w + 2(j-1) \neq 0$, where $(\otimes^w h^1(E)(1))_{et}(j))$ denotes the étale cohomology group $(\otimes^w H^1_{et}(E \times_K \overline{K}, \mathbb{Q}_p(1)))(j)$ which is the p-adic realization of the motive $(\otimes^w h^1(E))(w+j)$.

Proof: The integer idempotents e_θ give us a decomposition of the above motive

$$(\otimes^w h^1(E)(1))(j) = \prod_\theta e_\theta((\otimes^w h^1(E)(1))(j)),$$

thus a decomposition of the p-adic realization

$$H^2(\mathrm{Gal}(\underline{F}_{S^*}/\underline{F}), (\otimes^w h^1(E)(1))_{et}(j)) \\ = \oplus_\theta H^2(\mathrm{Gal}(\underline{F}_{S^*}/\underline{F}), e_\theta(\otimes^w h^1(E)(1))_{et}(j))$$

which is zero by Corollary 2.8. □

Corollary 2.10. *Let p be a regular prime for E and $\mathcal{F}$ and let be $w - 2n \neq 0$ and $2n - w - 2 \neq 0$. Let S be a finite set of primes of K containing the primes over p and those where E has bad reduction. Then*

$$\dim_{\mathbb{Q}_p} H^1(\mathrm{Gal}(K_S/K), M_{\theta\mathbb{Q}_p}(n)) \\ + \dim_{\mathbb{Q}_p} H^1(\mathrm{Gal}(K_S/K), M_{\overline{\theta}\mathbb{Q}_p}(w+1-n)) = [K:\mathbb{Q}] \dim_{\mathbb{Q}_p} M_{\theta\mathbb{Q}_p} = 4.$$

Moreover,

$$\dim_{\mathbb{Q}_p} H^1(\mathrm{Gal}(K_S/K), M_{\theta\mathbb{Q}_p}(w+l+1)) = 2 \\ = \dim_{\mathbb{Q}_p} H^1(\mathrm{Gal}(K_S/K), M_{\overline{\theta}\mathbb{Q}_p}(-l))$$

for $-l \leq \min(a,b)$ if $a \neq b$ (and $-l < a$ if $a = b$).

Proof: By the conditions on the weights and the regularity assumption we obtain $H^0(G_S, M_{\theta\mathbb{Q}_p}(m)) = 0$ and $H^2(G_S, M_{\theta\mathbb{Q}_p}(m)) = 0$, with $m = n$ and $m = w + 1 - n$ (and the same statements for $\overline{\theta}$ instead of θ). Thus, the first equality holds by [**9**, Corollary 1, proof of Lemma 2]. Observe that $M_{\theta\mathbb{Q}_p} \cong e_\theta(\otimes^w T_pE(-1)) \otimes \mathbb{Q}_p$ has $\mathbb{Q}_p$-rank equal to 2 (see for example [**1**, §2]); therefore the second equality follows. The last statement follows from the Beilinson conjecture for these characters (proved by Deninger [**5**]) and [**9**, Lemma 2] at the twist $w + l + 1$. □

Observe $M_\theta(n)$ is a submotive of $h^w(E^w)(n)$, therefore using Soulé's result (see Remark 1.7) we obtain, without the regularity assumption:

Corollary 2.11. *Let $\mathfrak{p}$ be a prime of K such that $\mathfrak{p} \mid p$. Suppose $w-2n \neq -2$ if $\mathfrak{p}$ is unramified, otherwise $n > w+1$ or $n < 1$. Then,*

$$H^2(K_\mathfrak{p}, M_{\theta\mathbb{Q}_p}(n))) = 0.$$

Moreover if also $w - 2n \neq 0$, then

$$\dim_{\mathbb{Q}_p} H^1(K_\mathfrak{p}, M_{\theta\mathbb{Q}_p}(n)) = [K_\mathfrak{p} : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} M_{\theta\mathbb{Q}_p}(n).$$

3. The conjecture in the critical situation

We follow the notations of Section 2, but now p does not necessarily split in K. The critical situation corresponds to realizations of the motive

$$M_\theta(n)$$

where θ has infinity type (a, b) and n satisfies:

$$\min(a, b) < n \leq \max(a, b).$$

Let us impose that the weight is ≤ -3; this means $a + b - 2n \leq -3$.

Tsuji in [**16**, §9, §10] proves the Jannsen conjecture in this case, p inert and θ of infinite type $(k-j, 0)$ with $\vartheta_\theta = (\lambda, \ldots, \lambda)$, $n = k$ and $0 \leq -j < k$. One obtains also the case p inert and θ is of infinite type $(0, j-k)$, $n = j$ and $0 \leq -k < j$ by complex conjugation. The p-adic realization corresponds to:

$$M_{(k-j,0)\mathbb{Q}_p}(k) \cong V_p(E)^{\otimes k} \otimes \overline{V_p(E)}^{\otimes j};$$

where $\otimes$ is $\otimes_{\mathcal{O}_p \otimes_{\mathcal{O}_K} K}$, $\mathcal{O}_p = \mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_K$ and the bar means complex conjugation. Observe then that any $M_\theta(n)$ in the critical situation has a p-adic realization (we write the situation $b = \min(a, b)$) isomorphic to

$$V_p(E)^{\otimes n-b} \otimes \overline{V_p(E)}^{\otimes n-a}.$$

Thus, if we write $k = n - b$ and $j = n - a$ we need to study Jannsen's conjecture only for the p-adic realizations $M_{(k-j,0)}(k)$ (the situation $a = \min(a, b)$ corresponds by complex conjugation to study the motives $M_{(0,j-k)}(j)$).

We have the following isomorphism,

$$(M_{\theta\mathbb{Q}_p}(n))^*(1) \cong M_{\overline{\theta}\mathbb{Q}_p}(-n + w + 1).$$

After all the above considerations we have in the critical situation:

Theorem 3.1 (Tsuji, [**16**, Theorem 9.1 and Proposition 10.2])**.** *Let p be a prime ≥ 5 which is inert in K and such that E has good reduction at the prime above p in K. Consider $M_\theta(n)$ with $\min(a,b) < n \leq \max(a,b)$ ($a \neq b$) satisfying $a+b-2n \leq -3$, then*

1. $H^2(G_S, M_{\theta\mathbb{Q}_p}(n)) = 0$, *and*
2. $H^2(G_S, M_{\overline{\theta}\mathbb{Q}_p}(1-n+w)) = 0$, *where $G_S = \mathrm{Gal}(K_S/K)$ with S a finite set of primes of K containing the primes dividing $p\mathfrak{f}$.*

Theorem 3.2 (Tsuji, [**16**, Lemma 10.1])**.** *Let $\min(a,b) < n \leq \max(a,b)$ ($a \neq b$) and $a+b-2n \leq -3$. Suppose p is inert in K and let $\mathfrak{p}$ be the prime of K above p.*

1. *Let $\mathfrak{q}$ be any non-zero prime ideal of $\mathcal{O}_K$ different to $\mathfrak{p}$. Then,* $H^i(K_\mathfrak{q}, M_{\theta\mathbb{Q}_p}(n)) = 0$ *and* $H^i(K_\mathfrak{q}, M_{\overline{\theta}\mathbb{Q}_p}(1-n+w)) = 0$ *for all i.*
2. $H^i(K_\mathfrak{p}, M_{\theta\mathbb{Q}_p}(n)) = 0$ *and* $H^i(K_\mathfrak{p}, M_{\overline{\theta}\mathbb{Q}_p}(1-n+w)) = 0$ *for $i = 0$ and 2.*

Remark 3.3. The main points needed to obtain Theorem 3.1 are [**16**, Theorem 6.1] and the main Iwasawa conjecture proved by Rubin. Both results are also known in the split situation by Han [**8**, §5.1] and Rubin [**15**] respectively. Suppose now that p splits in K, $p = \mathfrak{p}\mathfrak{p}^*$. In order to obtain Theorem 3.1 for the split case one could try to write in detail the second part of Tsuji's paper [**16**, II] replacing $\mathfrak{p}$ by $p = \mathfrak{p}\mathfrak{p}^*$. Let us indicate some steps: rewrite [**8**, Theorem 4, §5.1] with the unit $e_{1,S,p}$ in the notation of [**16**, §5] to obtain [**16**, Theorem 6.1] (one will need a result similar to [**1**, Lemma 3.3]), replace $\mathcal{O}$ by $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$ in §8 and §9 of [**16**] and check that the arguments follow (remember that in our case $L = K$ and $A_\mathfrak{p} = A_\beta = \mathcal{O}_p$).

4. The conjecture in the non-critical situation

Let us recall that we have fixed an imaginary quadratic field K with $cl(K) = 1$ and E has good reduction at the places above p in K.

From the specialization of the elliptic polylogarithm [**11**], one can prove the equality between the image by the Soulé regulator map r_p of a module $\mathcal{R}_\theta = \alpha_\theta \mathcal{O}_K$ in K-theory for the motive $e_\theta(\otimes^w h^1(E))(w+l+1)$ (see [**3**, 3.3] for the precise definition of α_θ) and the image of the elliptic units module by a Soulé map e_p (see [**3**, 4.6] for the definition of e_p) with $w - 2(w+l+1) \leq -3$ with $-l \leq \min(a,b)$ (we refer to [**3**, Corollary 5.9] for this equality). This result is obtained, under some restrictions, from the main conjecture of Iwasawa theory for imaginary quadratic fields.

Let us fix in this section the following assumptions: let $p \geq 5$ be a prime with $p > N_{K/\mathbb{Q}}\mathfrak{f}$ and θ with infinite type (a, b) such that $a \not\equiv b(\mathrm{mod}|\mathcal{O}_K^*|)$. Suppose also that $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective and the $\Delta = \mathrm{Gal}(K(E[p])/K)$-representation of $\mathrm{Hom}_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p)$ satisfies the Rubin's theorem on the Iwasawa main conjecture [**15**], where $\mathcal{O}_p$ denotes $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Proposition 4.1. *Suppose that* $H^2(\mathrm{Gal}(K_S/K), M_{\theta\mathbb{Q}_p}(w+l+1)) = 0$ *with S finite set of primes of K which contains the ones dividing $p\mathfrak{f}$. Then $r_p(\mathcal{R}_\theta \otimes_{\mathbb{Z}} \mathbb{Q}_p)$ has $\mathbb{Q}_p$-rank equal to 2, in particular e_p is injective.*

Proof: The same arguments of the proof of [**11**, Proposition 5.2.5] can be used with the diagram [**3**, Lemma 4.11] to obtain the result. ☐

As observed by Tsuji [**16**, (11.12)], Jannsen's conjecture can be obtained from the non-vanishing of the map e_p into local cohomology of the elliptic units module if [**16**, Question 11.15] has a positive answer. In [**3**] we prove that [**16**, Question 11.15] has a positive answer in the split case, and in this case one can show the same result of Tsuji by using the arguments of Remark 3.3. But since we do not want to write all the technical details, we make the following

Assumption. (I) The result [**16**, (11.12)] is true also when p splits in K.

In his thesis [**7**], Geisser studies this Soulé map e_p locally, i.e., with values in the local cohomology. In this study appear the values at non-critical values of the p-adic L-functions associated to ψ_{Ω_i} for $i = 1, 2$ (up to twist by cyclotomic character). The ψ_{Ω_i} comes from the Hecke character ψ_θ associated to θ as follows: $\psi_\theta \otimes \mathbb{Z}_p = \psi_{\Omega_1} \oplus \psi_{\Omega_2}$ (see for an extended explanation [**7**] or [**1**, §3]). Let us denote by $\iota\colon H^1(\mathrm{Gal}(\overline{K}/K), M_{\theta\mathbb{Z}_p}(w+l+1)) \to H^1(\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p}), M_{\theta\mathbb{Z}_p}(w+l+1))$ the restriction map.

Proposition 4.2 ([**1**, 3.4])**.** *Let p be a prime that splits in K.*

Suppose furthermore $w = a + b \geq 1$, $a + l > 0$, $b + l > 0$ and $p > 3w + 2l + w + 1$. Then the length of the coimage of $\iota \circ r_p(\mathcal{R}_\theta)$ in $H^1(K_\mathfrak{p}, M_{\theta\mathbb{Z}_p}(w+l+1))$ is equal to the p-adic valuation of

$$G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1} - 1, u_2^{-b_\theta-1} - 1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1} - 1, u_2^{-a_\theta-1} - 1),$$

where κ denotes the cyclotomic character of $\mathcal{G} = \mathrm{Gal}(K(E[p^\infty])/K)$ and G denotes the p-adic L-functions.

Corollary 4.3. *Let us assume (I) and the hypothesis of Proposition 4.2. If the non-critical values obtained in the p-adic L-functions* $G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1}-1, u_2^{-b_\theta-1}-1)$ *and* $G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1}-1, u_2^{-a_\theta-1}-1)$ *do not vanish, then Jannsen's conjecture is true for* $M_{\overline{\theta}}(-l)$*, i.e.*

$$H^2(G_S, M_{\overline{\theta}\mathbb{Q}_p}(-l)) = 0,$$

and the conclusions with $n = w+l+1$ *of Corollary 2.10 are also satisfied.*

Proof: By the local Jannsen conjecture we obtain from Corollary 2.11 that

$$\mathrm{rank}_{\mathbb{Z}_p} H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1)) = 2.$$

By the non-vanishing hypotheses and Proposition 4.2, we see that the module $\iota(r_p(\mathcal{R}_\theta))$ has finite index in $H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1))$; therefore $\dim_{\mathbb{Q}_p}\iota(r_p(\mathcal{R}_\theta\otimes\mathbb{Q}_p)) = \dim_{\mathbb{Q}_p}(\mathcal{R}_\theta\otimes\mathbb{Q}_p)$. Hence $\dim_{\mathbb{Q}_p}(r_p(\mathcal{R}_\theta\otimes\mathbb{Q}_p)) = 2$ and the Soulé map r_p does not vanish in $\mathcal{R}_\theta$. This is equivalent by [**16**, (11.12)] to Jannsen's conjecture for $H^2(G_S, M_{\overline{\theta}\mathbb{Q}_p}(-l))$ (using the Tate-Poitou long exact sequence). □

Remark 4.4. Moreover by [**16**, p. 171, (11.12)] (assuming (I)), the Jannsen conjecture for $M_{\overline{\theta}\mathbb{Q}_p}(-l)$ implies the non-vanishing of the product of the non-critical values of the p-adic L-functions which appear in Corollary 4.3.

5. The weak Jannsen conjecture

We want to consider now the following weak form of Jannsen's conjecture, which claims for our concrete realizations that: the Galois cohomology groups $H^2(G_S, M_{\theta\mathbb{Q}_p}(n))$ vanish for almost all Tate twist n. This weak Jannsen conjecture is equivalent to the weak Leopoldt conjecture [**14**].

Proposition 5.1. *Let us suppose that the cyclotomic* μ*-invariant of every abelian extension of* K *is zero. Let* S *be a set of primes of* K *containing the primes of* K *dividing* $\mathfrak{p}\mathfrak{f}$*. Then for almost all* n *we have*

$$H^2(G_S, M_{\theta\mathbb{Q}_p}(n)) = 0.$$

Proof: Since we have a Hecke character the Galois group $\mathrm{Gal}(K'/K)$ fixing the p-torsion of $M_{\theta\mathbb{Q}_p}(n)/M_{\theta\mathbb{Z}_p}(n)$ is an abelian extension of K. By [**14**, B.2, Corollary (i)] (here we use the hypothesis that μ vanishes for the cyclotomic extension of the abelian field K' over K) we obtain that

$$e_{\tilde{\chi}} H^2(\mathrm{Gal}(K_S/K(\mu_{p^\infty})), M_{\theta\mathbb{Z}_p}(n))$$

is a $\mathbb{Z}_p[[\mathrm{Gal}(K(\mu_{p^\infty})/K)]]^{\tilde{\chi}}$-torsion module, where $e_{\tilde{\chi}}$ are the idempotents associated to characters $\tilde{\chi}$ of $\mathrm{Gal}(K(\mu_{p^\infty})/C_\infty)$, with C_∞ the cyclotomic extension of K. This fact is equivalent to

$$H^2(\mathrm{Gal}(K_S/K(\mu_{p^\infty})), M^*_{\theta\mathbb{Q}_p}/M^*_{\theta\mathbb{Z}_p}) = 0,$$

where * means $\mathrm{Hom}(\ ,\mathbb{Q}_p)$ or $\mathrm{Hom}(\ ,\mathbb{Z}_p)$ respectively [**14**, Proposition 1.3.2]. Now use [**9**, Lemma 8] to obtain the result. □

Remark 5.2. The work of Kato for CM modular forms [**10**, §15] gives a proof of the weak Jannsen conjecture Proposition 5.1 without any assumption on the μ invariant. Moreover in [**4**] we obtain a different proof of Proposition 5.1 without the μ vanishing assumption using Iwasawa modules in two variables, imposing that $p+1 \nmid |b-a| = a-b$ if p is inert in K, or $p-1 \nmid |b-a| = a-b$ if p splits.

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