The modular automorphisms of quotient modular curves

Francesc Bars*and Tarun Dalal

May 14, 2025

Abstract

We obtain the modular automorphism group of any quotient modular curve of level N, with $4, 9 \nmid N$. In particular, we obtain some non-expected automorphisms of order 3 that appear for the quotient modular curves when the Atkin-Lehner involution w_{25} belongs to the quotient modular group, such automorphisms are not necessarily defined over $\mathbb{Q}$. As a consequence of the results, we obtain the full automorphism group of the quotient modular curve $X_0^*(N^2)$, for sufficiently large N.

1 Introduction

A curve C with non-trivial automorphism group encodes deep arithmetic information (in particular the twists of the curve C). The Fermat quartic or the Klein quartic are examples of curves with big automorphism groups extensively studied in the literature.

Some of the main curves in arithmetic geometry are the classical modular curves X over $\mathbb{Q}$, which are moduli spaces classifying elliptic curves with some N-level structure. A non-trivial automorphism group would have deep arithmetic meaning for such curves. For example, when N is prime, one expects that such a modular curve X has no rational point except the cusps and points associated to elliptic curves with complex multiplication, usually called CM points, (which is related with Serre's uniformity conjecture). In [Do16], the author related the existence (for certain X) of non-trivial automorphisms with the existence of rational points that are neither CM nor cusp. Thus modular curves with non-trivial automorphism group are of key interest.

Let X be a modular curve (we assume that it is defined over $\mathbb{Q}$), where its complex points correspond to the completion at certain cusps of the upper half plane $\mathcal{H}$ modulo the action by a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ (we assume $\pm I \in \Gamma$), and denote by $\mathrm{Aut}(X)$ the automorphism group over $\overline{\mathbb{Q}}$ of the modular curve X. In particular, the normalizer of Γ in $\mathrm{PSL}_2(\mathbb{R})$ (the automorphism group of $\mathcal{H}$) modulo Γ provides a subgroup of $\mathrm{Aut}(X)$ which is known as **the modular automorphism group** of X. For a group $G \leq \mathrm{PSL}_2(\mathbb{R})$, we denote its normalizer inside $\mathrm{PSL}_2(\mathbb{R})$ by $\mathcal{N}(G)$. In particular, the modular automorphism group of the modular curve associated to Γ corresponds to $\mathcal{N}(\Gamma)/\Gamma$.

Let $N \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of all positive integers), and consider the modular group $\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\}$. It is well known that the associated modular curve $X_0(N)$ is defined over $\mathbb{Q}$. Atkin-Lehner in [AtLeh70, Theorem 8], stated the result for $\mathcal{N}(\Gamma_0(N))$ modulo $\Gamma_0(N)$, (cf. [AkSi90], [Ba08] for the correct statement and the proof of the result). Such normalizer contains the Atkin-Lehner involutions defined by the matrices of the form $w_{d,N} = \frac{1}{\sqrt{d}}\left(\begin{smallmatrix} dx & y \\ Nz & dw \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $d > 1, d||N$ (i.e., $d|N$ and $(d, N/d) = 1$) and $x, y, z, w \in \mathbb{Z}$ such that $xwd - yz(N/d) = 1$ (we also use the notation w_d to denote $w_{d,N}$, the level N will be clear from the context). We denote the group generated by all such Atkin-Lehner involutions modulo $\Gamma_0(N)$ by $B(N)$, which is an abelian group with every non-trivial element of order 2. For $4, 9 \nmid N$, we known that

$$\mathcal{N}(\Gamma_0(N))/\Gamma_0(N) = B(N),$$

a group of order $2^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime divisors of N (loc.cit.). Later, Conway [Con96] gave a characterization of the normalizer of $\Gamma_0(N)$ in terms of a group action on lattices, which has deep interest and consequences in Group Theory. We emphasize here that the existence of such Atkin-Lehner automorphisms (involutions) play a crucial role in the understanding of the modular curves $X_0(N)$ and the theory of Hecke operators for $X_0(N)$ (cf. [AtLeh70]).

*First author is supported by PID2020-116542GB-I00 and CEX2020-001084-M

Now consider any subgroup W_N of $B(N)$ (by abuse of notations we denote the collection of distinct representatives of $B(N)$ by $B(N)$), and the group $\langle\Gamma_0(N), W_N\rangle$ (we use the notation $H+G$ or $\langle H, G\rangle$ to denote the group generated by the elements of H and G). The associated modular curve $X_0(N)/W_N$ is known as a quotient modular curve and it is defined over $\mathbb{Q}$. An automorphism of $X_0(N)/W_N$ is said to be **modular** if it is coming from an element of $\mathrm{PSL}_2(\mathbb{R})$ (note that such an element belongs to $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)$ and vice-versa). Since $\mathcal{N}(\Gamma_0(N))/\Gamma_0(N) = B(N)$ for $4, 9 \nmid N$, it is natural to ask whether the equality $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)/\langle\Gamma_0(N), W_N\rangle = B(N)/W_N$ is true or not for $4, 9 \nmid N$. When N is square-free, Lang in [Lan01] proved that the equality $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)/\langle\Gamma_0(N), W_N\rangle = B(N)/W_N$ is true for any subgroup W_N. The main motivation of this article is to study this question for general N with $4, 9 \nmid N$. More precisely, we completely determine the normalizer $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)$ and prove the following results.

Theorem 1.1. *[Theorem 3.12 in text] Let $N \in \mathbb{N}$ and W_N be a subgroup generated by the Atkin-Lehner involutions such that $4, 9 \nmid N$ and $w_{25} \notin W_N$. Then $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle) = \langle\Gamma_0(N), w_d : d||N\rangle$.*

Theorem 1.2. *[Theorem 3.13 in text] Let $N \in \mathbb{N}$ and W_N be a subgroup generated by the Atkin-Lehner involutions such that $4, 9 \nmid N$ and $w_{25} \in W_N$.*

1. *If there exists $w_d \in W_N$ such that $\frac{d}{(25,d)} \not\equiv \pm 1 \pmod 5$, then $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle) = \langle\Gamma_0(N), w_d : d||N\rangle$.*
2. *If $\frac{d}{(25,d)} \equiv \pm 1 \pmod 5$ for all $w_d \in W_N$, then $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle) = \langle\Gamma_0(N), \Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5, w_d : d||N\rangle$ where $\Upsilon_5 := \left(\begin{smallmatrix}1 & 0\\ 0 & 1/5\end{smallmatrix}\right)$, $B_j := \left(\begin{smallmatrix}\frac{N}{25}j+1 & -j\\ -\frac{N}{25} & 1\end{smallmatrix}\right)$, $C_i := \left(\begin{smallmatrix}1 & i\\ 0 & 1\end{smallmatrix}\right)$, $0 \le j, i \le 4$ such that $\frac{N}{25}j \equiv 2 \pmod 5$ and $i \equiv -j \pmod 5$. Moreover $\langle\Upsilon_5^{-1}B_jC_0\Upsilon_5 = (\Upsilon_5^{-1}B_0C_i\Upsilon_5)^{-1}\rangle$ has order 3 in $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)/\langle\Gamma_0(N), W_N\rangle$.*

As an immediate consequence of Theorem 1.1 and Theorem 1.2, we obtain that for $4, 9 \nmid N$, we have $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)/\langle\Gamma_0(N), W_N\rangle \supsetneq B(N)/W_N$ if and only if $w_{25} \in W_N$ and $\frac{d}{(25,d)} \equiv \pm 1 \pmod 5$ for all $w_d \in W_N$. Moreover, in such cases the group $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)/\langle\Gamma_0(N), W_N\rangle$ (and hence the group $\mathrm{Aut}(X_0(N)/W_N)$) may be non-abelian and contains elements of order 3. In particular, this explains the new automorphisms of order 3 that appear for the quotient curves $X_0(25q)/\langle w_{25}\rangle$ and $X_0(25q)/\langle w_{25}, w_q\rangle$ with q prime (under some assumptions), which is first observed in [BaDa24].

It is expected that when N is sufficiently large, the modular automorphism group of $X_0(N)/W_N$ coincides with the full automorphism group $\mathrm{Aut}(X_0(N)/W_N)$. This statement is true for the modular curve $X_0(N)$ (cf. [KenMom88]). Moreover, when N is either square-free (cf. [BaGo21]) or a perfect square (cf. [DLM22]), then this statement is true for the modular curve $X_0^*(N) := X_0(N)/B(N)$. In particular, combining Theorem 1.1 and Theorem 1.2 with [DLM22, Theorem 5.8] we get

Corollary 1.3. *Let $N \ge 10^{400}$ and $(6, N) = 1$. Then* $\mathrm{Aut}(X_0(N^2)/B(N^2)) \cong \begin{cases}\mathbb{Z}/3\mathbb{Z}, & \text{if } 5||N,\\ \{\mathrm{id}\}, & \text{otherwise.}\end{cases}$

In the last section of this paper, under some assumption we prove that the order 3 modular automorphisms are defined over $\mathbb{Q}(\sqrt{5})$.

2 The Conway Big Picture for quotient modular groups

For $N \in \mathbb{N}$ and a subgroup W_N generated by certain the Atkin-Lehner involutions, consider the group $\langle\Gamma_0(N), W_N\rangle$. We denote by $\Gamma_0^*(N)$, the subgroup generated by $\Gamma_0(N)$ and all the Atkin-Lehner involutions $w_{d,N}$ with $d||N$. The aim of this section is to prove that $\mathcal{N}(\langle\Gamma_0(N), W_N\rangle)$ is a subgroup of $\Gamma_0^*(M)$, for some positive divisor M of N. In order to do this, we will follow Conway's Big Picture introduced in [Con96].

2.1 The Big Picture

Two lattices $L(1)$ and $L(2)$ (commensurable with $\mathbb{Z}\times\mathbb{Z}$) are equivalent to each other if there exists $q \in \mathbb{Q}^*$ such that $L(1) = qL(2)$. This is an equivalence relation on the set of lattices that are commensurable with $\mathbb{Z}\times\mathbb{Z}$. Each equivalence class has a representative of the form $L_{s,g/t} := \langle(s, g/t), (0,1)\rangle_{Lat} = \langle se_1 + \frac{g}{t}e_2, e_2\rangle_{Lat}$, where $s > 0$ is a rational number and $0 \le g/t < 1$, with $g \ge 0$ and $t > 0$ coprime integers, $e_1 = (1,0)$ and $e_2 = (0,1)$; when $g = 0$ we denote L_s by $L_{s,0}$. For simplicity of notations we denote the equivalence class containing the lattice $L_{s,g/t}$ by $L_{s,g/t}$. The hyperdistance between two equivalence classes $L_{s_1,g_1/t_1}$ and $L_{s_2,g_2/t_2}$

is defined as follows: after a suitable base change one class corresponds to $\langle e_1, e_2\rangle_{Lat}$ and the other corresponds to $\langle ke_1, e_2\rangle_{Lat}$ for a certain $k \in \mathbb{N}$, the number k is the hyperdistance between $L_{s_1,g_1/t_1}$ and $L_{s_2,g_2/t_2}$.

The Big Picture of Conway is a graph defined as follows: the points (or vertices) correspond to the equivalence classes $L_{s,g/t}$, and two classes $\{L_{s_1,g_1/t_1}, L_{s_2,g_2/t_2}\}$ are connected by a non-oriented edge if and only if the hyperdistance between such two classes is a prime number.

There is a natural action of $\mathrm{PGL}_2(\mathbb{Q})$ on the Big Picture defined as follows: for $A := \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$, $A * L_{s,g/t}$ corresponds to the representative of the class containing the lattice $\langle s(ae_1+be_2)+\frac{g}{t}(ce_1+de_2), ce_1+de_2\rangle_{Lat}$, i.e., in terms of basis elements, this action can be written as:

$$se_1 + \frac{g}{t}e_2 \mapsto s(ae_1+be_2)+\frac{g}{t}(ce_1+de_2),\ e_2 \mapsto ce_1+de_2. \tag{2.1}$$

This action could be extended to $\mathrm{PSL}_2(\mathbb{R})$ with the same definition. We would like to remark that $w_{d,N} \in \mathrm{PSL}_2(\mathbb{Q})$ when d is a perfect square, and $w_{d,N} \notin \mathrm{PSL}_2(\mathbb{Q})$ if d is not a perfect square.

The following results are well-known (cf [Con96]).

Theorem 2.1 (Conway)**.** *The stabilizer of $X = L_{s,g/t}$ in $\mathrm{PSL}_2(\mathbb{R})$ is $\left(\begin{smallmatrix} s & g/t \\ 0 & 1\end{smallmatrix}\right)^{-1}\mathrm{PSL}_2(\mathbb{Z})\left(\begin{smallmatrix} s & g/t \\ 0 & 1\end{smallmatrix}\right) \subseteq \mathrm{PSL}_2(\mathbb{Q})$, and in the Big Picture literature such stabilizer is denoted $\Gamma_0(X|X)+$.*

Following the notation in the Big Picture, for a positive integer h with $h^2|N$, we define the group

$$\Gamma_0(N/h|h)+ := \left(\begin{smallmatrix} 1/h & 0 \\ 0 & 1\end{smallmatrix}\right)\Gamma_0^*(N/h^2)\left(\begin{smallmatrix} h & 0 \\ 0 & 1\end{smallmatrix}\right) \subset \mathrm{PGL}_2(\mathbb{Q}).$$

Theorem 2.2 (Conway)**.** *The point $L_{s,g/t}$ is fixed by $\Gamma_0(N)$ if and only if s is a positive integer and $t|24$ is the largest integer such that $t^2|N$ and $1|s|(N/t^2)$. The collection of all such points on the Big Picture is called the $(N|1)$-snake. Furthermore, $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ leaves the $(N|1)$-snake invariant as a set (not necessarily point-wise) if and only if $\sigma \in \Gamma_0(N/h|h)+$, where h is the largest positive integer such that $h|24$ and $h^2|N$. Thus $\mathcal{N}(\Gamma_0(N)) = \Gamma_0(N/h|h)+$.*

Example 2.3. If $4, 9 \nmid N$, then from Theorem 2.2 the $(N|1)$-snake corresponds to the set of all classes $L_{s,0}$ where s is a positive divisor of N. Furthermore, under such assumption we have $\mathcal{N}(\Gamma_0(N)) = \Gamma_0(N/1|1)+ = \Gamma_0^*(N)$.

2.2 The normalizer of $\Gamma := \Gamma_0(N) + W_N$ with $w_{u^2,N} \notin \Gamma$.

Let N be a positive integer. Following the ideas of [Lan01], we now study the normalizer of $\Gamma_0(N) + W_N$, where W_N is a subgroup generated by Atkin-Lehner involutions such that for any Atkin-Lehner involution $w_{d,N} \in \Gamma_0(N) + W_N$, d is not a perfect square. Note that under such assumption $w_{d,N} \notin \mathrm{PSL}_2(\mathbb{Q})$. For simplicity of notations we write $w_{d,N}$ by w_d. Let $d_1, \dots, d_n$ be exact divisors of N (i.e., $d_i||N$) such that $\Gamma_0(N) + W_N = \Gamma_0(N) + \langle w_{d_1}, \dots, w_{d_n}\rangle$. Now consider the action of $\Gamma_0(N) + W_N$ on the Big Picture. We use the following notations:

- `t`: a $(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n}\rangle)$-orbit of size 2^n,
- T_N: the set of all the `t`'s.

Lemma 2.4. *Let N be a positive integer, and W_N be a subgroup generated by Atkin-Lehner involutions such that for any Atkin-Lehner involution $w_d \in \Gamma_0(N) + W_N$, d is not a perfect square. For each $X \in (N|1)$-snake*

$$\{\sigma(X) : \sigma \in (\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n}\rangle)\} = \{X, w_d(X) : d||N, w_d \in \Gamma_0(N) + W_N\}$$

is a member of T_N.

Proof. Let $X \in (N|1)$-snake. Then X is fixed by $\Gamma_0(N)$. Since $w_{d_i} \in \mathcal{N}(\Gamma_0(N))$, the elements of $\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n}\rangle$ are of the form $\prod_{i=1}^n w_{d_i}^{k_i}\gamma$ for some $\gamma \in \Gamma_0(N)$ and $k_i \in \{0,1\}$. Moreover, we can write $\prod_{i=1}^n w_{d_i}^{k_i}\gamma = w_d^k\gamma'$, for some $\gamma' \in \Gamma_0(N)$, $w_d \in W_N$ and $k \in \{0,1\}$. Hence

$$\{\sigma(X) : \sigma \in (\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n}\rangle)\} = \{X, w_d(X) : d||N, w_d \in \Gamma_0(N) + W_N\}.$$

Let $\mathcal{C}_{W_N}$ denote the representatives of distinct left cosets of $\Gamma_0(N)$ in $\Gamma_0(N)+W_N$. Since $w_d^2 \in \Gamma_0(N)$ for $d||N$, $[\Gamma_0(N)+W_N : \Gamma_0(N)] = 2^n$ and $\Gamma_0(N)$ fixes X, the set $\{X, w_d(X) : d||N, w_d \in \Gamma_0(N)+W_N\} = \{X, \delta(X) : \delta \in \mathcal{C}_{W_N} \setminus \{\mathrm{id}\}\}$ has at most 2^n elements. Recall that the stabilizer of X is of the form $(\Gamma_0(X|X)+) \subseteq \mathrm{PSL}_2(\mathbb{Q})$. Let δ_1, δ_2 be two distinct elements of $\mathcal{C}_{W_N} \setminus \{\mathrm{id}\}$. Then there exist integers d_{m_1}, d_{m_2} with $d_{m_i}||N$ such that $\delta_i = w_{d_{m_i}} \in W_N$ for $i \in \{1,2\}$. By the assumption on W_N, it is easy to see that $w_{d_{m_1}}, w_{d_{m_2}}, w_{d_{m_1}}^{-1}w_{d_{m_2}} \notin \mathrm{PSL}_2(\mathbb{Q})$. Thus $\delta_i(X) \neq X$ and $\delta_1(X) \neq \delta_2(X)$. The result follows. $\square$

Lemma 2.5. *Under the assumptions of Lemma 2.4, for any $\mathtt{t} \in T_N$, $\mathtt{t}$ is a subset of the $(N|1)$-snake.*

Proof. Consider an element $\mathtt{t} \in T_N$, and $X = L_{s,g/t} \in \mathtt{t}$. Suppose $X \notin (N|1)$-snake. Then $\Gamma_0(X|X)+$ is not a supergroup of $\Gamma_0(N)$ (cf. [Lan01, p.33,(8)]). Consequently, $\Gamma_0(N) \cap \Gamma_0(X|X)+$ is a proper subgroup of $\Gamma_0(N)$. In particular we have

$$[\Gamma_0(N) : \Gamma_0(N) \cap \Gamma_0(X|X)+] \geq 2.$$

Recall that $\Gamma_0(X|X)+ \subseteq \mathrm{PSL}_2(\mathbb{Q})$, and $(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle) \cap \mathrm{PSL}_2(\mathbb{Q}) = \Gamma_0(N)$. Hence we obtain

$$(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle) \cap \Gamma_0(X|X)+ = \Gamma_0(N) \cap \Gamma_0(X|X)+.$$

Therefore,

$$\begin{aligned} &[\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle : (\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle) \cap \Gamma_0(X|X)+] \\ =&[\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle : \Gamma_0(N) \cap \Gamma_0(X|X)+] \\ =&[\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle : \Gamma_0(N)][\Gamma_0(N) : \Gamma_0(N) \cap \Gamma_0(X|X)+] \geq 2^{n+1}. \end{aligned}$$

Observe that $(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle) \cap \Gamma_0(X|X)+$ is the stabilizer of X in $\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle$. The last equality shows that the $(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle)$-orbit of X has at least 2^{n+1} elements, which contradicts that $X \in \mathtt{t}$. Therefore $X \in (N|1)$-snake. □

Lemma 2.6. *Let N, W_N be as in lemma 2.4. Then, $\mathcal{N}(\Gamma_0(N) + W_N)$ is a subgroup of $\Gamma_0(N/h|h)+$, where $h|24$ is the largest natural number such that $h^2|N$.*

Proof. By Lemma 2.4 and Lemma 2.5, $X \in (N|1)$-snake if and only if $X \in \mathtt{t} \in T_N$.

Now for each $\sigma \in \mathcal{N}(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle)$, and $\mathtt{t} \in T_N$, we have

$$(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle)\sigma(\mathtt{t}) = \sigma(\Gamma_0(N) + \langle w_{d_1}, \cdots, w_{d_n} \rangle)(\mathtt{t}) = \sigma(\mathtt{t}).$$

Thus σ fixes the $(N|1)$-snake. Now the result follows from the fact that $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ leaves the $(N|1)$-snake invariant if and only if $\sigma \in \Gamma_0(N/h|h)+$, where $h|24$ is the largest natural number such that $h^2|N$ (cf. Theorem 2.2). □

Corollary 2.7. *Let N, W_N be as in Lemma 2.4 with $4, 9 \nmid N$. Then, $\mathcal{N}(\Gamma_0(N) + W_N) = \Gamma_0^*(N)$. In particular the modular automorphism group of the quotient curve $X_0(N)/W_N$ is $B(N)/W_N$.*

Proof. Under the assumption $4, 9 \nmid N$ we have $h = 1$ in Lemma 2.6. Now the result follows from the facts that $\Gamma_0(N/1|1)+ = \Gamma_0^*(N)$ and $\mathcal{N}(\Gamma_0(N) + W_N) \supseteq \Gamma_0^*(N)$. □

2.3 Towards the normalizer of $\Gamma := \Gamma_0(N) + W_N$ with $w_{u^2,N} \in \Gamma$.

Next consider the group $\Gamma := \Gamma_0(N) + W_N$ such that $w_{u^2,N} \in \Gamma \setminus \{\mathrm{id}\}$ for some natural number $u \neq 1$. Inspired by [Con96] we try to obtain the points $L_{s,g/t}$ of the Big Picture which are fixed by Γ.

Consider the conjugation by $\Upsilon_u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/u \end{smallmatrix}\right)$ of $\Gamma_0(N) + W_N$, where we write once and for all in this subsection $N = M \cdot u^2$ with $(M, u^2) = 1$.

Lemma 2.8. *The conjugation by Υ_u satisfies the following properties:*

- $\Upsilon_u \Gamma_0(N) \Upsilon_u^{-1} = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(Mu) | b \equiv 0 \pmod u\}$, *and we denote such conjugate group by $\tilde{\Gamma}_0^u(Mu)$,*
- $\Upsilon_u w_{u^2,N} \Gamma_0(N) \Upsilon_u^{-1} = \{\left(\begin{smallmatrix} a & b \\ Mc & d \end{smallmatrix}\right) \in \Gamma_0(M) | a \equiv d \equiv 0 \pmod u\}$.
- *If $(d, u) = 1$, then $\Upsilon_u w_{d,N} \Upsilon_u^{-1}$ is equal to $w_{d,Mu}$ or $w_{d,M}$. If $(d, u^2) = u'$, then $\Upsilon_u w_{d,N} \Upsilon_u^{-1}$ is equal to $w_{d/u',Mu}$ also equal to $w_{d/u',M}$.*

We write $\tilde{\Gamma}_{uM}^u := \Upsilon_u(\Gamma_0(N) + \langle w_{u^2,N} \rangle)\Upsilon_u^{-1} = \langle \tilde{\Gamma}_u(M), \tilde{\Gamma}_0^u(uM) \rangle$. We now study the lattices $L_{s,g/t}$ fixed by $\tilde{\Gamma}_{uM}^u$. Note that $\tilde{\Gamma}_{uM}^u$ fixes the class containing the lattice $L_{s,g/t}$ if and only if it fixes the lattice $L_{s,g/t}$.

Lemma 2.9. *If the equivalence class $L_{s,g/t}$ is fixed by $\tilde{\Gamma}_0^u(Mu)$, then $su \in \mathbb{Z}$, and sut^2 is a divisor of u^2M.*

Proof. An arbitrary element of $\tilde{\Gamma}_0^u(Mu)$ can be written in the form $\left(\begin{smallmatrix} a & ub \\ Muc & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$ such that $ad - Mu^2bc = 1$. Assume that the lattice $L_{s,g/t}$ is fixed by $\tilde{\Gamma}_0^u(Mu)$. Recall that $L_{s,g/t}$ are generated by the vectors $v_1 := se_1 + \frac{g}{t}e_2$ and $v_2 := e_2$, and the smallest multiple of e_1 that it contains is $v_3 = ste_1 = tv_1 - gv_2$. Under the action (2.1), the matrix $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ sends the lattice $L_{s,g/t}$ to the lattice generated by $v_1 + suv_2$ and v_2. Since $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ fixes $L_{s,g/t}$, we must have $su \in \mathbb{Z}$.

For the second condition, consider the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -Mu & 1 \end{smallmatrix}\right)$ which sends the lattice $L_{s,g/t}$ to the lattice generated by $v_1' = v_1 + \frac{g}{t}(-Mu)e_1$ and $v_2' = v_2 - Mue_1$. Since $\left(\begin{smallmatrix} 1 & 0 \\ -Mu & 0 \end{smallmatrix}\right)$ fixes the lattice $L_{s,g/t}$, we have $L_{s,g/t} = \langle v_1', v_2' \rangle_{Lat}$. In particular, this implies $Mue_1, \frac{g}{t}Mue_1 \in L_{s,g/t}$. Since ste_1 is the smallest multiple of e_1 which belongs to $L_{s,g/t}$, there exist $k_1, k_2 \in \mathbb{Z}$ such that $gMu^2 = sut^2k_1$ and $Mu^2 = stuk_2$. Since $(g,t) = 1$, these relations give $sut^2 | Mu^2$ as claimed. □

Lemma 2.10. *Let $M, u \in \mathbb{N}$ such that $(u, M) = 1$ and $4, 9 \nmid N$, where $N = Mu^2$. If $\tilde{\Gamma}_{Mu}^u$ fixes the lattice $L_{s,\frac{g}{t}}$, then $L_{s,\frac{g}{t}}$ is of the form $L_{d,0}$ where d is positive divisor of M.*

Proof. Suppose $\tilde{\Gamma}_{uM}^u$ fixes the lattice $L_{s,\frac{g}{t}}$. By Lemma 2.9 we have $su \in \mathbb{Z}$ and $sut^2 | u^2M$. For simplicity of notations, in the proof we denote the group $\tilde{\Gamma}_{uM}^u$ by $\tilde{\Gamma}$.

Since $(u, M) = 1$, there exist $x, y \in \mathbb{Z}$ such that $u^2x - My = 1$, i.e., $\left(\begin{smallmatrix} ux & y \\ M & u \end{smallmatrix}\right) \in \tilde{\Gamma}$. Since $\left(\begin{smallmatrix} ux & y \\ M & u \end{smallmatrix}\right)$ fixes $L_{s,\frac{g}{t}}$, we must have $Me_1 \in L_{s,\frac{g}{t}}$. Thus there exist $c_1, c_2 \in \mathbb{Z}$ such that

$$Me_1 = c_1(se_1 + \frac{g}{t}e_2) + c_2e_2. \tag{2.2}$$

From (2.2), we get $M = c_1s$ and $c_1\frac{g}{t} \in \mathbb{Z}$. Since $(g,t) = 1$, we must have $t|c_1$ and there exists $N_1 \in \mathbb{Z}$ such that $M = stN_1$. Recall that if $t = 1$, then $g = 0$. Now assume that $g \neq 0$ equivalently $t > 1$.

<u>Case I</u>: First assume that s is an integer.

The condition $M = stN_1$ implies that $t|M$, $s|M$, and $(u,t) = (u,s) = 1$ (recall that $N = Mu^2$ and $(u, M) = 1$ by assumption). Furthermore, the condition $sut^2 | u^2M$ implies that $t^2 | M$. Since $4, 9 \nmid N$ there exists a prime $\ell \geq 5$ with $\ell | t | M$. Moreover, we can choose $w \in \mathbb{Z}$ such that $u^2w^2 \not\equiv 1 \pmod{\ell}$ (this is possible since $(u,t) = 1$ and $(\mathbb{Z}/\ell\mathbb{Z})^\times$ has an element of order more than 2). There exist $x, y, z \in \mathbb{Z}$ such that $u^2xw - Mtyz = 1$ i.e., $\left(\begin{smallmatrix} ux & y \\ Mtz & uw \end{smallmatrix}\right) \in \tilde{\Gamma}$. Since $\left(\begin{smallmatrix} ux & y \\ Mtz & uw \end{smallmatrix}\right)$ fixes the lattice $L_{s,\frac{g}{t}}$, we get

$$s(uxe_1 + ye_2) + \frac{g}{t}(Mtze_1 + uwe_2) \in L_{s,\frac{g}{t}}. \tag{2.3}$$

Since $s \in \mathbb{Z}$ and $e_2 \in L_{s,g/t}$, the last equation implies

$$suxe_1 + \frac{g}{t}uwe_2 \in L_{s,\frac{g}{t}}. \tag{2.4}$$

Thus there exist $c_1, c_2 \in \mathbb{Z}$ such that

$$suxe_1 + \frac{g}{t}uwe_2 = c_1(se_1 + \frac{g}{t}e_2) + c_2e_2. \tag{2.5}$$

Solving the above equation we get $c_1 = ux \in \mathbb{Z}$, $c_2 = \frac{g}{t}u(w - x) \in \mathbb{Z}$. Since $(g,t) = 1$ and $\frac{g}{t}u(w-x) \in \mathbb{Z}$, we must have $ux \equiv uw \pmod{t}$, in particular we have $ux \equiv uw \pmod{\ell}$. On the other hand, the relation $u^2xw - Mtyz = 1$ implies that $u^2w^2 \equiv 1 \pmod{\ell}$, which contradicts the assumption that $u^2w^2 \not\equiv 1 \pmod{\ell}$. Therefore $c_2 \notin \mathbb{Z}$. Hence s can not be an integer if $t > 1$.

<u>Case II</u>: Now suppose that s is not an integer. From the relation $su \in \mathbb{Z}$, it is clear that $s \in \mathbb{Q}$. Let p be a prime such that $v_p(s) < 0$ (for a prime p and $n \in \mathbb{N}$, we use the notation $v_p(n)$ to denote the unique integer n_p such that $p^{n_p} || n$). Since $su \in \mathbb{Z}$, we have $v_p(u) > 0$ and $v_p(s) + v_p(u) \geq 0$.

There exist $a, c, d \in \mathbb{Z}$ such that $u^2ad - Mc = 1$, i.e., $\left(\begin{smallmatrix} ua & 1 \\ Mc & ud \end{smallmatrix}\right) \in \tilde{\Gamma}$. Since $L_{s,\frac{g}{t}}$ is fixed by $\left(\begin{smallmatrix} ua & 1 \\ Mc & ud \end{smallmatrix}\right) \in \tilde{\Gamma}$, we have $s(uae_1 + e_2) + \frac{g}{t}(Mce_1 + ude_2) \in L_{s,\frac{g}{t}}$. Hence there exist $c_{11}, c_{12} \in \mathbb{Z}$ such that

$$s(uae_1 + e_2) + \frac{g}{t}(Mce_1 + ude_2) = c_{11}(se_1 + \frac{g}{t}e_2) + c_{12}e_2. \tag{2.6}$$

Solving the last equation we get

$$c_{11} = ua + gN_1c \in \mathbb{Z},\ c_{12} = s + \frac{g}{t}(ud - ua - gN_1c) \in \mathbb{Z}. \tag{2.7}$$

If $v_p(t) < -v_p(s)$, then from (2.7) it is easy to see that $v_p(c_{12}) = v_p(s) < 0$. Hence $v_p(t) \geq -v_p(s)$, consequently from the relations $M = stN_1$ and $(M, u) = 1$ we get $v_p(t) = -v_p(s)$ and $v_p(N_1) = 0$.

Since $v_p(u) > 0$, the assumption $4, 9 \nmid N$ forces that $p \geq 5$. In this case, there exists $y \in \mathbb{Z}$ such that $(u, y) = 1$ and $y^2 \not\equiv 1 \pmod{p^{-v_p(s)}}$ i.e., $v_p(y^2 - 1) < -v_p(s)$. By the choice of y, there exist $x, z, w \in \mathbb{Z}$ such that $u^2xw - Mzy = 1$, i.e., $\left(\begin{smallmatrix} ux & y \\ Mz & uw \end{smallmatrix}\right), \left(\begin{smallmatrix} ux & 1 \\ Mzy & uw \end{smallmatrix}\right) \in \tilde{\Gamma}$. Since $L_{s,\frac{g}{t}}$ fixed by the matrices $\left(\begin{smallmatrix} ux & y \\ Mz & uw \end{smallmatrix}\right), \left(\begin{smallmatrix} ux & 1 \\ Mzy & uw \end{smallmatrix}\right)$, we must have $s(uxe_1 + ye_2) + \frac{g}{t}(Mze_1 + uwe_2), s(uxe_1 + e_2) + \frac{g}{t}(Mzye_1 + uwe_2) \in L_{s,\frac{g}{t}}$. Thus there exist $c_1, c_2, d_1, d_2 \in \mathbb{Z}$ such that

$$s(uxe_1 + ye_2) + \frac{g}{t}(Mze_1 + uwe_2) = c_1(se_1 + \frac{g}{t}e_2) + c_2e_2, \text{ and} \tag{2.8}$$

$$s(uxe_1 + e_2) + \frac{g}{t}(Mzye_1 + uwe_2) = d_1(se_1 + \frac{g}{t}e_2) + d_2e_2. \tag{2.9}$$

Solving the previous equations we get

$$c_1 = ux + gN_1z \in \mathbb{Z},\ c_2 = sy + \frac{g}{t}(uw - ux - gN_1z) \in \mathbb{Z} \text{ and} \tag{2.10}$$

$$d_1 = ux + gN_1zy \in \mathbb{Z},\ d_2 = s + \frac{g}{t}(uw - ux - gN_1zy) \in \mathbb{Z}. \tag{2.11}$$

From (2.10) and (2.11) we get

$$\frac{g}{t}uw(1 - y) - \frac{g}{t}ux(1 - y) - \frac{g}{t}gN_1z(1 - y^2) \in \mathbb{Z}. \tag{2.12}$$

This is a contradiction since $v_p(\frac{g}{t}uw(1 - y)) \geq 0, v_p(\frac{g}{t}ux(1 - y)) \geq 0$ but $v_p(\frac{g}{t}gN_1z(1 - y^2)) < 0$ (it follows from the assumption on y). Therefore we must have $v_p(t) = 0$ i.e., $v_p(s) = 0$ which is not possible. Hence we conclude that $t = 1$. Therefore the lattice $L_{s,g/t}$ is of the form $L_{s,0}$.

Any matrix $\gamma = \left(\begin{smallmatrix} ux & yz \\ M & uw \end{smallmatrix}\right) \in \tilde{\Gamma}$ acts on $L_{s,0}$ as follows: $\gamma \cdot L_{s,0} = \langle s(uxe_1 + yze_2), Me_1 + uwe_2 \rangle_{Lat}$. If γ fixes $L_{s,0}$, then we must have

$$s(uxe_1 + yze_2) = d_1se_1 + d_2e_2 \text{ and} \tag{2.13}$$

$$Me_1 + uwe_2 = d_1'se_1 + d_2'e_2, \tag{2.14}$$

for some $d_1, d_2, d_1', d_2' \in \mathbb{Z}$. From the above equations we have $d_1 = ux$ and $d_2 = syz$. Recall that $d_2, su \in \mathbb{Z}$. If $s \notin \mathbb{Z}$ (observe that $s \in \mathbb{Q}$), then there exists a prime p such that $v_p(s) < 0$ but $v_p(u) > 0$ and $v_p(yz) > 0$. This contradicts that $u^2xw - Mzy = 1$. Hence $s \in \mathbb{Z}$. On other hand from equation (2.14) we have $M = d_1's$. Since $d_1', s \in \mathbb{Z}$, we must have $s|M$. This completes the proof. □

Corollary 2.11. *Let u, M, N be as in Lemma 2.10. The normalizer of $\Gamma := \langle \Gamma_0(N), w_{u^2,N} \rangle$ is a subgroup of $\Gamma_0^*(M)$ conjugated by the matrix Υ_u^{-1}.*

Proof. The conjugation of Γ by Υ_u is $\tilde{\Gamma} := \tilde{\Gamma}_{Mu}^u$. Recall that by Lemma 2.10, the lattices fixed under $\tilde{\Gamma}$ forms a $(M|1)$-snake. For $X \in (M|1)$-snake and $\sigma \in N(\tilde{\Gamma})$, we have

$$\sigma^{-1}\tilde{\Gamma}\sigma(X) = X,\ i.e.,\ \ \tilde{\Gamma}\sigma(X) = \sigma(X).$$

Thus $\tilde{\Gamma}$ fixes $\sigma(X)$, consequently $\sigma(X) \in (M|1)$-snake. Therefore we obtain that σ set-wise fixes the $(M|1)$-snake. By Theorem 2.2, we conclude that the normalizer of $\tilde{\Gamma}$ is contained in the group $\Gamma_0(M/1|1)+ = \Gamma_0^*(M)$. The result follows. □

Let us study the general situation. For certain Atkin-Lehner involutions $w_{d_1}, \ldots, w_{d_n}$, we write $|\langle w_{d_1}, \ldots, w_{d_n} \rangle|$ for $|\langle \Gamma_0(N), w_{d_1}, \ldots, w_{d_n} \rangle / \Gamma_0(N)|$.

Theorem 2.12. *Let $N \in \mathbb{N}$ such that $4, 9 \nmid N$ and $u_1^2, \ldots, u_k^2$ be divisors of N such that $u_i^2 || N$ for $i = 1, \ldots, k$ and $|\langle w_{u_1^2}, \ldots, w_{u_k^2} \rangle| = 2^k$. Then the normalizer of $\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$ is a subgroup of $\Gamma_0^*(\frac{N}{\mathrm{lcm}(u_1^2, u_2^2, \ldots, u_k^2)})$ conjugated by Υ^{-1}, where $\Upsilon := \Upsilon_{\mathrm{lcm}(u_1, \ldots, u_k)}$.*

Moreover, if $W_N = \langle w_{u_1^2}, \ldots, w_{u_k^2}, w_{v_{k+1}}, \ldots, w_{v_n} \rangle \leq B(N)$ such that $|\langle w_{v_{k+1}}, \ldots, w_{v_n} \rangle| = 2^{n-k}$ and for any Atkin-Lehner involution $w_d \in \langle \Gamma_0(N), w_{v_{k+1}}, \ldots, w_{v_n} \rangle$, d is not a perfect square, then the normalizer of $\Gamma_0(N) + W_N$ is a subgroup of $\Gamma_0^*(M)$ conjugated by Υ^{-1} where $M = \frac{N}{\mathrm{lcm}(u_1^2, u_2^2, \ldots, u_k^2)}$. In general, we have

$$\mathcal{N}(\langle \Gamma_0(N), W_N \rangle) \leq \mathcal{N}(\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle).$$

Proof. We first prove the statement regarding the normalizer of $\langle \Gamma_0(N), w_{u_1^2}, \dots, w_{u_k^2}\rangle$. The case $k=1$ follows from Corollary 2.11. Consider the group $\langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}\rangle$.

Taking conjugation by Υ_{u_1}, we get the group $\langle \tilde{\Gamma}^{u_1}_{Mu_1}, \Upsilon_{u_1} w_{u_2^2}\Upsilon_{u_1}^{-1}\rangle$, where $M=\frac{N}{u_1^2}$. Recall that by Lemma 2.10, the lattices fixed by $\tilde{\Gamma}^{u_1}_{Mu_1}$ forms a $(M|1)$-snake.

Now taking conjugation by $\Upsilon_{u_2/(u_1,u_2)}$, we obtain the group $\tilde{W}_{u_1,u_2} := \langle \Upsilon_{u_2/(u_1,u_2)}\tilde{\Gamma}^{u_1}_{Mu_1}\Upsilon^{-1}_{u_2/(u_1,u_2)}, \Upsilon_{\frac{u_1u_2}{(u_1,u_2)}} w_{u_2^2}\Upsilon^{-1}_{\frac{u_1u_2}{(u_1,u_2)}}\rangle$.

If $\tilde{W}_{u_1,u_2}$ fixes the lattice $L_{s,g/t}$, then the group $\tilde{\Gamma}^{u_1}_{Mu_1}$ fixes the lattice

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & \frac{(u_1,u_2)}{u_2}\end{smallmatrix}\right)\cdot L_{s,g/t} = \langle se_1 + \frac{(u_1,u_2)}{u_2}\frac{g}{t}e_2, \frac{(u_1,u_2)}{u_2}e_2\rangle_{Lat} = \langle s\frac{u_2}{(u_1,u_2)}e_1 + \frac{g}{t}e_2, e_2\rangle_{Lat} = L_{s\frac{u_2}{(u_1,u_2)},g/t}.$$

By Lemma 2.10, we have $t=1$ and $1|s\frac{u_2}{(u_1,u_2)}|M$. Hence the lattice $L_{s,g/t}$ is of the form $L_{s,0}$, where $1|s\frac{u_2}{(u_1,u_2)}|M$.

Let $x,y,w\in\mathbb{Z}$ such that $u_2^2xw - \frac{N}{u_2^2}y = 1$, then $\left(\begin{smallmatrix} u_2x & \frac{yu_1}{(u_1,u_2)} \\ \frac{N}{u_2}\frac{(u_1,u_2)}{u_1u_2} & u_2w\end{smallmatrix}\right)\in \tilde{W}_{u_1,u_2}$. If $\tilde{W}_{u_1,u_2}$ fixes the lattice $L_{s,0}$, then the matrix $\left(\begin{smallmatrix} u_2x & \frac{yu_1}{(u_1,u_2)} \\ \frac{N}{u_2}\frac{(u_1,u_2)}{u_1u_2} & u_2w\end{smallmatrix}\right)$ fixes the lattice $L_{s,0}$. Hence there exist $d_1,d_2,d_3,d_4\in\mathbb{Z}$ such that

$$s(u_2xe_1 + \frac{u_1y}{(u_1,u_2)}e_2) = d_1se_1 + d_2e_2, \tag{2.15}$$

$$\frac{N}{u_2}\frac{(u_1,u_2)}{u_1u_2}e_1 + u_2we_2 = d_3se_1 + d_4e_2. \tag{2.16}$$

From (2.15), we have $d_1 = u_2x$ and $d_2 = s\frac{u_1y}{(u_1,u_2)}$. Recall that $s\frac{u_2}{(u_1,u_2)}\in\mathbb{Z}$. If $s\notin\mathbb{Z}$, then there exists a prime p such that $v_p(s)<0$ but $v_p(\frac{u_2}{(u_1,u_2)})>0$ such that $v_p(s)+v_p(\frac{u_2}{(u_1,u_2)})\geq 0$. In particular we have $v_p(u_2)>0$. Since $s\frac{u_1}{(u_1,u_2)}y\in\mathbb{Z}$ and $\frac{u_1}{(u_1,u_2)}, \frac{u_2}{(u_1,u_2)}$ has no common factor, we must have $v_p(y)>0$. This contradict the assumption that $u_2^2xw-\frac{N}{u_2^2}y=1$. Hence $s\in\mathbb{Z}$.

From (2.16), we have $d_3s = \frac{N}{u_2}\frac{(u_1,u_2)}{u_1u_2}$, i.e., $s|\frac{N(u_1,u_2)}{u_1u_2^2}$. On the other hand we also have $s|\frac{N(u_1,u_2)}{u_1^2u_2}$. Since $\left(\frac{N(u_1,u_2)}{u_1u_2^2}, \frac{N(u_1,u_2)}{u_1^2,u_2}\right) = \frac{N}{\mathrm{lcm}(u_1^2,u_2^2)}$, we conclude that $s|\frac{N}{\mathrm{lcm}(u_1^2,u_2^2)}$. This completes the proof for the case $k=2$.

For the general case assume that $w_{u_t^2}\notin\langle\Gamma_0(N), w_{u_1^2},\dots,w_{u_{t-1}^2}\rangle$. Now the result follows by applying induction on $\Upsilon_{\frac{u_t\mathrm{lcm}(u_1,\dots,u_{t-1})}{(u_t,\mathrm{lcm}(u_1,\dots,u_{t-1}))}}\langle\Gamma_0(N), w_{u_1^2},\dots,w_{u_{t-1}^2}, w_{u_t^2}\rangle\Upsilon^{-1}_{\frac{u_t\mathrm{lcm}(u_1,\dots,u_{t-1})}{(u_t,\mathrm{lcm}(u_1,\dots,u_{t-1}))}}$ and proceeding similarly as in the case $k=2$.

We now prove the statement regarding the normalizer of $\langle\Gamma_0(N), W_N\rangle$, which is inspired from ideas of [Lan01], and follows from the arguments introduced in §1.2.

Consider the group $\tilde{\Gamma} := \Upsilon\langle\Gamma_0(N), w_{u_1^2},\dots,w_{u_k^2}\rangle\Upsilon^{-1}$, and denote by $\delta_{v_t} = \Upsilon w_{v_t,N}\Upsilon^{-1}$ for $t\in\{k+1,\dots,n\}$, and write $M = N/\mathrm{lcm}(u_1^2,\dots,u_k^2)$. Observe that the assumptions on W_N imply $\delta_{v_t}\notin\mathrm{PGL}_2(\mathbb{Q})$ and $\delta_{v_{t1}}^{-1}\delta_{v_{t2}}\notin\mathrm{PGL}_2(\mathbb{Q})$ for $t1\neq t2$, and $\delta_{v_t}^2\in\tilde{\Gamma}$.

Now consider the action of $\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle$ on the Big Picture. We use the following notations:

- $\tilde{t}$: $\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle$ orbit of size 2^{n-k}, • $\tilde{T}_N$: The set of all such $\tilde{t}$, orbits of size 2^{n-k},
- $\tilde{\mathcal{C}}_{W_N}$ denotes the representatives of distinct left cosets of $\tilde{\Gamma}$ in $\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle$.

Following the argument described in §2.2 we obtain:

- For $X\in(M|1)$-snake, $\{\sigma(X):\sigma\in\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle\} = \{X,\delta(X):\delta\in\tilde{\mathcal{C}}_{W_N}\setminus\{\mathrm{id}\}\}$ is a member of $\tilde{T}_N$.
- If $\tilde{t}\in\tilde{T}_N$, then $\tilde{t}$ is a subset of the $(M|1)$-snake.

Using these properties and arguing similarly as in the proofs of Lemma 2.4 and Lemma 2.5, we obtain $X\in(M|1)$-snake if and only if $X\in\tilde{t}\in\tilde{T}_N$.

Now for each $\sigma\in\mathcal{N}(\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle)$, and $\tilde{t}\in\tilde{T}_N$, we have

$$(\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle)\sigma(\tilde{t}) = \sigma(\langle\tilde{\Gamma},\delta_{v_{k+1}},\dots,\delta_{v_n}\rangle)(\tilde{t}) = \sigma(\tilde{t}).$$

Therefore $(M|1)$-snake is fixed by σ. Since $4,9\nmid N$, using Theorem 2.2 we conclude that $\sigma\in\Gamma_0(M|1)+$. This proves the second statement.

For the last statement, write $\Gamma_1 := \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle$ and $\Gamma_2 := \langle \Gamma_1, w_{v_{k+1}}, \cdots, w_{v_n}\rangle$.

Any element of Γ_2 can be written in the form $w_{u^2}^{n_0}\prod_{i=k+1}^{n} w_{v_i}^{n_i}\gamma$, where $n_0, n_i \in \{0,1\}$, $u^2||\mathrm{lcm}(u_1^2,\ldots,u_k^2), w_{u^2} \in \Gamma_1$ and $\gamma \in \Gamma_0(N)$. Any element of Γ_1 is of the form $w_{u'^2}^{m_0}\gamma'$, where $m_0 \in \{0,1\}$, $u'^2||\mathrm{lcm}(u_1^2,\ldots,u_k^2), w_{u'^2} \in \Gamma_1$ and $\gamma' \in \Gamma_0(N)$. Note that $w_{u^2}, \gamma \in \mathrm{PSL}_2(\mathbb{Q})$ for $\gamma \in \Gamma_0(N)$ and $u^2||\mathrm{lcm}(u_1^2,\ldots,u_k^2)$. By the assumptions on v_i's, we have $\prod_{i=k+1}^{n} w_{v_i}^{n_i} \notin \mathrm{PSL}_2(\mathbb{Q})$ for $n_i \in \{0,1\}$ with at least one of n_i's is non-zero.

Now consider $\tilde{\sigma} \in \mathcal{N}(\Gamma_2)$ and $w_{u'^2}^{m_0}\gamma' \in \Gamma_1$, where $m_0 \in \{0,1\}$, $u'^2||\mathrm{lcm}(u_1^2,\ldots,u_k^2)$ and $\gamma' \in \Gamma_0(N)$. Since $\mathcal{N}(\Gamma_2) \subseteq \Upsilon^{-1}\Gamma_0^*(M)\Upsilon$, we have $\tilde{\sigma}w_{u'^2}^{m_0}\gamma'\tilde{\sigma}^{-1} \in \mathrm{PSL}_2(\mathbb{Q})$, $\tilde{\sigma}w_{u'^2}^{m_0}\gamma'\tilde{\sigma}^{-1} \in \Gamma_2$. Thus we can write

$$\tilde{\sigma}w_{u'^2}^{m_0}\gamma'\tilde{\sigma}^{-1} = w_{u^2}^{n_0}\prod_{i=k+1}^{n} w_{v_i}^{n_i}\gamma \in \Gamma_2,\ \text{where } n_0, n_i \in \{0,1\},\ w_{u^2} \in \Gamma_1 \text{ and } \gamma \in \Gamma_0(N). \tag{2.17}$$

If $n_i > 0$ for some i, then from (2.17), we have $\tilde{\sigma}w_{u'^2}^{m_0}\gamma'\tilde{\sigma}^{-1} \in \mathrm{PSL}_2(\mathbb{Q})$ but $w_{u^2}^{n_0}\prod_{i=k+1}^{n} w_{v_i}^{n_i}\gamma \notin \mathrm{PSL}_2(\mathbb{Q})$, which is a contradiction. Hence $n_i = 0$ for $k+1 \le i \le n$. Therefore we conclude that $\tilde{\sigma}w_{u'^2}^{m_0}\gamma'\tilde{\sigma}^{-1} \in \Gamma_1$, i.e. $\tilde{\sigma} \in \mathcal{N}(\Gamma_1)$. The result follows. $\square$

3 Exact normalizer of $\langle \Gamma_0(N), W\rangle$

In this section we completely determine the normalizer of $\langle \Gamma_0(N), W\rangle$ where $4, 9 \nmid N$ and W is a subgroup generated by certain Atkin-Lehner involutions. We compute it in two steps. First we compute the exact normalizer of $\langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}, \ldots, w_{u_k^2}\rangle$. Then with the help of this result we compute the exact normalizer of $\langle \Gamma_0(N), W\rangle$ for any arbitrary subgroup W.

Throughout the section we assume that $4, 9 \nmid N$. We introduce the following notations:

For a matrix $A = \left(\begin{smallmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1}\end{smallmatrix}\right)$, we write $A[i,j] := \alpha_{i,j}$ for $i, j \in \{0,1\}$. For a prime p and $n \in \mathbb{N}$, we use the notation $v_p(n)$ to denote the unique integer n_p such that $p^{n_p}||n$. Consider the group $\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle$, where $u_i^2||N$. Since $\mathrm{lcm}(u_1^2,\ldots,u_k^2)||N$, we have $M := \frac{N}{\mathrm{lcm}(u_1^2,\ldots,u_k^2)} \in \mathbb{Z}$. We define

$$\Gamma_u := \left\{\left(\begin{smallmatrix} ux & y\cdot\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u} \\ M\cdot\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u}\cdot z & uw\end{smallmatrix}\right) \in \Gamma_0(M) : x,y,z,w \in \mathbb{Z}\right\} \text{ for any } u||\mathrm{lcm}(u_1,\ldots,u_k),\ u>1, \text{ and}$$

$$\Gamma_{(u_1,\ldots,u_k)} := \left\{\left(\begin{smallmatrix} x & y\cdot\mathrm{lcm}(u_1,\ldots,u_k) \\ M\cdot\mathrm{lcm}(u_1,\ldots,u_k)\cdot z & w\end{smallmatrix}\right) \in \Gamma_0(M) : x,y,z,w \in \mathbb{Z}\right\}.$$

Observe that any element of $\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle\backslash\Gamma_0(N)$ can be written in the form $w_{u'^2}\gamma'$, for some $u'||\mathrm{lcm}(u_1,\ldots,u_k)$ and $\gamma' \in \Gamma_0(N)$. Furthermore, for any $w_{u'^2}\gamma' \in \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle\backslash\Gamma_0(N)$ it is easy to check that $\Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)}w_{u'^2}\Gamma_0(N)\Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)}^{-1} = \Gamma_{u'}$ and $\Gamma_{(u_1,\ldots,u_k)} = \Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)}\Gamma_0(N)\Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)}^{-1}$. Therefore

$$\Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)}\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle\Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)}^{-1} = \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle.$$

Furthermore if $\delta \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle\backslash\Gamma_{(u_1,\ldots,u_k)}$, then $\delta \in \Gamma_{u'}$ for some $u'||\mathrm{lcm}(u_1,u_2,\ldots,u_k)$. In particular we have $\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u'}\rangle\backslash\Gamma_{(u_1,\ldots,u_k)} = \Gamma_{u'}$. We mention some basic facts about $w_{u_i^2}$'s and Γ_{u_i}'s.

- If $v_p(\mathrm{lcm}(u_1,u_2,\ldots,u_k)) > 0$ for some prime p, then $2v_p(\mathrm{lcm}(u_1,u_2,\ldots,u_k)) = v_p(N)$.
- For $i \neq j$, $w_{u_i^2}w_{u_j^2} \in w_{u^2}\Gamma_0(N)$ with $u||\mathrm{lcm}(u_i,u_j)$, in particular $u := u_iu_j/(u_1,u_2)^2$.
- If $v_p(u_i), v_p(u_j) > 0$, then $v_p(u_i) = v_p(u_j) = v_p(\mathrm{lcm}(u_1,u_2,\ldots,u_k))$.
- If $\delta \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle\backslash\Gamma_{(u_1,\ldots,u_k)}$, then $\delta \in \Gamma_{u'}$ for some $u'||\mathrm{lcm}(u_1,u_2,\ldots,u_k)$. In particular we have $\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u'}\rangle\backslash\Gamma_{(u_1,\ldots,u_k)} = \Gamma_{u'}$.
- Furthermore, if $(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle\backslash\Gamma_{(u_1,\ldots,u_k)}) \cap \Gamma_{u'}$ is non-empty and $v_p(u') > 0$, then
$$v_p(\mathrm{lcm}(u_1,u_2,\ldots,u_k)) = v_p(u').$$

We recall the following result from Theorem 2.12.

Lemma 3.1. *Let $N, u_i \in \mathbb{N}$ such that $4, 9 \nmid N$ and $u_i^2 || N$ for $i \in \{1, \ldots, k\}$. Then $\mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle)$ is a subgroup of $\Gamma_0^*(M)$ (recall that $M := \frac{N}{\mathrm{lcm}(u_1^2,\ldots,u_k^2)}$). Suppose $\{h_1, h_2, \ldots, h_n\}$ is a complete set of coset representatives of $\Gamma_0^*(M)/\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle$, and consider the set*

$$\Delta = \{h_i : h_i \gamma h_i^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle \ for\ every\ \gamma \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle\}.$$

Then $\mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle)$ is generated by $\{\gamma, h_i : \gamma \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle, h_i \in \Delta\}$.

Consider the set

$$S'_{(u_1,\ldots,u_k),M} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M) \backslash \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle : ac \equiv bd \equiv 0 \pmod{\mathrm{lcm}(u_1,\ldots,u_k)} \right\}.$$

Observe that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S'_{(u_1,\ldots,u_k),M}$, if $u' = (a, \mathrm{lcm}(u_1,\ldots,u_k))$ and $u'' = (c, \mathrm{lcm}(u_1,\ldots,u_k))$, then

$$\mathrm{lcm}(u_1,\ldots,u_k) = u'u'', (b, \mathrm{lcm}(u_1,\ldots,u_k)) = u'' \text{ and } (d, \mathrm{lcm}(u_1,\ldots,u_k)) = u'.$$

Let $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S'_{(u_1,\ldots,u_k),M}$ and $u' = (a, \mathrm{lcm}(u_1,\ldots,u_k))$ (note that this implies $u'||\mathrm{lcm}(u_1,\ldots,u_k)$ and ${u'}^2||N$). It is easy to check that $\Upsilon^{-1}_{\mathrm{lcm}(u_1,\ldots,u_k)} g \Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)} \in w_{u'^2,N}\Gamma_0(N)$. Since $w_{u'^2,N} \in \mathcal{N}(\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle)$, we have $g \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle)$. Therefore we obtain

Lemma 3.2. $S'_{(u_1,\ldots,u_k),M} \subseteq \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle)$.

For $e||M$, we fix $x_{(e,M,k)}, y_{(e,M,k)} \in \mathbb{Z}$ such that $ey_{(e,M,k)} - \frac{M}{e}\mathrm{lcm}(u_1,\ldots,u_k)^2 x_{(e,M,k)} = 1$ i.e.,

$$\delta_{(e,M,k)} := \frac{1}{\sqrt{e}} \left(\begin{smallmatrix} e & \mathrm{lcm}(u_1,\ldots,u_k)\cdot x_{(e,M,k)} \\ M\cdot \mathrm{lcm}(u_1,\ldots,u_k) & e\cdot y_{(e,M,k)} \end{smallmatrix}\right) \in w_{e,M}\Gamma_0(M).$$

Note that the set $\{\mathrm{id}, \delta_{(e,M,k)} : e||M\}$ forms a complete set of representatives for the left cosets of $\Gamma_0(M)$ in $\Gamma_0^*(M)$. If the set $S^M_{(u_1,\ldots,u_k)} := \{g_i : 1 \le i \le [\Gamma_0(M) : \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle]\}$ forms a complete set of representatives for the left cosets of $\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle$ in $\Gamma_0(M)$, then the set

$$S^{M,+}_{(u_1,\ldots,u_k)} := \{\delta^j_{(e,M,k)} g_i : 0 \le j \le 1, e||M, 1 \le i \le [\Gamma_0(M) : \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle]\}$$

forms a complete set of representatives for the left cosets of $\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle$ in $\Gamma_0^*(M)$.

Since $\Upsilon^{-1}_{\mathrm{lcm}(u_1,\ldots,u_k)} \delta_{(e,M,k)} \Upsilon_{\mathrm{lcm}(u_1,\ldots,u_k)} \in w_{e,N}\Gamma_0(N)$ (with $e||M$) and $w_{e,N} \in \mathcal{N}(\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle)$, we have $\delta_{(e,M,k)} \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle)$. Therefore by Lemma 3.1 it suffices to compute the g_i's such that $g_i \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle)$, i.e., we need to compute the set $\mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle) \cap \Gamma_0(M)$.

Proposition 3.3. *Let $N, u_1, u_2 \ldots, u_k \in \mathbb{N}$ such that $4, 9 \nmid N$ and $u_i^2||N$ for $i \in \{1, 2, \ldots, k\}$. If*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle) \cap \Gamma_0(M), \text{ then } ac \equiv bd \equiv 0 \pmod{\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{5^{v_5(\mathrm{lcm}(u_1,\ldots,u_k))}}}.$$

Proof. Let $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle) \cap \Gamma_0(M)$ and $p_1 > 5$ be a prime such that $p_1 | \mathrm{lcm}(u_1, u_2, \ldots, u_k)$. Without loss of generality we assume that $v_{p_1}(u_1) = v_{p_1}(\mathrm{lcm}(u_1, u_2, \ldots, u_k))$, and write $n_1 := v_{p_1}(u_1)$.

For $l \in \{1, 2, 3\}$, there exist $r_{u_1,l}, k_{u_1,l} \in \mathbb{Z}$ such that

$$u_1^2 k_{u_1,l} + lM \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} r_{u_1,l} = 1, \tag{3.1}$$

i.e., $\gamma_{u_1,l} := \left(\begin{smallmatrix} u_1 & -l\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_1} \\ M\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_1} r_{u_1,l} & u_1 k_{u_1,l} \end{smallmatrix}\right) \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}\rangle \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle$. Consequently, $\sigma\gamma_{u_1,l}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k}\rangle$. In particular, $\sigma\gamma_{u_1,l}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{v_l}\rangle$ for some $v_l||\mathrm{lcm}(u_1, u_2, \ldots, u_k)$.

Suppose there exist $l_1, l_2 \in \{1, 2, 3\}$ (with $l_1 \neq l_2$) such that $\sigma\gamma_{u_1,l_1}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u'_1}\rangle$ and $\sigma\gamma_{u_1,l_2}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u'_2}\rangle$ for some u'_1, u'_2 with $p_1 \nmid u'_1 u'_2$. Then from the constructions of $\Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u'_i}$ we have (see the discussions before Lemma 3.1)

$$\sigma\gamma_{u_1,l_i}\sigma^{-1}[0,1] \equiv \sigma\gamma_{u_1,l_i}\sigma^{-1}[1,0] \equiv 0 \pmod{\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u'_i}}, \text{ for } i \in \{1, 2\}. \tag{3.2}$$

Since $p_1 \nmid u_1' u_2'$, we have $p_1^{n_1} | \frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_1'}$ and $p_1^{n_1} | \frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_2'}$. From (3.2) we have

$$\sigma\gamma_{u_1,l_i}\sigma^{-1}[0,1] \equiv \sigma\gamma_{u_1,l_i}\sigma^{-1}[1,0] \equiv 0 \pmod{p_1^{n_1}}, \text{ for } i \in \{1,2\}. \tag{3.3}$$

Combining (3.1) with (3.3), we get

$$a^2 \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_i^2 + b^2 \equiv c^2 \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_i^2 + d^2 \equiv 0 \pmod{p_1^{n_1}}, \text{ for } i \in \{1,2\}. \tag{3.4}$$

From (3.4) we obtain

$$a^2 \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} (l_1^2 - l_2^2) \equiv c^2 \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} (l_1^2 - l_2^2) \equiv 0 \pmod{p_1^{n_1}}. \tag{3.5}$$

Recall that $p \nmid (l_1^2 - l_2^2)$ for any prime $p > 5$ and $u_1 || \mathrm{lcm}\{u_1,\ldots,u_k\}$. Since $p_1^{n_1} > 5$ and $p_1 \nmid \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2}$, (3.5) implies that $(a,c) > 1$. Which contradicts that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Therefore, for any two distinct elements $i_1, i_2 \in \{1,2,3\}$ we must have $\{\sigma\gamma_{u_1,i_1}\sigma^{-1}, \sigma\gamma_{u_1,i_2}\sigma^{-1}\} \not\subset \Gamma_{(u_1,\ldots,u_k)}$, $\sigma\gamma_{u_1,i_1}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{v_1'} \rangle$ and $\sigma\gamma_{u_1,i_2}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{v_2'} \rangle$ for some v_1', v_2' with $p_1 | v_1' v_2'$ i.e., either $p | v_1$ or $p | v_2$. Hence there exist $l_1, l_2 \in \{1,2,3\}$ (with $l_1 \neq l_2$) such that $\sigma\gamma_{u_1,l_1}\sigma^{-1} \in \Gamma_{u_3'}$ and $\sigma\gamma_{u_1,l_2}\sigma^{-1} \in \Gamma_{u_4'}$ for some u_3', u_4' with $p_1 | (u_3', u_4')$ (for example suppose that $i_1 \in \{1,2,3\}$ such that $\sigma\gamma_{u_1,i_1}\sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{v_1'} \rangle$ with $p \nmid v_1'$, then for the two remaining elements $i_2, i_3 \in \{1,2,3\} \backslash \{i_1\}$ we must have $\sigma\gamma_{u_1,i_2}\sigma^{-1} \in \Gamma_{v_2'}$ and $\sigma\gamma_{u_1,i_3}\sigma^{-1} \in \Gamma_{v_3'}$ for some v_2', v_3' with $p_1 | v_2'$ and $p | v_3'$). Therefore we have

$$\sigma\gamma_{u_1,l_i}\sigma^{-1}[0,0] \equiv \sigma\gamma_{u_1,l_i}\sigma^{-1}[1,1] \equiv 0 \pmod{u_{2+i}'}, \text{ for } i \in \{1,2\}, \tag{3.6}$$

$$\sigma\gamma_{u_1,l_i}\sigma^{-1}[0,1] \equiv \sigma\gamma_{u_1,l_i}\sigma^{-1}[1,0] \equiv 0 \pmod{\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_{2+i}'}}, \text{ for } i \in \{1,2\}. \tag{3.7}$$

Recall that from (3.1) we have

$$l_i M \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} r_{u_1,l_i} \equiv 1 \pmod{p_1^{n_1}} \text{ for } i \in \{1,2\}. \tag{3.8}$$

Using this congruence, from (3.6) we have

$$ac \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2 + bd \equiv ac \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_2^2 + bd \equiv 0 \pmod{p_1^{n_1}}. \tag{3.9}$$

Thus we obtain

$$ac \frac{\mathrm{lcm}(u_1,\ldots,u_k)^2}{u_1^2} (l_1^2 - l_2^2) \equiv 0 \pmod{p_1^{n_1}}, \tag{3.10}$$

equivalently we get

$$ac(l_1^2 - l_2^2) \equiv 0 \pmod{p_1^{n_1}}. \tag{3.11}$$

Since $(p_1^{n_1}, |l_1^2 - l_2^2|) = 1$, (3.11) implies that $ac \equiv 0 \pmod{p_1^{n_1}}$. Since p_1 is arbitrary, we conclude that $ac \equiv bd \equiv 0 \pmod{p^{v_p(\mathrm{lcm}(u_1,u_2,\ldots,u_k))}}$ for every prime $p > 5$. The result follows. $\square$

In order to compute the set $\mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle) \cap \Gamma_0(M)$ explicitly, first we consider the case $w_{25} \notin \langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}, \ldots, w_{u_k^2} \rangle$ and then we consider the case $w_{25} \in \langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}, \ldots, w_{u_k^2} \rangle$.

3.1 Exact normalizer of $\langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}, \ldots, w_{u_k^2} \rangle$ with $w_{25} \notin \langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}, \ldots, w_{u_k^2} \rangle$

The following result will be very useful for computing the normalizer when $5 | u_i$.

Lemma 3.4. *Let $N, u_1, u_2 \ldots, u_k \in \mathbb{N}$ such that $4, 9 \nmid N, u_i^2 || N$ and $w_{5^2} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$. If*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle) \cap \Gamma_0(M), \text{ then } v_5(abcd) \geq 2v_5(\mathrm{lcm}(u_1,u_2,\ldots,u_k)).$$

Proof. For simplicity of notations, we write $\eta := \mathrm{lcm}(u_1, u_2, \ldots, u_k)$ and $n_0 := v_5(\eta)$. If $n_0 = 0$, then the proposition is obvious, so we assume that $n_0 > 0$. Since $w_{5^2} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$, the set $\Gamma_5 \cap \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle$ is empty. Let $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle) \cap \Gamma_0(M)$. Since $(2, \eta) = 1$, there exists a prime p such that $p \equiv 2 \pmod{\eta}$ and $p \nmid N$. Since $u > 5$ for any $\Gamma_u \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \backslash \Gamma_{(u_1,\ldots,u_k)}$, we have $p^2 \equiv 4 \not\equiv \pm 1 \pmod{u}$ (because $u | \eta$ and $p^2 \equiv 4 \not\equiv \pm 1 \pmod{\eta}$). Moreover, there exist $k', r \in \mathbb{Z}$ such that $pk' - \eta^2 M r = 1$ i.e., $\left(\begin{smallmatrix} p & \eta \\ \eta M r & k' \end{smallmatrix}\right) \in \Gamma_{(u_1,\ldots,u_k)}$. Consequently, we have $pk' \equiv 2k' \equiv 1 \pmod{\eta}$.

Since $\sigma \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle)$, we have $E := \sigma \left(\begin{smallmatrix} p & \eta \\ \eta M r & k' \end{smallmatrix}\right) \sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle$. Therefore $E \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_u \rangle$ for some $\Gamma_u \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \backslash \Gamma_{(u_1,\ldots,u_k)}$. Suppose $E \in \Gamma_u$ for some $\Gamma_u \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \backslash \Gamma_{(u_1,\ldots,u_k)}$. Then $E[0,0] \equiv E[1,1] \equiv 0 \pmod{u}$. Thus

$$E[0,0] + E[1,1] \equiv (p + k')(ad - bc) \equiv 2 + k' \equiv 0 \pmod{u}. \tag{3.12}$$

The congruences $2k' \equiv 1 \pmod{u}$ and $2 + k' \equiv 0 \pmod{u}$, imply that $2^2 \equiv -1 \pmod{u}$, which is not possible. Therefore $E \notin \Gamma_u$ for any $\Gamma_u \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \backslash \Gamma_{(u_1,\ldots,u_k)}$.

Now suppose that $E \in \Gamma_{(u_1,\ldots,u_k)}$. Then $E[1,0] \equiv E[0,1] \equiv 0 \pmod{\eta}$. Consequently, we have

$$E[1,0] \cdot E[0,1] \equiv (p - k')^2 abcd \equiv 0 \pmod{\eta^2}. \tag{3.13}$$

Thus $v_5((p - k')^2 abcd) \geq 2v_5(\eta) = 2n_0$. Since $n_0 > 0$, we have $p \equiv 2 \pmod{5}$ and $pk' \equiv 2k' \equiv 1 \pmod{5}$. If $v_5(p - k') > 0$, then the congruence $pk' \equiv 2k' \equiv 1 \pmod{5}$ implies $4 \equiv 1 \pmod{5}$, which is not possible. Hence $v_5(p - k') = 0$. Consequently, we get $v_5(abcd) \geq 2n_0$. $\square$

Now we compute the set $\mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle) \cap \Gamma_0(M)$ when $w_{5^2} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$.

Proposition 3.5. *Let $N, u_1, u_2 \ldots, u_k \in \mathbb{N}$ such that $4, 9 \nmid N, u_i^2 || N$ and $w_{5^2} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$. If*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \cap \Gamma_0(M), \text{ then } ac \equiv bd \equiv 0 \pmod{\mathrm{lcm}(u_1, \ldots, u_k)}.$$

Proof. Let $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \cap \Gamma_0(M)$ and $n_0 := v_5(\mathrm{lcm}(u_1, u_2, \ldots, u_k))$. By Proposition 3.3, we know that $ac \equiv bd \equiv 0 \pmod{\frac{\mathrm{lcm}(u_1, u_2, \ldots, u_k)}{5^{n_0}}}$. We now prove that $ac \equiv bd \equiv 0 \pmod{5^{n_0}}$. If $n_0 = 0$, then this is clear. Hence we assume that $n_0 \geq 1$. Recall that the assumption $w_{5^2} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$ implies the set $\Gamma_5 \cap \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle$ is empty.

Without loss of generality we assume that $v_5(u_1) = n_0$. If $n_0 = 1$, then there exists a prime p_1 $(\neq 5)$ such that $n_1 := v_{p_1}(u_1) > 0$ (this is possible since $w_{25} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2} \rangle$). We define $\eta_0 := \begin{cases} p_1^{n_1}, & \text{if } n_0 = 1 \\ 5^{n_0}, & \text{otherwise} \end{cases}$, $\eta_1 := \begin{cases} 5^{n_0} p_1^{n_1}, & \text{if } n_0 = 1 \\ 5^{n_0}, & \text{otherwise} \end{cases}$ and $\eta_2 := \begin{cases} p_1, & \text{if } n_0 = 1 \\ 5, & \text{otherwise} \end{cases}$. Then $\eta_1 || u_1$. Recall that for any prime p if $v_p(u') > 0$ for some $\Gamma_{u'} \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \backslash \Gamma_{(u_1,\ldots,u_k)}$, then $v_p(\mathrm{lcm}(u_1, u_2, \ldots, u_k)) = v_p(u')$.

For $l \in \{1, 2, 3\}$, there exist $r_{u_1,l}, k_{u_1,l} \in \mathbb{Z}$ such that

$$u_1^2 k_{u_1,l} + lM \frac{\mathrm{lcm}(u_1, \ldots, u_k)^2}{u_1^2} r_{u_1,l} = 1, \tag{3.14}$$

i.e., $\gamma_{u_1,l} := \left(\begin{smallmatrix} u_1 & -l\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_1} \\ M\frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_1} r_{u_1,l} & u_1 k_{u_1,l} \end{smallmatrix}\right) \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1} \rangle \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle$.

Suppose there exist $l_1, l_2 \in \{1, 2, 3\}$ (with $l_1 \neq l_2$) such that $\sigma \gamma_{u_1,l_1} \sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1'} \rangle$ and $\sigma \gamma_{u_1,l_2} \sigma^{-1} \in \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_2'} \rangle$ for some u_1', u_2' with $\eta_2 \nmid u_1' u_2'$. Therefore we must have

$$\sigma \gamma_{u_1,l_i} \sigma^{-1}[0,1] \equiv \sigma \gamma_{u_1,l_i} \sigma^{-1}[1,0] \equiv 0 \pmod{\frac{\mathrm{lcm}(u_1, \ldots, u_k)}{u_i'}}, \quad \text{for } i \in \{1, 2\}. \tag{3.15}$$

Since $\eta_2 \nmid u_1' u_2'$, we have $\eta_0 | \frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_1'}$ and $\eta_0 | \frac{\mathrm{lcm}(u_1,\ldots,u_k)}{u_2'}$. From (3.15) we have

$$\sigma \gamma_{u_1,l_i} \sigma^{-1}[0,1] \equiv \sigma \gamma_{u_1,l_i} \sigma^{-1}[1,0] \equiv 0 \pmod{\eta_0}, \text{ for } i \in \{1, 2\}. \tag{3.16}$$

Combining (3.14) with (3.16), we get

$$a^2 \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_i^2 + b^2 \equiv c^2 \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_i^2 + d^2 \equiv 0 \pmod{\eta_0}, \text{ for } i \in \{1,2\}. \tag{3.17}$$

Thus we have

$$a^2 \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} (l_1^2 - l_2^2) \equiv c^2 \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} (l_1^2 - l_2^2) \equiv 0 \pmod{\eta_0}. \tag{3.18}$$

Recall that $5^2 \nmid (l_1^2 - l_2^2)$ and $p \nmid (l_1^2 - l_2^2)$ for any prime $p > 5$. Since $\eta_0^{n_1} > 5$ and $\eta_0 \nmid \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2}$, (3.18) implies that $(a,c) > 1$. Which contradicts that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Hence there exist $l_1, l_2 \in \{1,2,3\}$ (with $l_1 \neq l_2$) such that $\sigma\gamma_{u_1,l_1}\sigma^{-1} \in \Gamma_{u'_3}$ and $\sigma\gamma_{u_1,l_2}\sigma^{-1} \in \Gamma_{u'_4}$ for some u'_3, u'_4 with $\eta_2 | (u'_3, u'_4)$ (which automatically implies that $\eta_0 | (u'_3, u'_4)$) and $\Gamma_{u'_3}, \Gamma_{u'_4} \subseteq \langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle \backslash \Gamma_{(u_1,\ldots,u_k)}$. Therefore for $i \in \{1,2\}$ we must have

$$\sigma\gamma_{u_1,l_i}\sigma^{-1}[0,0] \equiv \sigma\gamma_{u_1,l_i}\sigma^{-1}[1,1] \equiv 0 \pmod{u'_{2+i}}, \text{ and} \tag{3.19}$$

$$\sigma\gamma_{u_1,l_i}\sigma^{-1}[0,1] \equiv \sigma\gamma_{u_1,l_i}\sigma^{-1}[1,0] \equiv 0 \pmod{\frac{\operatorname{lcm}(u_1,\ldots,u_k)}{u'_{2+i}}}. \tag{3.20}$$

If possible let $5 \nmid (u'_3, u'_4)$ and WLOG assume that $5 \nmid u'_3$. Then $v_5(\frac{\operatorname{lcm}(u_1,\ldots,u_k)}{u'_3}) = n_0$. From (3.20), we get

$$a^2 \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2 + b^2 \equiv c^2 \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2 + d^2 \equiv 0 \pmod{5^{n_0}}, \tag{3.21}$$

i.e.,

$$a^2 d^2 - b^2 c^2 \equiv 0 \pmod{5^{n_0}}. \tag{3.22}$$

Since $ad - bc = 1$, from the last equation we get

$$ad + bc \equiv 0 \pmod{5^{n_0}}. \tag{3.23}$$

Therefore $v_5(a) = v_5(b) = v_5(c) = v_5(d) = 0$, in particular this implies $v_5(abcd) = 0 < n_0$, which contradicts Lemma 3.4. Therefore $5 | (u'_3, u'_4)$. In particular we have $5^{n_0} | (u'_3, u'_4)$. Recall that from (3.14) we have

$$l_i M \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} r_{u_1,l_i} \equiv 1 \pmod{5^{n_0}} \text{ for } i \in \{1,2\}. \tag{3.24}$$

Using this congruence, from (3.19) we have

$$ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2 + bd \equiv ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_2^2 + bd \equiv 0 \pmod{5^{n_0}}. \tag{3.25}$$

Thus we obtain

$$ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} (l_1^2 - l_2^2) \equiv 0 \pmod{5^{n_0}}, \tag{3.26}$$

equivalently we get

$$ac(l_1^2 - l_2^2) \equiv 0 \pmod{5^{n_0}}. \tag{3.27}$$

Since $(5^{n_0}, |l_1^2 - l_2^2|) \in \{1, 5\}$, (3.27) implies that $5ac \equiv 0 \pmod{5^{n_0}}$.

Consider the case $(5^{n_0}, |l_1^2 - l_2^2|) = 5$, $5ac \equiv 0 \pmod{5^{n_0}}$ but $ac \not\equiv 0 \pmod{5^{n_0}}$, i.e., $v_5(ac) = n_0 - 1$. If $v_5(bd) \neq v_5(ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2)$, then

$$v_5(ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2 + bd) = \min\{v_5(bd), v_5(ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2)\} < n_0,$$

which contradicts (3.25). Thus we have

$$v_5(bd) = v_5(ac \frac{\operatorname{lcm}(u_1,\ldots,u_k)^2}{u_1^2} l_1^2) = v_5(ac) = n_0 - 1. \tag{3.28}$$

Consequently we get $v_5(abcd) < 2n_0$, which contradicts Lemma 3.4. Hence we must have $ac \equiv 0 \pmod{5^{n_0}}$. Consequently, from (3.25) we obtain $ac \equiv bd \equiv 0 \pmod{5^{n_0}}$. Thus, we obtain that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{N}(\langle \Gamma_{(u_1,\ldots,u_k)}, \Gamma_{u_1}, \ldots, \Gamma_{u_k} \rangle) \cap \Gamma_0(M)$, then $ac \equiv bd \equiv 0 \pmod{\operatorname{lcm}(u_1, u_2, \ldots, u_k)}$. $\square$

Corollary 3.6. *Let $N, u_1, u_2 \dots, u_k \in \mathbb{N}$ such that $4, 9 \nmid N, u_i^2 || N$ and $w_{5^2} \notin \langle \Gamma_0(N), w_{u_1^2}, \dots, w_{u_k^2} \rangle$. Then*

$$\mathcal{N}(\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle) = \langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k}, \delta_{(e,M,k)}, g : e||M, g \in S'_{(u_1,\dots,u_k),M} \cap S^M_{(u_1,\dots,u_k)} \rangle.$$

Consequently we have $\mathcal{N}(\langle \Gamma_0(N), w_{u_1^2}, \dots, w_{u_k^2} \rangle) = \Gamma_0^*(N)$.

Proof. Recall that the set $S^M_{(u_1,\dots,u_k)} := \{g_i : 1 \le i \le [\Gamma_0(M) : \langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle]\}$ forms a complete set of representatives for the left cosets of $\Gamma_0(M)$ in $\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle$, and the set

$$S^{M,+}_{(u_1,\dots,u_k)} := \{\delta^j_{(e,M,k)} g_i : 0 \le j \le 1, e||M, 1 \le i \le [\Gamma_0(M) : \langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle]\}$$

forms a complete set of representatives for the left cosets of $\Gamma_0^*(M)$ in $\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle$.

By previous discussions and Lemma 3.2, we know that

$$\mathcal{N}(\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle) \supseteq \langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k}, \delta_{(e,M,k)}, g : e||M, g \in S'_{(u_1,\dots,u_k),M} \cap S^M_{(u_1,\dots,u_k)} \rangle.$$

If $g_i \in \mathcal{N}(\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle) \cap S^M_{(u_1,\dots,u_k)}$, then by Proposition 3.5 we have $g_i \in S'_{(u_1,\dots,u_k),M}$. Since $\delta_{(e,M,k)} \in \mathcal{N}(\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle)$, by Lemma 3.1 we conclude that

$$\mathcal{N}(\langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k} \rangle) = \langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k}, \delta_{(e,M,k)}, g : e||M, g \in S'_{(u_1,\dots,u_k),M} \cap S^M_{(u_1,\dots,u_k)} \rangle.$$

This proves the first part. For the second part, it suffices to show that

$$\Upsilon^{-1}_{\mathrm{lcm}(u_1,\dots,u_k)} \langle \Gamma_{(u_1,\dots,u_k)}, \Gamma_{u_1}, \dots, \Gamma_{u_k}, \delta_{(e,M,k)}, g : e||M, g \in S'_{(u_1,\dots,u_k),M} \cap S^M_{(u_1,\dots,u_k)} \rangle \Upsilon_{\mathrm{lcm}(u_1,\dots,u_k)} = \Gamma_0^*(N).$$

This follows from the facts that

- $\Upsilon^{-1}_{\mathrm{lcm}(u_1,\dots,u_k)} \delta_{(e,M,k)} \Gamma_{(u_1,\dots,u_k)} \Upsilon_{\mathrm{lcm}(u_1,\dots,u_k)} = w_{e,N}\Gamma_0(N)$ (with $e||M$) and
- for any $g \in S'_{(u_1,\dots,u_k),M}$, we have $\Upsilon^{-1}_{\mathrm{lcm}(u_1,\dots,u_k)} g \Upsilon_{\mathrm{lcm}(u_1,\dots,u_k)} \in w_{u'^2,N}\Gamma_0(N)$ for some $u'||\mathrm{lcm}(u_1,\dots,u_k)$. Conversely, for any $u'||\mathrm{lcm}(u_1,\dots,u_k)$, we have $\Upsilon_{\mathrm{lcm}(u_1,\dots,u_k)} w_{u'^2,N} \Upsilon^{-1}_{\mathrm{lcm}(u_1,\dots,u_k)} \in S'_{(u_1,\dots,u_k),M}$. □

3.2 Exact normalizer of $\langle \Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2} \rangle$

Now suppose that $w_{25} \in \langle \Gamma_0(N), w_{u_1^2}, w_{u_2^2}, \dots, w_{u_k^2} \rangle$ (note that this assumption implies $25||N$). Without loss of generality we can assume that $u_1 = 5$ and $5 \nmid \prod_{i=2}^k u_i$. We first compute the normalizer of $\langle \Gamma_0(N), w_{25} \rangle$. Then with the help of this result and Proposition 3.3 we compute the normalizer of $\langle \Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2} \rangle$.

By Theorem 2.12, we know that $\mathcal{N}(\langle \Gamma_0(N), w_{25} \rangle) \subseteq \Upsilon_5^{-1} \Gamma_0^*(M') \Upsilon_5$, where $M' := \frac{N}{25}$ (note that $(5, M') = 1$). We introduce the following notations:

$$\tilde{\Gamma}_5(M') := \left\{ \left(\begin{smallmatrix} 5x & y \\ M'z & 5w \end{smallmatrix}\right) \in \Gamma_0(M') : x, y, z, w \in \mathbb{Z} \right\},$$

$$\tilde{\Gamma}_0^5(5M') := \left\{ \left(\begin{smallmatrix} x & 5y \\ 5M'z & w \end{smallmatrix}\right) \in \Gamma_0(M') : x, y, z, w \in \mathbb{Z} \right\}, \text{ and } \tilde{\Gamma}^5_{5M'} := \langle \tilde{\Gamma}_5(M'), \tilde{\Gamma}_0^5(5M') \rangle.$$

For $e||M'$, we fix $x_{(e,M',1)}, y_{(e,M',1)} \in \mathbb{Z}$ such that $ey_{(e,M',1)} - 25\frac{M'}{e} x_{(e,M',1)} = 1$ i.e.,

$$\delta_{(e,M',1)} := \frac{1}{\sqrt{e}} \left(\begin{smallmatrix} e & 5x_{(e,M',1)} \\ 5M' & e \cdot y_{(e,M',1)} \end{smallmatrix}\right) \in w_{e,M'}\Gamma_0(M').$$

The set $\{\mathrm{id}, \delta_{(e,M',1)} : e||M'\}$ forms a complete set of representatives for the left cosets of $\Gamma_0^*(M')$ in $\Gamma_0(M')$.

Let $B_j := \left(\begin{smallmatrix} M'j+1 & -j \\ -M' & 1 \end{smallmatrix}\right)$ and $C_i := \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$. First observe that $[\Gamma_0(M') : \tilde{\Gamma}_0^5(5M')] = [\Gamma_0(M') : \Gamma_0(25M')]$, and the set $S_{5,M'} := \{B_jC_i, C_i : 0 \le i, j \le 4\}$ forms a complete set of representatives for the left cosets of $\tilde{\Gamma}_0^5(5M')$ in $\Gamma_0(M')$. Let $S^{M'}_{(5)}$ (a subset of $S_{5,M'}$) be a complete set of representatives for the left cosets of $\tilde{\Gamma}^5_{5M'}$ in $\Gamma_0(M')$. Consequently the set $S^{M',+}_{(5)} := \left\{ \delta^l_{(e,M',1)} g : g \in S^{M'}_{(5)}, e||M', l \in \{0,1\} \right\}$ forms a complete set of representatives for the left cosets of $\tilde{\Gamma}^5_{5M'}$ in $\Gamma_0^*(M')$.

Proposition 3.7. *We have $\mathcal{N}(\tilde{\Gamma}^5_{5M'}) = \left\langle \tilde{\Gamma}^5_{5M'}, \delta_{(e,M',1)}, B_jC_0, B_0C_i : e||M' \right\rangle$, where $0 \le j, i \le 4$ such that $M'j \equiv 2 \pmod 5$ and $i + j \equiv 0 \pmod 5$.*

Proof. It is easy to check that $\delta_{(e,M',1)} \in \mathcal{N}(\tilde{\Gamma}^5_{5M'})$ and $C_i \notin \mathcal{N}(\tilde{\Gamma}^5_{5M'})$ for $i \neq 0$, hence it suffices to compute the B_j's and C_i's such that $B_jC_i \in S^{M'}_{(5)} \cap \mathcal{N}(\tilde{\Gamma}^5_{5M'})$ (cf. Lemma 3.1). We give a complete proof for the case $M' \equiv 1 \pmod 5$, the proofs are similar for other cases. Assume that $M' \equiv 1 \pmod 5$. There exist $r_1, k_1 \in \mathbb{Z}$ such that $\gamma := \left(\begin{smallmatrix} 5 & -1 \\ M'r_1 & 5k_1 \end{smallmatrix}\right) \in \tilde{\Gamma}_5(M')$. Let $B_jC_i \notin \tilde{\Gamma}^5_{5M'}$. If $B_jC_i \in \mathcal{N}(\tilde{\Gamma}^5_{5M'})$, then $E := B_jC_i\gamma(B_jC_i)^{-1} \in \mathcal{N}(\tilde{\Gamma}^5_{5M'})$, i.e., either $E[0,0] \equiv E[1,1] \equiv 0 \pmod 5$ or $E[0,1] \equiv E[1,0] \equiv 0 \pmod 5$.

Suppose $E[0,0] \equiv E[1,1] \equiv 0 \pmod 5$, then

$$i^2M'(M'j+1) - i(2jM'+1) + j + M'(M'j+1) \equiv i^2(j+1) - i(2j+1) + 2j + 1 \equiv 0 \pmod 5. \tag{3.29}$$

If (j,i) is a solution of (3.29) in $\mathbb{Z}/5\mathbb{Z}$, then it is easy to see that $(j,i) \in \{(2,0),(4,1),(1,2)\}$. Since $B_4C_1 \in \tilde{\Gamma}^5_{5M'}$, we get $B_jC_i \in \{B_1C_2, B_2C_0\}$. Observe that $(B_2C_0)\tilde{\Gamma}^5_{5M'} = (B_1C_2)\tilde{\Gamma}^5_{5M'}$.

Now suppose that $E[0,1] \equiv E[1,0] \equiv 0 \pmod 5$. The condition $E[1,0] \equiv 0 \pmod 5$ gives

$$i^2 - 2i + 2 \equiv 0 \pmod 5. \tag{3.30}$$

The solutions of (3.30) are $i \in \{3,4\}$. On the other hand, the relation $E[0,1] \equiv 0 \pmod 5$ gives

$$j^2(-i^2+2i-2) - j(2i^2-2i+2) - i^2 - 1 \equiv 0 \pmod 5. \tag{3.31}$$

For $i = 3$ (resp., $i = 4$), from (3.31) we get $j = 0$ (resp., $j = 3$). Therefore in this case $B_jC_i \in \{B_0C_3, B_3C_4\}$. It is easy to check that $(B_0C_3)\tilde{\Gamma}^5_{5M'} = (B_3C_4)\tilde{\Gamma}^5_{5M'}$. Thus we obtain that $\mathcal{N}(\tilde{\Gamma}^5_{5M'}) \subseteq \left\langle \tilde{\Gamma}^5_{5M'}, \delta_{(e,M',1)}, B_2C_0, B_0C_3 : e||M' \right\rangle$. Now to prove the equality, it suffices to show that $B_2C_0, B_0C_3 \in \mathcal{N}(\tilde{\Gamma}^5_{5M'})$.

First consider a matrix of the form $\gamma := \left(\begin{smallmatrix} 5x & y \\ M'z & 5w \end{smallmatrix}\right) \in \tilde{\Gamma}^5_{5M'}$. Since $M' \equiv 1 \pmod 5$, we have

$$\begin{aligned} &(B_2C_0\gamma(B_2C_0)^{-1})[0,0] \equiv -2(y+z) \pmod 5, \quad (B_2C_0\gamma(B_2C_0)^{-1})[1,1] \equiv 2(y+z) \pmod 5, \text{ and} \\ &(B_2C_0\gamma(B_2C_0)^{-1})[0,1] \equiv z-y \pmod 5, \quad (B_2C_0\gamma(B_2C_0)^{-1})[1,0] \equiv z-y \pmod 5. \end{aligned}$$

Since $yz \equiv -1 \pmod 5$, either $y+z \equiv 0 \pmod 5$ or $z-y \equiv 0 \pmod 5$. Thus $B_2C_0\gamma(B_2C_0)^{-1} \in \tilde{\Gamma}^5_{5M'}$.

Now consider the matrix of the form $\delta := \left(\begin{smallmatrix} x & 5y \\ 5M'z & w \end{smallmatrix}\right) \in \tilde{\Gamma}^5_{5M'}$. We have

$$\begin{aligned} &(B_2C_0\delta(B_2C_0)^{-1})[0,0] \equiv -2(x+w) \pmod 5, \quad (B_2C_0\delta(B_2C_0)^{-1})[1,1] \equiv -2(x+w) \pmod 5, \text{ and} \\ &(B_2C_0\delta(B_2C_0)^{-1})[0,1] \equiv x-w \pmod 5, \quad (B_2C_0\delta(B_2C_0)^{-1})[1,0] \equiv -(x-w) \pmod 5. \end{aligned}$$

Since $xw \equiv 1 \pmod 5$, either $x+w \equiv 0 \pmod 5$ or $x-w \equiv 0 \pmod 5$. Thus $B_2C_0\delta(B_2C_0)^{-1} \in \tilde{\Gamma}^5_{5M'}$. Therefore we conclude that $B_2C_0 \in \mathcal{N}(\tilde{\Gamma}^5_{5M'})$. A similar argument discussed so far shows that $B_0C_3 \in \mathcal{N}(\tilde{\Gamma}^5_{5M'})$. Thus $\mathcal{N}(\tilde{\Gamma}^5_{5M'}) \supseteq \left\langle \tilde{\Gamma}^5_{5M'}, \delta_{(e,M',1)}, B_2C_0, B_0C_3 : e||M' \right\rangle$. This completes the proof. $\square$

As an immediate consequence of Proposition 3.7, we obtain

Corollary 3.8. *Let $M' \in \mathbb{N}$ such that $(5, M') = 1$ and $4, 9 \nmid M'$. Then*

$$\mathcal{N}(\langle \Gamma_0(5^2M'), w_{5^2} \rangle) = \left\langle \Gamma_0(5^2M'), w_{5^2}, w_{e,5^2M'}, \Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5 : e||M' \right\rangle,$$

where $0 \le j, i \le 4$ such that $M'j \equiv 2 \pmod 5$ and $i + j \equiv 0 \pmod 5$.

We now prove that $\mathcal{N}(\langle \Gamma_0(N), w_{25}) \subseteq \mathcal{N}(\langle \Gamma_0(N), w_{25}, w_{u_2^2}, \ldots, w_{u_k^2} \rangle)$, and the following lemma plays a very important role in proving such result.

Lemma 3.9. *Let $M' \in \mathbb{N}$ such that $4, 9 \nmid M'$, $(5, M') = 1$ and W be any subgroup generated by the Atkin-Lehner involutions such that $w_{5^2,5^2M'} \in W$. Let $w_d := w_{d,5^2M'} \in \langle \Gamma_0(5^2M'), W \rangle$ be an Atkin-Lehner involution and $\sigma \in \{\Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5\}$, where i, j are defined as in Corollary 3.8. Then $\sigma w_d \sigma^{-1} \in \langle \Gamma_0(5^2M'), W \rangle$ if and only if $\frac{d}{(5^2,d)} \equiv \pm 1 \pmod 5$.*

Proof. Note that we have $(5^2, d) \in \{1, 5^2\}$. Since $\sigma w_{5^2} \sigma^{-1} \in \langle \Gamma_0(5^2M'), w_{5^2} \rangle$, without loss of generality we can assume that $(5^2, d) = 1$ (note that this also implies $d||M'$).

Since $\sigma \in \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])\backslash\mathrm{PSL}_2(\mathbb{Z})$, it is easy to observe that $\sigma w_d\sigma^{-1} \in \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}, \frac{1}{\sqrt{d}}])\backslash\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$. Any arbitrary element of $\langle\Gamma_0(5^2M'), W\rangle$ can be written in the form $w_{d'}^{m_1}w_{5^2}^{m_0}\gamma$, where $m_0, m_1 \in \{0, 1\}$, $\gamma \in \Gamma_0(5^2M')$ and $w_{d'} \in W$.

If $\sigma w_d\sigma^{-1} \in \langle\Gamma_0(5^2M'), W\rangle$, then we have $\sigma w_d\sigma^{-1} = w_{d'}^{n}w_{5^2}^{n_0}\gamma$ i.e., $w_{d'}^{-n}\sigma w_d\sigma^{-1} = w_{5^2}^{n_0}\gamma$, for some $n, n_0 \in \{0, 1\}$, $w_{d'} \in W$ and $\gamma \in \Gamma_0(5^2M')$. If $w_{d'}^{n} \neq w_d$, then $w_{d'}^{-n}\sigma w_d\sigma^{-1} \notin \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$ but $w_{5^2}^{n_0}\gamma \in \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$. Hence we have $w_d^{-1}\sigma w_d\sigma^{-1} = w_{5^2}^{n_0}\gamma$, equivalently we have $E := \Upsilon_5 w_d^{-1}\sigma w_d\sigma^{-1}\Upsilon_5^{-1} \in \tilde{\Gamma}^5_{5M'}$.

Recall that if $\delta \in \tilde{\Gamma}^5_{5M'}$, then we must have either $\delta[0,0] \equiv \delta[1,1] \equiv 0 \pmod 5$ or $\delta[1,0] \equiv \delta[0,1] \equiv 0 \pmod 5$ (note that we always have $\delta[1,0] \equiv 0 \pmod{M'}$). We give a complete proof for the case $M' \equiv 1 \pmod 5$. The proofs are similar for the other cases.

Let $M' \equiv 1 \pmod 5$. In this case, without loss of generality we can assume that $\sigma = \Upsilon_5^{-1}B_2C_0\Upsilon_5$, and consider a representative $w_d = \frac{1}{\sqrt{d}}\left(\begin{smallmatrix} xd & y \\ 5^2M'z & wd\end{smallmatrix}\right)$ such that $xwd^2 - 5^2M'yz = d$. Considering the modulo 5 reductions, we get

$$E[0,0] \equiv 2-2w^2d+1 \equiv -2(1+w^2d) \equiv -2(1\pm d) \pmod 5,\ E[0,1] \equiv 4+2-w^2d \equiv 1-w^2d \equiv 1-(\pm d) \pmod 5,$$

$$E[1,0] \equiv -x^2d+1 \equiv 1-x^2d \equiv 1-(\pm d) \pmod 5,\ E[1,1] \equiv -2x^2d+2+1 \equiv -2(1+x^2d) \equiv -2(1\pm d) \pmod 5.$$

From the last relations it is easy to see that either $E[0,0] \equiv E[1,1] \equiv 0 \pmod 5$ or $E[1,0] \equiv E[0,1] \equiv 0 \pmod 5$ if and only if $d \equiv \pm 1 \pmod 5$.

Therefore for $M' \equiv 1 \pmod 5$, we have $\sigma w_d\sigma^{-1} \in \langle\Gamma_0(5^2M'), W\rangle$ if and only if $d \equiv \pm 1 \pmod 5$. The result follows. □

Corollary 3.10. *Let $N, u_2, \dots, u_k \in \mathbb{N}$ such that $4, 9 \nmid N$, $u_i^2 || N$ and $5 \nmid u_i$ for $i \in \{2, \dots, k\}$. Then*

$$\mathcal{N}(\langle\Gamma_0(N), w_{25}\rangle) \subseteq \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle).$$

Proof. Recall that $w_{25}, w_{e,N} \in \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)$ for $e||\frac{N}{25}$. Any element of $\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle\backslash\Gamma_0(N)$ can be written in the form $w_{25^md^2}\gamma$ for some $\gamma \in \Gamma_0(N)$, $m \in \{0, 1\}$ and $d||\mathrm{lcm}(u_2, \dots, u_k)$. Let $\sigma \in \{\Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5\}$, where i, j are defined as in Corollary 3.8. Since $\frac{25^md^2}{(25, 25^md^2)} \equiv \pm 1 \pmod 5$, by Lemma 3.9 we get $\sigma w_{25^md^2}\sigma^{-1} \in \langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle$. Thus $\sigma \in \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)$. Now the result follows from Corollary 3.8. □

We are now ready to compute the normalizer of $\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle$.

Proposition 3.11. *Let $N, u_2, \dots, u_k \in \mathbb{N}$ such that $4, 9 \nmid N$, $u_i^2 || N$ and $5 \nmid u_i$ for $i \in \{2, \dots, k\}$. Then*

$$\mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle) = \mathcal{N}(\langle\Gamma_0(N), w_{25}\rangle).$$

Proof. By Corollary 3.10 we know that $\mathcal{N}(\langle\Gamma_0(N), w_{25}\rangle) \subseteq \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)$. We now prove the other inclusion. For simplicity of notation, we write $u := \mathrm{lcm}(u_2, \dots, u_k)$ and assume that $u > 1$.

Recall that $(5, u) = 1$. By Theorem 2.12 we know that $\mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle) \subseteq \Upsilon_{5u}^{-1}\Gamma_0^*(M)\Upsilon_{5u}$, where $M := \frac{N}{25u^2}$. As discussed in the beginning of §3, it suffices to compute the elements $\Upsilon_{5u}^{-1}\sigma\Upsilon_{5u}$ with $\sigma \in \Gamma_0(M)$ such that $\Upsilon_{5u}^{-1}\sigma\Upsilon_{5u} \in \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)$. Let $\sigma := \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(M)$ such that $\Upsilon_{5u}^{-1}\sigma\Upsilon_{5u} \in \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)$. Note that for such σ, we have

$$\sigma \in \Upsilon_{5u}\mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)\Upsilon_{5u}^{-1} = \mathcal{N}(\langle\Gamma_{(5,u_1,\dots,u_k)}, \Gamma_5, \Gamma_{u_2}, \dots, \Gamma_{u_k}\rangle).$$

By Proposition 3.3, we know that $ac \equiv bd \equiv 0 \pmod u$.

Claim: We now prove that $\gamma := \Upsilon_{5u}^{-1}\sigma\Upsilon_{5u} = \left(\begin{smallmatrix} a & \frac{b}{5u} \\ 5uc & d\end{smallmatrix}\right) \in \mathcal{N}(\langle\Gamma_0(N), w_{25}\rangle)$.

Since $(a, b) = (c, d) = 1$, the condition $ac \equiv bd \equiv 0 \pmod u$ implies $ab \equiv cd \equiv 0 \pmod u$.

Since $\gamma \in \mathcal{N}(\langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle)$, for any $\tilde{\gamma} \in \langle\Gamma_0(N), w_{25}\rangle$ we have $\gamma\tilde{\gamma}\gamma^{-1} \in \langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle$. Note that $\alpha \in \langle\Gamma_0(N), w_{25}, w_{u_2^2}, \dots, w_{u_k^2}\rangle \cap \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$ if and only if $\alpha \in \langle\Gamma_0(N), w_{25}\rangle$. We prove the claim by showing that $\gamma\tilde{\gamma}\gamma^{-1} \in \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$ for any $\tilde{\gamma} \in \langle\Gamma_0(N), w_{25}\rangle$.

First assume that $\gamma_1 \in w_{25}\Gamma_0(N)$, i.e., $\gamma_1 := \left(\begin{smallmatrix} 5x & \frac{y}{5} \\ \frac{N}{5}z & 5w\end{smallmatrix}\right)$ with $x, y, x, w \in \mathbb{Z}$ such that $25xw - \frac{N}{25}yz = 1$. Then

$$E_1[0,0] := -acuy + 5adx - 5bcw + bdMuz,\ E_1[0,1] := \frac{1}{5}a^2y - \frac{ab}{u}(x-w) - \frac{1}{5}Mb^2z,$$
$$E_1[1,0] := -5c^2u^2y + 25cdux - 25cduw + 5Mu^2d^2z,\ E_1[1,1] := acuy + 5adw - 5bcx - Mubdz,\ \text{where } E_1 := \gamma\gamma_1\gamma^{-1}.$$

Since $ab \equiv 0 \pmod u$, we get $E_1 \in \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$. Therefore $E_1 \in \langle \Gamma_0(N), w_{25}, w_{u_2^2}, \ldots, w_{u_k^2}\rangle \cap \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$. Thus we obtain $E_1 \in \langle \Gamma_0(N), w_{25}\rangle$.

Now consider $\gamma_2 := \left(\begin{smallmatrix} x & y \\ Nz & w \end{smallmatrix}\right) \in \Gamma_0(N)$ and let $E_2 := \gamma\gamma_2\gamma^{-1}$. Then

$$E_2[0,0] := -5acuy + adx - bcw + 5bdMuz,\ E_2[0,1] := a^2y - \frac{1}{5}\frac{ab}{u}(x-w) - b^2Mz,$$
$$E_2[1,0] := -25c^2u^2y + 5cdux - 5cduw + 25d^2Mu^2z,\ E_2[1,1] := 5acuy + adw - bcx - 5bdMuz.$$

Since $ab \equiv 0 \pmod u$, we get $E_2 \in \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$. Therefore $E_2 \in \langle \Gamma_0(N), w_{25}, w_{u_2^2}, \ldots, w_{u_k^2}\rangle \cap \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{5}])$. Thus we obtain $E_2 \in \langle \Gamma_0(N), w_{25}\rangle$.

Hence we conclude that $\gamma\langle \Gamma_0(N), w_{25}\rangle\gamma^{-1} \subseteq \langle \Gamma_0(N), w_{25}\rangle$, i.e., $\gamma \in \mathcal{N}(\langle \Gamma_0(N), w_{25}\rangle)$. Consequently, we obtain that $\mathcal{N}(\langle \Gamma_0(N), w_{25}, w_{u_2^2}, \ldots, w_{u_k^2}\rangle) \subseteq \mathcal{N}(\langle \Gamma_0(N), w_{25}\rangle)$. The result follows. $\square$

3.3 Exact normalizer of $\langle \Gamma_0(N), W\rangle$ for arbitrary subgroup W

Let $N \in \mathbb{N}$ such that $4, 9 \nmid N$ and W be a subgroup generated by certain Atkin-Lehner involutions. Then we can find positive integers $u_1, u_2 \ldots, u_k$ and $v_{k+1}, \ldots, v_n$ such that

$$\langle \Gamma_0(N), W\rangle = \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}, w_{v_{k+1}}, \ldots, w_{v_n}\rangle, \tag{3.32}$$

and for any Atkin-Lehner involution $w_d \in \langle \Gamma_0(N), w_{v_{k+1}}, \ldots, w_{v_n}\rangle$, d is not a perfect square[1].

It is well known that $\mathcal{N}(\langle \Gamma_0(N), W\rangle) \supseteq \Gamma_0^*(N)$. Moreover, by Theorem 2.12, we know that

$$\mathcal{N}(\langle \Gamma_0(N), W\rangle) \leq \mathcal{N}(\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle). \tag{3.33}$$

If $w_{25} \notin W$, then $w_{25} \notin \langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle$. Consequently, by Corollary 3.6 we have $\mathcal{N}(\langle \Gamma_0(N), w_{u_1^2}, \ldots, w_{u_k^2}\rangle) = \Gamma_0^*(N)$. Therefore using (3.33) we conclude that $\mathcal{N}(\langle \Gamma_0(N), W\rangle) = \Gamma_0^*(N)$. Thus obtain the following theorem.

Theorem 3.12. *Let $N \in \mathbb{N}$ and W be a subgroup generated by certain Atkin-Lehner involutions such that $4, 9 \nmid N$ and $w_{25} \notin W$. Then $\mathcal{N}(\langle \Gamma_0(N), W\rangle) = \Gamma_0^*(N)$.*

We now study the case where $w_{25} \in W$. More precisely, we prove the following theorem.

Theorem 3.13. *Let $N \in \mathbb{N}$ and W be a subgroup generated by certain Atkin-Lehner involutions such that $4, 9 \nmid N$ and $w_{25} \in W$.*

1. *If there exists $w_d \in W$ such that $\frac{d}{(25,d)} \not\equiv \pm 1 \pmod 5$, then $\mathcal{N}(\langle \Gamma_0(N), W\rangle) = \Gamma_0^*(N)$.*
2. *If $\frac{d}{(25,d)} \equiv \pm 1 \pmod 5$ for all $w_d \in W$, then $\mathcal{N}(\langle \Gamma_0(N), W\rangle) = \langle \Gamma_0^*(N), \Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5\rangle$ where $B_j := \left(\begin{smallmatrix} \frac{N}{25}j+1 & -j \\ -\frac{N}{25} & 1 \end{smallmatrix}\right)$, $C_i := \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$, $0 \leq j, i \leq 4$ such that $\frac{N}{25}j \equiv 2 \pmod 5$ and $i+j \equiv 0 \pmod 5$. Moreover, the subgroup $\langle \Upsilon_5^{-1}B_jC_0\Upsilon_5 = (\Upsilon_5^{-1}B_0C_i\Upsilon_5)^{-1}\rangle$ has order 3 in $\mathcal{N}(\langle \Gamma_0(N), W\rangle)/\langle \Gamma_0(N), W\rangle$.*

Proof. Recall that $\mathcal{N}(\langle \Gamma_0(N), W\rangle) \supseteq \Gamma_0^*(N)$. Since $w_{25} \in W$, without loss of generality in (3.32) we can assume that $u_1 = 5$ and $5 \nmid (\prod_{i'=2}^k u_{i'} \cdot \prod_{j'=k+1}^n v_{j'})$. By Proposition 3.11 and Corollary 3.8 we get

$$\mathcal{N}(\langle \Gamma_0(N), w_{25}, w_{u_2^2}, \ldots, w_{u_k^2}\rangle) = \langle \Gamma_0^*(N), \Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5\rangle, \tag{3.34}$$

where $B_j := \left(\begin{smallmatrix} \frac{N}{25}j+1 & -j \\ -\frac{N}{25} & 1 \end{smallmatrix}\right)$, $C_i := \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$, $0 \leq j, i \leq 4$ such that $\frac{N}{25}j \equiv 2 \pmod 5$ and $i + j \equiv 0 \pmod 5$.

Let $\sigma \in \{\Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5\}$. If there exists $w_d \in W$ such that $\frac{d}{(25,d)} \not\equiv \pm 1 \pmod 5$, then from Lemma 3.9 we get $\sigma w_d\sigma^{-1} \notin \langle \Gamma_0(N), W\rangle$. Consequently from (3.33) and (3.34) we obtain $\mathcal{N}(\langle \Gamma_0(N), W\rangle) = \Gamma_0^*(N)$. This proves the first part.

On the other hand if $\frac{d}{(25,d)} \equiv \pm 1 \pmod 5$ for all $w_d \in W$, then from Lemma 3.9 we get $\sigma w_d\sigma^{-1} \in \langle \Gamma_0(N), W\rangle$ for all $w_d \in \langle \Gamma_0(N), W\rangle$. Thus $\sigma \in \mathcal{N}(\langle \Gamma_0(N), W\rangle)$. Moreover, it is easy to check that σ has order 3 in $\mathcal{N}(\langle \Gamma_0(N), W\rangle)/\langle \Gamma_0(N), W\rangle$ and $\Upsilon_5^{-1}B_jC_0\Upsilon_5, \Upsilon_5^{-1}B_0C_i\Upsilon_5$ are inverse of each other in $\mathcal{N}(\langle \Gamma_0(N), W\rangle)/\langle \Gamma_0(N), W\rangle$. Now the second part follows from (3.33) and (3.34). $\square$

[1] This can be done as follows: let $H_1 := \langle \Gamma_0(N), W\rangle/\Gamma_0(N)$ and $H_2 := \langle \Gamma_0(N), w_{u^2} : w_{u^2} \in W\rangle/\Gamma_0(N)$. Then the generators of H_2 will give the u_i's, and the non-trivial generators of H_1/H_2 will give the v_j's.

4 On the modular automorphisms of order 3 of $X_0(N)/W_N$

For $N \in \mathbb{N}$ and a subgroup $W_N \leq B(N)$, consider the quotient curve $X_0(N)/W_N$. Clearly we have

$$B(N)/W_N \leq \mathcal{N}(\Gamma_0(N)+W_N)/(\Gamma_0(N)+W_N) \leq \mathrm{Aut}(X_0(N)/W_N)$$

where $\mathcal{N}(\Gamma_0(N)+W_N)/(\Gamma_0(N)+W_N)$ is the modular automorphism group of $X_0(N)/W_N$.

Assuming $4, 9 \nmid N$, by Theorem 3.12 and Theorem 3.13 we know that the modular automorphism group of $X_0(N)/W_N$ is exactly $B(N)/W_N$, except for the case $w_{25} \in W_N$ and $\frac{d}{(25,d)} \equiv \pm 1 \pmod 5$ for all $w_d \in W_N$; and in such situation we have

$$\mathcal{N}(\Gamma_0(N)+W_N)/(\Gamma_0(N)+W_N) = \langle \Gamma_0^*(N), \Upsilon_5^{-1} B_j C_0 \Upsilon_5, \Upsilon_5^{-1} B_0 C_i \Upsilon_5 \rangle/(\Gamma_0(N)+W_N),$$

where $0 \leq j, i \leq 4$ such that $\frac{N}{25} j \equiv 2 \pmod 5$ and $i + j \equiv 0 \pmod 5$.

In this section we restrict to $N = 25M$ where M is square-free, $(5, M) = 1$, and a subgroup $W_N \leq B(N)$ of the form $\langle w_{25}, w_{v_2}, \ldots, w_{v_n} \rangle$ with $v_l || M$ and $v_l \equiv \pm 1 \pmod 5$ for all $l \in \{2, \ldots, n\}$. Consider the order 3 element $\sigma_M := \Upsilon_5^{-1} B_j C_0 \Upsilon_5$ in $\mathcal{N}(\Gamma_0(25M)+W_{25M})/(\Gamma_0(25M)+W_{25M})$.

Lemma 4.1. *Under the assumptions and notations in this section,* $\mathcal{N}(\Gamma_0(25M)+W_{25M})/(\Gamma_0(25M)+W_{25M}) = \langle B(25M)/W_{25M}, \sigma_M \rangle$ *and* σ_M *has order 3. Then* σ_M *is defined over* $\mathbb{Q}(\sqrt{5})$ *(as an automorphism of* $X_0(N)/W_N$*), in particular* $\mathrm{Aut}(X_0(N)/W_N) = \mathrm{Aut}_{\mathbb{Q}(\sqrt{5})}(X_0(N)/W_N)$ *(where* $\mathrm{Aut}_K(X)$ *denotes the group of all automorphisms of* X *defined over the field* K*).*

Proof. Note that the elements of $\mathrm{Aut}(X_0(N)/W_N)$ can be thought of as automorphisms on the Jacobian variety of $X_0(N)/W_N$. Let ∞ be the cusp at infinity of $X_0(N)/W_N$. Then it is easy to check that $\sigma_M(\infty)$ is not a rational cusp (cf. [Ogg73] for the field of definition of the cusps). Therefore σ_M is not defined over $\mathbb{Q}$. Now the result follows from the fact that any automorphism of the Jacobian is defined over the compositum of the quadratic fields with discriminant D whose square divides N (cf. [KenMom88, Proposition 1.3, Lemma 1.5]). ☐

Remark 4.2. Assume $p \equiv 1 \pmod 4$ is a prime, M is a square-free positive integer coprime with p, and $W_{p^2M} = \langle w_{p^2}, w_{v_2}, \ldots, w_{v_n} \rangle$ with $v_l || M$. Then, by [KenMom88, Lemma 1.5] and [KenMom88, Proposition 1.3] any automorphism of $X_0(p^2M)/W_{p^2M}$ is defined either over $\mathbb{Q}$ or over $\mathbb{Q}(\sqrt{p})$ (the same conclusion is true if $p \equiv 3 \pmod 4$ and the Jacobian variety of $X_0(p^2M)/W_{p^2M}$ does not contain any subvariety with complex multiplication). For a prime $\ell \nmid p^2M$ we can reduce the curve $X_0(p^2M)/W_{p^2M}$ modulo ℓ, and denote such curve by $\mathcal{X}_0(p^2M)/W_{p^2M} \otimes \mathbb{F}_\ell$. Then we have an injection

$$\mathrm{Aut}(X_0(p^2M)/W_{p^2M}) \hookrightarrow \mathrm{Aut}_{\mathbb{F}_{\ell^2}}(\mathcal{X}_0(p^2M)/W_{p^2M} \otimes \mathbb{F}_\ell),$$

and using Magma in many cases (with small genus) we can compute the automorphism group over the finite field $\mathbb{F}_{\ell^2}$, via the instruction

```
Automorphisms(ChangeRing(X0NQuotient(p^2*M,[p^2,v_2,\ldots,v_n]),GF(\ell^2))).
```

Consequently, we have an upper bound for the order of the automorphism group, and a lower bound is given by the order of the modular automorphism group. For example, using Magma we obtain $|\mathrm{Aut}_{\mathbb{F}_4}(X_0(275)/\langle w_{25} \rangle)| = 6$, consequently we get $|\mathrm{Aut}(X_0(275)/\langle w_{25} \rangle)| \leq 6$. Furthermore by Theorem 3.13, we have $|\mathrm{Aut}(X_0(275)/\langle w_{25} \rangle)| \geq 6$. Therefore we conclude that $|\mathrm{Aut}(X_0(275)/\langle w_{25} \rangle)| = 6 = |\mathcal{N}(\Gamma_0(275)+\langle w_{25} \rangle)/(\Gamma_0(N)+\langle w_{25} \rangle)|$, i.e., $\mathrm{Aut}(X_0(275)/\langle w_{25} \rangle) = \mathcal{N}(\Gamma_0(275)+\langle w_{25} \rangle)/(\Gamma_0(275)+\langle w_{25} \rangle)$.

Now consider σ_M as an element of $X_0(25M)/\langle w_{25} \rangle$. We now give a theoretical explanation of the fact that for positive integers $v_2, \ldots, v_n$ with $v_l || M$ and $v_l \equiv \pm 1 \pmod 5$, σ_M induces an automorphism of order 3 on $X_0(25M)/\langle w_{25}, w_{v_2}, \ldots, w_{v_n} \rangle$.

We write the $\mathbb{Q}$-decomposition of the Jacobian of $X_0(25M)/\langle w_{25} \rangle$ by:

$$\mathrm{Jac}(X_0(25M)/\langle w_{25} \rangle) \sim_{\mathbb{Q}} \prod_{m=1}^{s} A_{f_m}^{n_m}, \tag{4.1}$$

where f_m is a newform of level N_m (with $N_m | 25M$) such that w_{25} acts as $+1$ on f_m if $25 | N_m$. Since $\mathrm{Aut}(X_0(25M)/\langle w_{25} \rangle)$ has an automorphism defined over $\mathbb{Q}(\sqrt{5})$ but not over $\mathbb{Q}$, there exist f_{l_1}, f_{l_2} (in (4.1)) such that $A_{f_{l_1}} \sim_{\mathbb{Q}(\sqrt{5})} A_{f_{l_2}}$ with $f_{l_2} = f_{l_1} \otimes \chi_5$ (where χ_5 is the quadratic Dirichlet character associated to $\mathbb{Q}(\sqrt{5})$). It is well-known that the abelian variety A_f is simple over $\overline{\mathbb{Q}}$ if and only if f does not have any inner twist, i.e. there is no quadratic Dirichlet character χ such that $f \otimes \chi$ is a Galois conjugated of f. This condition amounts to say that $\mathrm{End}_{\mathbb{Q}}(A_f) = \mathrm{End}_{\overline{\mathbb{Q}}}(A_f)$. Assume that f_{l_1} and f_{l_2} (appearing in (4.1)) do not have any inner twist (in particular $l_1 \neq l_2$). If $A_{f_{l_1}}$ and $A_{f_{l_2}}$ are isogenous over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$, then there exists a Dirichlet character χ such that $A_{f_{l_2}} = A_{f_{l_1}} \otimes \chi$ (cf. [GaJiU12, Proposition 4.2]), and if χ is the quadratic Dirichlet character attached to the quadratic number field $\mathbb{Q}(\sqrt{5})$, then there is an isogeny (defined over $\mathbb{Q}(\sqrt{5})$) between the abelian varieties $A_{f_{l_2}}$ and $A_{f_{l_1}}$.

Therefore assuming that all the f_m's appearing in (4.1) have no inner twist, we have that the modular automorphisms of order 3 in Corollary 3.8 are coming from matrices (acting on the canonical model obtained using the cusp forms appearing in (4.1)) defined over $\mathbb{Q}(\sqrt{5})$ (such matrices consist of blocks corresponding to the $\mathbb{Q}(\sqrt{5})$-isogeny factors $A_{f_{l_1}}^{n_{l_1}} \times A_{f_{l_2}}^{n_{l_2}} \sim_{\mathbb{Q}(\sqrt{5})} A_{f_{l_1}}^{n_{l_1}+n_{l_2}}$ where $f_{l_2} = f_{l_1} \otimes \chi_5$ and $f_{l_1} \neq f_{l_2}$).

In order that the modular automorphism σ_M of order 3 of $X_0(25M)/\langle w_{25}\rangle$ descends to an order 3 automorphism of $X_0(25M)/\langle w_{25}, w_{v_2}, \ldots, w_{v_n}\rangle$, a sufficient condition is that for any quadratic twist $f_{l_1} \otimes \chi_5 = f_{l_2}$ (in (4.1)) where the action of σ_M is non-trivial on the $\mathbb{Q}(\sqrt{5})$-isogeny factor $A_{f_{l_1}}^{n_{l_1}+n_{l_2}}$, the Atkin-Lehner involution w_{v_l} should act with the same sign on $A_{f_{l_1}}$ and $A_{f_{l_1}}$ (cf. [BaDa24, Lemma 18] for more detail).

We recall the following result of Atkin-Lehner in [AtLeh70, p.156] concerning quadratic twists:

Lemma 4.3. *Let p be a prime, $M' \in \mathbb{N}$ with $(p, M') = 1$, and χ_p be the quadratic Dirichlet character associated to $\mathbb{Q}(\sqrt{p})$. If f is a newform for $\Gamma_0(M')$ or $\Gamma_0(pM')$, then $f \otimes \chi_p$ is a newform for $\Gamma_0(p^2M')$. Furthermore*

- *for $d||M'$ we have $f \otimes \chi_p|w_d = \left(\frac{d}{p}\right)\epsilon_d(f)(f \otimes \chi_p)$, where $f|w_d = \epsilon_d(f)f$, and $\left(\frac{d}{p}\right)$ denotes the Kronecker symbol.*
- $f \otimes \chi_p|w_{p^2} = \left(\frac{-1}{p}\right) f \otimes \chi_p$

If f is a newform for $\Gamma_0(p^2M')$ and $f \otimes \chi_p$ is also a newform for $\Gamma_0(p^2M')$, then for any $d||M'$ we have $f \otimes \chi_p|w_d = \left(\frac{d}{p}\right)\epsilon_d(f)f \otimes \chi_p$.

We recall that (cf. [Ogg73]) if $f_{m_1} \otimes \chi_p = f_{m_2}$ where f_{m_1} and f_{m_2} are newforms of levels M_1 and M_2 respectively, both dividing p^2M with $(M, p) = 1$, then $M_1 = p^2M'$ or $M_2 = p^2M'$ for a natural number $M'|M$. Therefore, the level of the other quadratic twisted modular form involved is M', pM' or p^2M'.

Corollary 4.4. *Consider $N = 5^2M$, where M is a square-free positive integer with $(5, M) = 1$. Assume that the Jacobian of $X_0(N)/\langle w_{25}\rangle$ has no inner twist, and for each quadratic twist $f_{m_1} \otimes \chi_5 = f_{m_2}$ with $m_1 \neq m_2$ (where $A_{f_{m_1}}$ and $A_{f_{m_2}}$ are distinct $\mathbb{Q}$-isogeny factors of $\mathrm{Jac}(X_0(N)/\langle w_{25}\rangle)$) the conductor of f_{m_1} or f_{m_2} is equal to N. Then, an Atkin-Lehner involution w_d (with $d||M$) acts exactly by the same sign on $A_{f_{m_1}}$ and $A_{f_{m_2}}$ iff $\left(\frac{d}{5}\right) = 1$ iff $d \equiv \pm 1 \pmod 5$.*

Recall that any non-trivial $w_d \in B(p^2M)/W_{p^2M}$ ($p \equiv 1 \pmod 4$ is a prime and $p \nmid M$) acts by $\pm\mathrm{id}$ on each $\mathbb{Q}$-isogeny factor of the Jacobian of $X_0(p^2M)/W_{p^2M}$. Furthermore, if it acts exactly with the same sign on all distinct $\mathbb{Q}$-isogeny factors that become isogenous over $\mathbb{Q}(\sqrt{p})$, then any $w \in \mathrm{Aut}_{\mathbb{Q}(\sqrt{p})}(X_0(p^2M)/W_{p^2M}) \setminus (B(p^2M)/W_{p^2M})$ induces a non-trivial automorphism of $X_0(p^2M)/\langle W_{p^2M}, w_d\rangle$ over $\mathbb{Q}(\sqrt{p})$. (cf. [BaDa24, Lemma 18]).

Corollary 4.5. *Consider $N = 5^2M$, where M is a square-free positive integer with $(5, M) = 1$. Assume that the Jacobian of $X_0(N)/\langle w_{25}\rangle$ has no inner twist, and for each quadratic twist $f_{m_1} \otimes \chi_5 = f_{m_2}$ with $m_1 \neq m_2$ (where $A_{f_{m_1}}$ and $A_{f_{m_2}}$ are distinct $\mathbb{Q}$-isogeny factors of $\mathrm{Jac}(X_0(N)/\langle w_{25}\rangle)$) the conductor of f_{m_1} or f_{m_2} is equal to N. Let σ_M an element of order 3 in $\mathcal{N}(\Gamma_0(N)+\langle w_{25}\rangle)/(\Gamma_0(N)+\langle w_{25}\rangle)$. Then for positive integers $v_2, \ldots, v_n$ with $v_l||M$ and $v_l \equiv \pm 1 \pmod 5$, σ_M induces an automorphism of order 3 on $X_0(25M)/\langle w_{25}, w_{v_2}, \ldots, w_{v_n}\rangle$.*

Remark 4.6. By Lemma 4.3 it is easy to see that when M is a prime, the assumption that the conductor of f_{m_1} or f_{m_2} is equal to $N = 5^2M$ is always true.

Acknowledgments

The first author's work is supported by the Spanish State Research Agency through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), alongside PID2020-116542GB-I00, Ministerio de Ciencia y Universidades of Spanish government. Some part of the paper was written during the second author's visit to the Universitat Autònoma de Barcelona and the second author is grateful to the Department of Mathematics of Universitat Autònoma de Barcelona for its support and hospitality.

References

[AtLeh70] Atkin, A. O. L.; Lehner, J.: Hecke operators on $\Gamma_0(m)$. Math. Ann. 185 (1970), 134–160.

[AkSi90] Akbas, M.; Singerman, D.: The normalizer of $\Gamma_0(N)$ in $\mathrm{PSL}(2, \mathbf{R})$. Glasgow Math. J. 32 (1990), no. 3, 317–327.

[Ba08] Bars, F.: The group structure of the normalizer of $\Gamma_0(N)$ after Atkin-Lehner. Comm. Algebra 36 (2008), no. 6, 2160–2170.

[BaDa24] Bars, F.; Dalal, T.: On the automorphism group of quotient modular curves. Journal of Algebra (2025). `https://doi.org/10.1016/j.jalgebra.2025.02.037`

[BaGo21] Bars, F.; González, J.: The automorphism group of the modular curve $X_0^*(N)$ with square-free level. Trans. Amer. Math. Soc. 374 (2021), no. 8, 5783–5803.

[Con96] Conway, J. H.: Understanding groups like $\Gamma_0(N)$. Groups, difference sets, and the Monster (Columbus, OH, 1993), 327–343, Ohio State Univ. Math. Res. Inst. Publ., 4, de Gruyter, Berlin, 1996.

[Do16] Dose, V.: On the automorphisms of the nonsplit Cartan modular curves of prime level. Nagoya Math. J. 224 (2016), no.1, 74–92.

[DLM22] Dose, V.; Lido, G.; Mercuri, P.: Automorphisms of Cartan modular curves of prime and composite level. Algebar Number Theory 16(6), (2022), 1423–1461.

[GaJiU12] González, J.; Jiménez-Urroz, J.: The Sato-Tate distribution and the values of Fourier coefficients of modular newforms. Exp. Math., 21(1):84–102, 2012.

[KenMom88] Kenku,M.A. ; Momose, F.:Automorphism groups of the modular curves $X_0(N)$. Compositio Math. 65 (1988), no. 1, 51–80.

[Lan01] Lang, Mong-Lung: Normalizers of the congruence subgroups of the Hecke groups G_4 and G_6. J. Number Theory 90 (2001), no. 1, 31–43.

[Ogg73] Ogg, A.P.: Rational points on certain elliptic modular curves. Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 221–231, Proc. Sympos. Pure Math., Vol. XXIV, Amer. Math. Soc., Providence, RI, 1973.

Francesc Bars Cortina
Departament Matemàtiques, Edif. C, Universitat Autònoma de Barcelona
08193 Bellaterra, Catalonia;
Centre de Recerca Matemàtica (CRM), C. dels Til.lers,
08193 Bellaterra, Catalonia, Spain
Francesc.Bars@uab.cat

Tarun Dalal
Institute of Mathematical Sciences, ShanghaiTech University
393 Middle Huaxia Road, Pudong, Shanghai 201210, China
tarun.dalal80@gmail.com