ASPECTS OF IWASAWA THEORY OVER FUNCTION FIELDS

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Abstract. We consider $\mathbb{Z}_p^{\mathbb{N}}$-extensions $\mathcal{F}$ of a global function field F and study various aspects of Iwasawa theory with emphasis on the two main themes already (and still) developed in the number fields case as well. When dealing with the Selmer group of an abelian variety A defined over F, we provide all the ingredients to formulate an Iwasawa Main Conjecture relating the Fitting ideal and the p-adic L-function associated to A and $\mathcal{F}$. We do the same, with characteristic ideals and p-adic L-functions, in the case of class groups (using known results on characteristic ideals and Stickelberger elements for $\mathbb{Z}_p^d$-extensions). The final section provides more details for the cyclotomic $\mathbb{Z}_p^{\mathbb{N}}$-extension arising from the torsion of the Carlitz module: in particular, we relate cyclotomic units with Bernoulli-Carlitz numbers by a Coates-Wiles homomorphism.

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1. Introduction

The main theme of number theory (and, in particular, of arithmetic geometry) is probably the study of representations of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ - or, more generally, of the absolute Galois group $G_F := Gal(F^{sep}/F)$ of some global field F. A basic philosophy (basically, part of the yoga of motives) is that any object of arithmetic interest is associated with a p-adic realization, which is a p-adic representation ρ of G_F with precise concrete properties (and to any p-adic representation with such properties should correspond an arithmetic object). Moreover from this p-adic representation one defines the L-function associated to the arithmetic object. Notice that the image of ρ is isomorphic to a compact subgroup of $GL_n(\mathbb{Z}_p)$ for some n, hence it is a p-adic Lie group and the representation factors through $Gal(\mathcal{F}'/F)$, where $\mathcal{F}'$ contains subextensions $\mathcal{F}$ and F' such that $\mathcal{F}/F'$ is a pro-p extension and F'/F and $\mathcal{F}'/\mathcal{F}$ are finite.

Iwasawa theory offers an effective way of dealing with various issues arising in this context, such as the variation of arithmetic structures in p-adic towers, and is one of the main tools currently available for the knowledge (and interpretation) of zeta values associated to an arithmetic object when F is a number field [26]. This theory constructs some sort of elements, called p-adic L-functions, which provide a good understanding of both the zeta values and the arithmetic properties of the arithmetic object. In particular, the various forms of Iwasawa Main Conjecture provide a link between the zeta side and the arithmetic side.

The prototype is given by the study of class groups in the cyclotomic extensions $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$. In this case the arithmetic side corresponds to a torsion Λ-module X, where Λ is an Iwasawa algebra related to $Gal(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ and X measures the p-part of a limit of $cl(\mathbb{Q}(\zeta_{p^n}))$. As for the zeta side, it is represented by a p-adic version of the Riemann zeta function, that is, an element $\xi \in \Lambda$ interpolating the zeta values. One finds that ξ generates the characteristic ideal of X.

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For another example of Iwasawa Main Conjecture, take E an elliptic curve over $\mathbb{Q}$ and p a prime of good ordinary reduction (in terms of arithmetic objects, here we deal with the Chow motive $h^1(E)$, as before with $h^0(\mathbb{Q})$). Then on the arithmetic side the torsion Iwasawa module X corresponds to the Pontrjagin dual of the Selmer group associated to E and the p-adic L-function of interest here is an element $L_p(E)$ in an Iwasawa algebra Λ (that now is $\mathbb{Z}_p[[Gal(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]]$) which interpolates twists of the L-function of E by Dirichlet characters of $(\mathbb{Z}/p^n)^*$. As before, conjecturally $L_p(E)$ should be the generator of the characteristic ideal of X.

In both these cases, we had $F = \mathbb{Q}$ and $\mathcal{F}' = \mathbb{Q}(\zeta_{p^\infty})$. Of course there is no need for such a limitation and one can take as $\mathcal{F}'$ any p-adic extension of the global field F: for example one can deal with $\mathbb{Z}_p^n$-extensions of F. A more recent creation is non-commutative Iwasawa theory, which allows to deal with non-commutative p-adic Lie group, as the ones appearing from non-CM elliptic curves (in particular, this may include the extensions where the p-adic realization of the arithmetic object factorizes).

In most of these developments, the global field F was assumed to be a number field. The well-known analogy with function fields suggests that one should have an interesting Iwasawa theory also in the characteristic p side of arithmetic. So in the rest of this paper F will be a global function field, with $char(F) = p$ and constant field $\mathbb{F}_F$. Observe that there is a rich and well-developed theory of cyclotomic extension for such an F, arising from Drinfeld modules: for a survey on its analogy with the cyclotomic theory over $\mathbb{Q}$ see [47].

We shall limit our discussion to abelian Galois extension of F. One has to notice that already with this assumption, an interesting new phenomenon appears: there are many more p-adic abelian extensions than in the number field case, since local groups of units are $\mathbb{Z}_p$-modules of infinite rank. So the natural analogue of the $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is the maximal p-adic abelian extension $\mathcal{F}/F$ unramified outside a fixed place and we have $\Gamma = Gal(\mathcal{F}/F) \simeq \mathbb{Z}_p^{\mathbb{N}}$. It follows that the ring $\mathbb{Z}_p[[\Gamma]]$ is not noetherian; consequently, there are some additional difficulties in dealing with Λ-modules in this case. Our proposal is to see Λ as a limit of noetherian rings and replace characteristic ideals by Fitting ideals when necessary.

As for the motives originating the Iwasawa modules we want to study, to start with we consider abelian varieties over F and ask the same questions as in the number field case. Here the theory seems to be rich enough. In particular, various control theorems allow to define the algebraic side of the Iwasawa Main conjecture. As for the analytic part, we will sketch how a p-adic L-function can be defined for modular abelian varieties.

Then we consider the Iwasawa theory of class groups of abelian extensions of F. This subject is older and more developed: the Iwasawa Main Conjecture for $\mathbb{Z}_p^n$-extension was already proved by Crew in the 1980's, by geometric techniques. We concentrate on $\mathbb{Z}_p^{\mathbb{N}}$-extensions, because they are the ones arising naturally in the cyclotomic theory; besides they are more naturally related to characteristic p L-functions (a brave new world where zeta values have found another, yet quite mysterious, life). The final section, which should be taken as a report on work in progress, provides some material for a more cyclotomic approach to the Main Conjecture.

1.1. **Contents of the paper.** In section 2 we study the structure of Selmer groups associated with elliptic curves (and, more in general, with abelian varieties) and $\mathbb{Z}_p^d$-extensions of a global function field F. We use the different versions of control theorems avaliable at present to show that the Pontrjagin duals of such groups are finitely generated (sometimes torsion) modules over the appropriate Iwasawa algebra. These results allow us to define characteristic and Fitting ideals for those duals. In section 3, taking the $\mathbb{Z}_p^d$-extensions as a filtration of a $\mathbb{Z}_p^{\mathbb{N}}$-extension $\mathcal{F}$, we can use a limit argument to define a (pro)Fitting ideal for the Pontrjagin dual of the Selmer group associated with $\mathcal{F}$. This (pro)Fitting ideal (or, better, one of its generators) can be considered as a worthy candidate for an algebraic L-function in this setting. In section 4 we deal with the analytic counterpart, giving a brief description of the p-adic L-functions which have been defined (by various authors) for abelian varieties and the extensions $\mathcal{F}/F$. Sections 3 and 4 should provide the ingredients for the formulation of an Iwasawa Main Conjecture in this setting. In section 5 we move to the problem of class groups. We use some techniques of an (almost) unpublished work of Kueh, Lai and Tan to show that the characteristic ideals of the class groups of $\mathbb{Z}_p^d$-subextensions of a cyclotomic $\mathbb{Z}_p^{\mathbb{N}}$-extension are generated by some Stickelberger element. Such a result can be extended to the whole $\mathbb{Z}_p^{\mathbb{N}}$-extension via a limit process because, at least under a certain assumption, the characteristic ideals behave well with respect to the inverse limit (as Stickelberger elements do). This provides a new approach to the Iwasawa Main Conjecture for class groups. At the end of section 5 we briefly recall some results on what is known about class groups and characteristic p zeta values. Section 6 is perhaps the closest to the spirit of function field arithmetic. For simplicity we deal only with the Carlitz module. We study the Galois module of cyclotomic units by means of Coleman power series and show how it fits in an Iwasawa Main Conjecture. Finally we compute the image of cyclotomic units by Coates-Wiles homomorphisms: one gets special values of the Carlitz-Goss zeta function, a result which might provide some hints towards its interpolation.

1.2. **Some notations.** Given a field L, $\overline{L}$ will denote an algebraic closure and L^{sep} a separable closure; we shall use the shortening $G_L := Gal(L^{sep}/L)$. When L is (an algebraic extension of) a global field, L_v will be its completion at the place v, $\mathcal{O}_v$ the ring of integers of L_v and $\mathbb{F}_v$ the residue field. We are going to deal only with global fields of positive characteristic: so $\mathbb{F}_L$ shall denote the constant field of L.

As mentioned before, let F be a global field of characteristic $p > 0$, with field of constants $\mathbb{F}_F$ of cardinality q. We also fix algebraic closures $\overline{F}$ and $\overline{F_v}$ for any place v of F, together with embeddings $\overline{F} \hookrightarrow \overline{F_v}$, so to get restriction maps $G_{F_v} \hookrightarrow G_F$. All algebraic extensions of F (resp. F_v) will be assumed to be contained in $\overline{F}$ (resp. $\overline{F_v}$).

Script letters will denote infinite extensions of F. In particular, $\mathcal{F}$ shall always denote a Galois extension of F, ramified only at a finite set of places S and such that $\Gamma := Gal(\mathcal{F}/H)$ is a free $\mathbb{Z}_p$-module, with H/F a finite subextension (to ease notations, in some sections we will just put $H = F$); the associated Iwasawa algebra is $\Lambda := \mathbb{Z}_p[[\Gamma]]$. We also put $\tilde{\Gamma} := Gal(\mathcal{F}/F)$ and $\tilde{\Lambda} := \mathbb{Z}_p[[\tilde{\Gamma}]]$.

The Pontrjagin dual of an abelian group A shall be denoted as $A^{\vee}$.

Remark 1.1. Class field theory shows that, in contrast with the number field case, in the characteristic p setting $Gal(\mathcal{F}/F)$ (and hence Γ) can be very large indeed. Actually, it is well known that for every place v the group of 1-units $\mathcal{O}_{v,1}^* \subset F_v^*$ (which is identified with the inertia subgroup of the maximal abelian extension unramified outside v) is isomorphic to a countable product of copies of $\mathbb{Z}_p$: hence there is no bound on the dimension of Γ. Furthermore, the only $\mathbb{Z}_p^{finite}$-extension of F which arises somewhat naturally is the arithmetic one $\mathcal{F}^{arit}$, i.e., the compositum of F with the maximal p-extension of $\mathbb{F}_F$. This justifies our choice to concentrate on the case of a Γ of infinite rank: $\mathcal{F}$ shall mostly be the maximal abelian extension unramified outside S (often imposing some additional condition to make it disjoint from $\mathcal{F}^{arit}$).

We also recall that a $\mathbb{Z}_p$-extension of F can be ramified at infinitely many places [17, Remark 4]: hence our condition on S is a quite meaningful restriction.

2. Control theorems for abelian varieties

2.1. **Selmer groups.** Let A/F be an abelian variety, let $A[p^n]$ be the group scheme of p^n-torsion points and put $A[p^\infty] := \varinjlim A[p^n]$. Since we work in characteristic p we define the Selmer groups via flat cohomology of group schemes. For any finite algebraic extension L/F let

$X_L := Spec\,L$ and for any place v of L let L_v be the completion of L at v and $X_{L_v} := Spec\,L_v$. Consider the local Kummer embedding

$$\kappa_{L_v} \,:\, A(L_v)\otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \varinjlim_n H^1_{fl}(X_{L_v},A[p^n]) =: H^1_{fl}(X_{L_v},A[p^\infty]) \ .$$

Definition 2.1. The p part of the Selmer group of A over L is defined as

$$Sel_A(L)_p := Ker\left\{ H^1_{fl}(X_L,A[p^\infty]) \to \prod_v H^1_{fl}(X_{L_v},A[p^\infty])/Im\,\kappa_{L_v} \right\}$$

where the map is the product of the natural restrictions at all primes v of L. For an infinite algebraic extension $\mathcal{L}/F$ we define, as usual, the Selmer group $Sel_A(\mathcal{L})_p$ via the direct limit of the $Sel_A(L)_p$ for all the finite subextensions L of $\mathcal{L}$.

In this section we let $\mathcal{F}_d/F$ be a $\mathbb{Z}_p^d$-extension ($d<\infty$) with Galois group Γ_d and associated Iwasawa algebra Λ_d. Our goal is to describe the structure of $Sel_A(\mathcal{F}_d)_p$ (actually of its Pontrjagin dual) as a Λ_d-module. The main step is a control theorem proved in [2] for the case of elliptic curves and in [44] in general, which will enable us to prove that $\mathcal{S}(\mathcal{F}_d) := Sel_A(\mathcal{F}_d)_p^\vee$ is a finitely generated (in some cases torsion) Λ_d-module. The proof of the control theorem requires semi-stable reduction for A at the places which ramify in $\mathcal{F}_d/F$: this is not a restrictive hypothesis thanks to the following (see [35, Lemma 2.1])

Lemma 2.2. *Let F'/F be a finite Galois extension. Let $\mathcal{F}'_d := \mathcal{F}_dF'$ and $\Lambda'_d := \mathbb{Z}_p[[Gal(\mathcal{F}'_d/F')]]$. Put A' for the base change of A to F'. If $\mathcal{S}' := Sel_{A'}(\mathcal{F}'_d)_p^\vee$ is a finitely generated (torsion) Λ'_d-module then $\mathcal{S}$ is a finitely generated (torsion) Λ_d-module.*

Proof. From the natural embeddings $Sel_A(L)_p \hookrightarrow H^1_{fl}(X_L,A[p^\infty])$ (any L) one gets a diagram between duals

$$\begin{tikzcd}
H^1_{fl}(X_{\mathcal{F}'_d},A[p^\infty])^\vee \arrow[r, two heads] \arrow[d] & \mathcal{S}' \arrow[d] \\
H^1_{fl}(X_{\mathcal{F}_d},A[p^\infty])^\vee \arrow[r, two heads] \arrow[d, two heads] & \mathcal{S} \arrow[d, two heads] \\
H^1(Gal(\mathcal{F}'_d/\mathcal{F}_d),A[p^\infty](\mathcal{F}'_d))^\vee \arrow[r, two heads] & \mathcal{S}/Im\,\mathcal{S}'
\end{tikzcd}$$

(where in the lower left corner one has the dual of a Galois cohomology group and the whole left side comes from the dual of the Hochschild-Serre spectral sequence). Obviously $\mathcal{F}'_d/\mathcal{F}_d$ is finite (since F'/F is) and $A[p^\infty](\mathcal{F}'_d)$ is cofinitely generated, hence $H^1(Gal(\mathcal{F}'_d/\mathcal{F}_d),A[p^\infty](\mathcal{F}'_d))^\vee$ and $\mathcal{S}/Im\,\mathcal{S}'$ are finite as well. Therefore $\mathcal{S}$ is a finitely generated (torsion) Λ'_d-module and the lemma follows from the fact that $Gal(\mathcal{F}'_d/F')$ is open in Γ_d. □

2.2. **Elliptic curves.** Let E/F be an elliptic curve, non-isotrivial (i.e., $j(E)\notin \mathbb{F}_F$) and having good ordinary or split multiplicative reduction at all the places which ramify in $\mathcal{F}_d/F$ (assuming there is no ramified prime of supersingular reduction one just needs a finite extension of F to achieve this). We remark that for such curves $E[p^\infty](F^{sep})$ is finite (it is an easy consequence of the fact that the p^n-torsion points for $n\gg 0$ generate inseparable extensions of F, see for example [5, Proposition 3.8]).
For any finite subextension $F\subseteq L\subseteq \mathcal{F}_d$ we put $\Gamma_L := Gal(\mathcal{F}_d/L)$ and consider the natural restriction map

$$a_L \,:\, Sel_E(L)_p \longrightarrow Sel_E(\mathcal{F}_d)_p^{\Gamma_L} \ .$$

The following theorem summarizes results of [2] and [44].

Theorem 2.3. *In the above setting assume that $\mathcal{F}_d/F$ is unramified outside a finite set of places of F and that E has good ordinary or split multiplicative reduction at all ramified places. Then $Ker\, a_L$ is finite (of order bounded independently of L) and $Coker\, a_L$ is a cofinitely generated $\mathbb{Z}_p$-module (of bounded corank if $d=1$). Moreover if all places of bad reduction for E are unramified in $\mathcal{F}_d/F$ then $Coker\, a_L$ is finite as well (of bounded order if $d=1$).*

Proof. Let $\mathcal{F}_w$ be the completion of $\mathcal{F}_d$ at w and, to shorten notations, let

$$\mathcal{G}(X_L) := Im \left\{ H^1_{fl}(X_L, E[p^\infty]) \longrightarrow \prod_{v\in\mathcal{M}_L} H^1_{fl}(X_{L_v}, E[p^\infty])/Im\,\kappa_{L_v} \right\}$$

(analogous definition for $\mathcal{G}(X_{\mathcal{F}_d})$).
Consider the diagram

$$\begin{tikzcd}
Sel_E(L)_p \arrow[r, hook] \arrow[d, "a_L"] & H^1_{fl}(X_L, E[p^\infty]) \arrow[r, two heads] \arrow[d, "b_L"] & \mathcal{G}(X_L) \arrow[d, "c_L"] \\
Sel_E(\mathcal{F}_d)_p^{\Gamma_L} \arrow[r, hook] & H^1_{fl}(X_{\mathcal{F}_d}, E[p^\infty])^{\Gamma_L} \arrow[r] & \mathcal{G}(X_{\mathcal{F}_d})^{\Gamma_L} .
\end{tikzcd}$$

Since $X_{\mathcal{F}_d}/X_L$ is a Galois covering the Hochschild-Serre spectral sequence (see [33, III.2.21 a),b) and III.1.17 d)]) yields

$$Ker\, b_L = H^1(\Gamma_L, E[p^\infty](\mathcal{F}_d)) \quad \text{and} \quad Coker\, b_L \subseteq H^2(\Gamma_L, E[p^\infty](\mathcal{F}_d))$$

(where the H^i's are Galois cohomology groups). Since $E[p^\infty](\mathcal{F}_d)$ is finite it is easy to see that

$$|Ker\, b_L| \le |E[p^\infty](\mathcal{F}_d)|^d \quad \text{and} \quad |Coker\, b_L| \le |E[p^\infty](\mathcal{F}_d)|^{\frac{d(d-1)}{2}}$$

([2, Lemma 4.1]). By the snake lemma the inequality on the left is enough to prove the first statement of the theorem, i.e.,

$$|Ker\, a_L| \le |Ker\, b_L| \le |E[p^\infty](\mathcal{F}_d)|^d$$

which is finite and bounded independently of L (actually one can also use the upper bound $|E[p^\infty](F^{sep})|^d$ which makes it independent of $\mathcal{F}_d$ as well).
We are left with $Ker\, c_L$: for any place w of $\mathcal{F}_d$ dividing v define

$$d_w : H^1_{fl}(X_{L_v}, E[p^\infty])/Im\,\kappa_{L_v} \longrightarrow H^1_{fl}(X_{\mathcal{F}_w}, E[p^\infty])/Im\,\kappa_w .$$

Then

$$Ker\, c_L \hookrightarrow \prod_{v\in\mathcal{M}_L} \bigcap_{w|v} Ker\, d_w$$

(note also that $Ker\, d_w$ really depends on v and not on w). If v totally splits in $\mathcal{F}_d/L$ then d_w is obviously an isomorphism. Therefore from now on we study the $Ker\, d_w$'s only for primes which are not totally split. Moreover, because of the following diagram coming from the Kummer exact sequence

$$\begin{tikzcd}
H^1_{fl}(X_{L_v}, E[p^\infty])/Im\,\kappa_{L_v} \arrow[r, hook] \arrow[d, "d_w"] & H^1_{fl}(X_{L_v}, E) \arrow[d, "h_w"] \\
H^1_{fl}(X_{\mathcal{F}_w}, E[p^\infty])/Im\,\kappa_{\mathcal{F}_w} \arrow[r, hook] & H^1_{fl}(X_{\mathcal{F}_w}, E) ,
\end{tikzcd}$$

one has an injection

$$Ker\, d_w \hookrightarrow Ker\, h_w \simeq H^1(\Gamma_{L_v}, E(\mathcal{F}_w))$$

which allows us to focus on $H^1(\Gamma_{L_v}, E(\mathcal{F}_w))$.

2.2.1. *Places of good reduction.* If v is unramified let L_v^{unr} be the maximal unramified extension of L_v. Then using the inflation map and [34, Proposition I.3.8] one has

$$H^1(\Gamma_{L_v}, E(\mathcal{F}_w)) \hookrightarrow H^1(Gal(L_v^{unr}/L_v), E(L_v^{unr})) = 0 \ .$$

Let $\widehat{E}$ be the formal group associated with E and, for any place v, let E_v be the reduced curve. From the exact sequence

$$\widehat{E}(\mathcal{O}_{\overline{L}_v}) \hookrightarrow E(\overline{L}_v) \twoheadrightarrow E_v(\overline{\mathbb{F}}_v)$$

and the surjectivity of $E(L_v) \twoheadrightarrow E_v(\mathbb{F}_v)$ (see [33, Exercise I.4.13]), one gets

$$H^1(\Gamma_{L_v}, \widehat{E}(\mathcal{O}_w)) \hookrightarrow H^1(\Gamma_{L_v}, E(\mathcal{F}_w)) \to H^1(\Gamma_{L_v}, E_v(\mathbb{F}_w)) \ .$$

Using the Tate local duality (see [34, Theorem III.7.8 and Appendix C]) and a careful study of the p^n-torsion points in inseparable extensions of L_v, Tan proves that $H^1(\Gamma_{L_v}, \widehat{E}(\mathcal{O}_w))$ is isomorphic to the Pontrjagin dual of $E_v[p^\infty](\mathbb{F}_v)$ (see [44, Theorem 2]). Hence

$$|H^1(\Gamma_{L_v}, E(\mathcal{F}_w))| \leq |H^1(\Gamma_{L_v}, E_v(\mathbb{F}_w))| \, |E_v[p^\infty](\mathbb{F}_v)| \ .$$

Finally let L'_v be the maximal unramified extension of L_v contained in $\mathcal{F}_w$ (so that, in particular, $\mathbb{F}_w = \mathbb{F}_{L'_v}$) and let $\Gamma_{L'_v} := Gal(\mathcal{F}_w/L'_v)$ be the inertia subgroup of Γ_{L_v}. The Hochschild-Serre sequence reads as

$$\begin{array}{ll} H^1(\Gamma_{L_v}/\Gamma_{L'_v}, E_v(\mathbb{F}_{L'_v})) \hookrightarrow & H^1(\Gamma_{L_v}, E_v(\mathbb{F}_w)) \\ & \qquad\downarrow \\ & H^1(\Gamma_{L'_v}, E_v(\mathbb{F}_w))^{\Gamma_{L_v}/\Gamma_{L'_v}} \to H^2(\Gamma_{L_v}/\Gamma_{L'_v}, E_v(\mathbb{F}_{L'_v})) \ . \end{array}$$

Now $\Gamma_{L_v}/\Gamma_{L'_v}$ can be trivial, finite cyclic or $\mathbb{Z}_p$ and in any case Lang's Theorem yields

$$H^i(\Gamma_{L_v}/\Gamma_{L'_v}, E_v(\mathbb{F}_{L'_v})) = 0 \quad i = 1,\ 2 \ .$$

Therefore

$$H^1(\Gamma_{L_v}, E_v(\mathbb{F}_w)) \simeq H^1(\Gamma_{L'_v}, E_v(\mathbb{F}_{L'_v}))^{Gal(L'_v/L_v)} \simeq H^1(\Gamma_{L'_v}, E_v(\mathbb{F}_v))$$

and eventually

$$\begin{aligned} |H^1(\Gamma_{L_v}, E(\mathcal{F}_w))| &\leq |H^1(\Gamma_{L'_v}, E_v(\mathbb{F}_v))| \, |E_v[p^\infty](\mathbb{F}_v)| \\ &\leq |E_v[p^\infty](\mathbb{F}_v)|^{d(L'_v)+1} \leq |E_v[p^\infty](\mathbb{F}_v)|^{d+1} \end{aligned}$$

where $d(L'_v) := rank_{\mathbb{Z}_p}\Gamma_{L'_v} \leq d$ and the last bound (independent from $\mathcal{F}_d$) comes from [2, Lemma 4.1].

We are left with the finitely many primes of bad reduction.

2.2.2. *Places of bad reduction.* By our hypothesis at these primes we have the Tate curve exact sequence

$$q_{E,v}^{\mathbb{Z}} \hookrightarrow \mathcal{F}_w^* \twoheadrightarrow E(\mathcal{F}_w) \ .$$

For any subfield K of $\mathcal{F}_w/L_v$ one has a Galois equivariant isomorphism

$$K^*/\mathcal{O}_K^* q_{E,v}^{\mathbb{Z}} \simeq T_K$$

(coming from $E_0(\mathcal{F}_w) \hookrightarrow E(\mathcal{F}_w) \twoheadrightarrow T_{\mathcal{F}_w}$) where T_K is a finite cyclic group of order $-ord_K(j(E))$ arising from the group of connected components (see, for example, [2, Lemma 4.9 and Remark 4.10]). Therefore

$$H^1(\Gamma_{L_v}, E(\mathcal{F}_w)) = \varinjlim_K H^1(\Gamma_K, E(K)) \hookrightarrow \varinjlim_K H^1(\Gamma_K, T_K) \simeq \varinjlim_K (T_K)_p^{d(K)}$$

where $(T_K)_p$ is the p-part of T_K and $d(K) = \mathrm{rank}_{\mathbb{Z}_p}\Gamma_K$.

If v is unramified then all T_K's are isomorphic to T_{L_v} and $d(K) = d(L_v) = 1$ hence

$$|H^1(\Gamma_{L_v}, E(\mathcal{F}_w))| = |(T_{L_v})_p| = |(T_{F_\nu})_p|$$

where ν is the prime of F lying below v (note that the bound is again independent of $\mathcal{F}_d$).

If v is ramified then taking Galois cohomology in the Tate curve exact sequence, one finds

$$Ker\, h_w = H^1(\Gamma_{L_v}, E(\mathcal{F}_w)) \hookrightarrow H^2(\Gamma_{L_v}, q_{E,v}^{\mathbb{Z}})$$

where the injectivity comes from Hilbert 90.
Since Γ_{L_v} acts trivially on $q_{E,v}^{\mathbb{Z}}$, one finds that

$$Ker\, h_w \hookrightarrow H^2(\Gamma_{L_v}, q_{E,v}^{\mathbb{Z}}) \simeq H^2(\Gamma_{L_v}, \mathbb{Z}) \simeq (\Gamma_{L_v}^{ab})^\vee \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{d(L_v)} \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^d \ ,$$

i.e., $Ker\, h_w$ is a cofinitely generated $\mathbb{Z}_p$-module.

This completes the proof for the general case. If all ramified primes are of good reduction, then the $Ker\, d_w$'s are finite so $Coker\, a_L$ is finite as well. In particular its order is bounded by

$$\prod_{v\ ram,\ good} |E_v[p^\infty](\mathbb{F}_{L_v})|^{d+1} \times \prod_{v\ inert,\ bad} p^{ord_p(ord_v(j(E)))} \times |Coker\, b_L| \ .$$

If $d = 1$ the last term is trivial and in a $\mathbb{Z}_p$-extension the (finitely many) ramified and inert of bad reduction places admit only a finite number of places above them. If $d \geq 2$ such bound is not independent of L because the number of terms in the products is unbounded. In the case of ramified primes of bad reduction the bound for the corank is similar. □

Both $Sel_E(\mathcal{F}_d)_p$ and its Pontrjagin dual are modules over the ring Λ_d in a natural way. An easy consequence of the previous theorem and of Nakayama's Lemma (see [1]) is the following (see for example [2, Corollaries 4.8 and 4.13])

Corollary 2.4. *In the setting of the previous theorem, let $\mathcal{S}(\mathcal{F}_d)$ be the Pontrjagin dual of $Sel_E(\mathcal{F}_d)_p$. Then $\mathcal{S}(\mathcal{F}_d)$ is a finitely generated Λ_d-module. Moreover if all ramified primes are of good reduction and $Sel_E(F)_p$ is finite, then $\mathcal{S}(\mathcal{F}_d)$ is Λ_d-torsion.*

Remark 2.5.

1. We recall that, thanks to Lemma 2.2, the last corollary holds when there are no ramified primes of supersingular reduction for E.
2. The ramified primes of split multiplicative reduction are the only obstacle to the finiteness of $Coker\, a_L$ and this somehow reflects the number field situation as described in [31, section II.6], where the authors defined an extended Mordell-Weil group whose rank is $rank\, E(F) + N$ (where N is the number of primes of split multiplicative reduction and dividing p, i.e., totally ramified in the cyclotomic $\mathbb{Z}_p$-extension they work with) to deal with the phenomenon of exceptional zeroes.
3. A different way of having finite kernels and cokernels (and then, at least in some cases, torsion modules $\mathcal{S}(\mathcal{F}_d)$) consists in a modified version of the Selmer groups. Examples with trivial or no conditions at all at the ramified primes of bad reduction are described in [2, Theorem 4.12].
4. The available constructions of a p-adic L-function associated to $\mathbb{Z}_p^{\mathbb{N}}$-extensions require the presence of a totally ramified prime $\mathfrak{p}$ of split multiplicative reduction for E. Thus the theorem applies to that setting but, unfortunately, it only provides finitely generated Λ_d-modules $\mathcal{S}(\mathcal{F}_d)$.

2.3. **Higher dimensional abelian varieties.** We go back to the general case of an abelian variety A/F. For any finite subextension L/F of $\mathcal{F}_d$ we put $\Gamma_L := Gal(\mathcal{F}_d/L)$ and consider the natural restriction map

$$a_L \,:\, Sel_A(L)_p \longrightarrow Sel_A(\mathcal{F}_d)_p^{\Gamma_L} \ .$$

The following theorem summarizes results of [44].

Theorem 2.6. *In the above setting assume that $\mathcal{F}_d/F$ is unramified outside a finite set of places of F and that A has good ordinary or split multiplicative reduction at all ramified places. Then $Ker\, a_L$ is finite (of bounded order if $d=1$) and $Coker\, a_L$ is a cofinitely generated $\mathbb{Z}_p$-module. Moreover if all places of bad reduction for A are unramified in $\mathcal{F}_d/F$ then $Coker\, a_L$ is finite as well (of bounded order if $d=1$).*

Proof. We use the same notations and diagrams as in Theorem 2.3, substituting the abelian variety A for the elliptic curve E.
The Hochschild-Serre spectral sequence yields

$$Ker\, b_L = H^1(\Gamma_L, A[p^\infty](\mathcal{F}_d)) \quad \text{and} \quad Coker\, b_L \subseteq H^2(\Gamma_L, A[p^\infty](\mathcal{F}_d)) \ .$$

Let $L_0 \subseteq \mathcal{F}_d$ be the extension generated by $A[p^\infty](\mathcal{F}_d)$. The extension L_0/L is everywhere unramified (for the places of good reduction see [44, Lemma 2.5.1 (b)], for the other places note that the p^n-torsion points come from the p^n-th roots of the periods provided by the Mumford parametrization so they generate an inseparable extension while $\mathcal{F}_d/F$ is separable): hence $Gal(L_0/L) \simeq \Delta \times \mathbb{Z}_p^e$ where Δ is finite and $e=0$ or 1. Let γ be a topological generator of $\mathbb{Z}_p^e$ in $Gal(L_0/L)$ (if $e=0$ then $\gamma=1$) and let L_1 be its fixed field. Then $A[p^\infty](\mathcal{F}_d)^{\overline{<\gamma>}} = A[p^\infty](L_1)$ is finite and we can apply [3, Lemma 3.4] (with b the maximum between $|A[p^\infty](L_1)|$ and $|A[p^\infty](\mathcal{F}_d)/(A[p^\infty](\mathcal{F}_d))_{div}|$) to get

$$|Ker\, b_L| \leq b^d \quad \text{and} \quad |Coker\, b_L| \leq b^{\frac{d(d-1)}{2}} \ .$$

By the snake lemma the inequality on the left is enough to prove that $Ker\, a_L$ is finite (for the bounded order in the case $d=1$ see [44, Corollary 3.2.4]).
The bounds for the $Ker\, d_w$'s are a direct generalization of the ones provided for the case of the elliptic curve so we give just a few details. Recall the embedding

$$Ker\, d_w \hookrightarrow Ker\, h_w \simeq H^1(\Gamma_{L_v}, E(\mathcal{F}_w)) \ .$$

2.3.1. *Places of good reduction.* If v is unramified then

$$H^1(\Gamma_{L_v}, A(\mathcal{F}_w)) \hookrightarrow H^1(Gal(L_v^{unr}/L_v), A(L_v^{unr})) = 0 \ .$$

If v is ramified one has an exact sequence (as above)

$$H^1(\Gamma_{L_v}, \widehat{A}(\mathcal{O}_{\mathcal{F}_w})) \hookrightarrow H^1(\Gamma_{L_v}, A(\mathcal{F}_w)) \to H^1(\Gamma_{L_v}, A_v(\mathbb{F}_{\mathcal{F}_w})) \ .$$

By [44, Theorem 2]

$$H^1(\Gamma_{L_v}, \widehat{A}(\mathcal{O}_{\mathcal{F}_w})) \simeq B_v[p^\infty](\mathbb{F}_{L_v})$$

(where B is the dual variety of A) and the last group has the same order of $A_v[p^\infty](\mathbb{F}_{L_v})$. Using Lang's theorem as in 2.2.1 one finds

$$|H^1(\Gamma_{L_v}, A(\mathcal{F}_w))| \leq |A_v[p^\infty](\mathbb{F}_{L_v})|^{d+1} \ .$$

2.3.2. *Places of bad reduction.* If v is unramified let $\pi_{0,v}(A)$ be the group of connected components of the Néron model of A at v. Then, again by [34, Proposition I.3.8],

$$H^1(\Gamma_{L_v}, A(\mathcal{F}_w)) \hookrightarrow H^1(Gal(L_v^{unr}/L_v), A(L_v^{unr})) \simeq H^1(Gal(L_v^{unr}/L_v), \pi_{0,v}(A))$$

and the last group has order bounded by $|\pi_{0,v}(A)^{Gal(L_v^{unr}/L_v)}|$.
If v is ramified one just uses Mumford's parametrization with a period lattice $\Omega_v \subset L_v \times \cdots \times L_v$ (genus A times) to prove that $H^1(\Gamma_{L_v}, A(\mathcal{F}_w))$ is cofinitely generated as in 2.2.2. $\square$

We end this section with the analogue of Corollary 2.4.

Corollary 2.7. *In the setting of the previous theorem, let $\mathcal{S}(\mathcal{F}_d)$ be the Pontrjagin dual of $Sel_A(\mathcal{F}_d)_p$. Then $\mathcal{S}(\mathcal{F}_d)$ is a finitely generated Λ_d-module. Moreover if all ramified primes are of good reduction and $Sel_A(F)_p$ is finite, then $\mathcal{S}(\mathcal{F}_d)$ is Λ_d-torsion.*

Remark 2.8. In [35, Theorem 1.7], by means of crystalline and syntomic cohomology, Ochiai and Trihan prove a stronger result. Indeed they can show that the dual of the Selmer group is always torsion, with no restriction on the abelian variety A/F, but only in the case of the arithmetic extension $\mathcal{F}^{arit}/F$, which lies outside the scope of the present paper. Moreover in the case of a (not necessarily commutative) pro-p-extension containing $\mathcal{F}^{arit}$, they prove that the dual of the Selmer group is finitely generated (for a precise statement, see [35], in particular Theorem 1.9).

3. Λ-modules and Fitting ideals

We need a few more notations.

For any $\mathbb{Z}_p^d$-extension $\mathcal{F}_d$ as above let $\Gamma(\mathcal{F}_d) := Gal(\mathcal{F}_d/F)$ and $\Lambda(\mathcal{F}_d) := \mathbb{Z}_p[[\Gamma(\mathcal{F}_d)]]$ (the Iwasawa algebra) with augmentation ideal $I^{\mathcal{F}_d}$ (or simply Γ_d, Λ_d and I_d if the extension $\mathcal{F}_d$ is clearly fixed).

For any $d > e$ and any $\mathbb{Z}_p^{d-e}$-extension $\mathcal{F}_d/\mathcal{F}_e$ we put $\Gamma(\mathcal{F}_d/\mathcal{F}_e) := Gal(\mathcal{F}_d/\mathcal{F}_e)$, $\Lambda(\mathcal{F}_d/\mathcal{F}_e) := \mathbb{Z}_p[[\Gamma(\mathcal{F}_d/\mathcal{F}_e)]]$ and $I_{\mathcal{F}_e}^{\mathcal{F}_d}$ as the augmentation ideal of $\Lambda(\mathcal{F}_d/\mathcal{F}_e)$, i.e., the kernel of the canonical projection $\pi_{\mathcal{F}_e}^{\mathcal{F}_d} : \Lambda(\mathcal{F}_d) \to \Lambda(\mathcal{F}_e)$ (whenever possible all these will be abbreviated to Γ_e^d, Λ_e^d, I_e^d and π_e^d respectively).

Recall that $\Lambda(\mathcal{F}_d)$ is (noncanonically) isomorphic to $\mathbb{Z}_p[[T_1, \ldots, T_d]]$. A finitely generated torsion $\Lambda(\mathcal{F}_d)$-module is said to be *pseudo-null* if its annihilator ideal has height at least 2. If M is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module then there is a pseudo-isomorphism (i.e., a morphism with pseudo-null kernel and cokernel)

$$M \sim_{\Lambda(\mathcal{F}_d)} \bigoplus_{i=1}^{n} \Lambda(\mathcal{F}_d)/(g_i^{e_i})$$

where the g_i's are irreducible elements of $\Lambda(\mathcal{F}_d)$ (determined up to an element of $\Lambda(\mathcal{F}_d)^*$) and n and the e_i's are uniquely determined by M (see e.g. [7, VII.4.4 Theorem 5]).

Definition 3.1. In the above setting the *characteristic ideal* of M is

$$Ch_{\Lambda(\mathcal{F}_d)}(M) := \begin{cases} 0 & \text{if } M \text{ is not torsion} \\ (\prod_{i=1}^{n} g_i^{e_i}) & \text{otherwise} \end{cases} .$$

Let Z be a finitely generated $\Lambda(\mathcal{F}_d)$-module and let

$$\Lambda(\mathcal{F}_d)^a \xrightarrow{\varphi} \Lambda(\mathcal{F}_d)^b \twoheadrightarrow Z$$

be a presentation where the map φ can be represented by a $b \times a$ matrix Φ with entries in $\Lambda(\mathcal{F}_d)$.

Definition 3.2. In the above setting the *Fitting ideal* of Z is

$$Fitt_{\Lambda(\mathcal{F}_d)}(Z) := \begin{cases} 0 & \text{if } a < b \\ \text{the ideal generated by all the} & \\ \text{determinants of the } b \times b & \text{if } a \geq b \\ \text{minors of the matrix } \Phi & \end{cases} .$$

Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^{\mathbb{N}}$-extension with Galois group Γ. Our goal is to define an ideal in $\Lambda := \mathbb{Z}_p[[\Gamma]]$ associated with $\mathcal{S}$ the Pontrjagin dual of $Sel_A(\mathcal{F})_p$. For this we consider all the $\mathbb{Z}_p^d$-extensions $\mathcal{F}_d/F$ ($d \in \mathbb{N}$) contained in $\mathcal{F}$ (which we call $\mathbb{Z}_p$-finite extensions). Then $\mathcal{F} = \cup \mathcal{F}_d$ and $\Lambda = \varprojlim \Lambda(\mathcal{F}_d) := \varprojlim \mathbb{Z}_p[[Gal(\mathcal{F}_d/F)]]$. The classical characteristic ideal does not behave well (in general) with respect to inverse limits (because the inverse limit of pseudo-null modules is not necessarily pseudo-null). For the Fitting ideal, using the basic properties described in the Appendix of [32], we have the following

Lemma 3.3. *Let $\mathcal{F}_d \subset \mathcal{F}_e$ be an inclusion of multiple $\mathbb{Z}_p$-extensions, $e > d$. Assume that $A[p^\infty](\mathcal{F}) = 0$ or that $Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))$ is principal. Then*

$$\pi_{\mathcal{F}_d}^{\mathcal{F}_e}(Fitt_{\Lambda(\mathcal{F}_e)}(\mathcal{S}(\mathcal{F}_e))) \subseteq Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)) \ .$$

Proof. Consider the natural map $a_d^e : Sel_A(\mathcal{F}_d)_p \to Sel_A(\mathcal{F}_e)_p^{\Gamma_d^e}$ and dualize to get

$$\mathcal{S}(\mathcal{F}_e)/I_d^e\mathcal{S}(\mathcal{F}_e) \to \mathcal{S}(\mathcal{F}_d) \twoheadrightarrow (Ker\, a_d^e)^\vee$$

where (as in Theorem 2.6)

$$Ker\, a_d^e \hookrightarrow H^1(\Gamma_d^e, A[p^\infty](\mathcal{F}_e))$$

is finite. If $A[p^\infty](\mathcal{F}) = 0$ then $(Ker\, a_d^e)^\vee = 0$ and

$$\pi_d^e(Fitt_{\Lambda_e}(\mathcal{S}(\mathcal{F}_e))) = Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_e)/I_d^e\mathcal{S}(\mathcal{F}_e)) \subseteq Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_d)) \ .$$

If $(Ker\, a_d^e)^\vee \neq 0$ one has

$$Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_e)/I_d^e\mathcal{S}(\mathcal{F}_e))Fitt_{\Lambda_d}((Ker\, a_d^e)^\vee) \subseteq Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_d)) \ .$$

The Fitting ideal of a finitely generated torsion module contains a power of its annihilator, so let σ_1, σ_2 be two relatively prime elements of $Fitt_{\Lambda_d}((Ker\, a_d^e)^\vee)$ and θ_d a generator of $Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_d))$. Then θ_d divides $\sigma_1\alpha$ and $\sigma_2\alpha$ for any $\alpha \in Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_e)/I_d^e\mathcal{S}(\mathcal{F}_e))$ (it holds, in the obvious sense, even for $\theta_d = 0$). Hence

$$\pi_d^e(Fitt_{\Lambda_e}(\mathcal{S}(\mathcal{F}_e))) = Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_e)/I_d^e\mathcal{S}(\mathcal{F}_e)) \subseteq Fitt_{\Lambda_d}(\mathcal{S}(\mathcal{F}_d)) \ .$$

□

Remark 3.4. In the case $A = E$ an elliptic curve, the hypothesis $E[p^\infty](\mathcal{F}) = 0$ is satisfied if $j(E) \notin (F^*)^p$, i.e., when the curve is admissible (in the sense of [5]); otherwise $j(E) \in (F^*)^{p^n} - (F^*)^{p^{n+1}}$ and one can work over the field F^{p^n}. The other hypothesis is satisfied in general by elementary $\Lambda(\mathcal{F}_d)$-modules or by modules having a presentation with the same number of generators and relations.

Let $\pi_{\mathcal{F}_d}$ be the canonical projection from Λ to $\Lambda(\mathcal{F}_d)$ with kernel $I_{\mathcal{F}_d}$. Then the previous lemma shows that, as $\mathcal{F}_d$ varies, the $(\pi_{\mathcal{F}_d})^{-1}(Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)))$ form an inverse system of ideals in Λ.

Definition 3.5. Assume that $A[p^\infty](\mathcal{F}) = 0$ or that $Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))$ is principal for any $\mathcal{F}_d$. Define

$$\widetilde{Fitt}_\Lambda(\mathcal{S}(\mathcal{F})) := \varprojlim_{\mathcal{F}_d} (\pi_{\mathcal{F}_d})^{-1}(Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)))$$

to be the *pro-Fitting ideal* of $\mathcal{S}(\mathcal{F})$ (the Pontrjagin dual of $Sel_E(\mathcal{F})_p$).

Proposition 3.6. *Assume that $A[p^\infty](\mathcal{F}) = 0$ or that $Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))$ is principal for any $\mathcal{F}_d$. If $corank_{\mathbb{Z}_p} Sel_A(\mathcal{F}_1)_p^{\Gamma(\mathcal{F}_1)} \geq 1$ for any $\mathbb{Z}_p$-extension $\mathcal{F}_1/F$ contained in $\mathcal{F}$, then $\widetilde{Fitt}_\Lambda(\mathcal{S}(\mathcal{F})) \subset I$ (where I is the augmentation ideal of Λ).*

Proof. Recall that $I^{\mathcal{F}_d}$ is the augmentation ideal of $\Lambda(\mathcal{F}_d)$, that is, the kernel of $\pi^{\mathcal{F}_d} : \Lambda(\mathcal{F}_d) \to \mathbb{Z}_p$. By hypothesis $Fitt_{\mathbb{Z}_p}((Sel_A(\mathcal{F}_1)_p^{\Gamma(\mathcal{F}_1)})^\vee) = 0$. Thus, since $\mathbb{Z}_p = \Lambda(\mathcal{F}_1)/I^{\mathcal{F}_1}$ and $(Sel_A(\mathcal{F}_1)_p^{\Gamma(\mathcal{F}_1)})^\vee = \mathcal{S}(\mathcal{F}_1)/I^{\mathcal{F}_1}\mathcal{S}(\mathcal{F}_1)$,

$$0 = Fitt_{\mathbb{Z}_p}((Sel_A(\mathcal{F}_1)_p^{\Gamma(\mathcal{F}_1)})^\vee) = \pi^{\mathcal{F}_1}(Fitt_{\Lambda(\mathcal{F}_1)}(\mathcal{S}(\mathcal{F}_1))) \ ,$$

i.e., $Fitt_{\Lambda(\mathcal{F}_1)}(\mathcal{S}(\mathcal{F}_1)) \subset Ker\, \pi^{\mathcal{F}_1} = I^{\mathcal{F}_1}$.
For any $\mathbb{Z}_p^d$-extension $\mathcal{F}_d$ take a $\mathbb{Z}_p$-extension $\mathcal{F}_1$ contained in $\mathcal{F}_d$. Then, by Lemma 3.3,

$$\pi_{\mathcal{F}_1}^{\mathcal{F}_d}(Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) \subseteq Fitt_{\Lambda(\mathcal{F}_1)}(\mathcal{S}(\mathcal{F}_1)) \subset I^{\mathcal{F}_1} \ .$$

Note that $\pi^{\mathcal{F}_d} = \pi^{\mathcal{F}_1} \circ \pi^{\mathcal{F}_d}_{\mathcal{F}_1}$. Therefore

$$Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)) \subset I^{\mathcal{F}_d} \iff \pi^{\mathcal{F}_d}_{\mathcal{F}_1}(Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) \subset I^{\mathcal{F}_1} ,$$

i.e., $Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)) \subset I^{\mathcal{F}_d}$ for any $\mathbb{Z}_p$-finite extension $\mathcal{F}_d$. Finally

$$\widetilde{Fitt}_{\Lambda}(\mathcal{S}(\mathcal{F})) := \bigcap_{\mathcal{F}_d}(\pi_{\mathcal{F}_d})^{-1}(Fitt_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) \subset \bigcap_{\mathcal{F}_d}(\pi_{\mathcal{F}_d})^{-1}(I^{\mathcal{F}_d}) \subset I .$$

□

Remark 3.7. From the exact sequence

$$Ker\, a_{\mathcal{F}_1} \hookrightarrow Sel_A(F)_p \xrightarrow{a_{\mathcal{F}_1}} Sel_A(\mathcal{F}_1)_p^{\Gamma(\mathcal{F}_1)} \twoheadrightarrow Coker\, a_{\mathcal{F}_1}$$

and the fact that $Ker\, a_{\mathcal{F}_1}$ is finite one immediately finds out that the hypothesis on $corank_{\mathbb{Z}_p} Sel_A(\mathcal{F}_1)_p^{\Gamma(\mathcal{F}_1)}$ is satisfied if $rank_{\mathbb{Z}} A(F) \geq 1$ or $corank_{\mathbb{Z}_p} Coker\, a_{\mathcal{F}_1} \geq 1$. As already noted, when there is a totally ramified prime of split multiplicative reduction, the second option is very likely to happen. In the number field case, when $\mathcal{F}$ is the cyclotomic $\mathbb{Z}_p$-extension and, in some cases, $Sel_E(\mathcal{F})_p^\vee$ is known to be a torsion module, this is equivalent to saying that T divides a generator of the characteristic ideal of $Sel_E(\mathcal{F})_p^\vee$ (i.e., there is an exceptional zero). Note that all the available constructions of p-adic L-function for our setting require a ramified place of split multiplicative reduction and they are all known to belong to I.

4. Modular abelian varieties of GL_2-type

The previous sections show how to define the algebraic (p-adic) L-function associated with $\mathcal{F}/F$ and an abelian variety A/F under quite general conditions. On the analytic side there is, of course, the complex Hasse-Weil L-function $L(A/F, s)$, so the problem becomes to relate it to some element in an Iwasawa algebra. In this section we will sketch how this can be done at least in some cases; in order to keep the paper to a reasonable length, the treatment here will be very brief.

We say that the abelian variety A/F is of GL_2-type if there is a number field K such that $[K : \mathbb{Q}] = \dim A$ and K embeds into $\mathrm{End}_F(A) \otimes \mathbb{Q}$. In particular, this implies that for any $l \neq p$ the Tate module $T_l A$ yields a representation of G_F in $GL_2(K \otimes \mathbb{Q}_l)$. The analogous definition for $A/\mathbb{Q}$ can be found in [40], where it is proved that Serre's conjecture implies that every simple abelian variety of GL_2-type is isogenous to a simple factor of a modular Jacobian. We are going to see that a similar result holds at least partially in our function field setting.

4.1. Automorphic forms.

Let $\mathbf{A}_F$ denote the ring of adeles of F. By automorphic form for GL_2 we shall mean a function $f\colon GL_2(\mathbf{A}_F) \longrightarrow \mathbb{C}$ which factors through $GL_2(F)\backslash GL_2(\mathbf{A}_F)/\mathcal{K}$, where $\mathcal{K}$ is some open compact subgroup of $GL_2(\mathbf{A}_F)$; furthermore, f is cuspidal if it satisfies some additional technical condition (essentially, the annihilation of some Fourier coefficients). A classical procedure associates with such an f a Dirichlet sum $L(f, s)$: see e.g. [49, Chapters II and III].

The $\mathbb{C}$-vector spaces of automorphic and cuspidal forms provide representations of $GL_2(\mathbf{A}_F)$. Besides, they have a natural $\mathbb{Q}$-structure: in particular, the decomposition of the space of cuspidal forms in irreducible representations of $GL_2(\mathbf{A}_F)$ holds over $\overline{\mathbb{Q}}$ (and hence over any algebraically closed field of characteristic zero); see e.g. the discussion in [39, page 218]. We also recall that every irreducible automorphic representation π of $GL_2(\mathbf{A}_F)$ is a restricted tensor product $\otimes'_v \pi_v$, v varying over the places of F: the π_v's are representations of $GL_2(F_v)$ and they are called local factors of π.

Let W_F denote the Weil group of F: it is the subgroup of G_F consisting of elements whose restriction to $\overline{\mathbb{F}}_q$ is an integer power of the Frobenius. By a fundamental result of Jacquet and Langlands [23, Theorem 12.2], a two-dimensional representation of W_F corresponds to a cuspidal representation if the associated L-function and its twists by characters of W_F are entire functions bounded in vertical strips (see also [49]).

Let A/F be an abelian variety of GL_2-type. Recall that $L(A/F,s)$ is the L-function associated with the compatible system of l-adic representations of G_F arising from the Tate modules T_lA, as l varies among primes different from p. Theorems of Grothendieck and Deligne show that under certain assumptions $L(A/F,s)$ and all its twists are polynomials in q^{-s} satisfying the conditions of [23, Theorem 12.2] (see [14, §9] for precise statements). In particular all elliptic curves are obviously of GL_2-type and one finds that $L(A/F,s)=L(f,s)$ for some cusp form f when A is a non-isotrivial elliptic curve.

4.2. **Drinfeld modular curves.** From now on we fix a place ∞.

The main source for this section is Drinfeld's original paper [15]. Here we just recall that for any divisor $\mathfrak{n}$ of F with support disjoint from ∞ there exists a projective curve $M(\mathfrak{n})$ (the Drinfeld modular curve) and that these curves form a projective system. Hence one can consider the Galois representation

$$\underline{H} := \lim_{\rightarrow} H^1_{et}(M(\mathfrak{n})\times F^{sep},\overline{\mathbb{Q}}_l).$$

Besides, the moduli interpretation of the curves $M(\mathfrak{n})$ allows to define an action of $GL_2(\mathbf{A}_f)$ on $\underline{H}$ (where $\mathbf{A}_f$ denotes the adeles of F without the component at ∞). Let Π_∞ be the set of those cuspidal representations having the special representation of $GL_2(F_\infty)$ (i.e., the Steinberg representation) as local factor at ∞. Drinfeld's reciprocity law [15, Theorem 2] (which realizes part of the Langlands correspondence for GL_2 over F) attaches to any $\pi\in\Pi_\infty$ a compatible system of two-dimensional Galois representations $\sigma(\pi)_l\colon G_F\longrightarrow GL_2(\overline{\mathbb{Q}}_l)$ by establishing an isomorphism of $GL_2(\mathbf{A}_f)\times G_F$-modules

$$\underline{H}\simeq\bigoplus_{\pi\in\Pi_\infty}(\otimes'_{v\neq\infty}\pi_v)\otimes\sigma(\pi)_l\,. \tag{4.1}$$

As $\sigma(\pi)_l$ one obtains all l-adic representations of G_F satisfying certain properties: for a precise list see [39] [1]. Here we just remark the following requirement: the restriction of $\sigma(\pi)_l$ to G_{F_∞} has to be the special l-adic Galois representation sp_∞. For example, the representation originated from the Tate module T_lE of an elliptic curve E/F satisfies this condition if and only if E has split multiplicative reduction at ∞.

The Galois representations appearing in (4.1) are exactly those arising from the Tate module of the Jacobian of some $M(\mathfrak{n})$. We call modular those abelian varieties isogenous to some factor of $Jac(M(\mathfrak{n}))$. Hence we see that a necessary condition for abelian varieties of GL_2-type to be modular is that their reduction at ∞ is a split torus.

The paper [16] provides a careful construction of Jacobians of Drinfeld modular curves by means of rigid analytic geometry.

4.3. **The p-adic L-functions.** For any ring R let $Meas(\mathbb{P}^1(F_v),R)$ denote the R-valued measures on the topological space $\mathbb{P}^1(F_v)$ (that is, finitely additive functions on compact open subsets of $\mathbb{P}^1(F_v)$) and $Meas_0(\mathbb{P}^1(F_v),R)$ the subset of measures of total mass 0. A key ingredient in the proof of (4.1) is the identification of the space of R-valued cusp forms with direct sums of certain subspaces of $Meas_0(\mathbb{P}^1(F_\infty),R)$ (for more precise statements, see [39, §2] and [16, §4]). Therefore we can associate with any modular abelian variety A some measure

[1] Actually, [39] provides a thorough introduction to all this subject.

μ_A on $\mathbb{P}^1(F_\infty)$; this fact can be exploited to construct elements (our p-adic L-functions) in Iwasawa algebras in the following way.

Let K be a quadratic algebra over F: an embedding of K into the F-algebra of 2×2 matrices $M_2(F)$ gives rise to an action of the group $G := (K \otimes F_\infty)^*/F_\infty^*$ on the $PGL_2(F_\infty)$-homogeneous space $\mathbb{P}^1(F_\infty)$. Class field theory permits to relate G to a subgroup Γ of $\tilde{\Gamma} = Gal(\mathcal{F}/F)$, where $\mathcal{F}$ is a certain extension of F (depending on K) ramified only above ∞. Then the pull-back of μ_A to G yields a measure on Γ; this is enough because $Meas(\Gamma, R)$ is canonically identified with $R \otimes \Lambda$ (and $Meas_0(\Gamma, R)$ with the augmentation ideal). Various instances of the construction just sketched are explained in [30] for the case when A is an elliptic curve: here one can take $R = \mathbb{Z}$. Similar ideas were used in Pál's thesis [37], where there is also an interpolation formula relating the element in $\mathbb{Z}[[\Gamma]]$ so obtained to special values of the complex L-function. One should also mention [38] for another construction of p-adic L-function, providing an interpolation property for one of the cases studied in [30]. Notice that in all this cases the p-adic L-function is, more or less tautologically, in the augmentation ideal.

A different approach had been previously suggested by Tan [43]: starting with cuspidal automorphic forms, he defines elements in certain group algebras and proves an interpolation formula [43, Proposition 2]. Furthermore, if the cusp form is "well-behaved" his modular elements form a projective system and originate an element in an Iwasawa algebra of the kind considered in the present paper: in particular, this holds for non-isotrivial elliptic curves having split multiplicative reduction. In the case of an elliptic curve over $\mathbb{F}_q(T)$ Teitelbaum [46] re-expressed Tan's work in terms of modular symbols (along the lines of [31]); in [21] it is shown how this last method can be related to the "quadratic algebra" techniques sketched above.

A unified treatment of all of this will appear in [4].

Thus for a modular abelian variety A/F we can define both a Fitting ideal and a p-adic L-function: it is natural to expect that an Iwasawa Main Conjecture should hold, i.e., that the Fitting ideal should be generated by the p-adic L-function.

Remark 4.1. In the cases considered in this paper (involving a modular abelian variety and a geometric extension of the function field) the Iwasawa Main Conjecture is still wide open. However, recently there has been some interesting progress in two related settings: both results are going to appear in [29].

First, one can take A to be an isotrivial abelian variety (notice that [43, page 308] defines modular elements also for an isotrivial elliptic curve). Ki-Seng Tan has observed that then the Main Conjecture can be reduced to the one for class groups, which is already known to hold (as it will be explained in the next section).

Second, one can take as $\mathcal{F}$ the maximal arithmetic pro-p-extension of F, i.e., $\mathcal{F} = \mathcal{F}^{arit} = F\mathbb{F}_F^{(p)}$, where $\mathbb{F}_F^{(p)}$ is the subfield of $\overline{\mathbb{F}}_F$ defined by $Gal(\mathbb{F}_F^{(p)}/\mathbb{F}_F) \simeq \mathbb{Z}_p$ (notice that this is the setting of [35, Theorem 1.7]). In this case Trihan has obtained a proof of the Iwasawa Main Conjecture, by techniques of syntomic cohomology. He needs no assumption on the abelian variety A/F: his p-adic L-function is defined by means of cohomology and it interpolates the Hasse-Weil L-function.

5. Class groups

For any finite extension L/F, $\mathcal{A}(L)$ will denote the p-part of the group of degree zero divisor classes of L; for any $\mathcal{F}'$ intermediate between F and $\mathcal{F}$, we put $\mathcal{A}(\mathcal{F}') := \varprojlim \mathcal{A}(L)$ as L runs among finite subextensions of $\mathcal{F}'/F$ (the limit being taken with respect to norm maps). The study of similar objects and their relations with zeta functions is an old subject and was the starting point for Iwasawa himself (see [47] for a quick summary). The goal of this section is to say something on what is known about Iwasawa Main Conjectures for class groups in our setting.

5.1. **Crew's work.** A version of the Iwasawa Main Conjecture over global function fields was proved by R. Crew in [13]. His main tools are geometric: so he considers an affine curve X over a finite field of characteristic p (in the language of the present paper, F is the function field of X) and a p-adic character of $\pi_1(X)$, that is, a continuous homomorphism $\rho\colon \pi_1(X) \longrightarrow R^*$, where R is a complete local noetherian ring of mixed characteristic, with maximal ideal $\mathfrak{m}$ (notice that the Iwasawa algebras Λ_d introduced in section 2.1 above are rings of this kind). To such a ρ are attached $H(\rho, x) \in R[x]$ (the characteristic polynomial of the cohomology of a certain étale sheaf - see [12] for more explanation) and the L-function $L(\rho, x) \in R[[x]]$. The main theorem of [13] proves, by means of étale and crystalline cohomology, that the ratio $L(\rho, x)/H(\rho, x)$ is a unit in the $\mathfrak{m}$-adic completion of $R[x]$. An account of the geometric significance of this result (together with some of the necessary background) is provided by Crew himself in [12]; in [13, §3] he shows the following application to Iwasawa theory. Letting (in our notations) R be the Iwasawa algebra $\Lambda(\mathcal{F}_d)$, the special value $L(\rho, 1)$ can be seen to be a Stickelberger element (the definition will be recalled in section 5.3 below). As for $H(\rho, 1)$, [12, Proposition 3.1] implies that it generates the characteristic ideal of the torsion Λ_d-module $\varprojlim \mathcal{A}(L)^\vee$, L varying among finite subextensions of $\mathcal{F}_d/F$ [2]. The Iwasawa Main Conjecture follows.

Crew's cohomological techniques are quite sophisticated. A more elementary approach was suggested by Kueh, Lai and Tan in [28] (and refined, with Burns's contribution and different cohomological tools, in [8]). In the next two sections we will give a brief account of this approach (and its consequences) in a particularly simple setting, related to Drinfeld-Hayes cyclotomic extensions (which will be the main topic of section 6).

5.2. **Characteristic ideals for class groups.** In this section (which somehow parallels section 3) we describe an algebraic object which can be associated to the inverse limit of class groups in a $\mathbb{Z}_p^{\mathbb{N}}$-extension $\mathcal{F}$ of a global function field F. Since our first goal is to use this "algebraic L-function" for the cyclotomic extension which will appear in section 6.1, we make the following simplifying assumption.

Assumption 5.1. There is only one ramified prime in $\mathcal{F}/F$ (call it $\mathfrak{p}$) and it is totally ramified (in particular this implies that $\mathcal{F}$ is disjoint from $\mathcal{F}^{arit}$).

We shall use some ideas of [27] which, in our setting, provide a quite elementary approach to the problem. We maintain the notations of section 3: $\mathcal{F}/F$ is a $\mathbb{Z}_p^{\mathbb{N}}$-extension with Galois group Γ and associated Iwasawa algebra Λ with augmentation ideal I. For any $d \geq 0$ let $\mathcal{F}_d$ be a $\mathbb{Z}_p^d$-extension of F contained in $\mathcal{F}$, taken so that $\bigcup \mathcal{F}_d = \mathcal{F}$.

For any finite extension L/F let $\mathcal{M}(L)$ be the p-adic completion of the group of divisor classes $\mathcal{D}iv(L)/P_L$ of L, i.e.,

$$\mathcal{M}(L) := (L^* \backslash \mathbf{I}_L / \Pi_v \mathcal{O}_v^*) \otimes \mathbb{Z}_p$$

where $\mathbf{I}_L$ is the group of ideles of L. As before, when $\mathcal{L}/F$ is an infinite extension, we put $\mathcal{M}(\mathcal{L}) := \varprojlim \mathcal{M}(K)$ as K runs among finite subextensions of $\mathcal{L}/F$ (the limit being taken with

[2]Note that in [12] our $\mathcal{A}(L)$'s appear as Picard groups, so the natural functoriality yields $\mathcal{A}(L) \to \mathcal{A}(L')$ if $L \subset L'$ - that is, arrows are opposite to the ones we consider in this paper: hence Crew takes Pontrjagin duals and we don't.

respect to norm maps). For two finite extensions $L \supset L' \supset F$, the degree maps $\deg_L$ and $\deg_{L'}$ fit into the commutative diagram (with exact rows)

$$\begin{tikzcd} \mathcal{A}(L) \arrow[r, hook] \arrow[d, "N^L_{L'}"] & \mathcal{M}(L) \arrow[r, two heads, "\deg_L"] \arrow[d, "N^L_{L'}"] & \mathbb{Z}_p \arrow[d] \\ \mathcal{A}(L') \arrow[r, hook] & \mathcal{M}(L') \arrow[r, two heads, "\deg_{L'}"] & \mathbb{Z}_p \end{tikzcd} , \tag{5.1}$$

where $N^L_{L'}$ denotes the norm and the vertical map on the right is multiplication by $[\mathbb{F}_L : \mathbb{F}_{L'}]$ (the degree of the extension between the fields of constants). For an infinite extension $\mathcal{L}/F$ contained in $\mathcal{F}$, taking projective limits (and recalling Assumption 5.1 above), one gets an exact sequence

$$\begin{tikzcd} \mathcal{A}(\mathcal{L}) \arrow[r, hook] & \mathcal{M}(\mathcal{L}) \arrow[r, two heads, "\deg_{\mathcal{L}}"] & \mathbb{Z}_p \end{tikzcd} . \tag{5.2}$$

Remark 5.2. If one allows non-geometric extensions then the $\deg_{\mathcal{L}}$ map above becomes the zero map exactly when the $\mathbb{Z}_p$-extension $\mathcal{F}^{arit}$ is contained in $\mathcal{L}$.

It is well known that $\mathcal{M}(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module (see e.g. [19, Theorem 1]), so the same holds for $\mathcal{A}(\mathcal{F}_d)$ as well. Moreover take any $\mathbb{Z}_p^d$-extension $\mathcal{F}_d$ of F contained in $\mathcal{F}$: since our extension $\mathcal{F}/F$ is totally ramified at the prime $\mathfrak{p}$, for any $\mathcal{F}_{d-1} \subset \mathcal{F}_d$ one has

$$\mathcal{M}(\mathcal{F}_d)/I^{\mathcal{F}_d}_{\mathcal{F}_{d-1}}\mathcal{M}(\mathcal{F}_d) \simeq \mathcal{M}(\mathcal{F}_{d-1}) \tag{5.3}$$

(see for example [48, Lemma 13.15]). As in Section 3, to ease notations we will often erase the $\mathcal{F}$ from the indices (for example $I^{\mathcal{F}_d}_{\mathcal{F}_{d-1}}$ will be denoted by I^d_{d-1}), hoping that no confusion will arise. Consider the following diagram

$$\begin{tikzcd} \mathcal{A}(\mathcal{F}_d) \arrow[r, hook] \arrow[d, "\gamma-1"] & \mathcal{M}(\mathcal{F}_d) \arrow[r, two heads, "\deg"] \arrow[d, "\gamma-1"] & \mathbb{Z}_p \arrow[d, "\gamma-1"] \\ \mathcal{A}(\mathcal{F}_d) \arrow[r, hook] & \mathcal{M}(\mathcal{F}_d) \arrow[r, two heads, "\deg"] & \mathbb{Z}_p \end{tikzcd} \tag{5.4}$$

(where $\overline{\langle\gamma\rangle} = Gal(\mathcal{F}_d/\mathcal{F}_{d-1}) =: \Gamma^d_{d-1}$; note also that the vertical map on the right is 0) and its snake lemma sequence

$$\begin{tikzcd} \mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[r, hook] & \mathcal{M}(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[r, "\deg"] & \mathbb{Z}_p \arrow[d] \\ \mathbb{Z}_p & \mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d) \arrow[l, two heads, "\deg"'] & \mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d) \arrow[l] \end{tikzcd} . \tag{5.5}$$

For $d \geq 2$ the Λ_d-module $\mathbb{Z}_p$ is pseudo-null, hence (5.2) yields $Ch_{\Lambda_d}(\mathcal{M}(\mathcal{F}_d)) = Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d))$, and, using (5.3) and (5.5), one finds (for $d \geq 3$)

$$\begin{aligned} Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) &= Ch_{\Lambda_{d-1}}(\mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d)) \\ &= Ch_{\Lambda_{d-1}}(\mathcal{M}(\mathcal{F}_{d-1})) = Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_{d-1})) \end{aligned} \tag{5.6}$$

(where all the modules involved are Λ_{d-1}-torsion modules).

Let

$$N(\mathcal{F}_d) \hookrightarrow \mathcal{A}(\mathcal{F}_d) \xrightarrow{\iota} E(\mathcal{F}_d) \twoheadrightarrow R(\mathcal{F}_d) \tag{5.7}$$

be the exact sequence coming from the structure theorem for Λ_d-modules (see section 3) where

$$E(\mathcal{F}_d) := \bigoplus_i \Lambda_d/(f_{i,d})$$

is an elementary module and $N(\mathcal{F}_d)$, $R(\mathcal{F}_d)$ are pseudo-null. Let $Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d)) = (\Pi_i f_{i,d})$ be the characteristic ideal of $\mathcal{A}(\mathcal{F}_d)$: we want to compare $Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_{d-1}))$ with $\pi^{\mathcal{F}_d}_{\mathcal{F}_{d-1}}(Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d)))$ for some $\mathcal{F}_{d-1} \subset \mathcal{F}_d$ and show that these characteristic ideals form an inverse system (in Λ). Consider the module $B(\mathcal{F}_d) := N(\mathcal{F}_d) \oplus R(\mathcal{F}_d)$. We need the following hypothesis.

Assumption 5.3. There is a choice of the pseudo-isomorphism ι in (5.7) and a splitting of the projection $\Gamma_d \twoheadrightarrow \Gamma_{d-1}$ so that

i) $\Gamma_d = \overline{\langle \gamma_d \rangle} \oplus \Gamma_{d-1}$;
ii) $B(\mathcal{F}_d)$ is a finitely generated torsion $\mathbb{Z}_p[[\Gamma_{d-1}]]$-module.

As explained in [20] (see the remarks just before Lemma 3), for any $\mathcal{F}_d$ and ι one can find a subfield $\mathcal{F}_{d-1}$ so that Assumption 5.3 holds.

In order to ease notations, we put $\gamma = \gamma_d$, so that $\Gamma^d_{d-1} = \overline{\langle \gamma \rangle}$.

Lemma 5.4. *With the above notations, one has*

$$Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) = \pi^d_{d-1}\big(Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d))\big) \cdot Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) \ .$$

Proof. We split the previous sequence in two by

$$N(\mathcal{F}_d) \hookrightarrow \mathcal{A}(\mathcal{F}_d) \twoheadrightarrow C(\mathcal{F}_d) \ , \ C(\mathcal{F}_d) \hookrightarrow E(\mathcal{F}_d) \twoheadrightarrow R(\mathcal{F}_d)$$

and consider the snake lemma sequences coming from the following diagrams

$$\begin{tikzcd} N(\mathcal{F}_d) \arrow[r, hook] \arrow[d, "\gamma-1"] & \mathcal{A}(\mathcal{F}_d) \arrow[r, two heads] \arrow[d, "\gamma-1"] & C(\mathcal{F}_d) \arrow[d, "\gamma-1"] \\ N(\mathcal{F}_d) \arrow[r, hook] & \mathcal{A}(\mathcal{F}_d) \arrow[r, two heads] & C(\mathcal{F}_d) \end{tikzcd} \qquad \begin{tikzcd} C(\mathcal{F}_d) \arrow[r, hook] \arrow[d, "\gamma-1"] & E(\mathcal{F}_d) \arrow[r, two heads] \arrow[d, "\gamma-1"] & R(\mathcal{F}_d) \arrow[d, "\gamma-1"] \\ C(\mathcal{F}_d) \arrow[r, hook] & E(\mathcal{F}_d) \arrow[r, two heads] & R(\mathcal{F}_d) \end{tikzcd} \ , \tag{5.8}$$

i.e.,

$$\begin{tikzcd} N(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[r, hook] & \mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[r] & C(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[d] \\ C(\mathcal{F}_d)/I^d_{d-1}C(\mathcal{F}_d) & \mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d) \arrow[l, two heads] & N(\mathcal{F}_d)/I^d_{d-1}N(\mathcal{F}_d) \arrow[l] \end{tikzcd} \tag{5.9}$$

and

$$\begin{tikzcd} C(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[r, hook] & E(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[r] & R(\mathcal{F}_d)^{\Gamma^d_{d-1}} \arrow[d] \\ R(\mathcal{F}_d)/I^d_{d-1}R(\mathcal{F}_d) & E(\mathcal{F}_d)/I^d_{d-1}E(\mathcal{F}_d) \arrow[l, two heads] & C(\mathcal{F}_d)/I^d_{d-1}C(\mathcal{F}_d) \arrow[l] \end{tikzcd} \ . \tag{5.10}$$

From (5.7) we get an exact sequence

$$\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d) \longrightarrow \bigoplus_i \Lambda_d/(\gamma-1, f_{i,d}) \longrightarrow R(\mathcal{F}_d)/I^d_{d-1}R(\mathcal{F}_d)$$

where the last term is a torsion Λ_{d-1}-module. So is $\mathcal{A}(\mathcal{F}_{d-1})$ for $d \geq 3$ and, by (5.6), $Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_{d-1})) = Ch_{\Lambda_{d-1}}\big(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)\big)$. It follows that none of the $f_{i,d}$'s belong to I^d_{d-1}. Therefore:

1. the map $\gamma - 1 : \ E(\mathcal{F}_d) \longrightarrow E(\mathcal{F}_d)$ has trivial kernel, i.e., $E(\mathcal{F}_d)^{\Gamma^d_{d-1}} = 0$ so that $C(\mathcal{F}_d)^{\Gamma^d_{d-1}} = 0$ as well;

2. the characteristic ideal of the Λ_{d-1}-module $E(\mathcal{F}_d)/I^d_{d-1}E(\mathcal{F}_d)$ is generated by the product of the $f_{i,d}$'s modulo I^d_{d-1}, hence it is obviously equal to $\pi^d_{d-1}(Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d)))$.

Moreover, from the fact that $N(\mathcal{F}_d)$ and $R(\mathcal{F}_d)$ are finitely generated torsion Λ_{d-1}-modules [3] and the multiplicativity of characteristic ideals, looking at the left (resp. right) vertical sequence of the first (resp. second) diagram in (5.8), one finds

$$Ch_{\Lambda_{d-1}}(N(\mathcal{F}_d)^{\Gamma^d_{d-1}}) = Ch_{\Lambda_{d-1}}(N(\mathcal{F}_d)/I^d_{d-1}N(\mathcal{F}_d))$$

and

$$Ch_{\Lambda_{d-1}}(R(\mathcal{F}_d)^{\Gamma^d_{d-1}}) = Ch_{\Lambda_{d-1}}(R(\mathcal{F}_d)/I^d_{d-1}R(\mathcal{F}_d)) \ .$$

Hence from (5.9) one has

$$\begin{aligned} Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) &= Ch_{\Lambda_{d-1}}\big(C(\mathcal{F}_d)/I^d_{d-1}C(\mathcal{F}_d)\big) \cdot Ch_{\Lambda_{d-1}}(N(\mathcal{F}_d)^{\Gamma^d_{d-1}}) \\ &= Ch_{\Lambda_{d-1}}\big(C(\mathcal{F}_d)/I^d_{d-1}C(\mathcal{F}_d)\big) \cdot Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) \end{aligned}$$

(where the last line comes from the isomorphism $\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}} \simeq N(\mathcal{F}_d)^{\Gamma^d_{d-1}}$). The sequence (5.10) provides the equality

$$\begin{aligned} Ch_{\Lambda_{d-1}}(C(\mathcal{F}_d)/I^d_{d-1}C(\mathcal{F}_d)) &= Ch_{\Lambda_{d-1}}(E(\mathcal{F}_d)/I^d_{d-1}E(\mathcal{F}_d)) \\ &= \pi^d_{d-1}(Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d))) \ . \end{aligned}$$

Therefore one concludes that

$$Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) = \pi^d_{d-1}\big(Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d))\big) \cdot Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) \ . \tag{5.11}$$

□

Our next step is to prove that $\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}} = 0$ (note that it would be enough to prove that it is pseudo-null as a Λ_{d-1}-module). For this we need first a few lemmas.

Lemma 5.5. *Let G be a finite group and endow $\mathbb{Z}_p$ with the trivial G-action. Then for any G-module M we have*

$$H^i(G, M \otimes \mathbb{Z}_p) = H^i(G, M) \otimes \mathbb{Z}_p$$

for all $i \geq 0$.

This result should be well-known. Since we were not able to find a suitable reference, here is a sketch of the proof.

Proof. Let X be a G-module which has no torsion as an abelian group and put $Y := X \otimes \mathbb{Q}$. It is not hard to prove that $Y^G \otimes \mathbb{Z}_p = (Y \otimes \mathbb{Z}_p)^G$ and it follows that the same holds for X, since $X^G \otimes \mathbb{Z}_p$ is a saturated submodule of $X \otimes \mathbb{Z}_p$. Applying this to the standard complex by means of which the $H^i(G, M)$ are defined, one can prove the equality in the case M has no torsion as an abelian group. The general case follows because any G-module is the quotient of two such modules. □

Up to now we have mainly considered $\mathcal{M}(\mathcal{L})$ as an Iwasawa module (for various $\mathcal{L}$), now we focus on its interpretation as a group of divisor classes. Let L be a finite extensions of F and recall that we defined $\mathcal{M}(L) = (\mathcal{D}iv(L)/P_L) \otimes \mathbb{Z}_p$. From the exact sequence

$$\mathbb{F}_L^* \hookrightarrow L^* \twoheadrightarrow P_L$$

and the fact that $|\mathbb{F}_L^*|$ is prime with p, one finds an isomorphism between $L^* \otimes \mathbb{Z}_p$ and $P_L \otimes \mathbb{Z}_p$. Hence we can (and will) identify the two.

[3] It might be worth to notice that this is the only point where we use the hypothesis that Assumption 5.3 holds.

Lemma 5.6. *For any finite Galois extension L/F, the map*

$$\mathcal{D}iv(L)^{Gal(L/F)} \otimes \mathbb{Z}_p \longrightarrow \mathcal{M}(L)^{Gal(L/F)}$$

is surjective.

Proof. The sequence

$$L^* \otimes \mathbb{Z}_p \hookrightarrow \mathcal{D}iv(L) \otimes \mathbb{Z}_p \twoheadrightarrow \mathcal{M}(L) \tag{5.12}$$

is exact because $\mathbb{Z}_p$ is flat and $|\mathbb{F}_L^*|$ is prime with p. The claim follows by taking the $Gal(L/F)$-cohomology of (5.12) and applying Lemma 5.5 and Hilbert 90. ☐

For any finite subextension L of $\mathcal{F}/F$ let $\mathfrak{p}_L$ be the unique prime lying above $\mathfrak{p}$. In the following lemma, we identify $\mathfrak{p}_L$ with its image in $\mathcal{D}iv(L) \otimes \mathbb{Z}_p$. Moreover for any element $x \in \mathcal{M}(\mathcal{F})$ we let x_L denote its image in $\mathcal{M}(L)$ via the canonical norm map.

Lemma 5.7. *Let $x \in \mathcal{M}(\mathcal{F})^\Gamma$: then, for any L as above, x_L is represented by a Γ-invariant divisor supported in $\mathfrak{p}_L$.*

Proof. For any L, let y_L be the image of $\mathfrak{p}_L$ in $\mathcal{M}(L)$. Since $\mathbb{Z}_p y_L$ is a closed subset of $\mathcal{M}(L)$, to prove the lemma it is enough to show that $\big(x_L + p^n\mathcal{M}(L)\big) \cap \mathbb{Z}_p y_L \neq \emptyset$ for any n.

For any finite Galois extension K/L we have the maps

$$\iota_L^K : \mathcal{D}iv(L) \otimes \mathbb{Z}_p \longrightarrow \mathcal{D}iv(K) \otimes \mathbb{Z}_p$$

and

$$N_L^K : \mathcal{D}iv(K) \otimes \mathbb{Z}_p \longrightarrow \mathcal{D}iv(L) \otimes \mathbb{Z}_p$$

respectively induced by the inclusion and the norm. For any divisor whose support is unramifed in K/L we have

$$N_L^K(\iota_L^K(D)) = [K : L]D\,.$$

Also, Lemma 5.5 yields

$$(\mathcal{D}iv(K) \otimes \mathbb{Z}_p)^{Gal(K/L)} = \mathcal{D}iv(K)^{Gal(K/L)} \otimes \mathbb{Z}_p = \iota_L^K(\mathcal{D}iv(L) \otimes \mathbb{Z}_p)$$

(since in a $Gal(K/L)$-invariant divisor all places of K above a same place of L occur with the same multiplicity).

Choose n and let $K \subset \mathcal{F}$ be such that $[K : L] \geq p^n$. By Lemma 5.6, there exists a $Gal(K/L)$-invariant $E_K \in \mathcal{D}iv(K) \otimes \mathbb{Z}_p$ having image x_K. Write $E_K = D_K + a_K\mathfrak{p}_K$, where $a_K \in \mathbb{Z}_p$ and D_K has support disjoint from $\mathfrak{p}_K$. Then D_K is Galois invariant, so $D_K = \iota_L^K(D_L)$ and (using Assumption 5.1)

$$N_L^K(E_K) = [K : L]D_L + a_K\mathfrak{p}_L\,.$$

Projecting into $\mathcal{M}(L)$ we get $x_L \in a_K y_L + p^n\mathcal{M}(L)$. ☐

Corollary 5.8. $\mathcal{A}(\mathcal{F}_d)^{\Gamma_{d-1}^d} = 0$.

Proof. Taking Γ_{d-1}^d-invariants in (5.2) (with $\mathcal{L} = \mathcal{F}_d$), one finds a similar sequence

$$\mathcal{A}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \hookrightarrow \mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \xrightarrow{\deg_{\mathcal{F}_d}} \mathbb{Z}_p\,. \tag{5.13}$$

Lemma 5.7 holds, with exactly the same proof, also replacing $\mathcal{F}$ and Γ with $\mathcal{F}_d$ and Γ_d. Therefore any $x = (x_L)_L \in \mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d}$ can be represented by a sequence $(a_L\mathfrak{p}_L)_L$. Furthermore $N_L^K(a_K\mathfrak{p}_K) = a_L\mathfrak{p}_L$ implies that the value a_L is independent of L: call it a. Then

$$\deg_{\mathcal{F}_d}(x) = \lim\big(a_L \deg_L(\mathfrak{p}_L)\big) = a\deg_F(\mathfrak{p})\,.$$

Hence $x \in Ker(\deg_{\mathcal{F}_d}) = \mathcal{A}(\mathcal{F}_d)^{\Gamma_{d-1}^d}$ only if $a = 0$. ☐

Remark 5.9. The image of the degree map appearing in (5.13) is $(\deg \mathfrak{p})\mathbb{Z}_p$, so $\deg_{\mathcal{F}_d}$ always provides an isomorphism between $\mathcal{M}(\mathcal{F}_d)^{\Gamma^d_{d-1}}$ and $\mathbb{Z}_p$. Moreover, if p does not divide $\deg \mathfrak{p}$, one has surjectivity as well. In this case, looking back at the sequence (5.5), one finds a short exact sequence

$$\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d) \hookrightarrow \mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d) \xrightarrow{\deg} \mathbb{Z}_p \quad .$$

From (5.1), by taking the limit with L and L' varying respectively among subextensions of $\mathcal{F}_d$ and $\mathcal{F}_{d-1}$, one obtains a commutative diagram

$$\begin{tikzcd}
\mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d) \arrow[r, two heads, "\deg"] \arrow[d, "N"] & \mathbb{Z}_p \arrow[d, equal] \\
\mathcal{M}(\mathcal{F}_{d-1}) \arrow[r, two heads, "\deg_{\mathcal{F}_{d-1}}"] & \mathbb{Z}_p
\end{tikzcd}$$

where the map N is the isomorphism induced by the norm, i.e., the one appearing in (5.3). This and the exact sequence (5.2) for $\mathcal{L} = \mathcal{F}_{d-1}$, show that $\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d) \simeq \mathcal{A}(\mathcal{F}_{d-1})$ (for any $d \geq 1$).

From (5.11) one finally obtains

$$Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_{d-1})) = Ch_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) = \pi^d_{d-1}(Ch_{\Lambda_d}(\mathcal{A}(\mathcal{F}_d))) \ . \tag{5.14}$$

We remark that this equation holds for any $\mathbb{Z}_p$-extension $\mathcal{F}_d/\mathcal{F}_{d-1}$ satisfying Assumption 5.3. If the filtration $\{\mathcal{F}_d : d \in \mathbb{N}\}$ verifies that Assumption at any level d, then the inverse images of the $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))$ in Λ (with respect to the canonical projections $\pi_{\mathcal{F}_d} : \Lambda \to \Lambda(\mathcal{F}_d)$) form an inverse system and we can define

Definition 5.10. The *pro-characteristic ideal* of $\mathcal{A}(\mathcal{F})$ is

$$\widetilde{Ch}_\Lambda(\mathcal{A}(\mathcal{F})) := \varprojlim_{\mathcal{F}_d} (\pi_{\mathcal{F}_d})^{-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) \ .$$

Remark 5.11. Two questions naturally arise from the above definition:

a. is there a filtration verifying Assumption 5.3 at any level d?

b. (assuming *a* has a positive answer) *is the limit independent from the chosen filtration?*

In the next section we are going to show (in particular in (5.17) and Corollary 5.16) that there is an element $\theta \in \Lambda$, independent of the filtration and such that, for all $\mathcal{F}_d$, its image in $\Lambda(\mathcal{F}_d)$ generates $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))$. Hence Question *b* has a positive answer (and presumably so does Question *a*) and we only needed (5.14) as a first step and a natural analogue of (5.16).
Nevertheless we believe that these questions have some interest on their own and it would be nice to have a direct construction of a "good" filtration $\{\mathcal{F}_d : d \in \mathbb{N}\}$ based on a generalization of [20, Lemma 2]. Since our goal here is the Main Conjecture we do not pursue this subject further, but we hope to get back to it in a future paper.
We also observe that Assumption 5.3 was used only in one passage in the proof of Lemma 5.4, as we evidentiated in a footnote. It might be easier to show that in that passage one does not need the finitely generated hypothesis: if so, Definition 5.10 would makes sense for all filtrations $\{\mathcal{F}_d\}_d$. We are currently considering this approach.

Remark 5.12. We could have used Fitting ideals, just as we did in section 3, to provide a more straightforward construction (there would have been no need for preparatory lemmas). But, since the goal is a Main Conjecture, the characteristic ideals, being principal, provide a better formulation. We indeed expect equality between Fitting and characteristic ideals in all the cases studied in this paper but, at present, are forced to distinguish between them (but see Remark 5.17).

5.3. **Stickelberger elements.** We shall briefly describe a relation between the characteristic ideal of the previous section and Stickelberger elements. The main results on those elements are due to A. Weil, P. Deligne and J. Tate and for all the details the reader can consult [45, Ch. V]. Let S be a finite set of places of F containing all places where the extension $\mathcal{F}/F$ ramifies; since we are interested in the case where $\mathcal{F}$ is substantially bigger than the arithmetic extension, we assume $S \neq \emptyset$. We consider also another non-empty finite set T of places of F such that $S \cap T = \emptyset$. For any place outside S let Fr_v be the Frobenius of v in $\Gamma = Gal(\mathcal{F}/F)$.

Let

$$\Theta_{\mathcal{F}/F,S,T}(u) := \prod_{v\in T}(1 - Fr_v q^{\deg(v)} u^{\deg(v)}) \prod_{v \notin S}(1 - Fr_v u^{\deg(v)})^{-1}. \tag{5.15}$$

For any $n \in \mathbb{N}$ there are only finitely many places of F with degree n: hence we can expand (5.15) and consider $\Theta_{\mathcal{F}/F,S,T}(u)$ as a power series $\sum c_n u^n \in \mathbb{Z}[\Gamma][[u]]$. Moreover, it is clear that for any continuous character $\psi\colon \Gamma \longrightarrow \mathbb{C}^*$ the image $\psi(\Theta_{\mathcal{F}/F,S,T}(q^{-s}))$ is the L-function of ψ, relative to S and modified at T. For any subextension $F \subset L \subset \mathcal{F}$, let $\pi^{\mathcal{F}}_L : \mathbb{Z}[\Gamma][[u]] \to \mathbb{Z}[Gal(L/F)][[u]]$ be the natural projection and define

$$\Theta_{L/F,S,T}(u) := \pi^{\mathcal{F}}_L(\Theta_{\mathcal{F}/F,S,T}(u)).$$

For L/F finite it is known (essentially by Weil's work) that $\Theta_{L/F,S,T}(q^{-s})$ is an element in the polynomial ring $\mathbb{C}[q^{-s}]$ (see [45, Ch. V, Proposition 2.15] for a proof): hence $\Theta_{L/F,S,T}(u) \in \mathbb{Z}[Gal(L/F)][u]$. It follows that the coefficients c_n of $\Theta_{\mathcal{F}/F,S,T}(u)$ tend to zero in

$$\varprojlim \mathbb{Z}[Gal(L/F)] =: \mathbb{Z}[[\Gamma]] \subset \Lambda\,.$$

Therefore we can define

$$\theta_{\mathcal{F}/F,S,T} := \Theta_{\mathcal{F}/F,S,T}(1) \in \Lambda\,.$$

We also observe that the factors $(1 - Fr_v q^{\deg(v)} u^{\deg(v)})$ in (5.15) are units in the ring $\Lambda[[u]]$. Hence the ideal generated by $\theta_{\mathcal{F}/F,S,T}$ is independent of the auxiliary set T and we can define the Stickelberger element

$$\theta_{\mathcal{F}/F,S} := \theta_{\mathcal{F}/F,S,T} \prod_{v\in T}(1 - Fr_v q^{\deg(v)})^{-1}.$$

We also define, for $F \subset L \subset \mathcal{F}$,

$$\theta_{L/F,S,T} := \pi^{\mathcal{F}}_L(\theta_{\mathcal{F}/F,S,T}) = \Theta_{L/F,S,T}(1).$$

It is clear that these form a projective system: in particular, for any $\mathbb{Z}_p$-extension $\mathcal{F}_d/\mathcal{F}_{d-1}$ the relation

$$\pi^d_{d-1}(\theta_{\mathcal{F}_d/F,S,T}) = \theta_{\mathcal{F}_{d-1}/F,S,T} \tag{5.16}$$

clearly recalls the one satisfied by characteristic ideals (equation (5.14)). Also, to define $\theta_{L/F,S}$ there is no need of $\mathcal{F}$: one can take for a finite extension L/F the analogue of product (5.15) and reason as above.

Theorem 5.13 (Tate, Deligne)**.** *For any finite extension L/F one has that $|\mathbb{F}^*_L|\theta_{L/F,S}$ is in the annihilator ideal of the class group of L (considered as a $\mathbb{Z}[Gal(L/F)]$-module).*

Proof. This is [45, Ch. V, Théorème 1.2]. □

Remark 5.14. Another proof of this result was given by Hayes [22], by means of Drinfeld modules.

Corollary 5.15. *Let $\mathcal{F}_d/F$ be a $\mathbb{Z}_p^d$-extension as before and $S = \{\mathfrak{p}\}$, the unique (totally) ramified prime in $\mathcal{F}/F$: then*

1. $\theta_{\mathcal{F}_d/F,S}\mathcal{A}(\mathcal{F}_d) = 0$;

2. *if* $\theta_{\mathcal{F}_d/F,S}$ *is irreducible in* $\Lambda(\mathcal{F}_d)$*, then* $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d)) = (\theta_{\mathcal{F}_d/F,S})^m$ *for some* $m \geq 1$*;*

3. *if* $\theta_{\mathcal{F}_d/F,S}$ *is irreducible in* $\Lambda(\mathcal{F}_d)$ *for all* $\mathcal{F}_d$*'s, then* $\widetilde{Ch}_\Lambda(\mathcal{A}(\mathcal{F})) = (\theta_{\mathcal{F}/F,S})^m$ *for some* $m \geq 1$*.*

Proof. For **1** one just notes that $|\mathbb{F}_L^*|$ is prime with p. Part **2** follows from the structure theorem for torsion $\Lambda(\mathcal{F}_d)$-modules. Part **3** follows from **2** by taking limits (as in Definition 5.10) and noting that the m is constant through the $\mathcal{F}_d$'s because of equations (5.14) and (5.16). □

The exponent in **2** and **3** of the corollary above is actually $m = 1$. A proof of this fact is based on the following technical result of [28] (generalized in [9, Theorem A.1]). Once $\mathcal{F}_d$ is fixed it is always possible to find a $\mathbb{Z}_p^c$-extension of F containing $\mathcal{F}_d$, call it $\mathcal{L}_d$, such that:

a. the extension $\mathcal{L}_d/F$ is ramified at all primes of a finite set $\widetilde{S}$ containing S (moreover $\widetilde{S}$ can be chosen arbitrarily large);

b. the Stickelberger element $\theta_{\mathcal{L}_d/F,\widetilde{S}}$ is irreducible in the Iwasawa algebra $\Lambda(\mathcal{L}_d)$;

c. there is a $\mathbb{Z}_p$-extension $\mathcal{L}'$ of F contained in $\mathcal{L}_d$ which is ramified at all primes of $\widetilde{S}$ and such that the Stickelberger element $\theta_{\mathcal{L}'/F,\widetilde{S}}$ is monomial, i.e., congruent to $u(\sigma-1)^r$ modulo $(\sigma-1)^{r+1}$ (where σ is a topological generator of $Gal(\mathcal{L}'/F)$ and $u \in \mathbb{Z}_p^*$).

With condition **b** and an iteration of equation (5.14) one proves that

$$Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d)) = (\theta_{\mathcal{F}_d/F,S})^m \text{ for some } m \geq 1 \ .$$

The monomiality condition **c** (using $\mathcal{L}'$ as a first layer in a tower of $\mathbb{Z}_p$-extensions) leads to $m = 1$ (see [27, section 4] or [9, section A.1] which uses the possibility of varying the set $\widetilde{S}$, provided by **a**, more directly). We remark that the proof only uses the irreducibility of $\theta_{\mathcal{L}_d/F,\widetilde{S}}$, i.e.,

$$Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d)) = (\theta_{\mathcal{F}_d/F,S}) \tag{5.17}$$

holds in general for any $\mathcal{F}_d$.

Corollary 5.16 (Iwasawa Main Conjecture)**.** *In the previous setting we have*

$$\widetilde{Ch}_\Lambda(\mathcal{A}(\mathcal{F})) = (\theta_{\mathcal{F}/F,\mathfrak{p}}) \ .$$

Proof. From the main result of [27], one has that

$$Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d)) = (\theta_{\mathcal{F}_d/F,\mathfrak{p}})$$

and we take the limit in both sides. □

Remark 5.17. The equality between characteristic ideals and ideals generated by Stickelberger elements has been proved by K.-L. Kueh, K. F. Lai and K.-S. Tan ([27]) and by D. Burns ([8] and the Appendix coauthored with K.F. Lai and K-S. Tan [9]) in a more general situation. The $\mathbb{Z}_p^d$-extension they consider has to be unramified outside a finite set S of primes of F (but there is no need for the primes to be totally ramified). Moreover they require that none of the primes in S is totally split (otherwise $\theta_{\mathcal{F}_d/F,S} = 0$). The strategy of the proof is basically the same but, of course, many technical details are simplified by our choice of having just one (totally) ramified prime (just compare, for example, Lemma 5.7 with [27, Lemma 3.3 and 3.4]). Moreover, going back to the Fitting vs. characteristic ideal situation, it is worth noticing that Burns proves that the first cohomology group of certain complexes (strictly related to class groups, see [8, Proposition 4.4] and [9, section A.1]) are of projective dimension 1 ([8, Proposition 4.1]). In this case the Fitting and characteristic ideals are known to be equal to the inverse of the Knudsen-Mumford determinant (the ideal by which all the results of [8] are formulated).

5.4. **Characteristic p L-functions.** One of the most fascinating aspects of function field arithmetic is the existence, next to complex and p-adic L-functions, of their characteristic p avatars. For a thorough introduction the reader is referred to [18, Chapter 8]: here we just provide a minimal background.

Recall our fixed place ∞ and let $\mathbf{C}_\infty$ denote the completion of an algebraic closure of F_∞. Already Carlitz had studied a characteristic p version of the Riemann zeta function, defined on $\mathbb{N}$ and taking values in $\mathbf{C}_\infty$ (we will say more about it in section 6.6). More recently Goss had the intuition that, like complex and p-adic L-functions have as their natural domains respectively the complex and the p-adic (quasi-)characters of the Weil group, so one could consider $\mathbf{C}_\infty$-valued characters. In particular, the analogue of the complex plane as domain for the characteristic p L-functions is $S_\infty := \mathbf{C}_\infty^* \times \mathbb{Z}_p$, that can be seen as a group of $\mathbf{C}_\infty$-valued homomorphisms on F_∞^*, just as for $s \in \mathbb{C}$ one defines $x \mapsto x^s$ on $\mathbb{R}^+$. The additive group $\mathbb{Z}$ embeds discretely in S_∞. Similarly to the classical case, one can define $L(\rho, s)$ for ρ a compatible system of v-adic representation of G_F (v varying among places different from ∞) by Euler products converging on some "half-plane" of S_∞.

The theory of zeta values in characteristic p is still quite mysterious and at the moment we can at best speculate that there are links with the Iwasawa theoretical questions considered in this paper [4]. To the best of our knowledge, the main results available in this direction are the following. Let $F(\mathfrak{p})/F$ be the extension obtained from the $\mathfrak{p}$-torsion of a Drinfeld-Hayes module (in the simplest case, $F(\mathfrak{p})$ is the F_1 we are going to introduce in section 6.1). Goss and Sinnott have studied the isotypic components of $\mathcal{A}(F(\mathfrak{p}))$ and shown that they are non-zero if and only if $\mathfrak{p}$ divides certain characteristic p zeta values: see [18, Theorem 8.14.4] for a precise statement. Note that the proof given in [18], based on a comparison between the reductions of a p-adic and a characteristic p L-function respectively mod p and mod $\mathfrak{p}$ ([18, Theorem 8.13.3]), makes use of Crew's result. Okada [36] obtained a result of similar flavor for the class group of the ring of "integers" of $F(\mathfrak{p})$ when F is the rational function field, and Shu [42] extended it to any F; since Okada's result is strictly related with the subject of section 6.6 below, we will say more about it there.

6. Cyclotomy by the Carlitz module

6.1. **Setting.** From now on we take $F = \mathbb{F}_q(T)$ and let ∞ be the usual place at infinity, so that the ring of elements regular outside ∞ is $A := \mathbb{F}_q[T]$: this allows a number of simplifications, leaving intact the main aspects of the theory. The "cyclotomic" theory of function fields is obtained via Drinfeld-Hayes modules: in the setting of the rational function field the only one is the Carlitz module $\Phi\colon A \to A\{\tau\}$, $T \mapsto \Phi_T := T + \tau$ (here τ denotes the operator $x \mapsto x^q$ and, if R is an $\mathbb{F}_p$-algebra, $R\{\tau\}$ is the ring of skew polynomials with coefficients in R: multiplication in $R\{\tau\}$ is given by composition).

We also fix a prime $\mathfrak{p} \subset A$ and let $\pi \in A$ be its monic generator. In order to underline the fact that A and its completion at $\mathfrak{p}$ play the role of $\mathbb{Z}$ and $\mathbb{Z}_p$ in the Drinfeld-Hayes cyclotomic theory, we will often use the alternative notation $A_\mathfrak{p}$ for the ring of local integers $\mathcal{O}_\mathfrak{p} \subset F_\mathfrak{p}$. Let $\mathbf{C}_\mathfrak{p}$ be the completion of an algebraic closure of $F_\mathfrak{p}$.

As usual, if I is an ideal of A, $\Phi[I]$ will denote the I-torsion of Φ (i.e., the common zeroes of all Φ_a, $a \in I$). One checks immediately that if ι is the unique monic generator of I then

$$\Phi_\iota(x) = \prod_{u \in \Phi[I]} (x - u)\,.$$

[4][Note added in proof] This field is in rapid evolution. After this paper was written, L. Taelman introduced some important new ideas: see [L. Taelman, Special L-values of Drinfeld modules. To appear in *Annals of Math.* **175** (2012), 369–391] and [L. Taelman, A Herbrand-Ribet theorem for function fields. To appear in *Invent. Math.*, Online First DOI: 10.1007/s00222-011-0346-3].

We put

$$F_n := F(\Phi[\mathfrak{p}^n])$$

and

$$K_n := F_{\mathfrak{p}}(\Phi[\mathfrak{p}^n])\,.$$

As stated in §1.2, we think of the F_n's as subfields of $\mathbf{C}_{\mathfrak{p}}$, so that the K_n's are their topological closures. We shall denote the ring of A-integers in F_n by B_n and its closure in K_n by $\mathcal{O}_n$, and write $\mathcal{U}_n$ for the 1-units in $\mathcal{O}_n$. Let $\mathcal{F} := \cup F_n$ and $\tilde{\Gamma} := Gal(\mathcal{F}/F)$.

Consider the ring of formal skew power series $A_{\mathfrak{p}}\{\{\tau\}\}$: it is a complete local ring, with maximal ideal $\pi A_{\mathfrak{p}} + A_{\mathfrak{p}}\{\{\tau\}\}\tau$. It is easy to see that Φ extends to a continuous homomorphism $\Phi\colon A_{\mathfrak{p}} \to A_{\mathfrak{p}}\{\{\tau\}\}$ (i.e., a formal Drinfeld module) and this allows to define a "cyclotomic" character $\chi\colon \tilde{\Gamma} \xrightarrow{\sim} A_{\mathfrak{p}}^*$. More precisely, let $T_{\mathfrak{p}}\Phi := \varprojlim \Phi[\mathfrak{p}^n]$ (the limit is taken with respect to $x \mapsto \Phi_\pi(x)$) be the Tate module of Φ. The ring $A_{\mathfrak{p}}$ acts on $T_{\mathfrak{p}}\Phi$ via Φ, i.e., $a\cdot(u)_n := (\Phi_a(u_n))_n$, and the character χ is defined by $\sigma u =: \chi(\sigma)\cdot u$, i.e., $\chi(\sigma)$ is the unique element in $A_{\mathfrak{p}}^*$ such that $\Phi_{\chi(\sigma)}(u_n) = \sigma u_n$ for all n. From this it follows immediately that $\tilde{\Gamma} = \Delta \times \Gamma$, where $\Delta \simeq \mathbb{F}_{\mathfrak{p}}^*$ is a finite group of order prime to p and Γ is the inverse image of the 1-units.

Since Φ has rank 1, $T_{\mathfrak{p}}\Phi$ is a free $A_{\mathfrak{p}}$-module of rank 1. As in [6], we fix a generator $\omega = (\omega_n)_{n\geq 1}$: this means that the sequence $\{\omega_n\}$ satisfies

$$\Phi_{\pi^n}(\omega_n) = 0 \neq \Phi_{\pi^{n-1}}(\omega_n) \text{ and } \Phi_\pi(\omega_{n+1}) = \omega_n.$$

By definition $K_n = F_{\mathfrak{p}}(\omega_n)$. By Hayes's theory, the minimal polynomial of ω_n over F is Eisenstein: it follows that the extensions F_n/F and $K_n/F_{\mathfrak{p}}$ are totally ramified, ω_n is a uniformizer for the field K_n, $\mathcal{O}_n = A_{\mathfrak{p}}[[\omega_n]] = A_{\mathfrak{p}}[\omega_n]$. The extension F_n/F is unramified at all other finite places: this can be seen directly by observing that Φ_{π^n} has constant coefficient π^n. Furthermore F_n/F is tamely ramified at ∞ with inertia group $I_\infty(F_n/F) \simeq \mathbb{F}_q^*$.

The similarity with the classical properties of $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is striking.

The formula $N_{F_{n+1}/F_n}(\omega_{n+1}) = \omega_n$ shows that the ω_n's form a compatible system under the norm maps (the proof is extremely easy; it can be found in [6, Lemma 2]). This and the observation that $[F_{n+1} : F_n] = q^{\deg(\mathfrak{p})}$ for $n \geq 1$ imply

$$\varprojlim K_n^* = \omega^{\mathbb{Z}} \times \mathbb{F}_{\mathfrak{p}}^* \times \varprojlim \mathcal{U}_n\,. \tag{6.1}$$

Note that $\varprojlim \mathcal{U}_n$ is a $\tilde{\Lambda}$-module.

6.2. **Coleman's theory.** A more complete discussion and proofs of results in this section can be found in [6, §3]. Let R be a subring of $\mathbf{C}_{\mathfrak{p}}$: then, as usual, $R((x)) := R[[x]](x^{-1})$ is the ring of formal Laurent series with coefficients in R. Moreover, following [11] we define $R[[x]]_1$ and $R((x))_1$ as the subrings consisting of those (Laurent) power series which converge on the punctured open ball

$$B' := B(0,1) - \{0\} \subset \mathbf{C}_{\mathfrak{p}}\,.$$

The rings $R[[x]]_1$ and $R((x))_1$ are endowed with a structure of topological R-algebras, induced by the family of seminorms $\{\|\cdot\|_r\}$, where r varies in $|\mathbf{C}_{\mathfrak{p}}| \cap (0,1)$ and $\|f\|_r := \sup\{|f(z)| : |z| = r\}$.

All essential ideas for the following two theorems are due to Coleman [11].

Theorem 6.1. *There exists a unique continuous homomorphism*

$$\mathcal{N} : F_{\mathfrak{p}}((x))_1^* \to F_{\mathfrak{p}}((x))_1^*$$

such that

$$\prod_{u\in\Phi[\mathfrak{p}]} f(x+u) = (\mathcal{N}f) \circ \Phi_\pi.$$

Theorem 6.2. *The evaluation map* $ev : f \mapsto \{f(\omega_n)\}$ *gives an isomorphism*

$$(A_{\mathfrak{p}}((x))^*)^{\mathcal{N}=id} \simeq \varprojlim K_n^*$$

where the inverse limit is taken with respect to the norm maps.

We shall write Col_u for the power series in $A_{\mathfrak{p}}((x))^*$ associated to $u \in \varprojlim K_n^*$ by Coleman's isomorphism of Theorem 6.2.

Remark 6.3. An easily obtained family of $\mathcal{N}$-invariant power series is the following. Let $a \in A_{\mathfrak{p}}^*$: then

$$\prod_{u\in\Phi[\mathfrak{p}]} \Phi_a(x+u) = \prod_{u\in\Phi[\mathfrak{p}]} (\Phi_a(x) + \Phi_a(u)) = \Phi_\pi(\Phi_a(x))$$

(since Φ_a permutes elements in $\Phi[\mathfrak{p}]$) and from $\Phi_\pi\Phi_a = \Phi_a\Phi_\pi$ in $A_{\mathfrak{p}}\{\{\tau\}\}$ it follows that $\Phi_a(x)$ is invariant under the Coleman norm operator $\mathcal{N}$. (As observed in [6, page 797], this just amounts to replacing ω with $a\cdot\omega$ as generator of the Tate module.)

Following [11], we define an action of Γ on $F_{\mathfrak{p}}[[x]]_1$ by $(\sigma * f)(x) := f(\Phi_{\chi(\sigma)}(x))$. Then $Col_{\sigma u} = (\sigma * Col_u)$, as one sees from

$$(\sigma * Col_u)(\omega_n) = Col_u(\Phi_{\chi(\sigma)}(\omega_n)) = Col_u(\sigma\omega_n) = \sigma(Col_u(\omega_n)) = \sigma(u_n). \tag{6.2}$$

6.3. **The Coates-Wiles homomorphisms.** We introduce some operators on power series. Let $\operatorname{dlog} : F_{\mathfrak{p}}((x))_1^* \to F_{\mathfrak{p}}((x))_1$ be the logarithmic derivative, i.e., $\operatorname{dlog}(g) := \frac{g'}{g}$. Also, for any $j \in \mathbb{N}$ let $\Delta_j \colon F_{\mathfrak{p}}((x)) \to F_{\mathfrak{p}}((x))$ be the jth Hasse-Teichmüller derivative, defined by the formula

$$\Delta_j\left(\sum_{n=0}^{\infty} c_n x^n\right) := \sum_{n=0}^{\infty} \binom{n+j}{j} c_{n+j} x^n$$

(i.e., Δ_j "is" the differential operator $\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}x^j}$). A number of properties of the Hasse-Teichmüller derivatives can be found in [24]; here we just recall that the operators Δ_j are $F_{\mathfrak{p}}$-linear and that

$$f(x) = \sum_{j=0}^{\infty} \Delta_j(f)_{|x=0}\, x^j\,. \tag{6.3}$$

The last operator we need to introduce is composition with the Carlitz exponential $e_C(x) = x + \dots$, i.e., $f \mapsto f(e_C(x))$.

Definition 6.4. For any integer $k \geq 1$, define the kth Coates-Wiles homomorphism $\delta_k \colon \varprojlim \mathcal{O}_n^* \to F_{\mathfrak{p}}$ by

$$\delta_k(u) := \Delta_{k-1}\big((\operatorname{dlog} Col_u)(e_C(x))\big)_{|x=0} = \big(\Delta_{k-1}((\operatorname{dlog} Col_u)\circ e_C)\big)(0)\,.$$

Notice that by (6.3) this is equivalent to putting

$$(\operatorname{dlog} Col_u)(e_C(x)) = \sum_{k=1}^{\infty} \delta_k(u) x^{k-1}\,. \tag{6.4}$$

Lemma 6.5. *The Coates-Wiles homomorphisms satisfy*

$$\delta_k(\sigma u) = \chi(\sigma)^k \delta_k(u).$$

Proof. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\Phi_a(x) = a$ for any $a \in A_{\mathfrak{p}}$. Then from (6.2) it follows

$$\operatorname{dlog} Col_{\sigma u} = \operatorname{dlog}(Col_u \circ \Phi_{\chi(\sigma)}) = \chi(\sigma)(\operatorname{dlog} Col_u)\circ \Phi_{\chi(\sigma)}\,,$$

since $\mathrm{dlog}\,(f\circ g) = g'(\frac{f'}{f}\circ g)$. Composing with e_C and using $\Phi_a(e_C(x)) = e_C(ax)$, one gets, by (6.4),

$$(\mathrm{dlog}\, Col_{\sigma u})(e_C(x)) = \chi(\sigma)(\mathrm{dlog}\, Col_u)(e_C(\chi(\sigma)x)) = \chi(\sigma)\sum_{k=1}^{\infty}\delta_k(u)\chi(\sigma)^{k-1}x^{k-1}\,.$$

The result follows. □

6.4. **Cyclotomic units.**

Definition 6.6. The group C_n of cyclotomic units in F_n is the intersection of B_n^* with the subgroup of F_n^* generated by $\sigma(\omega_n)$, $\sigma\in Gal(F_n/F)$.

By the explicit description of the Galois action via Φ, one sees immediately that this is the same as $B_n^*\cap\langle\Phi_a(\omega_n)\rangle_{a\in A-\mathfrak{p}}$.

Lemma 6.7. *Let $\sum c_\sigma\sigma$ be an element in $\mathbb{Z}[Gal(F_n/F)]$: then*

$$\prod_{\sigma\in Gal(F_n/F)}\sigma(\omega_n)^{c_\sigma}\in C_n \Longleftrightarrow \sum c_\sigma = 0\,.$$

Proof. Obvious from the observation that ω_n is a uniformizer for the place above $\mathfrak{p}$ and a unit at every other finite place of F_n. □

Let $\mathcal{C}_n$ and $\mathcal{C}_n^1$ denote the closure respectively of $C_n\cap\mathcal{O}_n^*$ and of $C_n^1 := C_n\cap\mathcal{U}_n$.

Let $a\in A_{\mathfrak{p}}^*$. By Remark 6.3 $(\Phi_a(\omega_n))_n$ is a norm compatible system: hence one can define a homomorphism

$$\Upsilon\colon \mathbb{Z}[\tilde\Gamma]\longrightarrow \varprojlim K_n^*$$

$$\sum c_\sigma\sigma\mapsto\prod\left(\sigma(\omega_n)^{c_\sigma}\right)_n = \prod\left(\Phi_{\chi(\sigma)}(\omega_n)^{c_\sigma}\right)_n$$

Let $\widehat{\varprojlim K_n^*}$ be the p-adic completion of $\varprojlim K_n^*$. By (6.1) one gets the isomorphism $\widehat{\varprojlim K_n^*}\simeq \omega^{\mathbb{Z}_p}\times\varprojlim\mathcal{U}_n$.

Lemma 6.8. *The restriction of Υ to $\mathbb{Z}[\Gamma]$ can be extended to $\Upsilon\colon\Lambda\to\widehat{\varprojlim K_n^*}$*

Proof. If $a\in A_{\mathfrak{p}}$ is a 1-unit, then

$$\Phi_a(\omega_n) = \omega_n u_n \tag{6.5}$$

with $u_n\in\mathcal{U}_n$. Since by definition $\Gamma = \chi^{-1}(1+\pi A_{\mathfrak{p}})$, it follows that Υ sends $\mathbb{Z}[\Gamma]$ into $\omega^{\mathbb{Z}}\times\varprojlim\mathcal{U}_n$. To complete the proof it suffices to check that Υ is continuous with respect to the natural topologies on $\Lambda = \varprojlim(\mathbb{Z}/p^n\mathbb{Z})[Gal(F_n/F)]$ and $\widehat{\varprojlim K_n^*}$. But $a\equiv a' \bmod \pi^n$ in $A_{\mathfrak{p}}$ implies $\Phi_a(\omega_j) = \Phi_{a'}(\omega_j)$ for any $j\le n$; and the result follows from the continuity of χ. □

Proposition 6.9. *Let $I\subset\Lambda$ denote the augmentation ideal: then Υ induces a surjective homomorphism of Λ-modules $I\longrightarrow\varprojlim\mathcal{C}_n^1$. The kernel has empty interior.*

Proof. From Lemma 6.7 and (6.5) it is clear that $\Upsilon(\alpha)\in\varprojlim\mathcal{C}_n^1$ if and only if $\alpha\in I$. This map is surjective because I is compact and already the restriction to the augmentation ideal of $\mathbb{Z}[\Gamma]$ is onto C_n^1 for all n. A straightforward computation shows that it is a homomorphism of Λ-algebras: for $\gamma\in\Gamma$

$$\gamma\Upsilon\left(\sum c_\sigma\sigma\right) = \left(\gamma\left(\prod\Phi_{\chi(\sigma)}(\omega_n)^{c_\sigma}\right)_n\right) = \left(\prod\Phi_{\chi(\sigma\gamma)}(\omega_n)^{c_\sigma}\right)_n \tag{6.6}$$

because $\gamma(\Phi_a(\omega_n)) = \Phi_a(\gamma(\omega_n)) = \Phi_a(\Phi_{\chi(\gamma)}(\omega_n))$.

For the statement about the kernel, let $A^+ \subset A$ be the subset of monic polynomials and consider any function $A^+ \longrightarrow \mathbb{Z}$, $a \mapsto n_a$, such that $n_a = 0$ for almost all a. We claim that $\prod_{a \in A^+} \Phi_a(x)^{n_a} = 1$ only if $n_a = 0$ for all a. To see it, let u_a denote a generator of the cyclic A-module $\Phi[(a)]$. Then $x - u_a$ divides $\Phi_b(x)$ if and only if $b \in (a)$: hence the multiplicity m_a of u_a as root of $\prod \Phi_a(x)^{n_a} = 1$ is exactly $\sum_{b \in (a) \cap A^+} n_b$. For $b \in A$, let $\varepsilon(b)$ denote the number of primes of A dividing b (counted with multiplicities): then a simple combinatorial argument shows that

$$n_a = \sum_{b \in A^+} (-1)^{\varepsilon(b)} \sum_{c \in A^+} n_{abc} \,.$$

It follows that $m_a = 0$ for all $a \in A^+$ if and only if $n_a = 0$ for all a.

As in §5.3, for $v \neq \mathfrak{p}, \infty$ let $Fr_v \in \tilde{\Gamma}$ be its Frobenius. By [18, Proposition 7.5.4] one finds that $\chi(Fr_v)$ is the monic generator of the ideal in A corresponding to the place v: hence by Chebotarev $\chi^{-1}(A^+)$ is dense in $\tilde{\Gamma}$. Thus the isomorphism of Theorem 6.2 shows that we have proved that $\Upsilon \colon I \longrightarrow \varprojlim \mathcal{O}_n^*$ is injective on a dense subset: the kernel must have empty interior. □

Remark 6.10. Since $I \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta] = \oplus_{\delta \in \Delta} I\delta$, formula (6.6) shows that Υ can be extended to a homomorphism of $\tilde{\Lambda}$-modules $I \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta] \longrightarrow \varprojlim \mathcal{C}_n$.

Proposition 6.11. *We have:* $\varprojlim \mathcal{O}_n^* / \varprojlim \mathcal{C}_n \simeq \varprojlim \mathcal{U}_n / \varprojlim \mathcal{C}_n^1$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \varprojlim \mathcal{C}_n^1 & \longrightarrow & \varprojlim \mathcal{U}_n & \longrightarrow & \varprojlim \mathcal{U}_n / \varprojlim \mathcal{C}_n^1 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
1 & \longrightarrow & \varprojlim \mathcal{C}_n & \longrightarrow & \varprojlim \mathcal{O}_n^* & \longrightarrow & \varprojlim \mathcal{O}_n^* / \varprojlim \mathcal{C}_n & \longrightarrow & 1 \,.
\end{array}$$

All vertical maps are injective and by (6.1) the cokernel of α_2 is $\mathbb{F}_{\mathfrak{p}}^*$. For $\delta \in \Delta$ one has $\delta\omega_n = \Phi_{\chi(\delta)}(\omega_n) = \chi(\delta)\omega_n u_n$ for some $u_n \in \mathcal{U}_n$. By the injectivity part of the proof of Proposition 6.9, $C_n = C_n^1 \times \Upsilon(\mathbb{Z}[\Delta])$ and it follows that the cokernel of α_1 is also $\mathbb{F}_{\mathfrak{p}}^*$. □

6.5. **Cyclotomic units and class groups.** Let $F_n^+ \subset F_n$ be the fixed field of the inertia group $I_\infty(F_n/F)$. The extension F_n^+/F is totally split at ∞ and ramified only above the prime $\mathfrak{p}$. We shall denote the ring of A-integers of F_n^+ by B_n^+. Also, define $\mathcal{E}_n$ and $\mathcal{E}_n^1$ to be the closure respectively of $B_n^* \cap \mathcal{O}_n^*$ and $B_n^* \cap \mathcal{U}_n$.

We need to introduce a slight modification of the groups $\mathcal{A}(L)$ of section 5. For any finite extension L/F, $\mathcal{A}^\infty(L)$ will be the p-part of the class group of A-integers of L, so that, by class field theory, $\mathcal{A}^\infty(L) \simeq Gal(H(L)/L)$, where $H(L)$ is the maximal abelian unramified p-extension of L which is totally split at places dividing ∞. We shall use the shortening $\mathcal{A}_n := \mathcal{A}^\infty(F_n^+)$.

Also, let $\mathcal{X}_n := Gal(M(F_n^+)/F_n^+)$, where $M(L)$ is the maximal abelian p-extension of L unramified outside $\mathfrak{p}$ and totally split above ∞. As in the number field case, one has an exact sequence

$$1 \longrightarrow \mathcal{E}_n^1/\mathcal{C}_n^1 \longrightarrow \mathcal{U}_n/\mathcal{C}_n^1 \longrightarrow \mathcal{X}_n \longrightarrow \mathcal{A}_n \longrightarrow 1 \tag{6.7}$$

coming from the following

Proposition 6.12. *There is an isomorphism of Galois modules*

$$\mathcal{U}_n/\mathcal{E}_n^1 \simeq Gal(M(F_n^+)/H(F_n^+)) \,.$$

Proof. This is a consequence of class field theory in characteristic $p > 0$, as the analogous statement in the number field case: just recall that the role of archimedean places is now played by the valuations above ∞. Under the class field theoretic identification of idele classes $\mathbf{I}_{F_n^+}/(F_n^+)^*$ with a dense subgroup of $Gal((F_n^+)^{ab}/F_n^+)$, one finds a surjection

$$\prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{\mathfrak{P}}^* \twoheadrightarrow Gal(M(F_n^+)/H(F_n^+))$$

whose kernel contains the closure of

$$\prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{\mathfrak{P}}^* \cap (F_n^+)^* \prod_{w\nmid\mathfrak{p}} \mathcal{O}_w^* \prod_{w|\infty} (F_{n,w}^+)^* = \iota_{\mathfrak{p}}((B_n^+)^*)$$

(where $\iota_{\mathfrak{p}}$ denotes the diagonal inclusion). Reasoning as in [48, Lemma 13.5] one proves the proposition. □

Taking the projective limit of the sequence (6.7), we get

$$1 \longrightarrow \mathcal{E}_\infty^1/\mathcal{C}_\infty^1 \longrightarrow \mathcal{U}_\infty/\mathcal{C}_\infty^1 \longrightarrow \mathcal{X}_\infty \longrightarrow \mathcal{A}_\infty \longrightarrow 1\,. \tag{6.8}$$

Lemma 6.13. *The sequence* (6.8) *is exact.*

Proof. Taking the projective limit of the short exact sequence

$$1 \longrightarrow \mathcal{E}_n^1 \longrightarrow \mathcal{U}_n \longrightarrow Gal(M(F_n^+)/H(F_n^+)) \longrightarrow 1$$

we obtain

$$1 \longrightarrow \mathcal{E}_\infty^1 \longrightarrow \mathcal{U}_\infty \longrightarrow Gal(M(\mathcal{F}^+)/H(\mathcal{F}^+)) \longrightarrow \varprojlim{}^1 \mathcal{E}_n^1\ ,$$

where $M(\mathcal{F}^+)$ and $H(\mathcal{F}^+)$ are the maximal abelian p-extensions of $\mathcal{F}^+$ totally split above ∞ and unramified respectively outside the place above $\mathfrak{p}$ and everywhere.

To prove the lemma it is enough to show that $\varprojlim^1(\mathcal{E}_n^1) = 1$. By a well-known result in homological algebra, the functor $\varprojlim^1$ is trivial on projective systems satisfying the Mittag-Leffler condition. We recall that an inverse system (B_n, d_n) enjoys such property if for any n the images of the transition maps $B_{n+m} \to B_n$ are the same for large m. So we are reduced to check that this holds for the $\mathcal{E}_n^1$'s with the norm maps.

Observe first that $\mathcal{E}_n^1$ is a finitely generated Λ_n-module, thus noetherian because so is Λ_n. Consider now $\cap_k Image(N_{n+k,n})$, where $N_{n+k,n}\colon \mathcal{E}_{n+k}^1 \to \mathcal{E}_n^1$ is the norm map. This intersection is a Λ_n-submodule of $\mathcal{E}_n^1$, non-trivial because it contains the cyclotomic units. By noetherianity it is finitely generated, hence there exists l such that $Image(N_{n+k,n})$ is the same for $k \geq l$. Therefore $(\mathcal{E}_n^1)$ satisfies the Mittag-Leffler property. □

The exact sequence (6.8) lies at the heart of Iwasawa theory. Its terms are all Λ-modules and, in section 5.2, we have shown how to associate a characteristic ideal to $\mathcal{A}_\infty$ and its close relation with Stickelberger elements. In a similar way, i.e., working on $\mathbb{Z}_p^d$-subextensions, one might approach a description of $\mathcal{X}_\infty$, while, for the first two terms of the sequence, the filtration of the F_n^+'s seems more natural (as the previous sections show).

Assume for example that the class number of F is prime with p, then it is easy to see that $\mathcal{A}_n = 1$ for all n. Moreover, using the fact that, by a theorem of Galovich and Rosen, the index of the cyclotomic units is equal to the class number (see [41, Theorem 16.12]), one can prove that $\mathcal{E}_n^1/\mathcal{C}_n^1 = 1$ as well. These provide isomorphisms

$$\mathcal{U}_n/\mathcal{C}_n^1 \simeq \mathcal{X}_n$$

and

$$\mathcal{U}_\infty/\mathcal{C}_\infty^1 \simeq \mathcal{X}_\infty\ .$$

In general one expects a relation (at least at the level of $\mathbb{Z}_p^d$-subextensions, then a limit procedure should apply) between the pro-characteristic ideal of $\mathcal{A}_\infty$ and the (yet to be defined) analogous ideal for $\mathcal{E}^1_\infty/\mathcal{C}^1_\infty$ (the Stickelberger element might be a first hint for the study of this relation). Consequently (because of the multiplicativity of characteristic ideals) an equality of (yet to be defined) characteristic ideals of $\mathcal{X}_\infty$ and of $\mathcal{U}_\infty/\mathcal{C}^1_\infty$ is expected as well. Any of those two equalities can be considered as an instance of Iwasawa Main Conjecture for the setting we are working in.

6.6. **Bernoulli-Carlitz numbers.** We go back to the subject of characteristic p L-function. Let $A^+ \subset A$ be the subset of monic polynomials. The Carlitz zeta function is defined

$$\zeta_A(k) := \sum_{a\in A^+} \frac{1}{a^k}$$

for $k \in \mathbb{N}$.

Recall that the Carlitz module corresponds to a lattice $\xi A \subset \mathbf{C}_\infty$ and can be constructed via the Carlitz exponential $e_C(z) := z\prod_{a\in A'}(1 - z\xi^{-1}a^{-1})$ (where A' denotes $A - \{0\}$). Rearranging summands in the equality

$$\frac{1}{e_C(z)} = \mathrm{dlog}\,(e_C(z)) = \sum_{a\in A} \frac{1}{z-\xi a} = \frac{1}{z} - \sum_{a\in A'}\sum_{k=1}^{\infty} \frac{z^{k-1}}{(\xi a)^k}$$

(and using $A' = \mathbb{F}_q^* \times A^+$) one gets the well-known formula

$$\frac{1}{e_C(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\zeta_A(n(q-1))}{\xi^{n(q-1)}} z^{n(q-1)-1}. \tag{6.9}$$

From §6.4 it follows that for any $a, b \in A - \mathfrak{p}$, the function $\frac{\Phi_a(x)}{\Phi_b(x)}$ is an $\mathcal{N}$-invariant power series, associated with

$$c(a,b) := \frac{\Phi_a(\omega)}{\Phi_b(\omega)} = \left(\frac{\Phi_a(\omega_n)}{\Phi_b(\omega_n)}\right)_n \in \varprojlim \mathcal{O}_n^*\,. \tag{6.10}$$

Theorem 6.14. *The kth Coates-Wiles homomorphism applied to $c(a,b)$ is equal to:*

$$\delta_k(c(a,b)) = \begin{cases} 0 & if\ k \not\equiv 0 \mod q-1 \\ (a^k - b^k)\frac{\zeta_A(k)}{\xi^k} & if\ k \equiv 0 \mod q-1. \end{cases}.$$

We remark that the condition $k = n(q-1)$ here is the analogue of k being an even integer in the classical setting (since $q - 1 = |\mathbb{F}_q^*|$ just as $2 = |\mathbb{Z}^*|$).

Proof. Observe that (6.10) amounts to giving the Coleman power series $Col_{c(a,b)}$.
Let λ be the Carlitz logarithm: i.e., $\lambda \in F\{\{\tau\}\}$ is the element uniquely determined by $e_C \circ \lambda = 1$. Then $\Phi_a(x) = e_C(a\lambda(x))$ and by (6.10) and (6.9) one gets

$$\mathrm{dlog}\,Col_{c(a,b)}(x) = \frac{a}{\Phi_a(x)} - \frac{b}{\Phi_b(x)} = \sum_{n\geq 1}(a^{n(q-1)} - b^{n(q-1)})\frac{\zeta_A(n(q-1))}{\xi^{n(q-1)}}\lambda(x)^{n(q-1)-1}\,.$$

Since $\lambda(e_C(x)) = x$, we get

$$(\mathrm{dlog}\,Col_{c(a,b)})(e_C(x)) = \sum_{n\geq 1}(a^{n(q-1)} - b^{n(q-1)})\frac{\zeta_A(n(q-1))}{\xi^{n(q-1)}}x^{n(q-1)-1} \tag{6.11}$$

and the theorem follows comparing (6.11) with (6.4). $\square$

Remark 6.15. As already known to Carlitz, $\zeta_A(k)\xi^{-k}$ is in F when $q-1$ divides k. Note that by a theorem of Wade, $\xi \in F_\infty$ is transcendental over F. Furthermore, Jing Yu [50] proved that $\zeta_A(k)$ for all $k \in \mathbb{N}$ and $\zeta_A(k)\xi^{-k}$ for k "odd" (i.e., not divisible by $q-1$) are transcendental over F.

Theorem 6.14 can be restated in terms of the Bernoulli-Carlitz numbers BC_k [18, Definition 9.2.1]. They can be defined by

$$\frac{1}{e_C(z)} = \sum_{n\geq 0} \frac{BC_n}{\Pi(n)} z^{n-1}$$

(where $\Pi(n)$ is a function field analogue of the classical factorial $n!$); in particular $BC_n = 0$ when $n \not\equiv 0 \pmod{q-1}$. Then Theorem 6.14 becomes

$$\delta_k(c(a,b)) = (a^k - b^k)\frac{BC_k}{\Pi(k)}. \tag{6.12}$$

Theorem 6.14 and formula (6.12) can be seen as extending a result by Okada, who in [36] obtained the ratios $\frac{BC_k}{\Pi(k)}$ (for $k = 1, \ldots, q^{\deg(\mathfrak{p})} - 2$) as images of cyclotomic units under the Kummer homomorphisms (which are essentially a less refined version of the Coates-Wiles homomorphisms). From here one proves that the non-triviality of an isotypic component of $\mathcal{A}_1$ implies the divisibility of the corresponding "even" Bernoulli-Carlitz number by $\mathfrak{p}$: we refer to [18, §8.20] for an account. As already mentioned, Shu [42] generalized Okada's work to any F (but with the assumption $\deg(\infty) = 1$): it might be interesting to extend Theorem 6.14 to a "Coates-Wiles homomorphism" version of her results.

6.7. **Interpolation?** In the classical setting of cyclotomic number fields, the analogue of the formula in Theorem 6.14 can be used as a key step in the construction of the Kubota-Leopoldt zeta function (see e.g. [10]). Hence it is natural to wonder if something like it holds in our function field case. For now we have no answer and can only offer some vague speculation.

As mentioned in section 5.4, Goss found a way to extend the domain of ζ_A from $\mathbb{N}$ to S_∞. He also considered the analogue of the p-adic domain and defined it to be $\mathbf{C}_\mathfrak{p}^* \times S_\mathfrak{p}$, with $S_\mathfrak{p} := \mathbb{Z}_p \times \mathbb{Z}/(q^{\deg(\mathfrak{p})} - 1)$ (observe that $\mathbf{C}_\mathfrak{p}^* \times S_\mathfrak{p}$ is the $\mathbf{C}_\mathfrak{p}$-valued dual of $F_\mathfrak{p}^*$). Then functions like ζ_A enjoy also a $\mathfrak{p}$-adic life: for example, letting $\pi_v \in A^+$ be a uniformizer for a place v, $\zeta_{A,\mathfrak{p}}$ is defined on $\mathbf{C}_\mathfrak{p}^* \times S_\mathfrak{p}$ by

$$\zeta_{A,\mathfrak{p}}(s) := \prod_{v\nmid \mathfrak{p}\infty} (1 - \pi_v^{-s})^{-1},$$

at least where the product converges.

The ring $\mathbb{Z}$ embeds discretely in S_∞ and has dense image in $1 \times S_\mathfrak{p}$. So Theorem 6.14 seems to suggest interpolation of $\zeta_{A,\mathfrak{p}}$ on $1 \times S_\mathfrak{p}$. Another clue in this direction is the fact that $S_\mathfrak{p}$ is the "dual" of Γ, just as $\mathbb{Z}_p$ is the "dual" of $Gal(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$. (A strengthening of this interpretation has been recently provided by the main result of [25].)

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