CHARACTERISTIC IDEALS AND IWASAWA THEORY

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Abstract. For a noetherian Krull ring R, the formal power series ring $R[[t]]$ and a pseudo-null $R[[t]]$-module M, we prove the equality between the characteristic ideals of M_t and M/tM (considered as R-modules). We use this to define (pro-)characteristic ideals of Iwasawa modules for the non-noetherian Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(\mathcal{F}/F)]]$, where $\mathrm{Gal}(\mathcal{F}/F) \simeq \mathbb{Z}_p^\infty$. We also give a little application to the Generalized Greenberg Conjecture.

1. Introduction

Let A be a noetherian Krull domain and M a finitely generated torsion A-module. The structure theorem for such modules provides an exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i} A \longrightarrow Q \longrightarrow 0 \tag{1.1}$$

where the $\mathfrak{p}_i$'s are height 1 prime ideals of A and P and Q are pseudo-null A-modules (for more details and precise definitions of all the objects appearing in this Introduction see Section 2). This sequence defines an important invariant of the module M, namely its *characteristic ideal*

$$Ch_A(M) := \prod_{i=1}^{n} \mathfrak{p}_i^{e_i} \ .$$

Characteristic ideals play a major role in (commutative) Iwasawa theory for global fields: they provide the algebraic counterpart for the p-adic L-functions associated to Iwasawa modules (such as class groups or duals of Selmer groups). Here the Krull domain one works with is the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$, where Γ is some p-adic Lie group occurring as Galois group (we shall deal mainly with the case $\Gamma \simeq \mathbb{Z}_p^d$ for some $d \in \mathbb{N}$). Even if pseudo-null modules do not contribute to characteristic ideals, they are relevant in important conjectures such as the Generalized Greenberg's Conjecture (which states that a certain Iwasawa module is pseudo-null: the precise statement will be recalled in Section 3.2) and appear in the descent problem when one wants to compare the characteristic ideal of an Iwasawa module of a $\mathbb{Z}_p^d$-extension with the one of a $\mathbb{Z}_p^{d-1}$-extension contained in it. The last topic is particularly important when the global field has characteristic p, since in this case extensions $\mathcal{F}/F$ with $\mathrm{Gal}(\mathcal{F}/F) \simeq \mathbb{Z}_p^\infty$ occur quite naturally: in this situation the Iwasawa algebra is non-noetherian and there is no guarantee one can find a sequence such as (1.1). One strategy to overcome this difficulty is to take a filtration of $\mathcal{F}$ as a union of $\mathbb{Z}_p^d$-extensions, define the characteristic ideals at the $\mathbb{Z}_p^d$-level for all d and then pass to the limit. The main obstacle is that inverse limits of pseudo-null modules are not pseudo-null in general, hence one has to be quite careful with what happens passing from one level to the other.

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To deal with this problem, we develop the following tool. Let R be a noetherian Krull domain and consider the ring of formal power series $R[[t]]$ and an $R[[t]]$-module M. In Section 2 (more precisely, in Theorem 2.7 and Proposition 2.10) we shall prove

Theorem 1.1. *If M is a pseudo-null $R[[t]]$-module, then M_t (the kernel of multiplication by t) and M/tM are finitely generated torsion R-modules and*

$$Ch_R(M_t) = Ch_R(M/tM)\ . \tag{1.2}$$

Moreover, for any finitely generated torsion $R[[t]]$-module N, we have

$$Ch_R(N_t)\pi(Ch_{R[[t]]}(N)) = Ch_R(N/tN) \tag{1.3}$$

(where $\pi\colon R[[t]] \to R$ is the canonical projection).

As a first application, we shall obtain a criterion for an $R[[t]]$-module to be pseudo-null (Corollary 2.11).

The Iwasawa algebra associated with a $\mathbb{Z}_p^d$-extension of a global field F is (noncanonically) isomorphic to the Krull ring $\mathbb{Z}_p[[t_1, \dots, t_d]]$, hence descending to a $\mathbb{Z}_p^{d-1}$-extension corresponds to the passage from $R[[t]]$ to R. (Note that in Theorem 1.1 we have kept the hypotheses on R to a minimum: the only point in the proof where this generality requires a little more work is Proposition 2.3, which becomes quite obvious for $R \simeq \mathbb{Z}_p[[t_1, \dots, t_{d-1}]]$). So the results of Section 2 apply immediately to Iwasawa modules. In order to keep the paper short, we just consider the case of class groups.

Section 3.1 deals with the case when F has characteristic $p > 0$ (for an account of Iwasawa theory over such function fields see [5] and the references there). We already mentioned that here one is naturally led to work with extremely large abelian p-adic extensions: this comes from class field theory, since, in the completion F_v of F at some place v, the group of 1-units $U_1(F_v)$ is isomorphic to $\mathbb{Z}_p^\infty$. As hinted above, our strategy to tackle $\mathcal{F}/F$ with $\mathrm{Gal}(\mathcal{F}/F) \simeq \mathbb{Z}_p^\infty$ is to work first with $\mathbb{Z}_p^d$-extensions and then use limits. The usefulness of Theorem 1.1 in this procedure is illustrated in §3.1.1, where we can define the characteristic ideal $\widetilde{Ch}_\Lambda(\mathcal{A}(\mathcal{F}))$ dispensing with the crutch of the *ad hoc* hypothesis [5, Assumption 5.3]. The search for a "good" definition for it was one of the main motivations for this work.

Section 3.2 provides another application of Theorem 1.1 and Corollary 2.11, this time in characteristic 0: when F is a number field and the extension is the compositum of all the $\mathbb{Z}_p$-extensions of F, we obtain an equivalent formulation for the Generalized Greenberg's Conjecture (Theorem 3.13).

Next to class groups, [5] considers the case of Selmer groups of abelian varieties: in [5, §3], following [3], we employed Fitting ideals instead than characteristic ones so to avoid the difficulties of taking the inverse limit. As a consequence, the algebraic side of the Iwasawa Main Conjecture relative to this situation would be given by the pro-Fitting ideal $\widetilde{Fitt}_\Lambda(\mathcal{S}(\mathcal{F}))$ ([5, Definition 3.5]; here $\mathcal{S}(\mathcal{F})$ denotes the Pontrjagin dual of the Selmer group $Sel_A(\mathcal{F})_p$, where A/F is an abelian variety having good ordinary or split multiplicative reduction at those places which ramify in $\mathcal{F}/F$). By a little additional work, Theorem 1.1 permits to define a pro-characteristic ideal $\widetilde{Ch}_\Lambda(\mathcal{S}(\mathcal{F}))$ under rather mild hypotheses (e.g., when $\mathcal{F}$ is a $\mathfrak{p}$-ramified extension - as in §3.1.1 below - and the number of connected components of the reduction of A is never a multiple of p), allowing to formulate a more classical Iwasawa Main Conjecture; details shall be given in some future publication. One could further improve this result exploiting the careful analysis of the variation of characteristic ideals of Selmer groups, conducted by Tan in [21].

Remark 1.2. If a pseudo-null $R[[t]]$-module M is finitely generated over R as well, then the statement of Theorem 1.1 is trivially deduced from the exact sequence

$$0 \to M[t] \to M \xrightarrow{t} M \to M/tM \to 0$$

and the multiplicativity of characteristic ideals. As explained in [11] (see Lemma 2 and the discussion right after it), for any $\mathbb{Z}_p^d$-extension $\mathcal{F}_d$ and any pseudo-null $\Lambda(\mathcal{F}_d)$-module M it is always possible to find (at least one) $\mathbb{Z}_p^{d-1}$-subextension $\mathcal{F}_{d-1}$ such that M is finitely generated over $\Lambda(\mathcal{F}_{d-1})$. Our search for a characteristic ideal via a projective limit does not allow this freedom in the choice of subextensions, hence the need for an "unconditional" result like Theorem 1.1.

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2. Λ-modules and characteristic ideals

2.1. **Krull domains.** We begin by reviewing some basic facts and definitions we are going to need. A comprehensive reference is [6, Chapter VII].

An integral domain A is called a Krull domain if $A = \cap A_{\mathfrak{p}}$ (where $\mathfrak{p}$ varies among prime ideals of height 1 and $A_{\mathfrak{p}}$ denotes localization), all $A_{\mathfrak{p}}$'s are discrete valuation rings and any $x \in A - \{0\}$ is a unit in $A_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ [1]. In particular, one attaches a discrete valuation to any height 1 prime ideal. Furthermore, a ring is a unique factorization domain if and only if it is a Krull domain and all height 1 prime ideals are principal ([6, VII, §3.2, Theorem 1]).

2.1.1. *Torsion modules.* Let A be a noetherian Krull domain. A finitely generated torsion A-module is said to be *pseudo-null* if its annihilator ideal has height at least 2. A morphism with pseudo-null kernel and cokernel is called a pseudo-isomorphism: being pseudo-isomorphic is an equivalence relation between finitely generated torsion A-modules (torsion is essential here[2]) and we shall denote it by $\sim_A$. If M is a finitely generated torsion A-module then there is a pseudo-isomorphism

$$M \longrightarrow \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i} \tag{2.1}$$

where the $\mathfrak{p}_i$'s are height 1 prime ideals of A (not necessarily distinct) and the $\mathfrak{p}_i$'s, n and the e_i's are uniquely determined by M (see e.g. [6, VII, §4.4, Theorem 5]). A module like the one on the right-hand side of (2.1) will be called *elementary A-module* and

$$E(M) := \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i} \sim_A M$$

is the *elementary module attached to M*.

Definition 2.1. Let M be a finitely generated A-module: its *characteristic ideal* is

$$Ch_A(M) := \begin{cases} 0 & \text{if } M \text{ is not torsion;} \\ \prod_{i=1}^{n} \mathfrak{p}_i^{e_i} & \text{if } M \sim_A \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i}\,. \end{cases}$$

In particular, M is pseudo-null if and only if $Ch_A(M) = A$.

We shall denote by $\mathbf{Fgt}_A$ the category of finitely generated torsion A-modules.

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[1] This is not the definition in [6], but it is equivalent to it: see [6, VII, §1.6, Theorem 4].

[2] For example the map $(p,t) \hookrightarrow \mathbb{Z}_p[[t]]$ is a pseudo-isomorphism, but there is no such map from $\mathbb{Z}_p[[t]]$ to (p,t).

Remarks 2.2.

1. An equivalent definition of pseudo-null is that all localizations at primes of height 1 are zero. If $\mathfrak{p}$ and $\mathfrak{q}$ are two different primes of height 1 (and M is a torsion A-module) we have $M \otimes_A A_\mathfrak{p} \otimes_A A_\mathfrak{q} = 0$. By the structure theorem recalled in (2.1) it follows immediately that for a finitely generated torsion A-module M, the canonical map

$$M \longrightarrow \bigoplus_{\mathfrak{p}} (M \otimes_A A_\mathfrak{p}) \tag{2.2}$$

(where the sum is taken over all primes of height 1) is a pseudo-isomorphism. Actually, the right-hand side of (2.2) can be used to compute $Ch_A(M)$: a prime $\mathfrak{p}$ appears in $Ch_A(M)$ with exponent the length of the module $M \otimes_A A_\mathfrak{p}$.

2. The previous remark suggests a generalization of the definition of characteristic ideal by means of "supernatural divisors" [3]. Let M be any torsion A-module (we drop the "finitely generated" assumption) and define

$$Ch_A(M) := \prod_{\mathfrak{p}} \mathfrak{p}^{l_\mathfrak{p}(M \otimes_A A_\mathfrak{p})}$$

where the product is taken over all primes of height 1 and the exponent of $\mathfrak{p}$ (i.e., the "length" of the module $M \otimes_A A_\mathfrak{p}$) is a supernatural number (i.e., belongs to $\mathbb{N} \cup \{\infty\}$). More precisely, for N a finitely generated torsion $A_\mathfrak{p}$-module let $l_\mathfrak{p}(N)$ denote its length. Then we put

$$l_\mathfrak{p}(M \otimes_A A_\mathfrak{p}) := \sup\{l_\mathfrak{p}(M_\alpha \otimes_A A_\mathfrak{p})\}\,,$$

where M_α varies among all finitely generated submodules of M. Note that, since $A_\mathfrak{p}$ is flat, $M_\alpha \otimes_A A_\mathfrak{p}$ is still a submodule of $M \otimes_A A_\mathfrak{p}$; furthermore, the length $l_\mathfrak{p}$ is an increasing function on finitely generated torsion $A_\mathfrak{p}$-modules (partially ordered by inclusion).

2.1.2. *Power series.* In the rest of this section, R will denote a Krull domain and $\Lambda := R[[t]]$ the ring of power series in one variable over R.

Proposition 2.3. *Let R be a Krull domain and $\mathfrak{p} \subset R$ a height 1 prime. Then Λ is also a Krull domain and $\mathfrak{p}\Lambda$ is a height 1 prime of Λ.*

This is well-known (actually, one can prove the analogue even with infinitely many variables: see [7]). In order to make the paper as self-contained as possible, and for lack of a suitable reference for the second part of the proposition, we provide a quick proof.

Proof. Let Q be the fraction field of R. Since R is Krull, we have $R = \cap R_\mathfrak{q}$ as $\mathfrak{q}$ varies among all prime ideals of height 1. Furthermore, each $R_\mathfrak{q}$ is a discrete valuation ring: then [6, VII, §3.9, Proposition 8] shows that $R_\mathfrak{q}[[t]]$ is a unique factorization domain. In particular every $R_\mathfrak{q}[[t]][t^{-1}]$ is a Krull domain and we get

$$\Lambda = R[[t]] = Q[[t]] \cap \bigcap_{\mathfrak{q}} R_\mathfrak{q}[[t]][t^{-1}] = Q[[t]] \cap \bigcap_{\mathfrak{q}} \bigcap_{\mathfrak{P} \in S_\mathfrak{q}} \left(R_\mathfrak{q}[[t]][t^{-1}]\right)_\mathfrak{P} \tag{2.3}$$

(where $S_\mathfrak{q}$ denotes the set of height 1 primes in $R_\mathfrak{q}[[t]][t^{-1}]$). This shows that Λ is an intersection of discrete valuation rings. A power series $\lambda = t^h \sum_{i \geq 0} c_i t^i \in \Lambda$ (with $c_0 \neq 0$) is a unit in $R_\mathfrak{q}[[t]][t^{-1}]$ unless $c_0 \in \mathfrak{q}$ and, in the latter case, λ is still a unit in $(R_\mathfrak{q}[[t]][t^{-1}])_\mathfrak{P}$ unless it can be divided by the generator of $\mathfrak{P}$. This proves that Λ is a Krull domain.

Since $R_\mathfrak{p}$ is a discrete valuation ring, its maximal ideal $\mathfrak{p}R_\mathfrak{p}$ is principal: let π be a uniformizer. Then π is irreducible in $R_\mathfrak{p}[[t]]$, hence it generates a height 1 prime ideal

[3] We recall that the group of divisors of A is the free abelian group generated by the prime ideals of height 1 in A (see [6, VII, §1]).

$\mathfrak{P} := \pi R_\mathfrak{p}[[t]] = \mathfrak{p}R_\mathfrak{p}[[t]]$. By the general theory of Krull domains, $\mathfrak{P}$ corresponds to a discrete valuation $\nu_\mathfrak{P}$ on the fraction field $Frac(R_\mathfrak{p}[[t]])$; the restriction of $\nu_\mathfrak{P}$ to Q is precisely the discrete valuation associated with $\mathfrak{p}$. Similarly, restricting $\nu_\mathfrak{P}$ to $Frac(\Lambda)$ yields a discrete valuation, with ring of integers $D_\mathfrak{P}$ and maximal ideal $\mathfrak{m}_\mathfrak{P}$. The ring $D_\mathfrak{P}$ is the localization of Λ at $\mathfrak{m}_\mathfrak{P}$: hence it is flat over Λ and, by [6, VII, §1.10, Proposition 15], the prime ideal

$$\mathfrak{m}_\mathfrak{P} \cap \Lambda = \mathfrak{P} \cap \Lambda = \mathfrak{p}\Lambda \neq 0$$

has height 1. □

2.2. **Pseudo-null Λ-modules.** Now assume that R (and hence Λ) is Noetherian. In this section P will be a pseudo-null Λ-module. We denote by P_t the kernel of multiplication by t and remark that in the exact sequence

$$P_t \hookrightarrow P \xrightarrow{t} P \twoheadrightarrow P/tP\,, \tag{2.4}$$

P_t and P/tP are finitely generated Λ-modules, because so is P. The former ones are also finitely generated as R-modules, because t acts as 0 on them. Moreover they are torsion R-modules (just take two relatively prime elements f and g in $Ann_\Lambda(P)$: their projections in R via $t \mapsto 0$ belong to both $Ann_R(P_t)$ and $Ann_R(P/tP)$ and at least one of them is nonzero since otherwise t would divide both f and g). Therefore the characteristic ideals $Ch_R(P_t)$ and $Ch_R(P/tP)$ are given by Definition 2.1 (there is no need for supernatural divisors here) and both of them are nonzero.

2.2.1. *Preliminaries.* For $\mathfrak{p}$ a prime of height one in R, define

$$\widehat{R_\mathfrak{p}} := \lim_{\leftarrow} R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}\,.$$

By a slight abuse of notation, we shall denote by $\mathfrak{p}$ also the maximal ideals of $R_\mathfrak{p}$ and $\widehat{R_\mathfrak{p}}$.

The natural embedding of R into $\widehat{R_\mathfrak{p}}$ allows to identify Λ with a subring of $\widehat{R_\mathfrak{p}}[[t]]$.

Lemma 2.4. *The ring $\widehat{R_\mathfrak{p}}[[t]]$ is a flat Λ-algebra.*

Proof. Put $S_\mathfrak{p} := R - \mathfrak{p}$. We claim that $\widehat{R_\mathfrak{p}}[[t]]$ is the completion of $S_\mathfrak{p}^{-1}\Lambda$ with respect to the ideal generated by $\mathfrak{p}$ and t. This is enough, since formation of fractions and completion of a noetherian ring both generate flat algebras, and the composition of flat morphisms is still flat. To verify the claim consider the inclusions

$$R_\mathfrak{p}[t]/(\mathfrak{p},t)^n \subset S_\mathfrak{p}^{-1}\Lambda/(\mathfrak{p},t)^n \subset \widehat{R_\mathfrak{p}}[[t]]/(\mathfrak{p},t)^n$$

and note that they are preserved by taking the inverse limit with respect to n. To conclude observe that $\lim\limits_{\leftarrow} R_\mathfrak{p}[t]/(\mathfrak{p},t)^n = \widehat{R_\mathfrak{p}}[[t]]$. □

The advantage of working over $\widehat{R_\mathfrak{p}}[[t]]$ is that one can apply the Weierstrass Preparation Theorem (for a proof see e.g. [6, VII, §3.8, Proposition 6]): given $\alpha = \sum a_i t^i \in \widehat{R_\mathfrak{p}}[[t]]$ such that not all coefficients are in $\mathfrak{p}$, there exist $u \in \widehat{R_\mathfrak{p}}[[t]]^*$ and a monic polynomial $\beta \in \widehat{R_\mathfrak{p}}[t]$ such that $\alpha = u\beta$ (the degree of β is equal to the minimum of the indices i such that $a_i \notin \mathfrak{p}$). Actually, as it is going to be clear from the proof of Lemma 2.6 below, we shall need just a weaker form of this statement.

2.2.2. *Characteristic ideals.* Now we deal with the equality between $Ch_R(P_t)$ and $Ch_R(P/tP)$.

Lemma 2.5. *For any finitely generated torsion $R_\mathfrak{p}$-module N one has the equality of lengths*

$$l_{R_\mathfrak{p}}(N) = l_{\widehat{R_\mathfrak{p}}}(N \otimes_{R_\mathfrak{p}} \widehat{R_\mathfrak{p}}) \ .$$

Proof. Since both $R_\mathfrak{p}$ and $\widehat{R_\mathfrak{p}}$ are discrete valuation rings and

$$R_\mathfrak{p}/\mathfrak{p}^n \simeq \widehat{R_\mathfrak{p}}/\mathfrak{p}^n \simeq (R_\mathfrak{p}/\mathfrak{p}^n) \otimes \widehat{R_\mathfrak{p}} \ ,$$

the statement follows directly from the structure theorem for finitely generated torsion modules over principal ideal domains. □

Lemma 2.6. *Let P be a pseudo-null Λ-module. Then $P \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]]$ is a finitely generated $\widehat{R_\mathfrak{p}}$-module for any height 1 prime ideal $\mathfrak{p} \subset R$.*

Proof. We consider P as an R-module and work separately with primes $\mathfrak{p}$ belonging or not belonging to the support of P. If the prime $\mathfrak{p}$ is not in this support, there is some $r \in Ann_R(P)$ which becomes a unit in $R_\mathfrak{p} \subset \widehat{R_\mathfrak{p}}[[t]]$, hence $P \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]] = 0$ and the statement is trivial. Thus, from now on, we assume $\mathfrak{p} \in Supp_R(P)$ (i.e., $Ann_R(P) \subset \mathfrak{p}$). Since $\mathfrak{p}\Lambda$ is a height 1 prime ideal in Λ, the hypothesis on P yields $Ann_\Lambda(P) \not\subset \mathfrak{p}\Lambda$. Hence there exists $\alpha \in Ann_\Lambda(P) - \mathfrak{p}\Lambda$, i.e.,

$$\alpha = \sum_{i \geq 0} a_i t^i \in Ann_\Lambda(P) \qquad (\text{with } a_i \in R \text{ for any } i)$$

with at least one $a_i \notin \mathfrak{p}$. For such an α, let n be the smallest index such that $a_n \notin \mathfrak{p}$. Then, by [6, VII, §3.8, Proposition 5] (which is a key step in the proof of the Weierstrass Preparation Theorem), one has a decomposition

$$\widehat{R_\mathfrak{p}}[[t]] = \alpha \widehat{R_\mathfrak{p}}[[t]] \oplus \Big(\bigoplus_{i=0}^{n-1} \widehat{R_\mathfrak{p}} t^i \Big) \ .$$

Now one just uses $P \otimes_\Lambda \alpha \widehat{R_\mathfrak{p}}[[t]] = \alpha \cdot (P \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]]) = 0$. □

Theorem 2.7. *Let P be a pseudo-null Λ-module. Then $Ch_R(P_t) = Ch_R(P/tP)$.*

Proof. By Remark 2.2 and Lemma 2.5, we need to show that

$$l_{\widehat{R_\mathfrak{p}}}(P_t \otimes_R \widehat{R_\mathfrak{p}}) = l_{\widehat{R_\mathfrak{p}}}((P/tP) \otimes_R \widehat{R_\mathfrak{p}})$$

for any height 1 prime ideal $\mathfrak{p}$ of R. By Lemma 2.4, the functor $\otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]]$ is exact. Applying it to (2.4), we get an exact sequence

$$(2.5) \qquad P_t \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]] \hookrightarrow P \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]] \xrightarrow{t} P \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]] \twoheadrightarrow (P/tP) \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]] \ .$$

Lemma 2.6 shows that all terms of (2.5) are finitely generated $\widehat{R_\mathfrak{p}}$-modules. Hence, the first and last term of the sequence have the same length. Finally, just observe that if N is a Λ-module annihilated by t then

$$N \otimes_\Lambda \widehat{R_\mathfrak{p}}[[t]] = N \otimes_R \widehat{R_\mathfrak{p}} \ .$$

□

Example 2.8. If P happens to be finitely generated over R then the statement of the theorem is obvious. We give a few examples of pseudo-null $\Lambda := \mathbb{Z}_p[[s,t]]$-modules which are not finitely generated as $R := \mathbb{Z}_p[[s]]$-modules, providing non-trivial examples in which the above theorem applies. However we remark that the main consequence of Lemma 2.6 is exactly the fact that we can ignore the issue of checking whether a pseudo-null Λ-module is finitely generated over R or not.

1. $P = \Lambda/(p, s)$. Then $P \simeq \mathbb{F}_p[[t]]$ is not finitely generated over R. In this case $P_t = 0$ and $P/tP \simeq \mathbb{F}_p$ (both R-pseudo-null), so that

$$Ch_R(P_t) = Ch_R(P/tP) = R \ .$$

2. $P = \Lambda/(s, pt)$. Then $P \simeq \mathbb{Z}_p[[t]]/(pt)$ is not finitely generated over R and elements in P can be written as

$$m = \sum_{i \geq 0} a_i t^i \quad a_0 \in \mathbb{Z}_p \text{ and } a_i \in \{0, ..., p-1\}\ \forall\, i \geq 1 \ .$$

Moreover

$$P_t = p\mathbb{Z}_p[[t]]/(pt) \simeq p\mathbb{Z}_p \simeq \mathbb{Z}_p \simeq R/(s)$$

and

$$P/tP = \mathbb{Z}_p[[s,t]]/(t, s, pt) \simeq \mathbb{Z}_p \simeq R/(s) \ ,$$

so both have characteristic ideal (s) (as R-modules).

3. With $P = \Lambda/(p, st)$, a similar reasoning shows that $Ch_R(P/tP) = Ch_R(P_t) = (p)$.

Remark 2.9. The hypothesis that P is pseudo-null is necessary: if M is a torsion Λ-module then it is not true, in general, that $Ch_R(M_t) = Ch_R(M/tM)$. We give an easy example: let again $\Lambda = \mathbb{Z}_p[[s,t]]$ with $R = \mathbb{Z}_p[[s]]$, and consider $M = \mathbb{Z}_p[[s,t]]/(p^2 + s + t)$, which is a torsion Λ-module. Observe that M_t is trivial (so $Ch_R(M_t) = R$) and

$$M/tM = \mathbb{Z}_p[[s,t]]/(t, p^2 + s) \simeq R/(p^2 + s)$$

has characteristic ideal over R equal to $(p^2 + s)$. Moreover $Ch_R(M/tM) = (p^2 + s)$ is the image of $Ch_\Lambda(M) = (p^2 + s + t)$ under the projection $\pi \colon \Lambda \to R$, $t \mapsto 0$. Hence, in this case,

$$Ch_R(M_t)\, \pi\big(Ch_\Lambda(M)\big) = Ch_R(M/tM)$$

which anticipates the general formula of Proposition 2.10.

2.3. **An application.** We keep the assumptions of §2.2: R is a noetherian Krull domain and $\Lambda = R[[t]]$. As mentioned in the Introduction, the following proposition will be crucial in the study of characteristic ideals for Iwasawa modules under descent.

Proposition 2.10. *Let $\pi \colon \Lambda \to R$ be the projection given by $t \mapsto 0$ and let M be a finitely generated torsion Λ-module. Then*

$$Ch_R(M_t)\, \pi\big(Ch_\Lambda(M)\big) = Ch_R(M/tM)\,. \tag{2.6}$$

Moreover,

$$Ch_R(M_t) = 0 \iff \pi\big(Ch_\Lambda(M)\big) = 0 \iff Ch_R(M/tM) = 0$$

and in this case M_t and M/tM are R-modules of the same rank.

Proof. Recall that the structure theorem (2.1) provides a pseudo-isomorphism between M and its associated elementary module $E(M)$. As noted above, being pseudo-isomorphic is an equivalence relation for torsion modules: therefore one has a (non-canonical) sequence

$$E(M) \hookrightarrow M \twoheadrightarrow P$$

where P is Λ-pseudo-null and the injectivity on the left comes from the fact that elementary modules have no nontrivial pseudo-null submodules (just use the valuation on $\Lambda_\mathfrak{p}$ to check that the annihilator of any $x \in \Lambda/\mathfrak{p}^e - \{0\}$ must be contained in $\mathfrak{p}$). The snake lemma sequence coming from the diagram

$$\begin{array}{ccccc} E(M) & \hookrightarrow & M & \twoheadrightarrow & P \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ E(M) & \hookrightarrow & M & \twoheadrightarrow & P \end{array}$$

reads as

$$E(M)_t \hookrightarrow M_t \longrightarrow P_t \longrightarrow E(M)/tE(M) \longrightarrow M/tM \twoheadrightarrow P/tP \ . \tag{2.7}$$

As we remarked at the beginning of §2.2, both P_t and P/tP are finitely generated torsion R-modules. It is also easy to see that all modules in the sequence (2.7) are finitely generated over R. Now observe that $(\Lambda/\mathfrak{p}^e)_t$ is zero if $\mathfrak{p} \neq (t)$ and isomorphic to R if $\mathfrak{p} = (t)$; similarly, $(\Lambda/\mathfrak{p}^e)/t(\Lambda/\mathfrak{p}^e)$ is either pseudo-null or isomorphic to R. Thus, putting $E(M) = \oplus\Lambda/\mathfrak{p}_i^{e_i}$, we find $E(M)_t \simeq R^r$ and

$$E(M)/tE(M) = \oplus\Lambda/(\mathfrak{p}_i^{e_i}, t)\Lambda \simeq \oplus R/(\pi(\mathfrak{p}_i)^{e_i}) \simeq R^r \oplus \bullet \ ,$$

where $r := \#\{i \mid \mathfrak{p}_i = t\Lambda\}$ and $\bullet$ is a pseudo-null Λ-module. Moreover (2.7) shows that $E(M)/tE(M)$ is R-torsion if and only if M/tM is R-torsion and $E(M)_t$ is R-torsion if and only if M_t is R-torsion. Therefore we have two cases:

1. if $r > 0$, then (t) divides $Ch_\Lambda(M)$, so $\pi(Ch_\Lambda(M)) = 0$ and, since M_t and M/tM are not R-torsion, $Ch_R(M_t) = Ch_R(M/tM) = 0$ as well (the statement on R-ranks is immediate from (2.7): e.g., apply the exact functor $\otimes_R Frac(R)$);

2. if $r = 0$, then, because of the equivalent conditions above, all the characteristic ideals involved in (2.6) are nonzero; moreover we have

$$Ch_R(E(M)/tE(M)) = \pi(Ch_\Lambda(E(M))) = \pi(Ch_\Lambda(M))$$

and (2.6) follows from the sequence (2.7), Theorem 2.7 and the multiplicativity of characteristic ideals.

□

Corollary 2.11. *In the above setting assume that M/tM is a finitely generated torsion R-module. Then M is a pseudo-null Λ-module if and only if $Ch_R(M_t) = Ch_R(M/tM)$. Moreover if $M/tM \sim_R 0$, then $M \sim_\Lambda 0$.*

Proof. The "only if" part is provided by Theorem 2.7. For the "if" part we assume the equality of characteristic ideals (which are nonzero by hypothesis). By (2.6) we have $\pi(Ch_\Lambda(M)) = R$, hence there is some $f \in Ch_\Lambda(M)$ such that $\pi(f) = 1$. But then $f = \sum_{i\geq 0} c_i t^i$ with $c_0 = 1$, which is an obvious unit in $\Lambda = R[[t]]$. Therefore $Ch_\Lambda(M) = \Lambda$, i.e., M is Λ-pseudo-null.

For the last statement just note that $Ch_R(M/tM) = R$ yields $Ch_R(M_t)\,\pi(Ch_\Lambda(M)) = R$, so $Ch_R(M_t) = R$ as well. □

3. Class groups in global fields

For the rest of the paper we adjust our notations a bit to be more consistent with the usual ones in Iwasawa theory. We fix a prime number p and a global field F (note that for now we are not making any assumption on the characteristic of F). For any finite extension E/F let $\mathcal{M}(E)$ be the p-adic completion of the group of divisor classes of E, i.e.,

$$\mathcal{M}(E) := (E^*\backslash \mathbf{I}_E/\Pi_v \mathcal{O}_{E_v}^*) \otimes \mathbb{Z}_p$$

where $\mathbf{I}_E$ is the group of finite ideles of E, v varies over all non-archimedean places of E and $\mathcal{O}_{E_v}$ is the ring of integers of the completion of E at v. When $\mathcal{L}/F$ is an infinite extension, we put $\mathcal{M}(\mathcal{L}) := \varprojlim \mathcal{M}(E)$ as E runs among finite subextensions of $\mathcal{L}/F$ (the limit being taken with respect to norm maps). Class field theory yields a canonical isomorphism

$$\mathcal{M}(E) \xrightarrow{\sim} X(E) := \mathrm{Gal}(L(E)/E) \ , \tag{3.1}$$

where $L(E)$ is the maximal unramified abelian pro-p-extension of E. Passing to the limit shows that (3.1) is still true for infinite extensions.

Finally, for any infinite Galois extension $\mathcal{L}/F$, let $\Lambda(\mathcal{L}) := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{L}/F)]]$ be the associated Iwasawa algebra. We shall be interested in the situation where $\mathrm{Gal}(\mathcal{L}/F)$ is an abelian p-adic

Lie group: in this case, both $\mathcal{M}(\mathcal{L})$ and $X(\mathcal{L})$ are $\Lambda(\mathcal{L})$-modules (the action of $\mathrm{Gal}(\mathcal{L}/F)$ on $X(\mathcal{L})$ is the natural one via inner automorphisms of $\mathrm{Gal}(L(\mathcal{L})/F)$) and these structures are compatible with the isomorphism (3.1). Furthermore, if $\mathrm{Gal}(\mathcal{L}/F) \simeq \mathbb{Z}_p^d$ then $\Lambda(\mathcal{L}) \simeq \mathbb{Z}_p[[t_1, .., t_d]]$ is a Krull domain.

Lemma 3.1. *Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^d$-extension, ramified only at finitely many places. If $d > 2$, one can always find a $\mathbb{Z}_p$-subextension $\mathcal{F}_1/F$ such that none of the ramified places splits completely in $\mathcal{F}_1$.*

Proof. Let S denote the set of primes of F which ramify in $\mathcal{F}$ and, for any place v in S let $D_v \subset \mathrm{Gal}(\mathcal{F}/F) =: \Gamma$ be the corresponding decomposition group. Getting $\mathcal{F}_1$ amounts to finding $\alpha \in \mathrm{Hom}(\Gamma, \mathbb{Z}_p)$ such that $\alpha(D_v) \neq 0$ for all $v \in S$. By hypothesis, for such v's the vector spaces $D_v \otimes \mathbb{Q}_p$ are non-zero, hence their annihilators are proper subspaces of $\mathrm{Hom}(\Gamma \otimes \mathbb{Q}_p, \mathbb{Q}_p)$ and since a $\mathbb{Q}_p$-vector space cannot be union of a finite number of proper subspaces, we deduce that the required α exists. □

The following lemma is a mostly a restatement of [9, Theorem 1].

Lemma 3.2. *Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^d$-extension, ramified only at finitely many places, and $\mathcal{F}' \subset \mathcal{F}$ a $\mathbb{Z}_p^{d-1}$-subextension, with $d > 2$. Let I be the kernel of the natural projection $\Lambda(\mathcal{F}) \to \Lambda(\mathcal{F}')$. Then $X(\mathcal{F})/IX(\mathcal{F})$ is a finitely generated torsion $\Lambda(\mathcal{F}')$-module and $X(\mathcal{F})$ is a finitely generated torsion $\Lambda(\mathcal{F})$-module. This holds also for $d = 2$, provided that no ramified place in $\mathcal{F}/F$ is totally split in $\mathcal{F}'$.*

Proof. The idea is to proceed by induction on d. Choose a filtration

$$F =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{d-1} := \mathcal{F}' \subset \mathcal{F}_d := \mathcal{F}$$

so that $\mathrm{Gal}(\mathcal{F}_i/\mathcal{F}_{i-1}) \simeq \mathbb{Z}_p$ for all i and no ramified place in $\mathcal{F}/F$ is totally split in $\mathcal{F}_1$ (by Lemma 3.1, this can always be achieved when $d > 2$).

Now one proceeds as in [9, Theorem 1]. Namely, a standard argument yields that a $\Lambda(\mathcal{F}_i)$-module M is in $\mathbf{Fgt}_{\Lambda(\mathcal{F}_i)}$ if $M/I_{i-1}^i M$ is in $\mathbf{Fgt}_{\Lambda(\mathcal{F}_{i-1})}$ (where I_{i-1}^i is the kernel of the projection $\Lambda(\mathcal{F}_i) \to \Lambda(\mathcal{F}_{i-1})$) and Greenberg's proof shows that $X(\mathcal{F}_{i-1}) \in \mathbf{Fgt}_{\Lambda(\mathcal{F}_{i-1})}$ implies $X(\mathcal{F}_i)/I_{i-1}^i X(\mathcal{F}_i) \in \mathbf{Fgt}_{\Lambda(\mathcal{F}_{i-1})}$. So it is enough to prove that $X(\mathcal{F}_1)$ is a finitely generated torsion $\Lambda(\mathcal{F}_1)$-module. This follows from Iwasawa's classical proof ([13], exposed e.g. in [20]; the function field version can be found in [17]). □

Remarks 3.3.

1. In a $\mathbb{Z}_p$-extension of a global field, only places with residual characteristic p can ramify: thus the finiteness hypothesis on the ramification locus is automatically satisfied unless $char(F) = p$. Note, however, that in the latter case this hypothesis is needed (see, e.g. [8, Remark 4]).
2. Among all $\mathbb{Z}_p$-extensions of F there is a distinguished one, namely, the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} if F is a number field and the arithmetic $\mathbb{Z}_p$-extension F_{arit} (arising from the unique $\mathbb{Z}_p$-extension of the constant field) if F is a function field. The condition on $\mathcal{F}'$ (when $d = 2$) is satisfied if it contains either F_{cyc} or F_{arit}.
3. For $d = 1$, we have $\mathcal{F}' = F$ and $\Lambda(\mathcal{F}') = \mathbb{Z}_p$. Thus the analogue of Lemma 3.2 would state that $X(\mathcal{F})/IX(\mathcal{F})$ is finite. This holds quite trivially if F is a global function field and $\mathcal{F} = F_{arit}$ (note also that if $char(F) = \ell \neq p$ then F_{arit} is the only $\mathbb{Z}_p$-extension of F, see e.g. [4, Proposition 4.3]). In this case the maximal abelian extension of F contained in $L(\mathcal{F})$ is exactly $L(F)$, hence $X(\mathcal{F})/IX(\mathcal{F}) \simeq \mathrm{Gal}(L(F)/F_{arit})$ which is known (e.g. by class field theory) to be finite. For more on the case $d = 1$ with F a number field, see Subsection 3.2.2 below.

3.1. **Class groups in function fields.** In this section F will be a global function field of characteristic p and F_{arit} its arithmetic $\mathbb{Z}_p$-extension as defined above. Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^\infty$-extension unramified outside a finite set of places S, with $\Gamma := \mathrm{Gal}(\mathcal{F}/F)$ and associated Iwasawa algebra $\Lambda := \Lambda(\mathcal{F})$. We fix a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i\in\mathbb{N}}$ for Γ and for any $d \geq 0$ we let $\mathcal{F}_d \subset \mathcal{F}$ be the fixed field of $\{\gamma_i\}_{i>d}$. Also, we assume that our basis is such that no place in S splits completely in $\mathcal{F}_1$ (Lemma 3.1 shows that there is no loss of generality in this assumption).

Remark 3.4. If $\mathcal{F}$ contains F_{arit} we can take the latter as $\mathcal{F}_1$. The additional hypothesis on $\mathcal{F}_1$ appears also in [14, Definition 2.1 and Lemma 2.4]: the authors use it to get monomial Stickelberger elements, a crucial step in their proof of the Main Conjecture (see also [15, Theorem 1.1], where the goal is enlarging the set S and the extension $\mathcal{F}_d$ in order to get a $\mathbb{Z}_p$-extension verifying that hypothesis and again a monomial Stickelberger element).

For notational convenience, let $t_i := \gamma_i - 1$. Then the subring $\mathbb{Z}_p[[t_1,\dots,t_d]]$ of Λ is canonically isomorphic to $\Lambda(\mathcal{F}_d)$ and, by a slight abuse of notation, the two shall be identified in the following. In particular, for any $d \geq 1$ we have $\Lambda(\mathcal{F}_d) = \Lambda(\mathcal{F}_{d-1})[[t_d]]$ and we can apply the results of Section 2.3. We shall denote by π^d_{d-1} the canonical projection $\Lambda(\mathcal{F}_d) \to \Lambda(\mathcal{F}_{d-1})$ with kernel $I^d_{d-1} = (t_d)$ (the augmentation ideal of $\mathcal{F}_d/\mathcal{F}_{d-1}$) and by Γ^d_{d-1} the group $\mathrm{Gal}(\mathcal{F}_d/\mathcal{F}_{d-1})$.

For two finite extensions $L \supset L' \supset F$, the degree maps $\deg_L$ and $\deg_{L'}$ fit into the commutative diagram (with exact rows)

$$\begin{array}{ccccc} \mathcal{A}(L) & \hookrightarrow & \mathcal{M}(L) & \xrightarrow{\deg_L} & \mathbb{Z}_p \\ \downarrow{\scriptstyle N^L_{L'}} & & \downarrow{\scriptstyle N^L_{L'}} & & \downarrow \\ \mathcal{A}(L') & \hookrightarrow & \mathcal{M}(L') & \xrightarrow{\deg_{L'}} & \mathbb{Z}_p\,, \end{array} \tag{3.2}$$

where $N^L_{L'}$ denotes the norm and the vertical map on the right is multiplication by $[\mathbb{F}_L : \mathbb{F}_{L'}]$ (the degree of the extension between the fields of constants). For an infinite extension $\mathcal{L}/F$ contained in $\mathcal{F}$, taking projective limits one gets an exact sequence

$$\mathcal{A}(\mathcal{L}) \hookrightarrow \mathcal{M}(\mathcal{L}) \xrightarrow{\deg_{\mathcal{L}}} \mathbb{Z}_p \quad . \tag{3.3}$$

Remark 3.5. The map $\deg_{\mathcal{L}}$ above becomes zero exactly when $\mathcal{L}$ contains the unramified $\mathbb{Z}_p$-subextension F_{arit}.

By (3.1), Lemma 3.2 shows that $\mathcal{M}(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module, so the same holds for $\mathcal{A}(\mathcal{F}_d)$. Hence, by Proposition 2.10, one has, for all $d \geq 1$,

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)_{t_d}) \cdot \pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)/t_d\mathcal{A}(\mathcal{F}_d)) \ ,$$

i.e.,

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) \cdot \pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) \ . \tag{3.4}$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{A}(\mathcal{F}_d) & \hookrightarrow & \mathcal{M}(\mathcal{F}_d) & \xrightarrow{\deg} & \mathbb{Z}_p \\ \downarrow{\scriptstyle t_d} & & \downarrow{\scriptstyle t_d} & & \downarrow{\scriptstyle t_d} \\ \mathcal{A}(\mathcal{F}_d) & \hookrightarrow & \mathcal{M}(\mathcal{F}_d) & \xrightarrow{\deg} & \mathbb{Z}_p \end{array} \tag{3.5}$$

(note that the vertical map on the right is 0) and its snake lemma sequence

$$\begin{array}{ccccc} \mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}} & \hookrightarrow & \mathcal{M}(\mathcal{F}_d)^{\Gamma^d_{d-1}} & \xrightarrow{\deg} & \mathbb{Z}_p \\ & & & & \downarrow \\ \mathbb{Z}_p & \overset{\deg}{\twoheadleftarrow} & \mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d) & \longleftarrow & \mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)\ . \end{array} \tag{3.6}$$

For $d \geq 2$ (which implies that $\mathbb{Z}_p$ is a torsion $\Lambda(\mathcal{F}_{d-1})$-module), (3.6) and Lemma 3.2 show that $\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)$ is in $\mathbf{Fgt}_{\Lambda(\mathcal{F}_{d-1})}$ as well. By Proposition 2.10 it follows that no term in (3.4) is trivial.

3.1.1. $\mathfrak{p}$*-ramified extensions and the Main Conjecture.* The main example we have in mind are extensions satisfying the following

Assumption 3.6. There is only one ramified place $\mathfrak{p}$ in $\mathcal{F}/F$ and it is totally ramified.

In what follows an extension satisfying this assumption will be called a $\mathfrak{p}$*-ramified extension.* A prototypical example is the $\mathfrak{p}$-cyclotomic extension of $\mathbb{F}_q(T)$ generated by the $\mathfrak{p}$-torsion of the Carlitz module (see e.g. [19, Chapter 12]).

Under this assumption any $\mathbb{Z}_p$-subextension can play the role of $\mathcal{F}_1$. Moreover it implies an isomorphism

$$\mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d) \simeq \mathcal{M}(\mathcal{F}_{d-1}) \tag{3.7}$$

(one just adapts the proof of [22, Lemma 13.15] and uses the isomorphism (3.1)).

By [5, Corollary 5.8], for a $\mathfrak{p}$-ramified extension one has $\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}} = 0$ and, from (3.4), we obtain

Corollary 3.7. *Let $\mathcal{F}_d$ be a $\mathbb{Z}_p^d$-extension of F contained in a $\mathfrak{p}$-ramified extension. Then, for any $\mathbb{Z}_p^{d-1}$-extension $\mathcal{F}_{d-1}$ contained in $\mathcal{F}_d$, one has*

$$\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d))\ . \tag{3.8}$$

Remark 3.8. The previous corollary holds in the general setting where one has only finitely many places of $\mathcal{F}_1$ above S. Indeed from [14, Corollary 3.2] it can be easily deduced that, for $d \geq 3$, $\mathcal{M}(\mathcal{F}_d)^{\Gamma^d_{d-1}}$ (and hence $\mathcal{A}(\mathcal{F}_d)^{\Gamma^d_{d-1}}$) is $\Lambda(\mathcal{F}_{d-1})$-pseudo-null.

For $d \geq 2$ the $\Lambda(\mathcal{F}_d)$-module $\mathbb{Z}_p$ is pseudo-null, hence the sequence (3.3) yields

$$Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{M}(\mathcal{F}_d)) = Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))\,.$$

Moreover the isomorphism in (3.7) shows that

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{M}(\mathcal{F}_d)/I^d_{d-1}\mathcal{M}(\mathcal{F}_d)) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{M}(\mathcal{F}_{d-1}))\ . \tag{3.9}$$

Using (3.9) and (3.6), one finds (for $d \geq 3$, so to have $\mathbb{Z}_p \sim_{\Lambda(\mathcal{F}_{d-1})} 0$)

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)/I^d_{d-1}\mathcal{A}(\mathcal{F}_d)) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_{d-1}))\ , \tag{3.10}$$

where all the modules involved are torsion $\Lambda(\mathcal{F}_{d-1})$-modules. Substituting in (3.8) and iterating, we obtain

Corollary 3.9. *Let $\mathcal{F}/F$ a $\mathfrak{p}$-ramified extension and $\mathcal{F}_d$ a $\mathbb{Z}_p^d$-extension of F in $\mathcal{F}$. Then, for any $\mathbb{Z}_p^e$-extension $\mathcal{F}_e$ contained in $\mathcal{F}_d$ (with $e \geq 2$), one has*

$$\pi^d_e(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_e)}(\mathcal{A}(\mathcal{F}_e))\ . \tag{3.11}$$

Hence the inverse images of the $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))$ in Λ (with respect to the canonical projections $\pi_{\mathcal{F}_d} : \Lambda \to \Lambda(\mathcal{F}_d)$) form an inverse system and we can define

Definition 3.10. Let $\mathcal{F}/F$ be a $\mathfrak{p}$-ramified extension. The *pro-characteristic ideal* of $\mathcal{A}(\mathcal{F})$ is

$$\widetilde{Ch}_\Lambda(\mathcal{A}(\mathcal{F})) := \varprojlim_{\mathcal{F}_d} (\pi_{\mathcal{F}_d})^{-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) \ .$$

Remark 3.11. Definition 3.10 only depends on the extension $\mathcal{F}/F$ and not on the filtration of $\mathbb{Z}_p^d$-extension we choose inside it. Indeed take two different filtrations $\{\mathcal{F}_d\}$ and $\{\mathcal{F}'_d\}$ and define a new filtration containing both by putting

$$\mathcal{F}''_0 := F \quad \text{and} \quad \mathcal{F}''_n = \mathcal{F}_n\mathcal{F}'_n \quad \forall\, n \geq 1$$

(note that $\mathcal{F}''_n$ is not, in general, a $\mathbb{Z}_p^n$-extension and $\mathcal{F}''_n/\mathcal{F}''_{n-1}$ is a $\mathbb{Z}_p^i$-extension with $0 \leq i \leq 2$, but these details are irrelevant for the limit process we need here). By Corollary 3.9, the limits of the characteristic ideals of the filtrations we started with coincide with the limit on the filtration $\{\mathcal{F}''_n\}$. This answers questions a and b of [5, Remark 5.11]: we had a similar definition there but it was based on the particular choice of the filtration.

3.2. **Class groups in number fields: the generalized Greenberg's Conjecture.** In this section we let F be a number field. Consider the compositum $\widetilde{F}$ of all $\mathbb{Z}_p$-extensions of F. Then $\mathrm{Gal}(\widetilde{F}/F) \simeq \mathbb{Z}_p^d$ where $r_2 + 1 \leq d \leq [F : \mathbb{Q}]$ (see [22, Theorem 13.4]) and r_2 is the number of pairs of complex embeddings of F (if Leopoldt's Conjecture holds, we have $d = r_2 + 1$). Lemma 3.2 shows that $X(\widetilde{F})$ is a finitely generated torsion $\Lambda(\widetilde{F})$-module.

The following is [12, Conjecture 3.5].

Conjecture 3.12 (Generalized Greenberg's Conjecture - GGC)**.** *$X(\widetilde{F})$ is a pseudo-null $\Lambda(\widetilde{F})$-module.*

For some examples of fields for which GGC holds see [1] or [2] (and the references there). Other important (conditionally) equivalent formulations of GGC can be found in [16, Théorème 4.4].

Theorem 3.13. *Let K/F be any $\mathbb{Z}_p^{d-1}$-extension if $d > 2$, or a $\mathbb{Z}_p$-extension such that no ramified place in $\widetilde{F}/F$ is totally split in K if $d = 2$. Then*

$$\text{GGC holds for } F \iff Ch_{\Lambda(K)}(X(\widetilde{F})_t) = Ch_{\Lambda(K)}(X(\widetilde{F})/tX(\widetilde{F}))$$

where t is any generator of $\mathrm{Ker}\,\{\Lambda(\widetilde{F}) \to \Lambda(K)\}$.

Proof. The choice of a splitting ι of the canonical projection $\mathrm{Gal}(\widetilde{F}/F) \twoheadrightarrow \mathrm{Gal}(K/F)$ yields an isomorphism $\Lambda(\widetilde{F}) \simeq \Lambda(K)[[t]]$. Besides, for any $\Lambda(\widetilde{F})$-module M and any choice of the splitting, M_t and M/tM are invariants and coinvariants of $\mathrm{Gal}(\widetilde{F}/K)$, so the action of $\mathrm{Gal}(K/F)$ on them (and hence their structure as $\Lambda(K)$-modules) is independent of ι. Thus this is just Corollary 2.11, which we can apply because of Lemma 3.2. □

We have ample freedom in the choice of K/F, so Theorem 3.13 might be useful to check the conjecture in some particular cases.

3.2.1. *Imaginary quadratic fields.* For example for an imaginary quadratic field F it is quite easy to recover (well known) results (see, for example, [18, §4]).

(1) If $\lambda(F_{cyc}) = 0$, then GGC holds for F.
(2) If p splits in F and $\lambda(F_{cyc}) = 1$, then GGC holds for F.

Proof. Let I be the augmentation ideal between $\Lambda(\widetilde{F})$ and $\Lambda(F_{cyc})$ and let L_0 be the maximal abelian extension of F_{cyc} contained in $L(\widetilde{F})$. Then $X(\widetilde{F})/IX(\widetilde{F}) \simeq \mathrm{Gal}(L_0/\widetilde{F})$ and it is a submodule of $X(F_{cyc})$ (because the extension $L_0/\widetilde{F}$ is unramified). Now:

(1) by the Ferrero-Washington theorem ([22, Theorem 7.15]) and the hypothesis, $X(F_{cyc})$ is finite;

(2) by the splitting hypothesis $\widetilde{F}/F_{cyc}$ is unramified so, in this case, $L_0 = L(F_{cyc})$ and $X(\widetilde{F})/IX(\widetilde{F}) \simeq \mathrm{Gal}(L(F_{cyc})/\widetilde{F})$ is again finite (by the hypothesis on $\lambda(F_{cyc})$).

In both cases one finds $X(\widetilde{F})/IX(\widetilde{F}) \sim_{\Lambda(F_{cyc})} 0$ and Corollary 2.11 yields $X(\widetilde{F}) \sim_{\Lambda(\widetilde{F})} 0$. □

3.2.2. *The case $d = 1$ and totally real fields.* For $d = 1$, the statements of Proposition 2.10 and Corollary 2.11 still hold: we just take $K = F$, so that $R = \Lambda(K) = \mathbb{Z}_p$ and the characteristic ideal of a finitely generated torsion $\Lambda(K)$-module is its cardinality. For a totally real field (for which Leopoldt's conjecture implies $\widetilde{F} = F_{cyc}$), we let $\Gamma := \Gamma_{cyc} = \mathrm{Gal}(F_{cyc}/F)$ and $I := I_{cyc} = \mathrm{Ker}\,\{\Lambda(F_{cyc}) \to \mathbb{Z}_p\}$. Hence, assuming that $X(F_{cyc})/IX(F_{cyc})$ is finite (i.e., the characteristic ideal of $X(F_{cyc})$ is not contained in I), the (classical) Greenberg's Conjecture on the finiteness of $X(F_{cyc})$ (see [10]) is equivalent to the equality

$$(3.12) \qquad |H^0(\Gamma, X(F_{cyc}))| = |X(F_{cyc})^\Gamma| = |X(F_{cyc})/IX(F_{cyc})| = |H^1(\Gamma, X(F_{cyc}))| \ .$$

We can formulate (or reformulate) some conditions which imply (or are equivalent to) Greenberg's conjecture.

Theorem 3.14. *Let F be a totally real field for which Leopoldt's conjecture holds.*

1. *If $X(F_{cyc})/IX(F_{cyc})$ is trivial, then $X(F_{cyc})$ is finite.*

2. *Let $^\vee$ denote the Pontrjagin dual. If the natural restriction map*

$$res\colon (X(F_{cyc})^\vee)^\Gamma \longrightarrow (X(F_{cyc})^\Gamma)^\vee$$

is injective, then $X(F_{cyc})$ is finite.

3. *Denote by F_n the n-th layer of the $\mathbb{Z}_p$-extension F_{cyc}/F. The natural maps*

$$\nu_n\colon X(F_n)^\Gamma \to X(F_{n-1})^\Gamma$$

are surjective for $n \gg 0$ if and only if $X(F_{cyc})$ is finite.

Proof. Statement **1** follows immediately from the last assertion of Corollary 2.11. For the rest, first note that, by [10, Proposition 1], the groups $X(F_n)^\Gamma$ are of bounded order and, since $X(F_n)$ is finite for any n, the orders of the groups $X(F_n)/IX(F_n)$ are bounded as well. For n big enough, the extensions F_n/F_{n-1} are totally ramified at some prime, so the maps $X(F_n) \to X(F_{n-1})$ are surjective and induce surjections

$$X(F_n)/IX(F_n) \longrightarrow X(F_{n-1})/IX(F_{n-1}) \ .$$

Therefore $X(F_{cyc})/IX(F_{cyc}) = \varprojlim X(F_n)/IX(F_n)$ is finite of order equal to $|X(F_n)/IX(F_n)|$ for $n \gg 0$ and we can use the results of Section 2. Moreover applying the exact sequence (2.7) to $M = X(F_{cyc})$ one gets

$$(3.13) \qquad |X(F_{cyc})/IX(F_{cyc})| = |X(F_{cyc})^\Gamma| \cdot |E(X(F_{cyc}))/IE(X(F_{cyc}))| \geq |X(F_{cyc})^\Gamma| \,,$$

using $|P^\Gamma| = |P/IP|$ (because P is finite) and $E(X(F_{cyc}))^\Gamma = 0$ (since $Ch_{\Lambda(F_{cyc})}(X(F_{cyc}))$ is not in I). Now note that

$$|X(F_{cyc})/IX(F_{cyc})| = |((X(F_{cyc})^\vee)^\Gamma)^\vee| = |(X(F_{cyc})^\vee)^\Gamma|$$

and, obviously,

$$|X(F_{cyc})^\Gamma| = |(X(F_{cyc})^\Gamma)^\vee|$$

(all the modules involved are finite). Hence, if res is injective one gets

$$|X(F_{cyc})/IX(F_{cyc})| \leq |X(F_{cyc})^\Gamma| \,,$$

which proves that (3.13) is a chain of equalities, establishing **2** (and also, in this case, the isomorphism $(X(F_{cyc})^\vee)^\Gamma \simeq (X(F_{cyc})^\Gamma)^\vee$).

As for **3**, observe that, since $X(F_{cyc})^\Gamma = \lim\limits_{\leftarrow} X(F_n)^\Gamma$ is finite by (3.13), the maps ν_n are injective for $n \gg 0$ and so

$$|X(F_{cyc})/IX(F_{cyc})| = |X(F_n)/IX(F_n)| = |X(F_n)^\Gamma| \geq |X(F_{cyc})^\Gamma|$$

Equality holds if and only if the maps $X(F_n)^\Gamma \to X(F_{n-1})^\Gamma$ are surjective for n big enough. □

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