AUTOMORPHISM GROUPS OF NON-SINGULAR PLANE CURVES OF DEGREE 5

ESLAM BADR AND FRANCESC BARS

Abstract. Let M_g be the moduli space of smooth, genus g curves over an algebraically closed field K of zero characteristic. Denote by $M_g(G)$ the subset of M_g of curves δ such that G (as a finite non-trivial group) is isomorphic to a subgroup of $Aut(\delta)$, and let $\widetilde{M_g(G)}$ be the subset of curves δ such that $G \cong Aut(\delta)$, where $Aut(\delta)$ is the full automorphism group of δ. Now, for an integer $d \geq 4$, let M_g^{Pl} be the subset of M_g representing smooth, genus g, plane curves of degree d (in this case, $g = (d-1)(d-2)/2$) and consider the sets $M_g^{Pl}(G) := M_g^{Pl} \cap M_g(G)$ and $\widetilde{M_g^{Pl}(G)} := \widetilde{M_g(G)} \cap M_g^{Pl}$.

Henn in [7] and Komiya-Kuribayashi in [10], listed the groups G for which $\widetilde{M_3^{Pl}(G)}$ is non-empty. In this paper, we determine the loci $\widetilde{M_6^{Pl}(G)}$, corresponding to non-singular degree 5 projective plane curves, which are non-empty. Also, we present the analogy of Henn's results for quartic curves concerning non-singular plane model equations associated to these loci (see Table 2 for more details). Similar arguments can be applied to deal with higher degrees.

1. Introduction

It is classically known from Hurwitz [9] that, given any non-trivial finite group G, one can construct a Riemann surface X whose automorphism group $Aut(X)$ is isomorphic to G.

A natural question is to list the groups such that the associated Riemann surface will have a non-singular plane model. Harui in [6] determined the list of the finite groups G that could appear in such case. However, for a complete answer to the problem, it remains to introduce the exact list of such groups which might appear for a fixed degree and conversely, for an arbitrary but fixed group in the list, one need to determine the degrees which such a group occur. Therefore, there are the following two open problems:

(1) Fixing a group G, for which degrees $d \geq 4$ we have that $\widetilde{M_g^{Pl}(G)}$ is a non-empty set? For example, by the work of Crass in [3, p.28], we know that $\widetilde{M_g^{Pl}(A_6)}$ is non-empty exactly for $g = 10$, $g = 66$ and $g = 406$, where A_6 is the alternating group of 6 letters.
(2) Once the degree d is fixed, determine the groups G (up to isomorphism) where $\widetilde{M_g^{Pl}(G)}$ is non-empty.

This note is concerned with the second question. Henn in [7] and Komiya-Kuribayashi in [10] solved the question for $d = 4$.

Recall that any $\delta \in M_g^{Pl}(G)$ corresponds to a set of non-singular plane models C_δ in $\mathbb{P}^2(K)$ such that any two of them are related through a change of variables $P \in PGL_3(K)$ (where $PGL_n(K)$ is the classical projective linear group of $n \times n$ invertible matrices over K), and their automorphism groups are conjugate. We mean by C a plane non-singular model associated to δ. Observe that $Aut(C)$ is a subgroup of $PGL_3(K)$ which is isomorphic to G by an injective representation $\rho : G \hookrightarrow PGL_3(K)$, that is $Aut(C) = \rho(G)$ for some ρ.

We denote by $\rho(M_g^{Pl}(G))$ the set of all elements $\delta \in M_g^{Pl}(G)$ such that G acts on a plane model associated to δ as $P^{-1}\rho(G)P$ for some P. This gives us the following union decomposition:

$$M_g^{Pl}(G) = \cup_{[\rho] \in A} \rho(M_g^{Pl}(G))$$

where $A := \{\rho \mid \rho : G \hookrightarrow PGL_3(K)\}/\sim$ such that $\rho_a \sim \rho_b$ if and only if $\rho_a(G) = P^{-1}\rho_b(G)P$ for some $P \in PGL_3(K)$. A similar decomposition (which is now disjoint) follows for $\widetilde{M_g^{Pl}(G)}$.

2010 *Mathematics Subject Classification.* 14H37, 14H50, 14H45.

Key words and phrases. plane curves; automorphism groups.

E. Badr and F. Bars are supported by MTM2013-40680-P.

Henn in [7] determined the $[\rho]'s$ and G such that $\rho(\widetilde{M_3^{Pl}(G)})$ is non-trivial and associated to such locus (once ρ and G are fixed) a certain projective plane equation which depends on some parameters together with some algebraic restrictions to these parameters. More concretely, he obtained a plane non-singular model of any element of the locus by a certain specialization of the values of the parameters and vice versa.

In this paper, we obtain the analogy of the previous Henn's results for the loci $\rho(\widetilde{M_6^{Pl}(G)})$, see Table 2 for a compact form of the analogy, which is also the main result of the paper.
First, we classified in [2], for an arbitrary but a fixed degree d, the ρ's and the cyclic groups $\mathbb{Z}/m\mathbb{Z}$ of order m such that $\rho(M_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ is not empty. In particular, m should divide one of the integers

$$d, d-1, d(d-1), (d-1)^2, d(d-2) \ or \ d^2-3d+3.$$

Furthermore, we characterized the locus $\widetilde{M_6^{Pl}(G)}$ whenever G has an element of order m with m large enough. By large enough, we mean to be one of the following integers: $d(d-1), (d-1)^2, d(d-2), d^2-3d+3, \ell d$ $(\ell\geq 3)$ or $\ell(d-1)$ $(\ell\geq 2)$. Lastly, it remains to treat case by case the groups G that appeared in Harui's list [6] in order to investigate which of them must leave when the locus $\widetilde{M_6^{Pl}(G)}$ is non-trivial and G has no elements of (large enough) order m.
We thank the referee for his or her comments and suggestions that improved the paper in its present form.

2. Cyclic subgroups for degree 5 non-singular plane curve

Consider $\delta\in M_6^{Pl}$ such that the group $G\cong Aut(\delta)$ is non-trivial. Let $C:\ F(X;Y;Z)=0$ in $\mathbb{P}^2(K)$ be a non-singular plane model of degree 5 over an algebraic closed field K of characteristic zero, where $Aut(C)=\rho(G)\leq PGL_3(K)$ for some $\rho: G\hookrightarrow PGL_3(K)$ (any other model C of δ is given by $C_P:\ F(P(X;Y;Z))=0$ with $Aut(C_P)=P^{-1}Aut(C)P$ for some $P\in PGL_3(K)$, and we say that C_P is K-equivalent or K-isomorphic to C). Assume that $\sigma\in Aut(C)$ is an element of order m hence by a change of variables in $\mathbb{P}^2$ (in particular, changing the plane model to a K-equivalent one associated to δ), we can consider σ as the automorphism $(x:y:z)\mapsto(x:\xi_m^a y:\xi_m^b z)$ where ξ_m is a primitive m-th root of unity in K, and a,b are integers such that $0\leq a<b\leq m-1$. Moreover, if $ab\neq 0$ then m and $gcd(a,b)$ are coprime (we can reduce to $gcd(a,b)=1$) and if $a=0$ then $gcd(b,m)=1$. Also, such an automorphism is identified with type $m,(a,b)$.

Then, by a change of variables, we may have one of the following situations (see [2] for more details, in which we follow the same line of argument as Dolgachev for degree 4 in [4], but for a general degree $d\geq 4$).

Table 1. Quintics

Type: $m,(a,b)$	$F(X;Y;Z)$
$20,(4,5)$	$X^5+Y^5+XZ^4$
$16,(1,12)$	$X^5+Y^4Z+XZ^4$
$15,(1,11)$	$X^5+Y^4Z+YZ^4$
$13,(1,10)$	$X^4Y+Y^4Z+Z^4X$
$10,(2,5)$	$X^5+Y^5+\alpha XZ^4+\beta_{2,0}X^3Z^2$
$8,(1,4)$	$X^5+Y^4Z+\alpha XZ^4+\beta_{2,0}X^3Z^2$
$5,(1,2)$	$X^5+Y^5+Z^5+\beta_{3,1}X^2YZ^2+\beta_{4,3}XY^3Z$
$5,(0,1)$	$Z^5+L_{5,Z}$

$4,(1,3)$	$X^5+X\big(Z^4+\alpha Y^4+\beta_{4,2}Y^2Z^2\big)+\beta_{2,1}X^3YZ$
$4,(1,2)$	$X^5+X\big(Z^4+\alpha Y^4\big)+\beta_{2,0}X^3Z^2+\beta_{3,2}X^2Y^2Z+\beta_{5,2}Y^2Z^3$
$4,(0,1)$	$Z^4L_{1,Z}+L_{5,Z}$
$3,(1,2)$	$X^5+Y^4Z+\alpha YZ^4+\beta_{2,1}X^3YZ+X^2\big(\beta_{3,0}Z^3+\beta_{3,3}Y^3\big)+\beta_{4,2}XY^2Z^2$
$2,(0,1)$	$Z^4L_{1,Z}+Z^2L_{3,Z}+L_{5,Z}$

Here $L_{i,*}$ means a homogenous polynomial of degree i in the variables $\{X,Y,Z\}$ such that the variable $*$ does not appear. Also, $\alpha,\beta_{i,j}\in K$ and α is always non-zero and it can be transformed by a diagonal change of variables P to 1.

Remark 1. *It is to be noted that the above table lists all the possible situations for which $\rho(M_6(\mathbb{Z}/m\mathbb{Z}))$ is not empty where $P\rho(\mathbb{Z}/m\mathbb{Z})P^{-1}=<(diag(1,\xi_m^a,\xi_m^b)>$ for some $P\in PGL_3(K)$ and $\rho(M_6(\mathbb{Z}/m\mathbb{Z}))$ corresponds to Type $m,(a,b)$.*

3. General properties of the full automorphism group

Before a detailed study of the automorphism groups for degree 5, we recall the following results concerning $Aut(\delta)$ for $\delta\in M_g^{Pl}$ which will be useful throughout this paper. In some cases we will use the notation of the GAP library for finite small groups to indicate some of them.

Because linear systems g_d^2 are unique (up to multiplication by $P\in PGL_3(K)$ in $\mathbb{P}^2(K)$ [8, Lemma 11.28]), we always take C a plane non-singular model of δ, which is given by a projective plane equation $F(X;Y;Z)=0$ and $Aut(C)$ is a finite subgroup of $PGL_3(K)$ that fixes the equation F and is isomorphic to $Aut(\delta)$. Any other plane model of δ is given by $C_P: F(P(X;Y;Z))=0$ with $Aut(C_P)=P^{-1}Aut(C)P$ for some $P\in PGL_3(K)$ and C_P is K-equivalent or K-isomorphic to C. By an abuse of notation, we also denote a non-singular projective plane curve of degree d by C. Therefore, $Aut(C)$ satisfies one of the following situations (see Mitchell [11] for more details):

(1) fixes a point Q and a line L with $Q\notin L$ in $PGL_3(K)$,
(2) fixes a triangle (i.e. a set of three non-concurrent lines),
(3) $Aut(C)$ is conjugate to a representation inside $PGL_3(K)$ of one of the finite primitive groups namely, the Klein group $PSL(2,7)$, the icosahedral group A_5, the alternating group A_6, the Hessian group $Hess_{216}$ or to one of its subgroups $Hess_{72}$ or $Hess_{36}$.

It is classically known that if a subgroup H of automorphisms of a non-singular plane curve C fixes a point on C then H is cyclic [8, Lemma 11.44], and recently Harui in [6, §2] provided the lacked result in the literature on the type of groups that could appear for non-singular plane curves. Before introducing Harui's statement, we need to define the terminology of being a descendent of a plane curve. For a non-zero monomial $cX^iY^jZ^k$ with $c\in K\setminus\{0\}$, we define its exponent as $max\{i,j,k\}$. For a homogenous polynomial F, the core of F is defined to be the sum of all terms of F with the greatest exponent. Let C_0 be a smooth plane curve, a pair (C,H) with $H\leq Aut(C)$ is said to be a descendant of C_0 if C is defined by a homogenous polynomial whose core is a defining polynomial of C_0 and H acts on C_0 under a suitable change of the coordinate system (i.e. H is conjugate to a subgroup of $Aut(C_0)$).

We also recall that $PBD(2,1)$ denotes the subgroup of $PGL_3(K)$ that consists of all elements of the form $A'=\left(\begin{array}{cc|c} \multicolumn{2}{c|}{A''} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & \alpha\end{array}\right)$ for some $\alpha\in K^*$ and A'' an invertible 2×2 matrix. There is a natural map $\varrho: PBD(2,1)\to PGL_2(K)$ given by $[A']\mapsto[A'']$, where $[M]$ denotes the equivalence class of a matrix M in the projective linear group.

Theorem 2 (Harui). *If $H\preceq Aut(C)$ where C is a non-singular plane curve of degree $d\geq 4$ then H satisfies one of the following.*

(1) *H fixes a point on C and then is cyclic.*
(2) *H fixes a point not lying on C and it satisfies a short exact sequence of the form*

$$1\to N\to H\to G'\to 1,$$

where N a cyclic group of order dividing d and G' (which is a subgroup of $PGL_2(K)$) is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m with $m \leq d-1$, a Dihedral group D_{2m} of order $2m$ where $|N|=1$ or $m|(d-2)$, the alternating groups A_4, A_5 or the symmetry group S_4. Moreover, we have the following commutative diagram with exact rows and vertical injective morphisms:

$$\begin{array}{ccccccccc} 1 & \to & K^* & \to & PBD(2,1) & \to^{\varrho} & PGL_2(K) & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & N & \to & H & \to & G' & \to & 1. \end{array}$$

(3) *H is conjugate to a subgroup of $Aut(F_d)$ where F_d is the Fermat curve $X^d+Y^d+Z^d$. In particular, $|H|\,|\,6d^2$ and (C,H) is a descendant of F_d.*
(4) *H is conjugate to a subgroup of $Aut(K_d)$ where K_d is the Klein curve curve $XY^{d-1}+YZ^{d-1}+ZX^{d-1}$ hence $|H|\,|\,3(d^2-3d+3)$ and (C,H) is a descendant of K_d.*
(5) *H is conjugate to a finite primitive subgroup of $PGL_3(K)$ which are mentioned above.*

We mention also the following statement [6, Theorem 2.3].

Theorem 3. *Given C a non-singular plane curve of degree $d \neq 4,6$, then $|Aut(C)| \leq 6d^2$.*

In particular, for $d=5$ we conclude:

Corollary 4. *Given C, a non-singular plane curve of degree 5, then $Aut(C)$ is not conjugate to the Hessian group $Hess_{216}$, the Klein group $PSL(2,7)$ or the alternating group A_6.*

Moreover, we proved in [2] the following two results for cyclic subgroups inside $Aut(C)$ (see [2, Corollary 33] and [2, §4] respectively).

Proposition 5. *Given C, a non-singular plane curve of degree d and let $\sigma \in Aut(C)$ be of order m. Then m divides one of the following integers: $d-1$, d, $(d-1)^2$, $d(d-2)$, $d(d-1)$ or d^2-3d+3.*

Theorem 6. *Let C be a non-singular plane curve of degree d and $\sigma \in Aut(C)$. Then,*

(1) *if σ has order $d(d-1)$ then $Aut(C) = <\sigma>$, and C is K-isomorphic to $X^d+Y^d+XZ^{d-1}=0$.*
(2) *if σ has order $(d-1)^2$ then $Aut(C) = <\sigma>$, and C is K-isomorphic to $X^d+Y^{d-1}Z+XZ^{d-1}=0$.*
(3) *if σ has order $d(d-2)$ then C is K-isomorphic to $X^d+Y^{d-1}Z+YZ^{d-1}=0$, and for $d \neq 4,6$ we have $Aut(C) = <\sigma,\tau|\tau^2=\sigma^{d(d-2)}=1, \text{and}\, \tau\sigma\tau=\sigma^{-(d-1)}>$.*
(4) *if σ has order d^2-3d+3 then C is K-isomorphic to the Klein curve K_d, and for $d \geq 5$ we have*
$$Aut(C) = <\sigma,\tau|\sigma^{d^2-3d+3}=\tau^3=1\, \text{and}\, \sigma\tau=\tau\sigma^{-(d-1)}>.$$
(5) *if σ has order $\ell(d-1)$ with $\ell \geq 2$ then $Aut(C)$ is cyclic of order $\ell'(d-1)$ with $\ell|\ell'$. If $\ell=1$, the same conclusion holds if σ is a homology (by a homology we mean if $P\sigma P^{-1}=diag(1,\xi_m^a,\xi_m^b)$ such that exactly one of a and b is zero for some $P \in PGL_3(K)$).*
(6) *if σ has order ℓd with $\ell \geq 3$ then $Aut(C)$ fixes a line and a point off that line, and $Aut(C)$ is an exterior group as in Theorem 2 (2) with N of order d. When $\ell=2$, C may be a decendent of the Fermat curve or $Aut(C)$ is an exterior group as in Theorem 2 (2) where $|N|=d$.*

Now, assume as usual that C is a non-singular plane curve of degree $d=5$ with $\sigma \in Aut(C)$ of order m that acts on $F(X;Y;Z)=0$ as $(x,y,z) \mapsto (x,\xi_m^a y,\xi_m^b z)$ such that m is the maximal order in $Aut(C)$. Recall also that we can take $\alpha=1$ by a convenient change of variables P.

The following result determines the full automorphism group of a quintic curve C which admits automorphisms of large orders.

Corollary 7. *For non-singular plane curves of degree 5 over an algebraic closed field K of zero characteristic we have:*

(1) *The cyclic group $\mathbb{Z}/20\mathbb{Z}$ appears as $Aut(C)$ inside $PGL_3(K)$, and is generated by the transformation $(x,y,z) \mapsto (x,\xi_{20}^4 y,\xi_{20}^5 z)$ up to conjugation by $P \in PGL_3(K)$. Moreover, C is K-isomorphic (through*

P) to the plane non-singular curve $X^5+Y^5+XZ^4=0$. *In particular* $\rho(\widetilde{M_6^{Pl}(\mathbb{Z}/20\mathbb{Z})})$ *is an irreducible locus with one element, where* $\rho(\mathbb{Z}/20\mathbb{Z}) =< diag(1,\xi_{20}^4,\xi_{20}^5) >$.

(2) *The cyclic group* $\mathbb{Z}/16\mathbb{Z}$ *appears as* $Aut(C)$ *inside* $PGL_3(K)$, *and is generated by the transformation* $(x,y,z)\mapsto(x,\xi_{16}y,\xi_{16}^{12}z)$ *up to conjugation by* $P\in PGL_3(K)$. *Furthermore,* C *is* K*-isomorphic (through* P*) to the plane non-singular curve* $X^5+Y^4Z+XZ^4=0$. *In particular* $\rho(\widetilde{M_6^{Pl}(\mathbb{Z}/16\mathbb{Z})})$ *is an irreducible locus with one element, where* $\rho(\mathbb{Z}/16\mathbb{Z}) =< diag(1,\xi_{16},\xi_{16}^{12}) >$.

(3) *The group* $SmallGroup(30,1)\ \cong <\sigma,\tau|\tau^2=\sigma^{15}=(\tau\sigma)^2\sigma^3=1>$ *of order 30 appears as* $Aut(C)$ *inside* $PGL_3(K)$, *where* σ *and* τ *are given, up to conjugation through* $P\in PGL_3(K)$, *by* $\sigma:(x,y,z)\mapsto(x,\xi_{15}y,\xi_{15}^{11}z)$ *and* $\tau:(x,y,z)\mapsto(x,z,y)$. *Moreover,* C *is* K*-isomorphic (through* P*) to the curve* $X^5+Y^4Z+YZ^4=0$. *In particular* $\rho(\widetilde{M_6^{Pl}(SmallGroup(30,1))})$ *is an irreducible locus with one element and* ρ *is given by* σ,τ.

(4) *The group* $SmallGroup(39,1)\ \cong <\tau,\sigma|\sigma^{13}=\tau^3=1,\ \sigma\tau=\tau\sigma^3>$ *of order 39 appears as* $Aut(C)$ *inside* $PGL_3(K)$ *which is given by* $\sigma:(x,y,z)\mapsto(x,\xi_{13}y,\xi_{13}^{10}z)$ *and* $\tau:(x,y,z)\mapsto(y,z,x)$ *up to conjugation by* $P\in PGL_3(K)$. *Also,* C *is* K*-isomorphic (through* P*) to the curve* $K_5: X^4Y+Y^4Z+Z^4X=0$. *In particular* $\rho(\widetilde{M_6^{Pl}(SmallGroup(39,1))})$ *is an irreducible locus with one element, where* ρ *is determined by* σ,τ.

(5) *The cyclic group* $\mathbb{Z}/8\mathbb{Z}$ *appears as* $Aut(C)$ *inside* $PGL_3(K)$ *that is generated by the transformation* $(x,y,z)\mapsto(x,\xi_8y,\xi_8^4z)$ *up to conjugation by* $P\in PGL_3(K)$, *and* C *is* K*-isomorphic (through* P*) to the plane non-singular curve* $X^5+Y^4Z+XZ^4+\beta_{2,0}X^3Z^2$, *with* $\beta_{2,0}\neq 0,\pm2$. *The locus* $\rho(\widetilde{M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})})$ *has dimension one where* $\rho(\mathbb{Z}/8\mathbb{Z}) =< diag(1,\xi_8,\xi_8^4) >$.

Proof. Except the last statement on the loci, the proof is a direct consequence of Theorem 6, because one could apply the result for $d=5$ using the table in §2 when the curve C has a cyclic automorphism of order: $d(d-1)$, $(d-1)^2$, $d(d-2)$, d^2-3d+3 and $\ell(d-1)$ with $\ell=2$ respectively. It remains, for the last case, to observe that if $Aut(C)$ is bigger then it is always cyclic and should be the group of order 16. Therefore, $\beta_{2,0}\neq0$ is the only restriction to impose so that the curve has automorphism group exactly $\mathbb{Z}/8\mathbb{Z}$ (Here, we note that $\beta_{2,0}\neq\pm2$ to ensure non-singularity. Also α is converted to 1 through a diagonal change of the variables). Lastly, we refer to [1] for the proof of the fact on the dimension and the irreducibility over $\mathbb{C}$ of the locus $\rho(\widetilde{M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})})$. □

4. Determination of the automorphism group with small cyclic subgroups

In this section, following the abuse of notation of the previous section concerning models and curves, we study $Aut(C)$ for non-singular plane curves C of degree $d=5$ that appear in the table of §2 such that the maximal order for any element inside the automorphism group is $2d$ or $\leq d$.

Also, we denote by C_n the cyclic group $\mathbb{Z}/n\mathbb{Z}$ to emphasize the multiplication notation as a subgroup inside $PGL_3(K)$.

Proposition 8. *Suppose that* C *is a non-singular plane curve of degree 5 with* $\sigma\in Aut(C)$ *of order* 10 *as an automorphism of maximal order. Then, we reduce after conjugation by certain* $P\in PGL_3(K)$ *that* σ *acts on* $C: X^5+Y^5+XZ^4+\beta_{2,0}X^3Z^2=0$ *such that* $\beta_{2,0}\neq0$ *as* $\sigma:(x,y,z)\mapsto(x,\xi_{10}^2y,\xi_{10}^5z)$, *and one of the following situations happens:*

(1) *If* $\beta_{2,0}^2=20$ *then* C *is* K*-equivalent to the Fermat quintic* $F_5: X^5+Y^5+Z^5=0$ *and* $Aut(C)$ *is isomorphic to* $SmallGroup(150,5)$.

(2) *If* $\beta_{2,0}^2\neq20$ *then* $Aut(C)$ *is cyclic of order* 10. *Moreover,* C *is a descendant of the Fermat curve of the form* $C_P:\ X^5+Y^5+Z^5+u\left(\xi_{10}^6Y^4Z+YZ^4\right)+u'\left(\xi_{10}^2Y^3Z^2+Y^2Z^3\right)=0$, *with* $(u,u')\neq(0,0)$.

Proof. Because the maximal order is 10 then, by the results of the previous section, we reduce C to be, up to K-isomorphism, of the form $X^5+Y^5+\alpha XZ^4+\beta_{2,0}X^3Z^2=0$ with $\alpha\beta_{2,0}\neq0$, and by a diagonal change of variables we always can take $\alpha=1$. This curve admits a homology σ^2 of order $5>3$ therefore, by Theorem 6, $Aut(C)$ fixes a line and a point off that line or C is a descendent of Fermat curve. Moreover, the center $(0;1;0)$

of this homology is an outer Galois point (by Lemma 3.7 [6]) and if C is not K-isomorphic to the Fermat curve $F_5 : X^5 + Y^5 + Z^5$ then it is unique (Theorem 4' [12]). Hence it should be fixed by $Aut(C)$.

Assume first that $Aut(C)$ fixes a line and a point off that line and C is not a descendent of the Fermat quintic. Hence, $Aut(C)$ satisfies a short exact sequence $1 \to C_5 \to Aut(C) \to G' \to 1$, where C_5 is generated by $\sigma^2 = [X; \zeta_5^2 Y; Z]$. In particular, G' contains an element of order 2 which is obtained by the image of σ under the restriction of the natural map ϱ from $PBD(2,1)$ to $PGL_2(K)$. Consequently G' is conjugate to C_2, C_4, S_3, A_4, S_4 or A_5. We claim that G' is conjugate to C_2 (in particular $Aut(C)$ is cyclic of order 10).

Since there are no groups of order 30 (respectively, 60) which contain elements of order 10 and no higher orders then G' is not conjugate to S_3 or A_4. Also, if G' is conjugate to S_4 then $Aut(C)$ is conjugate to $SmallGroup(120,5)$ or $SmallGroup(120,35)$ because these are the only groups of order 120 with elements of order 10 and no higher orders appear. But one can verify that there are no elements $\tau \in Aut(C)$ of order 3 or 10 that commute with σ^5 therefore $Aut(C)$ is not conjugate to any of these two groups. In particular, G' is not conjugate to S_4. On the other hand, groups of order 20 that contain elements of order 10 and no higher orders are $SmallGroup(20,\ell)$ where $\ell = 1,3,4$ or 5. There is no element $\tau \in Aut(C)$ of order 4 such that $\sigma^2\tau\sigma^2 = \tau$ or $\sigma^2\tau\sigma^{-2} = \tau\sigma^2$, which implies that $\ell \neq 1,3$. Furthermore, there is no element τ of order 2 in $Aut(C)$ which commutes with σ^5 thus $\ell \neq 4,5$. This also implies that G' is not conjugate to C_4. Lastly, groups of order 300 that contain elements of order 10 and no higher orders are $SmallGroup(300,\ell)$ where $\ell = 25,26,27,41$ or 43. If $\ell = 43$ or 41 then $Aut(C)$ contains exactly 3 element of order 2 which contradicts the fact that $Aut(C)$ should have at least 15 such elements as A_5 does. Moreover, there are no elements of order 2 in $Aut(C)$ such that $\tau\sigma^5 = \sigma^5\tau$ hence $\ell \neq 25, 26$ or 27. Consequently, G' is not conjugate to A_5. This proves the claim in this situation.

Secondly, assume that C is a descendant of the degree 5 Fermat curve. This should happen through a transformation $P \in PGL_3(K)$ such that $P^{-1}\sigma P = \lambda\sigma_i'$ where $\sigma_1' := [X; \zeta_{10}^{2b}Z; \zeta_{10}^{2a}Y]$, $\sigma_2' := [\zeta_{10}^{2a}Y; X; \zeta_{10}^{2b}Z]$ and $\sigma_3' := [\zeta_{10}^{2b}Z : \zeta_{10}^{2a}Y; X]$ with $5 \nmid (a+b)$. In each case, we get a Fermat descendant of the form

$$C_P :\ X^5 + Y^5 + Z^5 + u\left(\xi_{10}^{a'}A^4B + AB^4\right) + u'\left(\xi_{10}^{b'}A^3B^2 + A^2B^3\right) = 0,$$

where $\{A, B\} \subset \{X, Y, Z\}$. Moreover, C_P is the Fermat curve only if $\beta_{2,0}^2 = 20$, and $Aut(C_P)$ is cyclic of order 10 otherwise. For example, if $P^{-1}\sigma P = \lambda\sigma_1'$ then $\lambda = \zeta_{10}^2,\ 5|a{+}b{+}2$, and $P = [\zeta_{10}^{2a+2}\alpha_3 Y + \alpha_3 Z; X; -\zeta_{10}^{2a+2}\gamma_3 Y + \gamma_3 Z]$. Therefore C is transformed into C_P of the form

$$X^5 + Y^5 + Z^5 + u\left(\xi_{10}^{6(a+1)}Y^4Z + YZ^4\right) + u'\left(\xi_{10}^{2(a+1)}Y^3Z^2 + Y^2Z^3\right) = 0$$

such that $\alpha_3\left(\alpha_3^4 + \beta_{2,0}\gamma_3^2\alpha_3^2 + \gamma_3^4\right) = 1$. Now, C_P is the Fermat quintic curve only if $u = 0 = u'$ or equivalently

$$5\alpha_3^4 + \beta_{2,0}\gamma_3^2\alpha_3^2 - 3\gamma_3^4 = 0 = 5\alpha_3^4 - \beta_{2,0}\gamma_3^2\alpha_3^2 + \gamma_3^4.$$

Consequently, $\beta_{2,0}^2 = 20$. (For instance when $K = \mathbb{C}$, one can take $\alpha_3 = -\frac{(-1)^{3/5}}{2^{4/5}}$, and $\gamma_3 = \frac{(-1)^{3/5}\sqrt[4]{5}}{2^{4/5}}$ for $\beta_{2,0} = 2\sqrt{5}$. Also, for $\beta_{2,0} = -2\sqrt{5}$, one can assume $\alpha_3 = \frac{1}{2^{4/5}}$, and $\gamma_3 = -\frac{i\sqrt[4]{5}}{2^{4/5}}$). Otherwise (i.e. $u \neq 0$ or $u' \neq 0$), one could take $a = 0$ and $b = 3$ because all solutions (recall that $[X; \zeta_{10}^{2b}Z; \zeta_{10}^{2a}Y] \in Aut(C_P)$) are K-isomorphic through a change of variables of the form $X \mapsto X'$, $Y \mapsto \xi_{10}^{\ell}Y'$ and $Z \mapsto \xi_{10}^{\ell'}Z'$. Hence C_P admits no more automorphisms inside $Aut(F_5)$, that is, $Aut(C_P)$, as a subgroup of $Aut(F_5)$, is cyclic of order 10.

For the other situations (i.e. by replacing σ_1' with σ_2' and σ_3'), one can reduce to some concrete (a,b) and obtain exactly the same system to solve involving $\beta_{2,0}$ as before. Hence the same conclusion follows. □

Remark 9. *Recall that $Aut(F_5)$ is generated by $\eta_1 := [X; Z; Y], \eta_2 := [Y; Z; X], \eta_3 := [\xi_5 X; Y; Z]$ and $\eta_4 := [X; \xi_5 Y; Z]$ of orders $2, 3, 5$ and 5 respectively such that*

$$(\eta_1\eta_2)^2 = (\eta_1\eta_3)(\eta_3\eta_1)^{-1} = (\eta_3\eta_4)(\eta_4\eta_3)^{-1} = \eta_1\eta_4^2\eta_1(\eta_3\eta_4)^{-3} = \eta_2\eta_3\eta_2^{-1}(\eta_3\eta_4)^{-4} = 1.$$

The following lemma is very useful to discard all the groups with a subgroup isomorphic to $C_2 \times C_2$ for non-singular plane curves of degree 5.

Lemma 10. *There is no non-singular plane curve C of degree 5 with $C_2 \times C_2 \preceq Aut(C)$. In particular, the full automorphism group $Aut(C)$ is not isomorphic to any of the groups: $C_2 \times C_2$, A_4, S_4 or A_5.*

Proof. By Mitchell [11] and Harui [6], the group $C_2 \times C_2$ inside $PGL_3(K)$ which gives invariant a non-singular plane curve C of degree d should fix a point not lying on C or C is a descendant of either the Fermat or the Klein curve. For $d = 5$, it could not be a descendant of the Fermat or the Klein curve because 4 does not divide $|Aut(F_5)| = 150$ or $|Aut(K_5)| = 39$. Therefore, the automorphism subgroup $C_2 \times C_2$ fixes a point not in C. Moreover, because 2 does not divide the degree $d = 5$, then by Harui's main theorem [6], we can think about the elements of $C_2 \times C_2$ in a short exact sequence: $1 \to N = 1 \to H \to H \to 1$, where H is conjugate to $C_2 \times C_2$ inside $PGL_2(K)$. We can assume that H acts only on the variables Y, Z because N is the subgroup of $Aut(C)$ that acts on X. Now, let $\sigma, \tau \in H \subseteq PGL_2(K)$ be of order two such that $\sigma\tau = \tau\sigma$ then, we can suppose, up to a coordinate change inside $\mathbb{P}^2$, that $\sigma = diag(1, -1)$ and $\tau = [aY + bZ, cY - aZ] \neq \sigma$. Consequently, the curve C has a model of type $2, (0, 1)$. But all possible τ does not retain invariant the equation of the type $2, (0, 1)$ for any choice of the free parameters and hence the result follows. Indeed, because τ commutes with σ then $\tau = diag(-1, 1)$ or $[X, bZ, cY]$ with $bc \neq 0$ and therefore C has simultaneously the expressions: $Z^4L_{1,Z} + Z^2L_{3,Z} + L_{5,Z}$ and $Y^4L_{1,Y} + Y^2L_{3,Y} + L_{5,Y}$. Thus, C has the form $X \cdot G(X; Y; Z)$ with G of degree 4 if $L_{5,Z}$ and $L_{5,Y}$ are non-zero, a contradiction to irreducibility. □

Now, we deal with quintic curves whose automorphism group admits a cyclic element of order 5 (respectively 4) as an automorphism of maximal order.

Proposition 11. *If C is a degree 5, non-singular plane curve with an automorphism σ of maximal order 5, then we reduce, up to projective equivalence, to one of the following situations: $Aut(C)$ is cyclic of order 5 and C is K-equivalent to the type $5, (0, 1)$ of the form $Z^5 + L_{5,Z}$ or $Aut(C)$ is isomorphic to the Dihedral group D_{10} of order 10 where $Aut(C) =< \sigma, \tau >$ with $\sigma(x, y, z) = (x, \xi_5 y, \xi_5^2 z)$ and $\tau(x, y, z) = (z, y, x)$, and the curve C has the form $X^5 + Y^5 + Z^5 + \beta_{3,1}X^2YZ^2 + \beta_{4,3}XY^3Z = 0$ such that $(\beta_{3,1}, \beta_{4,3}) \neq (0, 0)$.*

Proof. We consider the situations in §1 concerning types $5, (a, b)$.

(1) Type $5, (1, 2)$: $Aut(C)$ is not conjugate to any of the Hessian groups $Hess_{36}$ or $Hess_{72}$ and is not conjugate to a subgroup of $Aut(K_5)$ since there are no elements of order 5. On the other hand, always C admits a bigger automorphism group isomorphic to D_{10} through the transformation $[Z; Y, X]$ (in particular, $Aut(C)$ is not cyclic). Moreover, by the previous Lemma 10, $Aut(C)$ is not conjugate to A_5 (as a finite primitive subgroup of $PGL_3(K)$). Consequently, C is a descendant of the Fermat quintic or $Aut(C)$ fixes a line and a point off that line.
   - If $Aut(C)$ fixes a line and a point off that line then this line should be $Y = 0$ and the point is $(0; 1; 0)$ because $< \sigma, \tau > \subseteq Aut(C)$ with $\sigma(x, y, z) = (x, \xi_5 y, \xi_5^2 z)$ and $\tau(x, y, z) = (z, y, x)$. In particular, elements of $Aut(C)$ are of the form $[\alpha_1 X + \alpha_3 Z; Y; \gamma_1 X + \gamma_3 Z]$. Hence, from the coefficients of Y^3Z^2 and Y^3X^2 (respectively, X^4Y and YZ^4), we should have $\alpha_1 = 0 = \gamma_3$ or $\alpha_3 = 0 = \gamma_1$. Moreover $\alpha_{3,1}^5 = \gamma_{1,3}^5 = 1$ and $\alpha_{3,1}\gamma_{1,3} = 1$ or $(\alpha_{3,1}\gamma_{1,3})^2 = 1$ since $(\beta_{3,1}, \beta_{4,3}) \neq (0, 0)$. This implies that $Aut(C)$ has order 10.
   - If C is a descendant of the Fermat curve F_5 through a transformation $P \in PGL_3(K)$ and neither a line nor a point is leaved invariant. Then, $P^{-1}[X; \xi_5 Y; \xi_5^2 Z]P = [X; \xi_5 Y; \xi_5^2 Z]$, because elements of order 5 in $Aut(F_5)$ are of the form $\sigma_{a,b} := [X; \xi_5^a Y; \xi_5^b Z]$, and if $P^{-1}[X; \xi_5 Y; \xi_5^2 Z]P = \lambda\sigma_{a,b}$ then $(a, b) \in \{(1, 2), (2, 1), (3, 4), (4, 3), (1, 4), (4, 1)\}$, but all are conjugate in $Aut(F_5)$. Now, P has one of the forms $[X; \beta Y; \gamma Z], [Y; \alpha Z; \beta X]$ or $[Z; \alpha X; \beta Y]$ and it is straightforward to verify that there are no more automorphisms in $Aut(F_5) \cap Aut(C_P)$ that is, $Aut(C)$ is conjugate to D_{10}.

(2) Type $5, (0, 1)$: This curve has a homology σ of order d with center $(0; 0; 1)$ and axis $Z = 0$ then (by Lemma 3.7 in [6]), this point is an outer Galois point of C. Moreover, C is not K-isomorphic to the Fermat curve because automorphisms of C has orders ≤ 5. Then (by Yoshihara [12]) this Galois point is unique and hence should be fixed by $Aut(C)$. In particular, $Aut(C)$ fixes a line ($Z = 0$) and a point off that line ($(0; 0; 1)$) that is, elements of $Aut(C)$ have the form $[\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$. Furthermore, $Aut(C)$ satisfies a short exact sequence $1 \to N \to Aut(C) \to G' \to 1$ with N is cyclic of order dividing 5 and G' is conjugate to C_m, D_{2m}, A_4, S_4 or A_5 where $m \leq 4$ and for the case $G' = D_{2m}$ we have $m|3$ or N is trivial.

If N is trivial then G' should be conjugate to A_5 (because none of the other groups contains elements of order 5) then $C_2 \times C_2$ is a subgroup of $Aut(C)$ which is not possible by Lemma 10. Hence, N can not be the trivial group.

If N has order 5 then for any value of G' (except possibly the trivial group, C_2, C_4 or A_4 such that $Aut(C)$ is conjugate to D_{10}, $SmallGroup(20,3)$ or A_5) there are elements of order > 5 in $Aut(C)$ a contradiction. Again, by Lemma 10, we conclude that G' can not be A_4. On the other hand, there exists no elements $\tau \in Aut(C)$ of order 2 such that $\tau\sigma\tau = \sigma^{-1}$ hence G' is not C_2. Moreover, there are no elements $\tau \in Aut(C)$ of order 4 such that $(\tau\sigma)^2 = 1$ and $\sigma\tau\sigma^{-1} = \tau\sigma$ thus $G^{'}$ is not conjugate to C_4. Consequently, $Aut(C)$ is cyclic of order 5. □

Proposition 12. *Suppose that C is a non-singular plane curve of degree 5 with $\sigma \in Aut(C)$ of order 4 as an element of maximal order, then we reduce, up to K-isomorphism, to one of the following two situations: $Aut(C)$ is cyclic of order 4 which is generated by $\sigma(x,y,z) = (x,y,\xi_4 z)$, and C is given by $Z^4Y + L_{5,Z}(X,Y) = 0$ such that $L_{5,Z}(X,\zeta_m Y) \neq \zeta_m^r L_{5,Z}(X,Y)$ where $(m,r) \in \{(8,1),(16,1),(20,4)\}$ or $\sigma(x,y,z) = (x,\xi_4 y,\xi_4^2 z)$ and C is defined by $X^5 + X(Z^4 + \alpha Y^4) + \beta_{2,0}X^3Z^2) + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3 = 0$ such that $\alpha\beta_{5,2} \neq 0$.*

Proof. We consider the situations in §1 concerning types $4,(a,b)$.

First, we observe that C can not be a descendant of the Fermat curve F_5 or the Klein curve K_5 because $|Aut(F_5)| = 150$ and $|Aut(K_5)| = 39$ and $4 \nmid |Aut(F_5)|$ or $|Aut(K_5)|$, and $Aut(C)$ is not conjugate to A_5 since there are no elements of order 4. Consequently, $Aut(C)$ is conjugate to $Hess_{36}, Hess_{72}$ or it should fix a line and a point off that line by the result of Harui. Moreover, for the last case, we need to consider the situation of a short exact sequence of the form $1 \to N = 1 \to Aut(C) \to G' \to 1$, where G' should contain an element of order 4. That is, G' is conjugate to a cyclic group C_4 or a Dihedral group D_8 (by use of Lemma 10).

(1) Type $4,(1,3)$; All such curves decompose into $X \cdot G(X,Y,Z)$ and therefore are reducible.
(2) Type $4,(0,1)$; This curve admits a homology of order $d-1$ with center $(0;0;1)$ then it follows by Harui [6], that this point is an inner Galois point of C, and moreover it is unique by Yoshihara [12]. Therefore, this point should be fixed by $Aut(C)$ consequently, $Aut(C)$ is cyclic. It follows by the assumption that C is not conjugate to any of the above that $Aut(C)$ is cyclic of order 4. More precisely, we can rewrite type $4,(0,1)$ as $Z^4Y' + L'_{5,Z}(X,Y') = 0$ and it is necessary to impose the condition that $L'_{5,Z}(X,\zeta_m Y') \neq \zeta_m^r L'_{5,Z}(X,Y')$ where $(m,r) \in \{(8,1),(16,1),(20,4)\}$ (otherwise; we get a bigger automorphism group conjugate to those for types $8,(1,4)$, $16,(1,12)$ or $20,(4,5)$).
(3) Type $4,(1,2)$; First, by the same reason as type $4,(1,3)$, we need to assume that $\beta_{5,2} \neq 0$. Secondly, we'll show that $Aut(C)$ is not conjugate to any of the Hessian subgroups $Hess_{36}$ or $Hess_{72}$ as follows. Both groups contains reflections but no four groups hence all reflections in the group will be conjugate to $[Z;Y;X]$ (see Theorem 11 in [11]). Therefore, we can take $P \in PGL_3(K)$ such that $P^{-1}\sigma^2 P = \lambda[Z;Y;X]$ and $Aut(C_P) \subset PGL_3(K)$ is given by the usual presentation inside $PGL_3(K)$ of the above Hessian groups, in particular always $Aut(C_P)$ have the following five elements of $Hess_{36}$ and $Hess_{72}$: $[Z;Y;X]$, $[X;Z;Y]$, $[Y;X;Z]$, $[Y;Z;X]$ and $[X;\omega Y;\omega^2 Z]$, where ω is a primitive 3rd root of unity. Because C_P is invariant through $[Z;Y;X]$, $[X;Z;Y]$, $[Y;X;Z]$ and $[Y;Z;X]$, then C_P should be of the form: $u(X^5+Y^5+Z^5)+a(X^4Z+X^4Y+Y^4X+Y^4Z+Z^4X+Z^4Y)+H(X;Y;Z)$, where $u,a \in K$ and $H(X,Y,Z)$ is a homogenous polynomial of degree 5 such that the degree of any of the variables is at most three. Now, impose that C_P is fixed by $[X,\omega Y,\omega^2 Z]$, we obtain $u=0$ and $a=0$ a contradiction, because C_P is non-singular. Therefore there is no degree 5 curve with Hessian group $Hess_{36}$ or $Hess_{72}$.

Consequently, the claim follows and $Aut(C)$ should fix a line and a point off that line.

Now, if C admits a bigger non-cyclic automorphism group then it should be non-commutative by Harui and contain an element of order 2 which does not commute σ. We can reduce to a subgroup which is conjugate to the dihedral group D_8, and moreover $Aut(C)$ fixes the point $(1;0;0)$ and the line $X=0$. That is, automorphisms of C are of the form $[X; vY + wZ; sY + tZ]$. There is no element $\tau \in Aut(C)$ of order 2 such that $\tau\sigma\tau = \sigma^{-1}$. Hence $Aut(C)$ is not conjugate to D_8 or S_4. In particular, it is cyclic of order 4.

□

Remains yet the study of curves C where their automorphisms have orders at most 3. In particular, $Aut(C)$ is not conjugate to $A_5, Hess_{36}$ or $Hess_{72}$, because each of these groups contains elements of order > 3. Therefore, $Aut(C)$ should fix a line and a point off that line or it is conjugate to a subgroup of $Aut(F_5)$ or $Aut(K_5)$.

Proposition 13. *Let C be a non-singular plane curve of type $3,(1,2)$ such that elements inside $Aut(C)$ have orders ≤ 3. Then, C is K-isomorphic to $X^5+Y^4Z+YZ^4+\beta_{2,1}X^3YZ+X^2(\beta_{3,0}Z^3+\beta_{3,3}Y^3)+\beta_{4,2}XY^2Z^2=0$. Moreover, if $\beta_{3,0}\neq\beta_{3,3}$ then $Aut(C)$ is cyclic of order 3 and is conjugate to S_3 otherwise.*

Proof. If C is a descendant of the Klein curve then $Aut(C)$ is conjugate to a subgroup of $Aut(K_5)$. Hence can not be of order > 3, since $|Aut(K_5)| = 3\cdot 13$ (otherwise; $Aut(C)$ should contain an element of order $13 > 3$ by Sylow's theorem).

If C is a descendant of the Fermat curve then $Aut(C)$ is cyclic of order 3 or conjugate to S_3 inside $Aut(F_5)$. Indeed, $|Aut(F_5)| = 2\cdot 3\cdot 5^2$ hence any subgroup of order > 3 is conjugate to S_3 (note that $Aut(F_5)$ contains no elements of order 6) or it contains elements of order $5 > 3$. Now, if $Aut(C)$ is conjugate to S_3 then there exists $\tau\in Aut(C)$ of order 2 such that $\tau\sigma\tau=\sigma^{-1}$ which reduces τ to be of the form $[X;\beta Z;\beta^{-1}Y]$. But, $[X;\beta Z;\beta^{-1}Y]\in Aut(C)$ iff $\beta^3=1$ and $\beta_{3,0}=\beta_{3,3}$.

If $Aut(C)$ fixes a point then should be one of the reference points $P_1 := (1;0;0)$, $P_2 := (0;1;0)$ or $P_3 := (0;0;1)$, since these are the only points which are fixed by σ. If the fixed points is P_2 or P_3 then $Aut(C)$ is cyclic of order 3 because both points lie on C. If the fixed point is P_1 then the line that is leaved invariant should be $X=0$, hence automorphisms of C have the form $[X;\beta_2Y+\beta_3Z;\gamma_2Y+\gamma_3Z]$. Moreover, it follows by Theorem 2(2), Lemma 10 and the assumption that there are no elements in $Aut(C)$ of order > 3 that $Aut(C)$ satisfies a short exact sequence of the form $1\to N=1\to Aut(C)\to G'\to 1$, where G' is conjugate to C_3, S_3. This completes the proof. $\square$

Proposition 14. *Let C be a non-singular plane curve of type $2,(0,1)$ such that elements inside $Aut(C)$ have orders ≤ 2. Then, C is K-isomorphic to $C: Z^4L_{1,Z}+Z^2L_{3,Z}+L_{5,Z}=0$ and $Aut(C)$ is cyclic of order 2.*

Proof. C is not a descendant of the Klein curve because $2\nmid |Aut(K_5)|(=39)$. Also, if C is a descendant of the Fermat curve then $Aut(C)$ can not be conjugate to a bigger subgroup of $Aut(F_5)$, because $|Aut(F_5)| = 2.3.5^2$, thus subgroups of order > 2 should contain an element of order 3 or 5 which is a contradiction. Finally, if $Aut(C)$ fixes a line and a point off that line then, by [6] and our assumption that there are no automorphisms of order > 2, we get that $Aut(C)$ satisfies a short exact sequence of the form $1\to N=1\to Aut(C)\to G'\to 1$, where G' contain an element of order 2 and no higher orders. Thus $Aut(C)$ should be conjugate to C_2 or $C_2\times C_2$. Consequently, the result follows by Lemma 10. $\square$

Lastly, we need to ensure the existence of a non-singular plane curve C, via certain specializations of the parameters, for which the maximal order of the elements in its full automorphism group is exactly m where $m\leq 5$.

This is a tedious computation because we do not know a priori the dimension of the locus $\rho(\widetilde{M_6(G)})$, see for example the situations with $m=4$ in [1]. To treat the case $m\neq 4$, we can apply similar arguments as the situation $m=4$, which will not be reproduced here (nevertheless, we know all the possible groups and the representations that could appear such that m divides their order). This in turns simplifies the computations, in order to conclude,

Lemma 15. *Take $F(X;Y;Z)$ as the equation of degree 5 associated to Type $m,(a,b)$ where $m\leq 5$ in §2, table 1 with $m,(a,b)\neq 4,(1,3)$. Then, there exists a non-singular plane curve C obtained by a concrete specialization of the parameters of the equation $F(X;Y;Z)$, such that all the elements of $Aut(C)$ are of order $\leq m$. Moreover, for type $3,(1,2)$, we have curves with this property on the elements of $Aut(C)$ of order ≤ 3 satisfying $\beta_{3,0}\neq\beta_{3,3}$ and also satisfying $\beta_{3,0}=\beta_{3,3}$.*

Lastly, we remark the following.

In the next table, we list the exact groups G (some of them are given in the GAP notations) that appear as an $Aut(\delta)$ of a non-singular plane curve δ of genus 6. The second column corresponds to the injective representation ρ of the group G inside $PGL_3(K)$. The third column corresponds to a homogenous equation of degree 5 associated with certain parameters such that for any $\delta \in \rho(\widetilde{M_6^{Pl}}(G))$, a plane non-singular model associated to δ can be obtained by a specialization of the parameters and vice versa.

It remains to assign to each equation the parameters' restrictions to ensure that the equation is geometrically irreducible, non-singular and it does not have a bigger automorphism group (so that, any $\delta \in \rho(\widetilde{M_6^{Pl}}(G))$ corresponds to some specialization of the parameters with respect to the restrictions, and conversely, any specialization of the parameters with respect the restrictions gives a non-singular plane model of some element in $\rho(\widetilde{M_6^{Pl}}(G)))$. We recall that, always $\alpha \neq 0$ can be transformed by a diagonal change of variables to 1.

TABLE 2. Full Automorphism of quintics

G	$\rho(G)$	$F_{\rho(G)}(X;Y;Z)$
$(150,5)$	$[\xi_5X;Y;Z], [X;\xi_5Y;Z]$ $[X;Z;Y]$, $[Y;Z;X]$	$X^5+Y^5+Z^5$
$(39,1)$	$[X;\xi_{13}Y;\xi_{13}^{10}Z], [Y;Z;X]$	$X^4Y+Y^4Z+Z^4X$
$(30,1)$	$[X;\xi_{15}Y;\xi_{15}^{11}Z], [X;Z;Y]$	$X^5+Y^4Z+YZ^4$
$\mathbb{Z}/20\mathbb{Z}$	$[X;\xi_{20}^4Y;\xi_{20}^5Z]$	$X^5+Y^5+XZ^4$
$\mathbb{Z}/16\mathbb{Z}$	$[X;\xi_{16}Y;\xi_{12}^5Z]$	$X^5+Y^4Z+XZ^4$
$\mathbb{Z}/10\mathbb{Z}$	$[X;\xi_{10}^2Y;\xi_{10}^5Z]$	$X^5+Y^5+XZ^4+\beta_{2,0}X^3Z^2$ $\beta_{2,0}\neq 0$ and $\beta_{2,0}^2\neq 20$
D_{10}	$[X;\xi_5Y;\xi_5^2Z]$, $[Z,Y,X]$	$X^5+Y^5+Z^5+\beta_{3,1}X^2YZ^2+\beta_{4,3}XY^3Z$ $(\beta_{3,1},\beta_{4,3})\neq(0,0)$
$\mathbb{Z}/8\mathbb{Z}$	$[X;\xi_8Y;\xi_8^4Z]$	$X^5+Y^4Z+XZ^4+\beta_{2,0}X^3Z^2$ $(\beta_{2,0}\neq 0,\pm 2)$
S_3	$[X;\xi_3Y;\xi_3^2Z]$ $[X;Z;Y]$	$X^5+Y^4Z+YZ^4+\beta_{2,1}X^3YZ+\beta_{3,3}X^2(Z^3+Y^3)+$ $+\beta_{4,2}XY^2Z^2$ (not above)
$\mathbb{Z}/5\mathbb{Z}$	$[X;Y;\xi_5Z]$	$Z^5+L_{5,Z}$ (not above)
$\mathbb{Z}/4\mathbb{Z}$	$[X;\xi_4Y;\xi_4^2Z]$	$X^5+X(Z^4+\alpha Y^4)+\beta_{2,0}X^3Z^2+\beta_{3,2}X^2Y^2Z+\beta_{5,2}Y^2Z^3$ $(\beta_{5,2}\neq 0)$ (not above)
$\mathbb{Z}/4\mathbb{Z}$	$[X;Y;\xi_4Z]$	$Z^4L_{1,Z}+L_{5,Z}$ (not above)
$\mathbb{Z}/3\mathbb{Z}$	$[X;\xi_3Y;\xi_3^2Z]$	$X^5+Y^4Z+\alpha YZ^4+\beta_{2,1}X^3YZ+$ $+X^2(\beta_{3,0}Z^3+\beta_{3,3}Y^3)+\beta_{4,2}XY^2Z^2$ (not above)
$\mathbb{Z}/2\mathbb{Z}$	$[X;Y;\xi_2Z]$	$Z^4L_{1,Z}+Z^2L_{3,Z}+L_{5,Z}$ (not above)

Remark 16. *It should be noted that it appears here a new phenomena that did not occur for degree $d=4$. Henn in [7] observed that for each finite group G that appeared as an automorphism group of a non-singular plane curve $\delta \in M_3^{Pl}(G)$, there is an unique equation $F_G(X,Y,Z)$ (endowed with a set of restrictions on the parameters), up to change of variables, where the specializations of such equation is a plane non-singular model for the elements of $M_3^{Pl}(G)$ and vice versa. This is not the case any more for degree 5, since from the above table, with the group $\mathbb{Z}/4\mathbb{Z}$, we obtain two ρ's where their equations $F_\rho(X;Y;Z)$ are not K-isomorphic, and corresponds to the disjoint decomposition of $\widetilde{M_6^{Pl}}(\mathbb{Z}/4\mathbb{Z})$ in terms of non-empty $\rho(\widetilde{M_6^{Pl}}(\mathbb{Z}/4\mathbb{Z}))$. Similar situations happen for higher degrees, for more details, we refer to [1].*

References

[1] E. Badr, F.Bars, *On the locus of smooth plane curves with a fixed automorphism group*. Preprint. See a first version in chapter 1 of "On the automorphism group of non-singular plane curves fixing the degree", arXiv:1503.01149 (2015).

[2] E. Badr, F.Bars, *Plane non-singular curves with large cyclic automorphism group*. Preprint. See a first version in chapter 2 of "On the automorphism group of non-singular plane curves fixing the degree", arXiv:1503.01149 (2015).

[3] S. Crass, *Solving the sextic by iteration:A study in complex geometry and dinamics.* Experimental Mathematics 8(3)(1999), 209-240

[4] Dolgachev, I., *Classical Algebraic Geometry: a modern view*, Cambridge Univ.Press 2012, see also Private Lecture Notes in: http://www.math.lsa.umich.edu/∼idolga/ .

[5] GAP, The GAP Group: Groups, Algorithms, and Programming, a system for computational discrete algebra (2008), available at http://www.gap-system.org. Version 4.4.11.

[6] T. Harui, *Automorphism groups of plane curves*, arXiv: 1306.5842v2[math.AG] 7 Jun 2014

[7] P. Henn, *Die Automorphismengruppen dar algebraischen Functionenkorper vom Geschlecht 3*, Inagural-dissertation, Heidelberg, 1976.

[8] J.W.P.Hirschfeld, G.Korchmáros, F.Torres, *Algebraic Curves over Finite Fields*, Princeton Series in Applied Mathematics, 2008.

[9] A. Hurwitz, *Über algebraische Gebilde mit eindeutigen Transformationen in sich*,Math.Ann. 41 (1893),403-442.

[10] A. Kuribayashi, K. Komiya, *On Weierstrass points and automor- phisms of curves of genus three*. Algebraic geometry (Proc. Summer Meeting, Copenhagen 1978), LNM 732, 253-299, Springer (1979).

[11] H. Mitchell, *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc. 12, no. 2 (1911), 207242.

[12] H. Yoshihara, *Function field theory of plane curves by dual curves*, J. Algebra 239, no. 1 (2001), 340-355

• Eslam Badr

Departament Matemàtiques, Edif. C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia

Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt
E-mail address: `eslam@mat.uab.cat`

• Francesc Bars

Departament Matemàtiques, Edif. C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia
E-mail address: `francesc@mat.uab.cat`