Universitat Autònoma de Barcelona

Bachelor's Thesis

On the critical strip of the Riemann zeta-function

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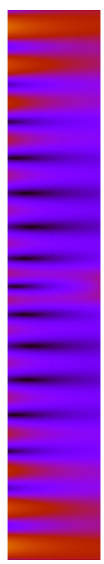

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1 Preface

The Riemann zeta-function $\zeta(s)$ is a meromorphic function. Many of the results about $\zeta(s)$ that can be found with the usual tools of complex analysis have very important consequences on number theory (see [4, p.43]). The growth of $\pi(x)$ for example can be related to the zeros of $\zeta(s)$, where $\pi(x)$ is defined as the amount of prime numbers less or equal to x (see [5, p.183]).
The first chapter will be devoted to basic results about $\zeta(s)$. In the second and third chapter we will see some results about the zeros of $\zeta(s)$ and the behavior of $\zeta(s)$ on the part of the complex plane where $0 < Re(s) < 1$, which are both of special interest to number theorists (see [5, p.133]).

I would like to thank my supervisor Dr.Francesc Bars for supporting me extensively.

2 Introduction

Definition. *For $s \in \mathbb{C}$ we define*

$$\zeta_k(s) := \sum_{n=1}^{k} n^{-s}. \tag{1}$$

Definition. *If $Re(s) > 1$, then the Riemann zeta-function is defined as*

$$\zeta(s) = \lim_{n\to\infty} \zeta_n(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{2}$$

We note that for $Re(s) > 1$ the sum on the right hand side converges uniformly and absolutely, since

$\left| \sum\limits_{n=1}^{\infty} n^{-s} \right| \leq \sum\limits_{n=1}^{\infty} |n^{-s}| = \sum\limits_{n=1}^{\infty} n^{-Re(s)}.$

Proposition. *$\zeta(s)$ is holomorphic in $Re(s) > 1$.*

Proof. Apply theorem A.1 to $\zeta_n(s)$, with Ω being the half-plane $Re(s) > 1$. □

Definition 2. *Let U_1 and U be regions of $\mathbb{C}$ such that U contains U_1. If $f : U_1 \to \mathbb{C}$ and $F : U \to \mathbb{C}$ are analytic functions such that $F(s) = f(s), \forall s \in U_1$, then F is called an analytic continuation of f.*

The next proposition shows that analytic continuations are unique in a certain way.

Proposition. *Let $F_1, F_2 : U \to \mathbb{C}$ be analytic continuations of $f : U_1 \to \mathbb{C}$. Then $F_1(s) = F_2(s), \forall s \in U$.*

Proof. Since F_1 and F_2 are analytic on U, $F_1 - F_2$ is analytic too and we can develop it as a Taylor-series around any point of U. Since $F_1 - F_2 = 0$ in U_1 the coefficients of the Taylor-series around a point on the border of U_1 are identically 0 and therefore $F_1 - F_2 = 0$ on an open ball around any point of the border. On the border of these open balls we can develop $F_1 - F_2$ again as a Taylor series and again the coefficients have to be 0. Repeating this process until we have covered the whole of U with open balls, we see that $F_1 - F_2 = 0$ on U. □

2.1 The Euler product

Now we introduce the Euler Product representation of $\zeta(s)$.

Theorem (Euler Product). *If $Re(s) > 1$, then*

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}, \tag{3}$$

where P is the set of all prime numbers.

Proof. We will prove the identity for real s with $Re(s) > 1$ and conclude by analytic continuation that it holds for all s with $Re(s) > 1$.
Obviously for every natural number $n \leq N$, there is one and only one way to write $n^{-s} = (p_1^{m_1} \cdots p_k^{m_k})^{-s}$, with $p_1, ..., p_k$ being different prime numbers and in that representation $m_1, ..., m_k, p_1, ..., p_k \leq N$.
Therefore the finite product

$$\prod_{p \in P, p \leq N} (1 + p^{-s} + p^{-2s} + ... + p^{-Ns}), \tag{4}$$

is a finite sum of terms of the form m^{-s}, which are all positive, since m are natural and s real numbers. Furthermore, for every $n \leq N, n^{-s}$ is 'contained' in this sum and therefore

$$\zeta_N(s) = 1 + 2^{-s} + ... + N^{-s} \leq \prod_{p \in P, p \leq N} (1 + p^{-s} + p^{-2s} + ... + p^{-Ns}), \tag{5}$$

and on the other hand

$$\begin{aligned} &\prod_{p \in P, p \leq N} (1 + p^{-s} + p^{-2s} + ... + p^{-Ns}) \leq \prod_{p \in P, p \leq N} \sum_{i=0}^{\infty} (p^{-s})^i \\ &= \prod_{p \in P, p \leq N} (\frac{1}{1 - p^{-s}}) \leq \prod_{p \in P} (\frac{1}{1 - p^{-s}}). \end{aligned} \tag{6}$$

Hence

$$\zeta(s) = \lim_{N \to \infty} \zeta_N(s) \leq \prod_{p \in P} (\frac{1}{1 - p^{-s}}). \tag{7}$$

Now we have to proof $\zeta(s) \geq \prod_{p \in P} (\frac{1}{1-p^{-s}})$. Therefore we consider the finite product

$$\prod_{p \in P, p \leq N} (1 + p^{-s} + p^{-2s} + ... + p^{-Ms}), M \geq N \tag{8}$$

and observe that this is a finite sum of terms of the form n^{-s} in which no term is contained more than once, because for any n^{-s} there is only one way to write it as a product of $p_1^{-m_1 s}, ..., p_k^{-m_k s}$, with $p_1, ..., p_k$ being different prime numbers. Hence

$$\prod_{p\in P, p\leq N}(1+p^{-s}+p^{-2s}+...+p^{-Ms})\leq\sum_{n=1}^{\infty}n^{-s}=\zeta(s), \tag{9}$$

and letting M and N tend to infinity

$$\prod_{p\in P}\left(\frac{1}{1-p^{-s}}\right)=\lim_{N\to\infty}\prod_{p\in P,p\leq N}\left(\frac{1}{1-p^{-s}}\right) \tag{10}$$

$$=\lim_{N\to\infty}\lim_{M\to\infty}\prod_{p\in P,p\leq N}(1+p^{-s}+p^{-2s}+...+p^{-Ms}) \tag{11}$$

$$\leq\sum_{n=1}^{\infty}n^{-s}=\zeta(s), \tag{12}$$

which completes the proof. □

2.2 Riemann's functional equation for $\zeta(s)$

In order to prove Riemann's functional equation we will introduce the functions $\psi(s)$ and $\Gamma(s)$ and discuss some of their properties. To prove a certain property of $\psi(s)$ that has an important role in the proof of Riemann's functional equation we will introduce Poisson summation formula. Furthermore we will introduce the Mellin transform and its inversion formula, since in the proof of Riemann's functional equation there will be several integrals that are precisely Mellin transforms. The function $\Gamma(s)$, for example, turns out to be the Mellin transform of e^{-s}.

2.2.1 The Poisson Summation Formula

Definition. *If it exists, the Fourier transform of f is defined as*

$$\widehat{f}(t)=\int_{-\infty}^{\infty}f(x)e^{2\pi itx}dx, \forall t\in\mathbb{R}. \tag{13}$$

Definition. *A differentiable function $f:\mathbb{R}\to\mathbb{C}$ is said to be of bounded total variation, if $\int_{-\infty}^{\infty}|f'(x)|dx$ converges.*

Theorem (Poisson Summation Formula). *If $f:\mathbb{R}\to\mathbb{R}$ is an integrable contiuous function of bounded total variation, such that there exists a bounded function $g:\mathbb{R}\to\mathbb{R}_+$ that satisfies:*
1)$g(x)=g(-x)$
2)$g(x)$ is decreasing in $\mathbb{R}_+$
3)$\int_{-\infty}^{\infty}g(x)dx$ converges
4)$|f(x)|<g(x),\forall x\in\mathbb{R}$,
then:

$$\sum_{n\in\mathbb{Z}}f(x+n)=\lim_{N\to\infty}\sum_{m\in\mathbb{Z},|m|<N}\widehat{f}(m)e^{2\pi imx} \tag{14}$$

and in particular, with $x = 0$, we obtain the formula which is usually called the Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{N\to\infty} \sum_{m\in\mathbb{Z},|m|<N} \widehat{f}(m) \tag{15}$$

Proof. As $\int_{-\infty}^{\infty} g(x)dx$ converges and $g(x)$ is decreasing in $\mathbb{R}_+$, $\sum_{n\in\mathbb{Z}} g(x+n)$ converges. Since $|f(x)| < g(x), \forall x \in \mathbb{R}$, by Weierstrass criterion $\sum_{n\in\mathbb{Z}} f(x+n)$ converges to a continuous function $F(x)$. Furthermore $F(x) = \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} f(x+1+n) = F(x+1), \forall x \in \mathbb{R}$ and therfore $F(x)$ is periodic. The Fourier coefficients are:

$$\begin{aligned}
\widehat{F}(m) &= \int_{[0,1]} F(x)e^{2\pi imx}dx = \int_{[0,1]} \sum_{n\in\mathbb{Z}} f(x+n)e^{2\pi imx}dx \\
&= \sum_{n\in\mathbb{Z}} \int_{[0,1]} f(x+n)e^{2\pi imx}dx = \sum_{n\in\mathbb{Z}} \int_{[n-1,n]} f(x)e^{2\pi imx}dx \\
&= \int_{-\infty}^{\infty} f(x+n)e^{2\pi imx}dx = \widehat{f}(m)
\end{aligned} \tag{16}$$

And finally:

$$\sum_{n\in\mathbb{Z}} f(x+n) = F(x) = \lim_{N\to\infty} \sum_{m\in\mathbb{Z},|m|<N} \widehat{F}(m)e^{2\pi imx} = \lim_{N\to\infty} \sum_{m\in\mathbb{Z},|m|<N} \widehat{f}(m)e^{2\pi imx} \tag{17}$$

□

Corollary. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function of total bounded variation such that $\int_{-\infty}^{\infty} |f(x)|dx$ and $\int_{-\infty}^{\infty} |\widehat{f}(s)|ds$ converge and $\widehat{f}$ is also of total bounded variation. For $u \neq 0$ the following holds:*

$$\sum_{n\in\mathbb{Z}} f(nu) = |u|^{-1} \lim_{N\to\infty} \sum_{m\in\mathbb{Z},|m|<N} \widehat{f}(mu) \tag{18}$$

Proof. If we define $f_u(x) = f(ux)$, then $\widehat{f_u}(x) = |u|^{-1}\widehat{f}(\frac{x}{u})$. Substituting this in the proof of the previous theorem we get the result. □

2.2.2 The Γ-function and ψ-function

Since $\int_0^{\infty} x^{s-1}e^{-x}dx = \int_0^1 x^{s-1}e^{-x}dx + \int_1^{\infty} x^{s-1}e^{-x}dx = A + B$ and $A \sim \int_0^1 x^{s-1}dx$ and $B \sim \int_1^{\infty} e^{-x}dx$, the integral in the following definition converges whenever $Re(s) > 0$.

Definition. *For $s \in \mathbb{C}$ with $Re(s) > 0$, we define $\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx$.*

Theorem. *The function $\Gamma(s)$ initially defined for $Re(s) > 0$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at the negative integers $s = 0, -1, \ldots$.*

A proof of this can be found in $[7, p.161]$.

Proposition. Γ*(s) has no zeros.*

Proof. According to $[7, p.165]$, $\frac{1}{\Gamma(s)}$ is an entire function. □

Proposition. *The sum $\sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$ converges absolutely when $Re(s) > 0$.*

Proof.

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \left| e^{-\pi n^2 s} \right| &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 Re(s)} < 2\sum_{n=0}^{\infty} e^{-\pi n^2 Re(s)} \\ &< 2\sum_{n=0}^{\infty} e^{-\pi n Re(s)} = 2\frac{1}{1-e^{-\pi Re(s)}}.\end{aligned} \tag{19}$$

□

Definition. *For $s \in \mathbb{C}, Re(s) > 0$ we define the function $\psi(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s} = \sum_{-\infty}^{\infty} f_s(n)$, where $f_s(n) = e^{-\pi n^2 s}$.*

Theorem.

$$\psi(s) = \frac{1}{\sqrt{s}}\psi(\frac{1}{s}), \forall s \in \mathbb{C}, Re(s) > 0 \tag{20}$$

where $\sqrt{s}$ is taken over the branch that makes $\sqrt{s}$ a real positive number when s is a real positive number.

Proof. Due to Poisson summation formula $\psi(s) = \sum_{-\infty}^{\infty} \widehat{f}_s(n)$. So in order to apply Poisson summation formula we calculate the Fourier coefficients

$$\begin{aligned}\widehat{f}_s(n) &= \int_{-\infty}^{\infty} e^{-\pi x^2 s} e^{-2\pi i n x} dx = e^{-\frac{\pi n^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s (x + \frac{in}{s})^2} dx \\ &= e^{-\frac{\pi n^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s z^2} dz = \frac{e^{-\frac{\pi n^2}{s}}}{\sqrt{\pi s}} \int_{-\infty}^{\infty} e^{-t^2} dt.\end{aligned} \tag{21}$$

It is well known that the value of the last integral is $\sqrt{\pi}$ and therefore

$$\widehat{f}_s(n) = \frac{e^{-\frac{\pi n^2}{s}}}{\sqrt{s}}. \tag{22}$$

So now we are in position to apply Poisson summation formula and obtain

$$\psi(s) = \sum_{-\infty}^{\infty} f_s(n) = \sum_{-\infty}^{\infty} \widehat{f}_s(n) = \sum_{-\infty}^{\infty} \frac{e^{-\frac{\pi n^2}{s}}}{\sqrt{s}} = \frac{1}{\sqrt{s}} \sum_{-\infty}^{\infty} e^{-\frac{\pi n^2}{s}} = \frac{1}{\sqrt{s}}\psi(\frac{1}{s}). \tag{23}$$

□

2.2.3 The Mellin Transform

Definition. *If it exists, the Mellin transform of a function $f : U \to \mathbb{C}, \mathbb{R} \subset U$ is defined as*

$$M(f,s) = \int_0^\infty f(x)x^{s-1}dx, \tag{24}$$

where the integral is taken over the positive real line.

Observation. *By the way we defined $\Gamma(s)$ in the previous section it is obvious that it is just the Mellin transform of e^{-x}.*

Definition. *A function $f : U \to \mathbb{C}, \mathbb{R} \subset U$ is said to be of type (α, β), if for all $s = \sigma + it$ with $\alpha < \sigma < \beta$ the integral*

$$M(f,s) = \int_0^\infty f(x)x^{s-1}dx \tag{25}$$

converges.

Let us study of what type the function e^{-x} is. Since

$$\int_0^\infty e^{-x}x^{\sigma+it-1}dx = \Gamma(\sigma+it) \tag{26}$$

converges whenever $\sigma > 0$, e^{-x} is of type $(0, \infty)$.

Mellin inversion theorem. *If f is of type (α, β), continuous, and for $\alpha < \sigma < \beta$ the function $x \to f(x)x^{\sigma-1}$ is of bounded total variation, then the Mellin transform can be inverted in the following way*

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M(f,s)x^{-s}ds. \tag{27}$$

See [6, p.122].

Notation: In this paper the path of integration of integrals as the one in Mellin inversion theorem have to be understood as

$$\int_{\sigma-i\infty}^{\sigma+i\infty} f(s)ds = \lim_{t\to\infty}\int_{\sigma-it}^{\sigma+it} f(s)ds. \tag{28}$$

We note that since f is of type (α, β), the function $M(f,s)x^{-s}$ is analytic on the strip $\alpha < \sigma < \beta$ and by Cauchy's theorem the value of the integral does not depend on σ.
In particular, since $\Gamma(s)$ is the Mellin transform of e^{-x} and e^{-x} is of type $(0, \infty)$ we have

$$e^{-x} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)x^{-s}ds, \tag{29}$$

for any $\sigma > 0$. We will use identity 29 in the proof of Riemann's functional equation.

2.2.4 Riemann's functional equation for $\zeta(s)$

In this subsection we will find an analytic continuation of $\zeta(s)$ and present and prove Riemann's functional equation.

Theorem. *For $s \in \mathbb{C}$ with $Re(s) > 1$ the following equality holds*

$$\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}} = -\frac{1}{s} - \frac{1}{1-s} + \frac{1}{2}\int_1^\infty (\psi(v)-1)(v^{\frac{1-s}{2}-1} + v^{\frac{s}{2}-1})dv. \quad (30)$$

Proof. We have seen, that $\Gamma(s)$ is the Mellin transform of e^{-x}. Inverting the Mellin transform, which we can do with any $\sigma > 0$, and making the change of variable $x = \pi n^2 u$, we obtain

$$e^{-\pi n^2 u} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)(\pi n^2 u)^{-s} ds, \forall \sigma > 0. \quad (31)$$

Since $e^{-\pi n^2 u} = e^{-\pi(-n)^2 u}$,

$$\begin{aligned}\psi(u) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u} = \sum_{n=-\infty}^{-1} e^{-\pi n^2 u} + 1 + \sum_{n=1}^{\infty} e^{-\pi n^2 u} \\ &= 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 u} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)(\pi n^2 u)^{-s} ds, \forall \sigma > 0.\end{aligned} \quad (32)$$

But as the sum and the integral converge absolutely, we can change the order. With the restriction $\sigma > 1$ we have

$$\begin{aligned}\psi(u) &= 1 + \frac{2}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)(\sum_{n=1}^{\infty} n^{-2s})(\pi u)^{-s} ds \\ &= 1 + \frac{2}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\zeta(2s)(\pi u)^{-s} ds\end{aligned} \quad (33)$$

and

$$\psi(u) - 1 = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} 2\Gamma(s)\zeta(2s)\pi^{-s}u^{-s} ds, \quad (34)$$

which tells us that $2\Gamma(s)\zeta(2s)\pi^{-s}$ is the Mellin transform of $\psi(u) - 1$ (applying Mellin inversion theorem).
So by the definition of the Mellin transform

$$2\Gamma(s)\zeta(2s)\pi^{-s} = \int_0^\infty (\psi(u)-1)u^{s-1} du. \quad (35)$$

With the simple change of variable $s = z/2$ we obtain

$$
\begin{aligned}
2\Gamma(\frac{z}{2})\zeta(z)\pi^{-\frac{z}{2}} &= \int_0^\infty (\psi(u)-1)u^{\frac{z}{2}-1}du \\
&= \int_0^1 \psi(u)u^{\frac{z}{2}-1}du - \int_0^1 u^{\frac{z}{2}-1}du + \int_1^\infty (\psi(u)-1)u^{\frac{z}{2}-1}du \\
&= \int_0^1 \psi(u)u^{\frac{z}{2}-1}du - \frac{2}{z} + \int_1^\infty (\psi(u)-1)u^{\frac{z}{2}-1}du.
\end{aligned} \tag{36}
$$

Substituting $\psi(u) = \frac{1}{\sqrt{u}}\psi(\frac{1}{u})$ and taking the change of variable $u = \frac{1}{v}$ we have

$$
\begin{aligned}
\int_0^1 \psi(u)u^{\frac{z}{2}-1}du &= \int_0^1 \psi(\frac{1}{u})u^{\frac{z}{2}-\frac{3}{2}}du \\
&= \int_1^\infty \psi(v)v^{-\frac{z}{2}-\frac{1}{2}}dv \\
&= \int_1^\infty (\psi(v)-1)v^{-\frac{z}{2}-\frac{1}{2}}dv + \int_1^\infty v^{-\frac{z}{2}-\frac{1}{2}}dv \\
&= \int_1^\infty (\psi(v)-1)v^{\frac{1-z}{2}-1}dv - \frac{2}{1-z}
\end{aligned} \tag{37}
$$

and consequently

$$
\begin{aligned}
&\Gamma(\frac{z}{2})\zeta(z)\pi^{-\frac{z}{2}} = \frac{1}{2}\int_0^1 \psi(u)u^{\frac{z}{2}-1}du - \frac{1}{2}\frac{2}{z} + \frac{1}{2}\int_1^\infty (\psi(u)-1)u^{\frac{z}{2}-1}du \\
&= \frac{1}{2}\int_1^\infty (\psi(v)-1)v^{\frac{1-z}{2}-1}dv - \frac{1}{2}\frac{2}{1-z} - \frac{1}{z} + \frac{1}{2}\int_1^\infty (\psi(u)-1)u^{\frac{z}{2}-1}du \\
&= -\frac{1}{z} - \frac{1}{1-z} + \frac{1}{2}\int_1^\infty (\psi(v)-1)(v^{\frac{1-z}{2}-1} + v^{\frac{z}{2}-1})dv.
\end{aligned} \tag{38}
$$

□

Theorem. *There exists an analytic continuation of $\zeta(s)$ to the whole complex plane.*

Proof. Since $\psi(v) - 1 = O(e^{-\pi v})$ as $v \to +\infty$,[1] the integral

$$
\int_1^\infty (\psi(v)-1)(v^{\frac{1-s}{2}-1} + v^{\frac{s}{2}-1})dv \tag{39}
$$

converges absolutely and uniformly on the half-plane $Re(s) > x$ for any x.

[1] We write $f(x) = O(g(x))$ as $x \to \infty$, if there exist positive real constants C and x_0, such that $|f(x)| < C\,|g(x)|\,, \forall x > x_0$.

Therefore we can express the integral as

$$
\begin{aligned}
\int_1^\infty (\psi(v)-1)(v^{\frac{1-s}{2}-1}+v^{\frac{s}{2}-1})dv &= \int_1^\infty (\lim_{N\to\infty} 2\sum_{n=1}^N e^{-\pi n^2 v})(v^{\frac{1-s}{2}-1}+v^{\frac{s}{2}-1})dv \\
&= \lim_{N\to\infty} 2\sum_{n=1}^N \int_1^\infty e^{-\pi n^2 v}(v^{\frac{1-s}{2}-1}+v^{\frac{s}{2}-1})dv \\
&= \lim_{N\to\infty} f_N(s),
\end{aligned} \tag{40}
$$

with each $f_N(s)$ being holomorphic. Now applying theorem A.1 we see that

$$\int_1^\infty (\psi(v)-1)(v^{\frac{1-s}{2}-1}+v^{\frac{s}{2}-1})dv \tag{41}$$

is holomorphic on $\mathbb{C}$. Consequently

$$\frac{\pi^{s/2}}{\Gamma(s/2)}\left(-\frac{1}{s}-\frac{1}{1-s}+\frac{1}{2}\int_1^\infty (\psi(v)-1)(v^{\frac{1-s}{2}-1}+v^{\frac{s}{2}-1})dv\right) \tag{42}$$

is a meromorphic function on the whole complex plane. Since for s with $Re(s)>1$ it takes the same values as $\zeta(s)$, it provides an analytic continuation of $\zeta(s)$. □

An important point about 30 is, that its right hand side has the same value in s and $1-s$. That implies

Corollary (Riemann's functional equation). *For all $s\in\mathbb{C}$*

$$\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}} = \Gamma(\frac{1-s}{2})\zeta(1-s)\pi^{-\frac{1-s}{2}}. \tag{43}$$

Proof.

$$
\begin{aligned}
\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}} &= -\frac{1}{s}-\frac{1}{1-s}+\frac{1}{2}\int_1^\infty (\psi(v)-1)(v^{\frac{1-s}{2}-1}+v^{\frac{s}{2}-1})dv \\
&= -\frac{1}{1-s}-\frac{1}{s}+\frac{1}{2}\int_1^\infty (\psi(v)-1)(v^{\frac{s}{2}-1}+v^{\frac{1-s}{2}-1})dv \\
&= \Gamma(\frac{1-s}{2})\zeta(1-s)\pi^{-\frac{1-s}{2}}
\end{aligned} \tag{44}
$$

. □

Definition. *We define Riemann's ξ-function as*

$$\xi(s) := \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}}, \forall s\in\mathbb{C}. \tag{45}$$

Note. *Using $\xi(s)$, Riemann's functional equation can be expressed as*

$$\xi(s)=\xi(1-s), \forall s\in\mathbb{C}. \tag{46}$$

2.3 Extending $\zeta(s)$ via $\eta(s)$

In this section we are going to introduce the Dirichlet eta-function $\eta(s)$ and discuss its relation with $\zeta(s)$.

Proposition. *The sum*

$$\sum_{n=1}^{\infty} n^{-s}(-1)^{n+1} \tag{47}$$

is uniformly convergent in any closed subset of $Re(s) > 0$.

Proof. When s_0 is a positive real number, n^{-s_0} is a strictly decreasing and real sequence that tends to zero and therefore $\sum_{n=1}^{\infty} n^{-s_0}(-1)^{n+1}$ converges. Applying theorem A.4 we conclude the proof. □

Definition. *For $s \in \mathbb{C}$ with $Re(s) > 0$ we define the Dirichlet η-function as*

$$\eta(s) = \sum_{n=1}^{\infty} n^{-s}(-1)^{n+1}. \tag{48}$$

Proposition. *$\zeta(s)$ can be extended to $Re(s) > 0$ via*

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}. \tag{49}$$

Proof. For s with $Re(s) > 1$ we have

$$\begin{aligned}
(1 - 2^{1-s})\zeta(s) &= \zeta(s) - 2^{1-s}\zeta(s) \\
&= \sum_{n=1}^{\infty} n^{-s} - 2^{1-s}\sum_{n=1}^{\infty} n^{-s} \\
&= \sum_{n=1}^{\infty} n^{-s} - 2\sum_{n=1}^{\infty} (2n)^{-s} \\
&= 1 - 2^{-s} + 3^{-s} - 4^{-s} + ... \\
&= \sum_{n=1}^{\infty} n^{-s}(-1)^{n+1} := \eta(s).
\end{aligned} \tag{50}$$

However $\eta(s)$ converges for every s with $Re(s) > 0$ and consequently

$$\zeta(s) = \frac{\sum_{n=1}^{\infty} n^{-s}(-1)^{n+1}}{1 - 2^{1-s}} = \frac{\eta(s)}{1 - 2^{1-s}} \tag{51}$$

gives an analytic continuation of $\zeta(s)$ to $Re(s) > 0$. □

Proposition. *Besides a simple pole in $s = 1$, $\zeta(s)$ has no other poles.*

Proof. Since $\eta(1) = \sum_{n=1}^{\infty} n^{-1}(-1)^{n+1} = log(2) \neq 0$ and $\frac{1}{1-2^{1-s}}$ has a simple pole in $s = 1$ we deduce that $\zeta(s) = \eta(s)\frac{1}{1-2^{1-s}}$ has a simple pole in $s = 1$.
Furthermore as $\frac{1}{1-2^{1-s}}$ has no other poles in $Re(s) > 0$ and $\eta(s) = \sum_{n=1}^{\infty} n^{-s}(-1)^{n+1}$ converges for any s with $Re(s) > 0$, we deduce that $\zeta(s)$ has no other poles in $Re(s) > 0$ and due to Riemann's functional equation it has no other poles on the whole complex plane. □

As this expression of $\zeta(s)$ converges on the critical strip ($0 < Re(s) < 1$) and it is 'only' an infinite sum, which is in a certain way easier to approximate numerically than the infinite products or improper integrals which arise in other analytic continuations of $\zeta(s)$, it could be a candidate to approximate values of $\zeta(s)$ on the critical strip.
Another good thing about this expression is that for real x with $0 < x < 1$, the series $\sum_{n=1}^{N} n^{-x}(-1)^{n+1}$ alternates and therefore it complies

$$\sum_{n=1}^{2N} n^{-x}(-1)^{n+1} < \eta(x) < \sum_{n=1}^{2N+1} n^{-x}(-1)^{n+1} \tag{52}$$

which gives a bound for the error of approximating $\eta(x)$ (and therefore $\zeta(x)$) by $\sum_{n=1}^{2N+1} n^{-x}(-1)^{n+1}$.

Lemma. *Suppose $Re(s) > 0$. Then*

$$\sum_{n\equiv 1(mod2)} \left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right] = a < \infty \tag{53}$$

if and only if

$$\sum_{n\in\mathbb{N}} \left[\frac{1}{n^s}(-1)^{n+1}\right] = b < \infty. \tag{54}$$

In that case $a = b$.

Proof. If

$$\sum_{n\equiv 1(mod2)} \frac{1}{n^s} - \frac{1}{(n+1)^s} = a < \infty \tag{55}$$

for any ϵ there exists a natural $N(\frac{\epsilon}{2})$ such that

$$\left| a - \sum_{n\equiv 1(mod2), n\leq K} \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| < \frac{\epsilon}{2} \tag{56}$$

for any natural $K \geq N(\frac{\epsilon}{2})$. Now since $Re(s) > 0$, $\frac{1}{(n+1)^s}$ tends to 0 as n goes to infinity. Therefore we can find $N'(\frac{\epsilon}{2})$ such that $\left|\frac{1}{(n+1)^s}\right| < \frac{\epsilon}{2}$ for any $n > N'(\frac{\epsilon}{2})$.

So if $K \geq N(\frac{\epsilon}{2}), N'(\frac{\epsilon}{2})$ we have:
case 1: If $K = 2k + 1$

$$\left| a - \sum_{n \leq 2k+1} \frac{1}{n^s}(-1)^{n+1} \right| = \left| a - \left(\frac{1}{(2k+1)^s} + \sum_{n \equiv 1 (mod 2), n \leq 2k} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right) \right|$$
$$\leq \left| a - \sum_{n \equiv 1 (mod 2), n \leq 2k} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right| + \left| \frac{1}{(2k+1)^s} \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \tag{57}$$

case 2: If K=2k

$$\left| a - \sum_{n \leq 2k} \frac{1}{n^s}(-1)^{n+1} \right| = \left| a - \sum_{n \equiv 1 (mod 2), n \leq 2k} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right| \leq \frac{\epsilon}{2}. \tag{58}$$

$\square$

Now we will see a bound for the error, when estimating $\eta(s)$ for any complex number s with $Re(s) > 0$ by the sum of its first 2N terms:

$$\eta(s) - \sum_{n=1}^{2N} \frac{1}{n^s}(-1)^{n+1} = \sum_{n \geq 2N+1} \frac{1}{n^s}(-1)^{n+1} = \sum_{n \geq 2N+1, n \equiv 1 (mod 2)} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right). \tag{59}$$

So we will search an upper bound for $\left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right|$.

Lemma. *If $s = a + bi, a > 0$ and $-1 < log(\frac{n}{n+1}) |b| < 0$, then*

$$\left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| < \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right) + \frac{\sqrt{2}\,|b|}{n^{1+a}}. \tag{60}$$

Proof.

$$\begin{aligned}
&\left|\frac{1}{n^s}-\frac{1}{(n+1)^s}\right|^2=\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\overline{\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)}\\
&=\left(\frac{(n+1)^s-n^s}{(n(n+1))^s}\right)\overline{\left(\frac{(n+1)^s-n^s}{(n(n+1))^s}\right)}\\
&=\left(\frac{(n+1)^{a+bi}-n^{a+bi}}{(n(n+1))^{a+bi}}\right)\left(\frac{(n+1)^{a-bi}-n^{a-bi}}{(n(n+1))^{a-bi}}\right)\\
&=\left(\frac{(n+1)^{a+bi}(n+1)^{a-bi}+n^{a+bi}n^{a+bi}-n^{a+bi}(n+1)^{a-bi}-n^{a-bi}(n+1)^{a+bi}}{(n(n+1))^{2a}}\right)\\
&=\left(\frac{(n+1)^{2a}+n^{2a}-(n(n+1))^a\left[(\frac{n}{n+1})^{bi}+(\frac{n}{n+1})^{-bi}\right]}{(n(n+1))^{2a}}\right)\\
&=\left(\frac{(n+1)^{2a}+n^{2a}-(n(n+1))^a\left[2Re((\frac{n}{n+1})^{bi}))\right]}{(n(n+1))^{2a}}\right)\\
&=\left(\frac{(n+1)^{2a}+n^{2a}-(n(n+1))^a\left[2cos(log(\frac{n}{n+1})b)\right]}{(n(n+1))^{2a}}\right)=(*)
\end{aligned} \tag{61}$$

Now we are going to use Taylor series in order to find a lower bound of $cos(log(\frac{n}{n+1})b)$ and therefore an upper bound of (*).
For $-1<x<1$ we have

$$cos(x)=1-x^2+x^4-x^6...>1-x^2. \tag{62}$$

Furthermore for $n\geq 1$

$$log(\frac{n}{n+1})=-log(1+\frac{1}{n})=-\left(\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n}\right)^2+\frac{1}{3}\left(\frac{1}{n}\right)^3...\right)>-\frac{1}{n}. \tag{63}$$

Since $cos(x)$ is strictly increasing in $(-1,0)$ and by hypothesis $-1<log(\frac{n}{n+1})\,|b|<0$, we have

$$cos(log(\frac{n}{n+1})b)>cos(-\frac{b}{n})>1-\frac{b^2}{n^2}. \tag{64}$$

If we plug this inequality into (*) we obtain

$$\left|\frac{1}{n^s}-\frac{1}{(n+1)^s}\right|^2 < \left(\frac{(n+1)^{2a}+n^{2a}-(n(n+1))^a\left[2\left(1-\frac{b^2}{n^2}\right)\right]}{(n(n+1))^{2a}}\right)$$
$$=\frac{\left(((n+1)^a-n^a)^2+2(n(n+1))^a\frac{b^2}{n^2}\right)}{(n(n+1))^{2a}} \tag{65}$$
$$=\left(\frac{(n+1)^a-n^a}{(n(n+1))^a}\right)^2+\frac{2b^2}{(n(n+1))^a\, n^2}$$
$$\leq\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)^2+\frac{2b^2}{(n(n))^a\, n^2}$$

Since $\leq \left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)^2$ and $\frac{2b^2}{(n(n))^a n^2}$ are both positive and $\sqrt{x}$ is an increasing and concave function for real positive x

$$\left|\frac{1}{n^s}-\frac{1}{(n+1)^s}\right| < \sqrt{\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)^2+\frac{2b^2}{(n(n))^a\, n^2}}$$
$$<\sqrt{\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)^2}+\sqrt{\frac{2b^2}{(n(n))^a\, n^2}} \tag{66}$$
$$=\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)+\frac{\sqrt{2}\,|b|}{n^{1+a}}$$

□

We define $\eta_N(s) := \sum_{n=1}^{N}\frac{1}{n^s}(-1)^{n+1}$ and $\zeta_N(s)=\sum_1^N\frac{1}{n^s}$

Proposition. *If $s=a+bi, a>0$ and $-1<log(\frac{N}{N+1})\,|b|\leq 0$, then*

$$|\eta(s)-\eta_{2N}(s)| < \eta(a)-\eta_{2N}(a)+\sqrt{2}\,|b|\,(\zeta(1+a)-\zeta_{2N}(1+a)) \tag{67}$$

Proof.

$$
\begin{aligned}
|\eta(s)-\eta_{2N}(s)| &= \left| \sum_{n\geq 2N+1, n\equiv 1(mod2)} \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| \\
&\leq \sum_{n\geq 2N+1, n\equiv 1(mod2)} \left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| \\
&< \sum_{n\geq 2N+1, n\equiv 1(mod2)} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right) + \frac{\sqrt{2}\,|b|}{n^{1+a}} \\
&= \sum_{n\geq 2N+1, n\equiv 1(mod2)} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right) + \sum_{n\geq 2N+1, n\equiv 1(mod2)} \frac{\sqrt{2}\,|b|}{n^{1+a}} \\
&< \sum_{n\geq 2N+1, n\equiv 1(mod2)} \left(\frac{1}{n^a} - \frac{1}{(n+1)^a} \right) + \sum_{n\geq 2N+1} \frac{\sqrt{2}\,|b|}{n^{1+a}} \\
&= \eta(a) - \eta_{2N}(a) + \sqrt{2}\,|b| \left(\zeta(1+a) - \zeta_{2N}(1+a)\right).
\end{aligned}
\tag{68}
$$

The inequality between the second and the third line comes from the previous Lemma. □

We note that since $log(\frac{N}{N+1})$ is negative and tends to zero when N goes to infinity, given b we can always chose N sufficiently large in order to have $-1 < log(\frac{N}{N+1})\,|b| \leq 0$.

3 Zeros of $\zeta(s)$

3.1 Weierstrass infinite products

Notation: Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$. We write

$$f(R) <<_\epsilon F(R,\epsilon) \tag{69}$$

if for any ϵ there exists C_ϵ such that

$$f(R) < C_\epsilon F(R,\epsilon), \forall R. \tag{70}$$

Theorem. *Suppose $g(s)$ is an entire function and not identically 0. Let Ω denote the set of zeros of $g(s)$ and v_α the multiplicity of the zero α (let $v_0 = 0$ if g has no zero in $s = 0$). Suppose also that*

$$log\left(\max_{|s|=R} |g(s)|\right) <<_\epsilon (R+1)^{1+\epsilon}. \tag{71}$$

Then:

$$g(s) = s^{v_0} e^{A+Bs} \prod_{\alpha\in\Omega, \alpha\neq 0} \left(\left(1-\frac{s}{\alpha}\right) e^{\frac{s}{\alpha}} \right)^{v_\alpha} \tag{72}$$

A proof of this theorem is given in $[4, p.328]$.
The previous theorem does not apply directly to $\zeta(s)$ since $\zeta(s)$ is not an entire function. However we know that $\zeta(s)$ has only one pole ($s = 1$) and we have also seen that this pole has order 1. Hence $(s-1)\zeta(s)$ is an entire function. In $[4, p.20]$ it is shown that the second condition of the previous theorem is also met and therefore we can write

$$\zeta(s) = \frac{s^{v_0}}{s-1} e^{A+Bs} \prod_{\alpha \in \Omega, \alpha \neq 0} \left(\left(1 - \frac{s}{\alpha}\right) e^{\frac{s}{\alpha}} \right)^{v_\alpha}, \tag{73}$$

where Ω denotes the set of zeros of $\zeta(s)$. Since $\zeta(0) = -\frac{1}{2} \neq 0$ we can specify A and v_0 from the previous expression to obtain

$$\zeta(s) = \frac{1}{2(s-1)} e^{Bs} \prod_{\alpha \in \Omega, \alpha \neq 0} \left(\left(1 - \frac{s}{\alpha}\right) e^{\frac{s}{\alpha}} \right)^{v_\alpha}. \tag{74}$$

3.2 The Riemann hypothesis

Proposition. *If $s \in \mathbb{R}$ then $\zeta(s) \in \mathbb{R}$.*

Proof. For real s with $Re(s) > 0$ we have $\zeta(s) = \frac{\eta(s)}{1-2^{1-s}}$ and since $\eta(s)$ is a sum of real numbers, $\zeta(s)$ is real. Due to Riemann's functional equation $\zeta(s)$ is real whenever s is real. $\square$

Proposition. *For all $s \in \mathbb{C}$ we have $\zeta(\overline{s}) = \overline{\zeta(s)}$.*

Proof. Apply Schwarz reflection principle (see theorem A.4) to $\zeta(s)(s-1) = p(s)$ (which is an entire function, since the only pole of $\zeta(s)$ is a simple one in $s = 1$). $\square$

Proposition. *$\zeta(s)$ has simple zeros at the negative even integers and nowhere else on the real line. These zeros are called the **trivial zeros** of $\zeta(s)$.*

Proof. From Riemann's functional equation we know that

$$\Gamma(\frac{z}{2})\zeta(z)\pi^{-\frac{z}{2}} = \Gamma(\frac{1-z}{2})\zeta(1-z)\pi^{-\frac{1-z}{2}}. \tag{75}$$

If z is a negative even integer then the right hand side of the previous equation takes a finite value different from 0, since each of the terms $\Gamma(\frac{1-z}{2}), \zeta(1-z)$ and $\pi^{-\frac{1-z}{2}}$ takes a finite value different from 0. But since Γ has simple poles at each negative integer (but nowhere else on the real line), $\Gamma(\frac{z}{2})$ has simple poles at the negative even integers and consequently ζ has simple zeros at the negative even integers (and nowhere else on the real line). $\square$

In the next theorem we will see that besides the trivial zeros $\zeta(s)$ has more zeros. They are called the **non-trivial zeros**.

Theorem. *ζ has only zeros at the negative even integers and in a set of complex numbers α_n that lie in the critical strip ($0 < Re(s) < 1$). Furthermore the zeros on the critical strip are situated symmetrically with respect to the real line and to $Re(s) = \frac{1}{2}$.*

Proof. Define the Möbius function $\mu(s)$ by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 ... p_k, \text{ and } p_1, ..., p_k \text{ are distinct primes} \\ 0 & \text{otherwise.} \end{cases} \tag{76}$$

From the Euler-product we know that

$$\frac{1}{\zeta(s)} = \prod_{p \in P} (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{77}$$

and therefore

$$\left| \frac{1}{\zeta(s)} \right| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| < \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < 1 + \sum_{n=1}^{\infty} \int_n^{n+1} x^{-\sigma} dx = 1 + \int_1^{\infty} x^{-\sigma} dx = \frac{\sigma}{\sigma - 1} \tag{78}$$

From the previous inequality we deduce that $\zeta(s)$ has no zeros with $Re(s) > 1$ and due to Riemann's functional equation it has no zeros with $Re(s) < 0$, except for the zeros at the negative even integers.
Now we need to proof the symmetries of the zeros.
Since $\overline{\zeta(s)} = \zeta(\overline{s})$, s is a zero if and only if $\overline{s}$ is a zero. Finally, due to Riemann's functional equation $1 - \overline{s}$ (the reflection of s on $Re(z) = \frac{1}{2}$) is a zero too. □

The Riemann hypothesis. *If s is a non-trivial zero of $\zeta(s)$, then $Re(s) = \frac{1}{2}$.*

At the present, the Riemann hypothesis is considered one of the most important unsolved problems in mathematics, as its verification would imply a lot of other results in mathematics, especially in number theory.

3.3 Some results about the distribution of the zeros of $\zeta(s)$

From now on $log(s)$ is always taken in such a way, that it is a real number, when s is a real positive number.

Theorem. *Let Ω be the set of zeros of $\zeta(s)$. Then the following equation holds*

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + B + \sum_{\alpha \in \Omega, Im(\alpha) \neq 0} \left(\frac{1}{s - \alpha} + \frac{1}{\alpha} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{s + 2n} - \frac{1}{2n} \right) \tag{79}$$

Proof. We have seen that

$$\zeta(s) = \frac{1}{2(s-1)} e^{Bs} \prod_{\alpha \in \Omega} \left(\left(1 - \frac{s}{\alpha} \right) e^{\frac{s}{\alpha}} \right)^{v_\alpha} \tag{80}$$

and therefore

$$\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} = (log(\zeta(s)))' &= \left(log\left(\frac{1}{2(s-1)}e^{Bs}\prod_{\alpha\in\Omega,\alpha\neq 0}\left(\left(1-\frac{s}{\alpha}\right)e^{\frac{s}{\alpha}}\right)\right)\right)' \\
&= \left(c' - log(s-1) + Bs + \sum_{\alpha\in\Omega} log\left(1-\frac{s}{\alpha}\right) + \frac{s}{\alpha}\right)' \\
&= \frac{1}{1-s} + B + \sum_{\alpha\in\Omega}\frac{1}{s-\alpha} + \frac{1}{\alpha} \\
&= \frac{1}{1-s} + B + \sum_{\alpha\in\Omega, Im(\alpha)\neq 0}\left(\frac{1}{s-\alpha}+\frac{1}{\alpha}\right) + \sum_{\alpha\in\Omega, Im(\alpha)=0}\left(\frac{1}{s-\alpha}+\frac{1}{\alpha}\right) \\
&= \frac{1}{1-s} + B + \sum_{\alpha\in\Omega, Im(\alpha)\neq 0}\left(\frac{1}{s-\alpha}+\frac{1}{\alpha}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right).
\end{aligned} \tag{81}$$

The last step is justified, because we know that the real zeros of $\zeta(s)$ are precisely the negative, even integers. □

Theorem. *Let Ω be the set of zeros of $\zeta(s)$ and $T \geq 2$. Then*

$$\sum_{\alpha\in\Omega, Im(\alpha)\neq 0}\frac{1}{1+(T-Im(\alpha))^2} < c\log(T). \tag{82}$$

For a proof see [4, p.23].

Definition. *We define the function N(T) as*

$$N(T) = \left|\{\alpha\in\Omega : 0 \leq Re(\alpha) \leq 1, 0 \leq Im(\alpha) \leq T\}\right|. \tag{83}$$

Corollary. *There exists an absolute constant c such that $N(T+1) - N(T) \leq c\log(T)$ when $T \geq 2$.*

Proof. For $\alpha \in \Omega$ with $T \leq Im(\alpha) \leq T+1$ we have

$$\frac{1}{2} = \frac{1}{1+1} \leq \frac{1}{1+(T-Im(\alpha))^2}. \tag{84}$$

Hence

$$\begin{aligned}
(N(T+1)-N(T)) &= 2\sum_{\alpha\in\Omega, T\leq Im(\alpha)\leq T+1}\frac{1}{2} \\
&\leq 2\sum_{\alpha\in\Omega, T\leq Im(\alpha)\leq T+1}\frac{1}{1+(T-Im(\alpha))^2} \\
&\leq 2\sum_{\alpha\in\Omega, Im(\alpha)\neq 0}\frac{1}{1+(T-Im(\alpha))^2} \\
&\leq 2c\log(T).
\end{aligned} \tag{85}$$

The last inequality is due to the previous theorem. □

The theorem of De la Vallée-Poussin bounding the zeros of $\zeta(s)$. *There exists a positive real constant c such that there are no zeros in the region:*

$$Re(s) > 1 - \frac{c}{log(|Im(s)|+2)}. \tag{86}$$

For a proof see [4, p.25].

This theorem does not tell us a lot about zero-free regions in the interior of the critical strip ($0 < Re(s) < 1$) since c could be arbitrary small. However it tells us that there are no zeros with $Re(s) = 1$, which is essential in the proof of the prime-number theorem.

3.4 Numerical computation of zeros of $\zeta(s)$

3.4.1 Using Euler-Maclaurin summation to compute values of $\zeta(s)$

Definition. *We define the Bernoulli functions, $P_n(x)$, recursively.*

$$\begin{aligned} &P_0(x) = 1, P_1(x) = x - [x] - \frac{1}{2}, \\ &P_{n-1}(x) = \frac{1}{n} P'_n(x), \\ &\int_k^{k+1} P_n(x)dx = 0, \forall k \in \mathbb{Z}, \end{aligned} \tag{87}$$

where $[x]$ denotes the greatest integer that is smaller or equal to x.

Note that the condition $P_{n-1}(x) = \frac{1}{n}P'_n(x)$ determines $P_n(x)$ up to a constant c, which is fixed by the condition $\int_k^{k+1} P_n(x)dx = 0$.
Since $P_1(x)$ is periodic (the value in x and $x+1$ are obviously the same), the two previous conditions let $P_n(x)$ be periodic too.

Definition. *We define the Bernoulli numbers as $B_n := P_n(0)$.*

The previous definition is equivalent to $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \cdot \frac{x^n}{n!}$. Furthermore It can be shown that $B_3 = B_5 = B_7 = ... = 0$.

The Euler-McLaurin summation formula. *Let $f(x)$ be a differentiable function and $k \in \mathbb{Z}, q \in \mathbb{N}$. The following equation holds*

$$\int_k^{k+1} f(x)dx = \frac{1}{2}(f(k+1)+f(k)) - \sum_{m \leq \frac{(q+1)}{2}} \frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(k+1) - f^{(2m-1)}(k)\right] + R_q \tag{88}$$

where

$$R_q = \frac{(-1)^{q+1}}{(q+1)!} \int_k^{k+1} f^{(q+1)}(x) P_{q+1}(x)dx. \tag{89}$$

See [3, p.119].

Theorem. *For $s \in \mathbb{C}$, $\widehat{n} \in \mathbb{N}$ the following holds*

$$\zeta(s) = \sum_{k=1}^{\widehat{n}} k^{-s} + \frac{\widehat{n}^{1-s}}{s-1} - \frac{\widehat{n}^{-s}}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} s(s+1)(s+2)...(s+2m-2)\widehat{n}^{-s-2m+1} \tag{90}$$

Proof. Applying Euler-McLaurin summation formula to the function $f(x) = x^{-s}$ we obtain

$$\int_k^{k+1} x^{-s}dx = \frac{1}{2}((k+1)^{-s}+k^{-s}) - \sum_{m \le \frac{(q+1)}{2}} \frac{B_{2m}}{(2m)!} \left\{ \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{x=k+1} - \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{x=k} \right\} + R_q \tag{91}$$

where

$$R_q = \frac{(-1)^{q+1}}{(q+1)!} \int_k^{k+1} \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right] P_{q+1}(x)dx. \tag{92}$$

Since

$$\lim_{q\to\infty} R_q = \lim_{q\to\infty} \frac{(-1)^{q+1}}{(q+1)!} \int_k^{k+1} \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right] P_{q+1}(x)dx = 0 \tag{93}$$

we have

$$\int_k^{k+1} x^{-s}dx = \frac{1}{2}((k+1)^{-s}+k^{-s}) - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \left\{ \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{x=k+1} - \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{x=k} \right\}. \tag{94}$$

Summing this expression over all $k \ge \widehat{n}$ we obtain

$$\begin{aligned} \frac{\widehat{n}^{1-s}}{1-s} &= \int_{\widehat{n}}^{\infty} x^{-s}dx = \sum_{k=\widehat{n}}^{\infty} \int_k^{k+1} x^{-s}dx \\ &= \sum_{k=\widehat{n}+1}^{\infty} k^{-s} + \frac{1}{2}\widehat{n}^{-s} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{x=\widehat{n}} \\ &= \zeta(s) - \sum_{k=1}^{\widehat{n}} k^{-s} + \frac{1}{2}\widehat{n}^{-s} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{x=\widehat{n}}. \end{aligned} \tag{95}$$

Substituting

$$\left[\frac{d^{(2m-1)}}{dx^{(2m-1)}}(x^{-s})\right]_{\widehat{n}} = s(s+1)(s+2)...(s+2m-2)\widehat{n}^{-s-2m+1} \tag{96}$$

and rearanging the terms we obtain the result. ∎

3.4.2 Finding zeros on the critical line

Proposition. *$\xi(s)$ takes real values on the critical line.*

Proof. Because of Riemann's functional equation and Schwarz reflection principle we have

$$\xi(\frac{1}{2}+it) = \xi(1-(\frac{1}{2}+it)) = \xi(\frac{1}{2}-it) = \xi(\overline{\frac{1}{2}+it}) = \overline{\xi(\frac{1}{2}+it)}. \tag{97}$$

□

Observation. *Since $\xi(s)$ takes real values on the critical line, we can use much more simple methods to find its zeros.*

Definition. *For $t \in \mathbb{R}$ and $s = \frac{1}{2}+it$ we define*

$$\vartheta(t) := Im(log(\Gamma(\frac{s}{2}))) - \frac{t}{2}\log(\pi) \tag{98}$$

and

$$Z(t) := e^{iIm(log(\Gamma(\frac{s}{2})))}\pi^{-i\frac{t}{2}}\zeta(\frac{1}{2}+it) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it), \tag{99}$$

where $s = \frac{1}{2}+it$ and the $\log(s)$ is taken in such a way that it is real, when s is a real positive number.

Proposition. *For $t \in \mathbb{R}$, the sign of $\xi(\frac{1}{2}+it)$ is always opposite to the sign of $Z(t)$.*

Proof. If we set and $s = \frac{1}{2}+it$ we have

$$\begin{aligned}\xi(\frac{1}{2}+it) &:= \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}} = e^{(log(\Gamma(\frac{s}{2})))}\pi^{-i\frac{t}{2}}\frac{s(s-1)}{2}\zeta(\frac{1}{2}+it)\\ &= \left[e^{Re(log(\Gamma(\frac{s}{2})))}\pi^{-\frac{1}{4}}\frac{-t^2-\frac{1}{4}}{2}\right]Z(t),\end{aligned} \tag{100}$$

where the term in the brackets is always negative. □

So in order to see if there is a zero between two points on the critical line, we have to check the sign of $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$ at these points, but we do not need to know exact values of $Z(t)$. For that purpose the following approximation of $\vartheta(t)$ will be good enough.

Proposition. *For $t \in \mathbb{R}$*

$$\vartheta(t) = \frac{t}{2}\log(\frac{t}{2\pi}) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + e(t) \tag{101}$$

where $|e(t)| < \frac{7}{5760t^3} + \frac{2}{t^5}$.See [2, page.120].

3.4.3 Example

$\zeta(s)$ has a zero at $p \approx 0.5 + 14.156875i$.

To prove that result we will search a rough estimate of $Z(15) = e^{i\vartheta(15)}\zeta(\frac{1}{2}+15i)$ and check its sign. Therefore we will use the approximation of $\vartheta(t)$ of the previous proposition. To approximate $\zeta(\frac{1}{2}+15i)$ we will use the representation of $\zeta(s)$ that we derived from the Euler-Maclaurin summation formula (90), truncating the infinite series at j such that the difference between the estimates truncating at j and $j-1$ has absolute value less than 0.001. Clearly the greater we choose $\widehat{n}$ the faster the infinite series converges. We will set $\widehat{n} = 30$ and obtain

$$\begin{aligned} &\vartheta(15) \approx -1.36501 \\ &j = 1 : \zeta(\frac{1}{2} + 15i) \approx 0.147082 + 0.704736i \\ &j = 2 : \zeta(\frac{1}{2} + 15i) \approx 0.14711 + 0.704752i. \end{aligned} \tag{102}$$

Since the absolute value of the difference of the estimates that we obtained with $j = 1$ and $j = 2$ is less than 0.001, we will use the approximation of $\zeta(\frac{1}{2} + 15i)$ that we obtained with $j = 2$. With these approximations we obtain an approximation of $Z(15)$

$$Z(15) = \zeta(\frac{1}{2}+15i)e^{\vartheta(15)i} \approx (0.14711+0.704752i)e^{-1.36501i} \approx 0.719942-3.99976\cdot 10^{-7}i, \tag{103}$$

which can be interpreted as a positive real number.

Now we will do the same for $Z(14)$.

$$\begin{aligned} &\vartheta(14) \approx -1.78294 \\ &j = 1 : \zeta(\frac{1}{2} + 14i) \approx 0.0222604 - 0.10324i \\ &j = 2 : \zeta(\frac{1}{2} + 14i) \approx 0.0222413 - 0.103258i. \end{aligned} \tag{104}$$

Again we can use the approximation that we obtained with $j = 2$, since it is close enough to the one with $j = 1$. So we have

$$Z(14) = \zeta(\frac{1}{2}+14i)e^{\vartheta(14)i} \approx (0.02224-0.103258i)e^{-1.78294i} \approx -0.105626-1.32961\cdot 10^{-7}i, \tag{105}$$

which can be interpreted as a negative real number.
Since the sign of $Z(14)$ is opposite to the sign of $Z(15)$, there is at least one zero between $t = 14$ and $t = 15$.
Now we will search a smaller interval for that zero. Therefore we have to compute $Z(14.5) \approx 0.297351 - 1.18712 \cdot 10^{-7}i$, which has an opposite sign to $Z(14)$. We conclude that there has to be a zero between $t = 14$ and $t = 14.5$. We repeat

this process of dividing intervals until the intervals are smaller than 0.1.

$$\begin{aligned}
&Z(14.25) \approx 0.0922651 - 3.88103 \cdot 10^{-8} i \Rightarrow I = (14, 14.25) \\
&Z(14.125) \approx -0.00770742 + 3.32897 \cdot 10^{-9} i \Rightarrow I = (14.125, 14.25) \qquad (106) \\
&Z(14.18875) \approx 0.0430373 - 1.83389 \cdot 10^{-8} i \Rightarrow I = (14.125, 14.18875)
\end{aligned}$$

We could repeat this process until we have the accuracy that we want. For now we have the estimation

$$p \approx 0.5 + \frac{(14.125 + 14.18875)}{2} i = 0.5 + 14.156875i. \qquad (107)$$

The 10 first zeros that we find with this method with an accuracy of 0.00001 are

$$\begin{aligned}
p_1 &\approx 0.5 + 14,134725i \\
p_2 &\approx 0.5 + 21.022040i \\
p_3 &\approx 0.5 + 25.010858i \\
p_4 &\approx 0.5 + 30.424876i \\
p_5 &\approx 0.5 + 32.935061i \\
p_6 &\approx 0.5 + 37.586178i \\
p_7 &\approx 0.5 + 40.918719i \\
p_8 &\approx 0.5 + 43.327073i \\
p_9 &\approx 0.5 + 48.005151i \\
p_{10} &\approx 0.5 + 49.773832i.
\end{aligned}$$

Actually we know the first 20 zeros now, since the conjugate of each zero is a zero too.

3.4.4 The number of zeros in a given set

Considering a set $S_t = \{s \in \mathbb{C}, 0 < Re(s) < 1, 0 < Im(s) < t\}$ such that there are no zeros on its boundary, we will discuss the following question: If we have found n zeros in S_t using the same method as in the previous example, how do we know if that are all the zeros in S_t?

The answer is theoretically simple: Compare n with the number given by the argument principle

$$m = \int_{\partial\{S_t \setminus B(1,\epsilon)\}} \frac{\zeta'(s)}{\zeta(s)} ds. \qquad (108)$$

The interpretation of m is not that simple. We have to distinguish different possibilities:

1.) If $n = m$, then we have found all the zeros in S_t, since $\zeta(s)$ has no poles in the critical strip and therefore m is equal to the number of zeros in S_t.

2.) There are three possible reasons why $n < m$,
2.1) In one of the intervals there was a even number of zeros and therefore we did not detect a change of sign. This can be avoided by taking other intervals.
2.2) One of the zeros that we detected has multiplicity greater than one.
2.3) The Riemann-hypothesis is false and there is a zero beyond the critical line.

3.4.5 A simple program in C to compute $\zeta(s)$

Using the formula 90 we can make a program in C which computes $\zeta(s)$ over a grid:

```
#include <stdio.h>
#include <math.h>
#include <complex.h>

long double B[5]=
{0.1666666666666666666666666666666666666,
-0.0333333333333333333333333333333333333,
0.0238095238095238095238095238095238095,
-0.0333333333333333333333333333333333333,
0.0757575757575757575757575757575757575757};

void zeta(double a,double b){
int i, N=150;
long double complex q=0+0*I;
long double complex c=a+b*I;
long double complex S1=1+0*I, S2=0+0*I;
int F=2, m=1;
long double complex sf;
while(cabs(S1-S2)>0.0000000000001){
sf=1+0*I;
for(i=0; i<2*m-1;i++){
    sf=sf*(c+i);
}
     S1=S2;
     S2=S2+B[m]*sf*cpow(N,-c-2*m+1)/F;
     m=m+1;
     F=F*2*m*(2*m-1);
}

for(i=1; i<N+1 ; i++){
  q=q+cpow(i,-c);
}
q=q+cpow(N,1-c)/(c-1)+cpow(N,-c)/2+S2;
printf("%g %g %g %g \n",a, b, creal(q), cimag(q));
```

```
}

void print(int nx, int ny)
{

  double dx=0.01, dy=0.01;
  int i,j;
  for(j=0;j<ny+1;j++){
    for(i=0;i<nx+1;i++){
      zeta(dx*i,1+dy*j);
    }
    printf("\n");
  }

}

main()
{

 print(100,2000);
  return 0;
}.
```

With the data that we obtain from that program we can generate interesting plots in GnuPlot.

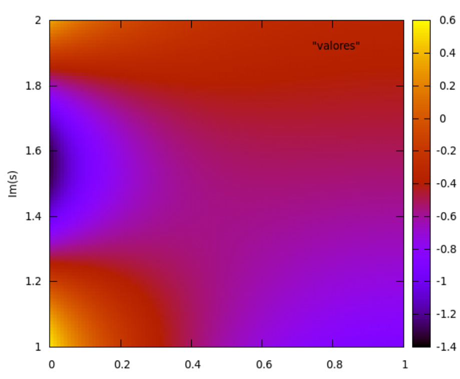
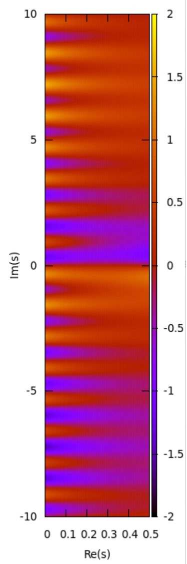
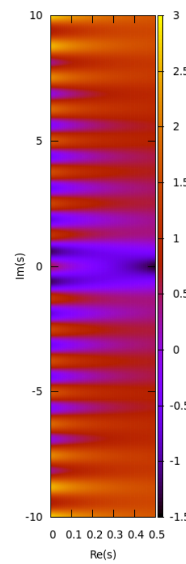


Figure 1: $Re(\zeta(s))$





-2cm (a) $Im(\zeta(s))$

(b) $Re(\zeta(s))$

Figure 3: (c) $|\zeta(s)|$

4 Universality of $\zeta(s)$

4.1 The universality theorem

In this section we will discuss a very interesting theorem, that states that any analytic function different from 0 can be approximated in a certain way by $\zeta(s)$. This property is called universality.

Theorem (Universality of $\zeta(s)$). *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is a function which is analytic for $|s| < r$ and continuous for $|s| \leq r$. If $f(s) \neq 0$ for $|s| < r$, then for any $\epsilon > 0$ there exists $T(\epsilon)$ such that*

$$\max_{|s|\leq r} |f(s) - \zeta(s + \frac{3}{4} + iT(\epsilon))| < \epsilon. \tag{109}$$

For a proof of this theorem see $[4, p.241]$.

Example 1 If we take the constant function $f(s) = \epsilon_1 \neq 0$, for any $\epsilon_2 > 0$ we can find a $T(\epsilon_2)$ such that

$$\max_{|s|\leq r} |\epsilon_1 - \zeta(s + \frac{3}{4} + iT(\epsilon_2))| < \epsilon_2, \tag{110}$$

which implies

$$\max_{|s|\leq r} |\zeta(s + \frac{3}{4} + iT(\epsilon_2))| < \epsilon_1 + \epsilon_2 := \epsilon. \tag{111}$$

Since ϵ_1 and ϵ_2 can be chosen arbitrary small, ϵ can be chosen arbitrary small too. That means that for an arbitrary small number, for example $\epsilon = \frac{1}{1000000}$, and an arbitrary $r < 0.25$, somewhere on the critical strip there exists a closed ball of radius r such that $|\zeta(s)| < \frac{1}{1000000}$ on this closed ball.

Example 2 For any function $f(s)$, analytic and zero-free on a closed ball of radius R, $\frac{f(\frac{s}{5R})}{5R}$ (which can be seen as a miniature-version of f) is analytic too, and therefore there exists a closed ball of radius $\frac{1}{5}$, such that ζ is very close to $\frac{f(\frac{s}{5R})}{5R}$ on this ball. So for any analytic function f there exists an approximated miniature-version of this function somewhere on the critical strip.

Example 3 Consider any continuous, zero-free function $f : [-r, r] \to \mathbb{C}$, where $0 < r < \frac{1}{4}$. Since f does not need to be analytic, we can also consider functions like $f(s) = 1 + |s|$ or $f(s) = s\cos(\frac{1}{s}) + 1$.
It is well known that there exists a sequence of polynomials $p_n(s)$ that converges uniformly to $f(s)$ on $[-r, r]$. Therfore given a $\epsilon_1 > 0$ there is a $p_n(s)$ such that $max_{s\in[-r,r]}|f(s) - p_n(s)| < \epsilon_1$. Since $p_n(s)$ is a polynomial, it is analytic and we can apply the universality theorem on it: For any $\epsilon_2 > 0$ there exists $T(\epsilon_2)$ such that $max_{s\in[-r,r]}|p_n(s) - \zeta(s + \frac{3}{4} + iT(\epsilon_2))| < \epsilon_2$. Consequently, for any

$\epsilon := \epsilon_1 + \epsilon_2$ there exists $T(\epsilon) = T(\epsilon_2)$ such that

$$max_{s\in[-r,r]}|f(s) - \zeta(s + \frac{3}{4} + iT(\epsilon_2))|$$
$$< max_{s\in[-r,r]}|f(s) - p_n(s)| + max_{s\in[-r,r]}|p_n(s) - \zeta(s + \frac{3}{4} + iT(\epsilon_2))| < \epsilon_1 + \epsilon_2. \tag{112}$$

Now we are going to discuss some theorems that can be derived from the universality theorem.

Theorem. *Let $f_\sigma : \mathbb{R} \longmapsto \mathbb{C}^N$ be defined as*

$$f_\sigma(t) = (log(\zeta(\sigma + it)), (log(\zeta(\sigma + it)))', ..., (log(\zeta(\sigma + it)))^{(N-1)}), \tag{113}$$

where $log(s)$ is taken in such a way, that it is a real number, when s is a real positive number. Then for σ with $\frac{1}{2} < \sigma < 1$, $f_\sigma(R)$ is dense in $\mathbb{C}^N$.

For a proof see $[4, p.252]$.

Theorem. *Let $f_\sigma : \mathbb{R} \longmapsto \mathbb{C}^N$ be defined as*

$$f_\sigma(t) = (\zeta(\sigma + it), (\zeta(\sigma + it))', ..., (\zeta(\sigma + it))^{(N-1)}). \tag{114}$$

Then for σ with $\frac{1}{2} < \sigma < 1$, $f_\sigma(R)$ is dense in $\mathbb{C}^N$.

For a proof see $[4, p.253]$.

Theorem. *Let $F : \mathbb{C}^N \longmapsto \mathbb{C}$ be a continuous function. If*

$$F(\zeta(s), (\zeta(s))', ..., (\zeta(s))^{(N-1)}) = 0, \forall s \in \mathbb{C}, \tag{115}$$

then $F \equiv 0$.

Proof. If F was not identically 0, then there would exist a point $s \in \mathbb{C}^N$ such that $F(s) \neq 0$. Since F is continuous by hypothesis, there would also exist an open neighborhood A of s, such that F is different from 0 for each point of A. However from the previous theorem we know that $(\zeta(s), (\zeta(s))', ..., (\zeta(s))^{(N-1)})$ is dense in $\mathbb{C}^N$ and therefore there would exist $s \in \mathbb{C}$ such that $(\zeta(s), (\zeta(s))', ...,$ $(\zeta(s))^{(N-1)}) \in A$ and consequently $F(\zeta(s), (\zeta(s))', ..., (\zeta(s))^{(N-1)}) \neq 0$. But that is a contradiction! □

4.2 Self-similarity and the Riemann hypothesis

In order to link the universality theorem to the topics of the previous chapters we may ask if this theorem could be used to find zeros of $\zeta(s)$ or to prove the non-existence of zeros beyond the critical line. In this section we will see a theorem that shows that indeed there is a surprising connection.

Consider an open ball $B = B(s', r)$ where $r < \frac{1}{4}$ and $1 \notin B$. Then $\zeta(s)$ itself is an analytic function on B and $\zeta(s - s')$ is analytic on $B(0, r)$. Due to the universality theorem we know that for any $\epsilon > 0$ there exists $T(\epsilon)$ such that

$$\max_{|s| \leq r} |\zeta(s - s') - \zeta(s + \frac{3}{4} + iT(\epsilon))| < \epsilon. \tag{116}$$

So any shape that $\zeta(s)$ takes is repeated somewhere else on the critical strip, which makes it self-similar.
In the following theorem we will see, that for a given compact set K with $Re(K) > 1$ there are not only some places where $\zeta(s)$ repeats the behavior that it had at K, but there are actually quite a lot places where it does this.
First we need a

Definition. *The Lebesgue measure of the set A is defined as*

$$meas(A) = \inf \left\{ \sum_{1}^{\infty} (b_i - a_i) : \bigcup_{1}^{\infty} [a_i, b_i] \supset A \right\}. \tag{117}$$

Theorem 4. *Let U be a compact set such that $Re(U) > 1$ and $\epsilon > 0$. Then*

$$\liminf_{T \to \infty} \frac{1}{T} meas \left\{ t \in [0, T], \max_{u \in U} |\zeta(u + it) - \zeta(u)| < \epsilon \right\} > 0. \tag{118}$$

For a proof see [8, p.57].

It turns out that the Riemann hypothesis is true if and only if the previous theorem can be extended to certain sets U of the critical strip.

Theorem 4. *The Riemann hypothesis is true if and only if for any compact subset U with connected complement, such that $\frac{1}{2} < Re(U) < 1$ and for any $\epsilon > 0$ the following holds:*

$$\liminf_{T \to \infty} \frac{1}{T} meas \left\{ t \in [0, T], \max_{u \in U} |\zeta(u + it) - \zeta(u)| < \epsilon \right\} > 0. \tag{119}$$

For a proof see [8, p.54].

5 Appendix

Theorem A.1. *If $\{f_n\}$ is a sequence of holomorphic functions that converges uniformely to a function f in every compact subset of U, then f is holomorphic on U.*

For a proof see [1, p.53].

Let Ω denote an open subset of $\mathbb{C}$ such that $s \in \Omega \Leftrightarrow \overline{s} \in \Omega$. Let $\Omega^+ = \{s \in \Omega : Re(s) > 0\}$ and $I = \Omega \bigcap \mathbb{R}$. Then

Theorem A.2 (Symmetry principle). *If $f^+(s)$ and $f^-(s)$ are holomorphic functions on Ω^+ and Ω^- respectively, that extend continuously to I and $f^+(s) = f^-(s), \forall s \in I$, then the function*

$$f(s) = \begin{cases} f^+(s) & \text{if } s \in \Omega^+ \\ f^+(s) = f^-(s) & \text{if } s \in I \\ f^-(s) & \text{if } s \in \Omega^-. \end{cases} \tag{120}$$

is holomorphic on Ω.

For a proof see [7, p.58].

Theorem A.3 (Schwarz reflection principle). *Suppose that f is a holomorphic function on Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function $F(s)$ holomorphic in all of Ω such that $F(s) = f(s), \forall s \in \Omega^+$. Furthermore the function $F(s)$ is given by*

$$F(s) = \overline{f(\overline{s})}, s \in \Omega^-. \tag{121}$$

For a proof see [7, p.60].

Since analytic continuations are unique we conclude that if $F : \Omega \to \mathbb{C}$ is an analytic function and it is real-valued on I, then

$$F(s) = \overline{F(\overline{s})}, \forall s \in \Omega. \tag{122}$$

Theorem A.4. *Suppose the series $\sum_{n=1}^{\infty} a_n n^{-s_0}$ is convergent. Then the series $\sum_{n=1}^{\infty} a_n n^{-s}$ is uniformly convergent in every closed region contained in the region $Re(s) > Re(s_0)$. The function $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is analytic for $Re(s) > Re(s_0)$.*

For a proof see $[4, p.354]$

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