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Automorphisms groups of genus 3 curves

Francesc Bars *

Introduction

This note mainly reproduces a sketch of two proofs ([8],[3]) of the determination of the non-trivial groups G that appear as the automorphism group of a non-hyperelliptic genus 3 curve over an algebraic closed field of characteristic zero. We display the tables obtained in these different approaches and the table given by Henn [6]too. These tables are refined in order to obtain in some situations the existence of curves having as exact automorphism group the one predicted in the table and not a strictly bigger group, see theorem 16, remark 17 and the last paragraph of §2.4.

Initially I believed that the result was first obtained by Komiya and Kuribayashi [8] (1978). Later I discovered the existence of the manuscript of Henn [6] in the subject (1976) (but I did not find yet any copy of Henn's manuscript).

More generally, we can ask: which groups appear as the automorphism group of a curve of genus $g \geq 4$? The reader interested in these questions can have a look at [9], where there is a list of groups for genus ≤ 10, but is incomplete. We suggest also to check Breuer's book [1], which studies all possible signatures.

Notation

We fix now some notation to be used in the rest of the chapter.

Let C be a non-singular, projective curve of genus $(g) \geq 2$ defined over an algebraic closed field K of characteristic 0.

By $WP(C)$ we mean the set of all Weierstrass points of $C(K)$ (see the definition and basic properties of Weierstrass points in [10, §1]).

Is classically known that

$$2g + 2 \leq \#WP(C) \leq (g-1)g(g+1),$$

and $\#WP(C)$ is exactly $2g+2$ if and only if C is an hyperelliptic curve.

We denote by $Aut(C)$ the group of all K-automorphism of the curve C. WP denotes a single Weierstrass point of C. For $\varphi \in Aut(C)$, $v(\varphi)$ is the number of points of C fixed by φ.

Consider a separable covering of smooth non-singular curves

$$\pi : C \to C'$$

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and denote by g' the genus of C'. We can write Hurwitz's formula as follows:

$$2g-2 = deg(\pi)(2g'-2) + \sum_{P\in C}(e_P - 1) = deg(\pi)(2g'-2) + \sum_{i=1}^{r}\frac{deg(\pi)}{v_j}(v_j - 1)$$

$$= deg(\pi)(2g'-2) + deg(\pi)\sum_{i=1}^{r}(1 - v_j^{-1}),$$

where r is the number of ramified points of$\tilde{P}_1, \ldots, \tilde{P}_r$ of C' and for each $\tilde{P}_j$ there are $\frac{deg(\pi)}{v_j}$ branch points in C: P_j^1,. . .,$P_j^{deg(\pi)/v_j}$ each of them with ramification index $v_j = e_{P_j^l}$.

1 General facts on the group $Aut(C)$

Lemma 1. *Let φ be any element of $Aut(C)$ with $\varphi \neq id$. Then φ fixes at most 2g+2 points (i.e. $v(\varphi) \leq 2g+2$).*

Proof. Denote by S the finite set of points of $C(K)$ fixed by φ. Take $P \in C(K)$ a non-fixed point by φ. We know that exist a meromorphic function f of C, with $(f)_\infty = rP$ (the divisor of poles of f) for some r with $1 \leq r \leq g+1$ (we need to take $r = g+1$ if P is not a Weierstrass point).

Let us denote by $h := f - f\varphi$, whose divisor of poles is $(h)_\infty = rP + r(\varphi^{-1}P)$, thus h has $2r(\leq 2g+2)$ zeroes. To obtain the result, we need only to mention that every fixed point of C by φ is by construction a zero of h. □

Lemma 2. *Let be $\varphi \in Aut(C)$. If P is a WP of C then $\varphi(P)$ is a WP of C.*

Proof. φ^* transforms regular differentials into regular differentials, therefore the gap sequences (with respect to differentials) are preserved by φ^*.Thus φ maps any WP (of some fixed weight) to another WP (of the same weight). □

Let us denote by $S_{WP(C)}$ the permutation group on the set of Weierstrass points. We have a group homomorphism (lemma 2):

$$\lambda : Aut(C) \to S_{WP(C)}.$$

Lemma 3. *λ is injective unless C is hyperelliptic. If C is hyperelliptic, then $ker(\lambda) = \{id, w\}$ where w denotes the hyperelliptic involution of $Aut(C)$.*

Proof. Take $\phi \in ker(\lambda)$. If C is non-hyperelliptic, we have strictly more than $2g+2$ WP points fixed by ϕ, thus by lemma 1, ϕ is the id automorphism.
If C is hyperelliptic, we know that $w \in ker(\lambda)$. We can suppose $\phi \neq w$ with $\phi \in ker(\lambda)$. We follow now the proof of lemma 1 with $\phi = \varphi$. In the hypereliptic case we can take $r = 2$, therefore we have at most 4 fixed points for ϕ if it is not the identity. We know that the number of WP is $2g+2(\geq 6)$, therefore $\phi \neq id$ and $\neq w$ does not belong to $ker(\lambda)$. □

If C is non-hyperelliptic we have a canonical immersion [10, §1, Thm.1.3.],

$$\phi : C \to \mathbb{P}^{g-1},$$

and then we have a canonical model of C, $\phi(C)$, inside the projective space $\mathbb{P}^{g-1}$.

Proposition 4. *If C is a non-hyperelliptic curve, then any automorphism of C is represented by a projective transformation on $\mathbb{P}^{g-1}$ leaving $\phi(C)$ invariant.*

Proof. For any morphism between two non-singular non-hyperelliptic curves, the pullback of the regular differentials maps to regular differentials; therefore any morphism lifts to a morphism between the projective spaces where the curves are embedded by the canonical immersions (both non-singular curves). □

Proposition 4 useful to obtain the exact automorphism group associated to a fixed non-hyperelliptic curve of genus 3. Proposition 4 and lemma 2 are key results in order to obtain the automorphism groups appearing on genus 3 curves, §2.2 (see for example theorem 20 of this notes).

Let us now list some general results using the Hurwitz's formula. We need the separability condition in the following results of this subsection, which is no problem since we work in $char(K) = 0$

Lemma 5. *Let be $\varphi \in Aut(C)$ of prime order p. Then $p \leq g$ or $p = g + 1$ or $p = 2g + 1$.*

Proof. Consider the Galois covering

$$\pi : C \to C/ < \varphi > .$$

Denote by $\tilde{g}$ the genus of $C/ < \varphi >$, from Hurwitz formula we obtain:

$$2g - 2 = p(2\tilde{g} - 2) + v(\varphi)(p - 1).$$

To prove our statement is enough to assume $p \geq g + 1$ and prove under this assumption that the only possible values for p are $g + 1$ or $2g + 1$.
If $\tilde{g} \geq 2$ then we have $2g-2 \geq p(2\tilde{g}-2) \geq 2p \geq 2g+2$, and this cannot happen.
If $\tilde{g} = 1$ then we have $2g - 2 = v(\varphi)(p - 1) \geq v(\varphi)g$. Since $v(\varphi) \geq 2$(any automorphism of prime order of $Aut(C)$ which has one fixed point, must have at least two, see [5, V.2.11]), this cannot happen either.
If $\tilde{g} = 0$ then if $v(\varphi) \geq 5$ we have $2g-2 = -2p+v(\varphi)(p-1) \geq 3p-5 \geq 3g-2$, and this cannot happen. If $v(\varphi) = 4$ then from Hurwitz formula $2g - 2 = -2p + 4(p - 1) = 2p - 4$ and this can happen only with $g = p + 1$. If $v(\varphi) = 3$ then $2g - 2 = -2p + 3(p - 1) = p - 3$ which happens only for $p = 2g + 1$. □

Applying Hurwitz formula one obtains:

Theorem 6 (Hurwitz, 1893)**.** *For any C non-singular curve C of genus $g \geq 2$ we have*

$$\#Aut(C) \leq 84(g - 1).$$

The proof of this result deals with the Galois cover $C \to C/Aut(C)$ and Hurwitz's formula on it, see [5, V.1.3].

Let us recall that we follow the notation of Hurwitz's formula introduced in the beginning of this notes.

Proposition 7 (Hurwitz, 1893)**.** *Let be H a cyclic subgroup of $Aut(C)$ and denote by $\tilde{g}$ the genus of C/H and $m = \#H$. Then:*

1. *if $\tilde{g} \geq 2$ then $m \leq g - 1$.*

2. *if $\tilde{g} = 1$ then $m \leq 2(g - 1)$.*

3. *if $\tilde{g} = 0$ and* $\begin{cases} r \geq 4 \Rightarrow m \leq 2(g-1). \\ r = 4 \Rightarrow m \leq 6(g-1). \\ r = 3 \Rightarrow m \leq 10(g-1). \end{cases}$

The proof deals with Hurwitz's formula in the Galois cover $\pi : C \to C/H$.

Remark 8. *Wiman in 1895 improved the bound $m \leq 10(g-1)$ to $m \leq 2(2g+1)$ and showed this is the best possible. Homma (1980) obtains that this bound is attained if and only if the curve C is birational equivalent to $y^{m-s}(y-1)^s = x^q$ for $1 \leq s < m \leq g + 1$.*

Let us finally collect some other properties that follow from an application of Hurwitz's formula:

Proposition 9 (Accola)**.** *Let be H and H_j $1 \leq j \leq k$ subgroups of $Aut(C)$ such that $H = \bigcup_{j=1}^{k} H_j$ and $H_i \cap H_l = \{id\}$ if $i \neq l$. Denote by $m_j := \#H_j$, $m := \#H$, $\tilde{g}$ the genus of C/H and $\tilde{g}_j$ the genus of C/H_j. Then,*

$$(k-1)g + m\tilde{g} = \sum_{j=1}^{k} m_j \tilde{g}_j.$$

For a proof we refer to [5, V.1.10].

Corollary 10. *Let C be a genus 3 curve which is non-hyperelliptic. Then any involution σ on C is a bielliptic involution (i.e. the genus of $C/ < \sigma >$ is 1)(the researchers on Riemann surfaces instead of bielliptic involution use the terminology 2-hyperelliptic involution).*

Proof. Suppose that σ is an involution which is not a bielliptic involution, so that the genus of $C/ < \sigma >$ is two (because C is not hyperelliptic). Then, the Galois covering $\pi : C \to C/ < \sigma >$ is unramified (Hurwitz). We know that any genus 2 curve is hyperelliptic, therefore there exists $\tau \in Aut(C/ < \sigma >)$ such that the curve $(C/\sigma)/ < \tau >$ has genus 0. These covers are Galois, extend τ to a morphism on $K(C)$ the field corresponding to C this gives joint with σ a subgroup of order 4 in $Aut(C)$, $\mathbb{Z}/4$ is not possible for the ramification index of the covers, therefore we have $H = \mathbb{Z}/2 \times \mathbb{Z}/2 \leq Aut(C)$ where the genus of C/H is equal to zero. We have three involutions in H, one is σ, applying the above Accola result (proposition 9, know $k = 3, m_i = 2, m = 4$ if $H_1 =< \sigma >$ $\tilde{g}_1 = 2$ and $\tilde{g} = 0$) we obtain:

$$(3-1)3 + 30 = 2(2 + g_2 + g_3).$$

We have then that $g_2 + g_3 = 1$, therefore $g_2 = 0$ or $g_3 = 0$ which implies C is hyperelliptic, a contradiction. $\square$

Let us make explicit the following straightforward consequence of the above proof.

Corollary 11. *Suppose that C has genus 3 and exists a subgroup H of $Aut(C)$ isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ such that the genus of C/H is zero. Impose moreover that one element of H fixes no point of C, then C is an hyperelliptic curve.*

Let us write down some results on fixed points using basically Hurwitz's formula on the covering given by fixing C by φ;

Lemma 12. *Let be $\varphi \in Aut(C)$ not the identity. Then $v(\varphi) \leq 2 + \frac{2g}{ord(\varphi)-1}$ where $ord(\varphi)$ is the order of this element in the group.*

See a proof in [5, V.1.5].

Proposition 13. *Let be $\varphi \in Aut(C)$ not the identity. If $v(\varphi) > 4$ then every fixed point of φ is a WP.*

We refer for a proof of this result to [5, V.1.7].

For more particular results on automorphism groups (for example the extension of the concept of WP to q-Weiertrass points which is useful to extend proposition 13 with $v(\varphi) > 2$ instead of 4; results around the question: when the involutions are in the center of $Aut(C)$?,...) we refer the interested reader to [5, chapter V].

Let us make explicit some of the general facts on $Aut(C)$ when $g = 3$:

Situation $g = 3$:
$\#Aut(C) \leq 168$
Only the primes $2, 3, 7$ can divide the order of $Aut(C)$
$Aut(C)$ is a finite subgroup of $PGL_3(K)$.

2 Automorphism groups of genus 3 curves

In this section we let C be a non-hyperelliptic genus 3 curve.We can think C embedded in $\mathbb{P}^2$ as a non-singular plane quartic.

Who determined first the list of groups appearing as automorphism groups on genus 3 curves over K? This is no clear to me.The result is published by Komiya and Kuribayashi in 1979 in an international available book [8], based in a talk deal by the authors in 1978 in Copenhagen. Recently, I noticed that the result is claimed (see [12, p.62]) to be published in the year 1976 in a publication of Heidelberg University [6].

We present two approaches (at §2.2. the one given by Komiya and Kuribayashi [8] and at §2.1 another given by Dolgachev [3]). Both approaches study first a cyclic subgroup of $Aut(C)$ in order to obtain a model for C, and latter from this equation, obtain its fuller automorphism group. We reproduce also in §2.4 the tables and the result obtained by Henn [6].By the form of the statement, it seems that Henn's result is close to the approach given in §2.1, but I do not have Henn's manuscript [6] to check this.

In §2.3 we give some results in terms of signature, we will think our curves with automorphism as points in the moduli space of genus 3 curve, and we determine which genus 3 curve has a big group of automorphism relating with the theory of "dessin d'enfant".

We want to warn the reader to be careful with the results of [3] (or [8] in some concrete situations mainly in the hyperelliptic situation) because some restrictions on the values of the parameters or other minor details are missing or misprinted in the statements of results. Here we try to fix some of them, and hopefully this is complete at least in §2.1 following Dolgachev's approach (he introduced some of my corrections in his Lecture Note after I sent him an e-mail noticing inaccuracies in the table. In other cases he did not believe me and remain unchanged in Dolgachev's table, February 2005). In order to fix the minor inaccuracies on [8] with the hyperelliptic (and non-hyperelliptic situation, see remark 24) we refer to §6 of the paper [9] of Magaard-Shaska-Shpectorov-Völklein (see also the work [9] for lists of groups which appear in curves of genus ≤ 10).

2.1 Determination of the finite subgroups of PGL_3

We want to consider finite subgroups of PGL_3 up to conjugation. We restrict our attention to groups with less than 169 elements and the only prime orders 2,3 or 7. For each of these subgroups in PGL_3 we shall study which ones have as fixed set in $\mathbb{P}^2$ a non-singular plane quartic. These are the automorphism groups we are looking for. Moreover we shall obtain equations for the quartics.

The idea to obtain the results is to use a cyclic subgroup H of order m, $H \leq Aut(C)$, in order to obtain a model equation for C, and from this model, find the full automorphism group. Let us review here the process used by Dolgachev in [3]. We remind the reader that we try to fix some of the inaccuracies in [3], for this reason we reproduce the proofs. When we write "He", we mean Dolgachev.

Proposition 14. *Let φ be an automorphism of order $m > 1$ of a non-singular plane quartic $C = V(F(X,Y,Z))$. Let us choose coordinates such that the generator of the cyclic group $H =< \varphi >$ is represented by the diagonal matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi_m^a & 0 \\ 0 & 0 & \xi_m^b \end{pmatrix},$$

where ξ_m is a primitive $m-th$ root of unity. Then $F(X,Y,Z)$ is in the following list:

Cyclic automorphism of order m.
$\varphi = diag[1, \xi_m^a, \xi_m^b]$ *we denote its Type by:* $m, (a, b)$
$C = V(F)$ *where* C *denotes the quartic.*
L_i *denotes a generic homogenous polynomial of degree* i

	Type	$F(X,Y,Z)$
(i)	$2,(0,1)$	$Z^4 + Z^2L_2(X,Y) + L_4(X,Y)$
(ii)	$3,(0,1)$	$Z^3L_1(X,Y) + L_4(X,Y)$
(iii)	$3,(1,2)$	$X^4 + \alpha X^2YZ + XY^3 + XZ^3 + \beta Y^2Z^2$
(iv)	$4,(0,1)$	$Z^4 + L_4(X,Y)$
(v)	$4,(1,2)$	$X^4 + Y^4 + Z^4 + \delta X^2Z^2 + \gamma XY^2Z$
(vi)	$6,(3,2)$	$X^4 + Y^4 + \alpha X^2Y^2 + XZ^3$
(vii)	$7,(3,1)$	$X^3Y + Y^3Z + Z^3X$
(viii)	$8,(3,7)$	$X^4 + Y^3Z + YZ^3$
(ix)	$9,(3,2)$	$X^4 + XY^3 + Z^3Y$
(x)	$12,(3,4)$	$X^4 + Y^4 + XZ^3$

Remark 15. *Note that, in the above list, the equation $F(X,Y,Z)$ that we attach to some concrete type can have another type for some specific values of the parameters. For example in the situation (i) the case $L_2 = 0$ has type $4,(0,1)$; another example is (vi) with $\alpha = 0$, the equation having also type $12,(3,4)$.*

Proof. (Dolgachev proof) Take a non-singular plane quartic (i.e. with degree ≥ 3 in each variable) and let φ act by

$$(X : Y : Z) \mapsto (X : \xi_m^a Y : \xi_m^b Z).$$

Suppose first that $ab = 0$. Assume $a = 0$, (otherwise with the change of variables $Y \leftrightarrow Z$ we should obtain the same results). Write:
$F = \beta Z^4 + Z^3L_1(X,Y) + Z^2L_2(X,Y) + ZL_3(X,Y) + L_4(X,Y)$,

If $\beta \neq 0$,then $4b \equiv 0 \bmod m$, thus $m = 2$ or $m = 4$. If $m = 2$ then $L_1 = L_3 = 0$ and we obtain Type $2,(0,1)$. If $m = 4$ ($b \neq 2$), then $L_1 = L_2 = L_3 = 0$ and we get Type $4,(0,1)$ (because type $4,(0,3)$ can be reduced to this situation by change of variables $X \leftrightarrow Z$ multiplying the matrix by ξ_4).

If $\beta = 0$, then $3b = 0 \bmod m$, then $m = 3$ and thus $L_2 = L_3 = 0$ and we get Type $3,(0,1)$ (the type $3,(0,2)$ we can obtain with a change of variables type $3,(0,1)$).

If $ab \neq 0$, we can suppose that $a \neq b$ and $mcd(a,b) = 1$(otherwise by scaling we could reduce to the first situation). Then necessarily $m > 2$. Let $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$ be the reference points.

1. All reference points lie in the non-singular plane quartic.

The possibilities for the equation are now:

$$F = X^3 L_{1,X}(Y,Z) + Y^3 L_{1,Y}(X,Z) + Z^3 L_{1,Z}(X,Y) +$$
$$+ X^2 L_{2,X}(Y,Z) + Y^2 L_{2,Y}(X,Z) + Z^2 L_{2,Z}(X,Y)$$

where $L_{i,B}$ denotes a homogenous polynomial of degree i with variables different from the variable B. It is easy to check that B_i can not appear in both L_{1,B_j} $j \neq i$ where $B_1 = X, B_2 = Y$ and $B_3 = Z$. By change of the variables X,Y,Z, he assumes that:

$$F = X^3Y + Y^3Z + Z^3X + X^2 L_{2,X}(Y,Z) + Y^2 L_{2,Y}(X,Z) + Z^2 L_{2,Z}(X,Y).$$

We see from the first 3 factors that $a = 3a + b = 3b$ mod m therefore $m = 7$ and we can take a generator of H such that $(a,b) = (3,1)$. By checking each monomial's invariance we obtain that no other monomial enters in F; thus, we obtain Type $7,(3,1)$.

2. Two reference points lie in the plane quartic.

   By re-scaling the matrix φ and permuting the coordinates we can assume that $(1:0:0)$ does not lie in C. The equation is then:

   $$F = X^4 + X^2 L_2(Y,Z) + X L_3(Y,Z) + L_4(Y,Z)$$

   because L_1 is not invariant by φ $(a, b \neq 0)$. Moreover Y^4 and Z^4 are not in L_4 because by assumption only $(1:0:0)$ does not lie in C.

   Assume first that Y^3Z is in L_4. We have $3a + b = 0$ mod m. Suppose Z^3Y is also in L_4 then $a + 3b = 0$ therefore $8b = 0$ mod m and then $m = 8$, we can take a generator φ with $(a,b) = (3,7)$ and we obtain Type $8,(3,7)$. If Z^3Y is not in L_4 then Z^3 is in L_3 (because non-singularity) and $3b = 0$ mod m; this condition, together with 3a+b=0 mod$m$, provides two situations: $m = 3$ and $(a,b) = (1,2)$ or $m = 9$ and $(a,b) = (3,2)$, but the first can not happen under the condition that Y^3Z is in L_4 and the second type is equal to $9,(3,2)$ of the table.

   Up to a permutation of $Y \leftrightarrow Z$ we can assume now that Y^3Z and Z^3Y are not in L_4. By non-singularity we have that Y^3 and Z^3 should be in L_3, then $3b = 0$ and $3a = 0$ mod m, therefore $m = 3$ and $(a,b) = (1,2)$ is the Type $3,(1,2)$ in the table.

3. One reference point lies in the plane quartic.
   By normalizing the matrix and permuting the coordinates we assume that $P_1 = (1:0:0)$ and $P_2 = (0:1:0)$ do not lie in C. We can write

$$F = X^4 + Y^4 + X^2 L_2(Y,Z) + X L_3(Y,Z) + L_4(Y,Z),$$

where Z^4 does not enter in L_4 for the hypotheses on which references points lie or not lie in the quartic, L_1 does not appear because $ab \neq 0$. We have then $4a = 0 \bmod m$. By non-singularity Z^3 is in L_3, therefore $3b = 0 \bmod m$, hence $m = 6$ or $m = 12$. Imposing the invariance by φ we obtain

$$(*) F = X^4 + Y^4 + \alpha X^2 Y^2 + X Z^3,$$

if $m = 6$ then $(a,b) = (3,2)$ (and α may be different from 0), this is Type $6,(3,2)$. If $m = 12$ then $(a,b) = (3,4)$ from the above equation $(*)$ and $\alpha = 0$, this is Type $12,(3,4)$.

4. None of the reference points lie in the plane quartic.

   In this situation

$$F = X^4 + Y^4 + Z^4 + X^2 L_2(Y,Z) + X L_3(Y,Z) + \alpha Y^3 Z + \beta Y Z^3 + \iota Y^2 Z^2,$$

   where L_1 does not appears because $ab \neq 0$. Clearly $4a = 4b = 0 \bmod m$, therefore $m = 4$ and we can take $(a,b) = (1,2)$ or $(1,3)$ both situation define isomorphic curves (only by a renaming which is X, Y, Z in the equations), this is type $4,(1,2)$.

□

Let us now introduce some notations. Let G be a subgroup of the general linear group $GL(V)$ of a complex vector space of dimension 3. G is named intransitive if the representation of G in $GL(V)$ is reducible. Otherwise it is named transitive. An intransitive G is called imprimitive if G contains an intransitive normal subgroup G'; in this situation V decomposes into direct sum of G'-invariant proper subspaces and the set of representatives of G of G/G' permutates them. Let C_m denote the cyclic group of order m, S_i by the symmetric group of i-elements, A_i the alternate group of i-elements, D_i the dihedral group which has order $2i$. Denote by H_8 the group of order 8 given by $< \tau, \iota | \tau^4 = \iota^2 = 1, \tau\iota = \iota\tau^3 >$ which is an element of $Ext^1(C_2, C_4)$ and also an element of $Ext^1(C_2, C_2 \times C_2)$ (observe that H_8 is isomorphic to D_4). Q_8 denotes the quaternion group. Denote by $C_4 \circledcirc A_4$ the group given by $\{(\delta, g) \in \mu_{12} \times H : \delta^4 = \chi(g)\}/\pm 1$, where μ_n is the set of n-th roots of unity, H is the group A_4 and let take S, T a generators of H of order 2 and 3 respectively and χ is the character $\chi : H \to \mu_3$ defined by $\chi(S) = 1$ and $\chi(T) = \rho$ with ρ a fixed 3-primitive root of unity. Observe that this group is an element of $Ext^1(A_4, C_4)$ by projecting in the second component, which corresponds in the

GAP library of small groups to the group identified by $(48, 33)$. You can found also a representation of this group of order 48 inside $PGL_3(\mathbb{C})$ in the table in §2.4. We denote by $C_4 \circledcirc (C_2 \times C_2)$ the group $(16, 13)$ in GAP library of small groups which is a group in $Ext^1(C_2 \times C_2, C_4))$, see this group inside the group $PGL_3(\mathbb{C})$ in the table given in §2.4.

Theorem 16. *In the following table we list all the groups G for which there exists a non-singular plane quartic with automorphism group G. Moreover, we list for each group a plane quartic having exactly this group as automorphism group. These equations cover up to isomorphism all plane non-singular quartics having some non-trivial automorphism.*

Full automorphism group G.

$\|G\|$	G	$F(X,Y,Z)$	$P.M.$
168	$PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2)$	$Z^3Y + Y^3X + X^3Z$	
96	$(C_4 \times C_4) \rtimes S_3$	$Z^4 + Y^4 + X^4$	
48	$C_4 \circledcirc A_4$	$X^4 + Y^4 + Z^3X$	
24	S_4	$Z^4 + Y^4 + X^4 + 3a(Z^2Y^2 + Z^2X^2 + Y^2X^2)$	$a \neq 0, \frac{-1\pm\sqrt{-7}}{2}$
16	$C_4 \circledcirc (C_2 \times C_2)$	$X^4 + Y^4 + Z^4 + \delta Z^2Y^2$	$\delta \neq 0, \pm 2, \pm 6, \pm(2\sqrt{-3})$
9	C_9	$Z^4 + ZY^3 + YX^3$	
8	$H_8 = D_4$	$Z^4 + Y^4 + X^4 + \alpha Z^2(Y^2 + X^2) + \gamma Y^2X^2$	$\alpha \neq \gamma, \alpha \neq 0$
6	C_6	$Z^4 + aZ^2Y^2 + Y^4 + ZX^3$	$a \neq 0$
6	S_3	$Z^4 + Z(Y^3 + X^3) + \alpha Z^2YX + \beta Y^2X^2$	$\alpha \neq \beta, \alpha\beta \neq 0$
4	$C_2 \times C_2$	$Z^4 + Y^4 + X^4 + Z^2(\alpha Y^2 + \beta X^2) + \gamma Y^2X^2$	$\alpha \neq \gamma, \beta \neq \gamma$ $\alpha \neq \beta$
3	C_3	$Z^3L_1(Y,X) + L_4(Y,X)$	*not above*
2	C_2	$Z^4 + uZ^2L_2(Y,X) + L_4(Y,X)$	$u \neq 0$,*not above*

where P.M. means parameter restriction. "not above" means not $K-$isomorphic to any other model above it in the table.

Remark 17. *Any non-singular plane quartic over K with automorphism group G is K-isomorphic to the curve in the line of the group G, for some concrete values of the parameters. Moreover, for the lines with $|G| \geq 9$, the written equations have automorphism group exactly G. In (§2.4) we show how one can ensure that an equation has exact group of automorphism the one predicted for the tables. We do this for the group $C_2 \times C_2$ but other situations can be implemented as well. (Information on Weierstrass points simplifies calculations).*

See §2.3 (or the table in §2.4) for the dimension of the subvariety of the moduli space of genus 3 curves representing curves with a fixed automorphism group G.

Remark 18. *The above table differs from Dolgachev's in some situations. For the reader's convenience we reproduce here Dolgachev's table in [3] (in December 2004):*

$\|G\|$	G	$F(X,Y,Z)$	P.M.
168	$PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2)$	$Z^3Y + Y^3X + X^3Z$	
96	$(C_4 \times C_4) \rtimes S_3$	$Z^4 + Y^4 + X^4$	
48	$C_4 \circledcirc A_4$	$Z^4 + YX^3 + YX^3$	
24	S_4	$Z^4 + Y^4 + X^4 + a(Z^2Y^2 + Z^2X^2 + Y^2X^2)$	$a \neq \frac{-1\pm\sqrt{-7}}{2}$
16	$C_4 \times C_4$	$Z^4 + \alpha(Y^4 + X^4) + \beta Z^2X^2$	$\alpha, \beta \neq 0$
9	C_9	$Z^4 + ZY^3 + YX^3$	
8	Q_8	$Z^4 + \alpha Z^2(Y^2 + X^2) + Y^4 + X^4 + \beta Y^2X^2$	$\alpha \neq \beta$
7	C_7	$Z^3Y + Y^3X + X^3Z + aZY^2X$	$a \neq 0$
6	C_6	$Z^4 + aZ^2Y^2 + Y^4 + YX^3$	$a \neq 0$
6	S_3	$Z^4 + \alpha Z^2YX + Z(Y^3 + X^3) + \beta Y^2X^2$	$a \neq 0$
4	$C_2 \times C_2$	$Z^4 + Z^2(\alpha Y^2 + \beta X^2) + Y^4 + X^4 + \gamma Y^2X^2$	$\alpha \neq \beta$
3	C_3	$Z^4 + \alpha Z^2YX + Z(Y^3 + X^3) + \beta Y^2X^2$	$\alpha, \beta \neq 0$
2	C_2	$Z^4 + Z^2L_2(Y,X) + L_4(Y,X)$	not above

Typing errors explain the equations of the group of order 48 and C_6. Moreover the equation corresponding to C_3 can not be the same as S_3, some parameters on PM are not appearing in the equation for example in the curves with automorphism group S_3.

The group $C_4 \times C_4$ appears in Dolgachev's table with the equation $Z^4 + \alpha(Y^4 + X^4) + \beta Z^2X^2 = 0$. Observe that the change of variable of Y and Z with a 4-th root of α can reduce to the equation $Z^4 + Y^4 + X^4 + \beta' Z^2X^2 = 0$. It is clear that the last curve has $C_4 \times C_4$ as a subgroup of automorphisms given by diagonal matrices in $SL_3(K)$: $diag[\xi_4, \xi_4^2, \xi_4]$ and $diag[\xi_4, 1, \xi_4^3]$ where ξ_4 is a fixed 4-th root of unity of 1; however, this curve has more automorphism, for example $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ which is an order two automorphism different from the above ones. Therefore this curve has a bigger group of automorphism and the equation is isomorphic to one above on it in the table. In particular, the group $C_4 \times C_4$ does not appear as an exact group of automorphism for a non-hyperelliptic genus 3 curve. Nevertheless, it appears another group of 16 elements which is not initially in Dolgachev's table.

The other big difference is the following one: Dolgachev writes that the cyclic group C_7 is the automorphism group for $Z^3Y + Y^3X + X^3Z + aZY^2X = 0$ with $a \neq 0$. This comes from a missprint calculation of the equations of Type 7(3,1) in proposition 14, observe that with $a \neq 0$ does not has Type $7(3,1)$ this equation. From proposition 14 the only curve with a cyclic group of order 7 is isomorphic to $X^3Z + Y^3X + Z^3Y = 0$.

Finally, he claims that the group of 8 *elements is* Q_8 *but I obtain the Dihedral group* $H_8 = D_4$*, instead. Henn's result [6] corroborates my calculations.*

Proof. (sketch, following Dolgachev)
Case 1: G an intransitive group realized as a group of automorphisms.
Case 1.a.:$V = V_1 \oplus V_2 \oplus V_3$.
Choose (X, Y, Z) such that V_1 spanned by $(1, 0, 0)$ and so on.
$\varphi \in G$ of order m, after scaling $\varphi = diag(1, a, b)$, we know models of equations and restrictions for m, a, b from above proposition 14.
Suppose $h \in G$ but $h \notin <\varphi>$, (choose m maximal with the property that G has an element of order m).
Study now situation by situation the equations on cyclic subgroups (i)-(x) (table in theorem 14):
Take $m = 12$, (x); we think $h = diag(1, \xi_{m'}^c, \xi_{m'}^d)$ then $4c = 3d = 0 \bmod m'$, then $12|m'$ and $h \in <\varphi>$.
Nevertheless situation (x) has bigger automorphisms group which we will observe in case 1.b.
Similar arguments in the cases (v)-(x) to conclude: there are no other automorphism appearing as an intransitive group with $V = V_1 \oplus V_2 \oplus V_3$.(We need to observe here that in case (v) the situation $\gamma = 0$ is included in situation also (iv), by a change of name of the variables, given already bigger commutative subgroup inside the automorphism group, see next situation (iv)).

Case (iv) and suppose $h \notin <\varphi>$, let

$$L_4 = aX^4 + bY^4 + cX^3Y + dXY^3 + eX^2Y^2$$

assume $ab \neq 0$, $h = diag(\xi_{m'}^p, \xi_{m'}^q, 1)$, then $m' = 2$ or 4. If $m' = 2$ the only possibility is $(p, q) = (0, 1)$ or $(1, 0)(h \notin <\varphi>)$ where $c = d = 0$, but in this possibility we obtain a bigger group of automorphism.

If $m' = 4$, the only possibilities are:

$$(p, q) = (1, 0),\ (0, 1),\ (1, 3),\ (3, 1),\ (1, 2),\ (2, 1).$$

If $(p, q) = (1, 3)$ or $(3, 1)$ we have $c = d = 0$, so that this equation has bigger group and appears in the next cases (interchanging X and Y). If (p, q)= $(1, 2)$ or $(2, 1)$ similar as the case $(1, 3)$. The situation$(1, 0)$ implies $c = d = e = 0$, this is the Fermat quartic and it has a bigger automorphism group.
Assume now $a \neq 0$ and $b = 0$. $d \neq 0$ (non-singularity). One has $4p = 3p + q = 0$ mod m', then $c = e = 0$. But then we obtain the group $\mathbb{Z}/12$ situation (x) considered before.
Assume now $a = b = 0$. $cd \neq 0$(non-singularity). $3p + q = p + 3q = 0$ mod (m'), but then $m' = 8$ (studied above).
Similar argument applied:
Case (iii) One checks that no other element arises except when 1)$\alpha = \beta = 0$ which is the situation (ix), already studied;2)$\alpha = \beta$ then C_6 is a subgroup of

the group an is already studied (vi),3)$\beta = 0$,$\alpha \neq 0$ no-reduced,4)$\alpha = 0$, $\beta \neq 0$ is C_6 in the group.
Case (ii): Since $L_1 \neq 0$ no h can exist.
Case (i): Only need to study when $h = diag(1, -1, 1)$ (i.e. we have $C_2 \times C_2$). We have that L_4 does not contain Y^3X and X^3Y and L_2 does not contain XY. In this situation one could have a bigger group of automorphism when $\alpha = \beta$ (see table).

Case 1.b. $V = V_1 \oplus V_2$ with dimV_2=2, where V_2 irreducible representation of G (G is then non-abelian).
Choose coordinates s.t. $(1, 0, 0) \in V_1$, V_2 spanned by $(0, 1, 0), (0, 0, 1)$. $\overline{\varphi}$ restriction of φ to $W = V(Z) = \mathbb{P}(V_2)$, choose in SL_2. Write:

$$F = \alpha Z^4 + Z^3 L_1(Y, X) + Z^2 L_2(Y, X) + Z L_3(Y, X) + L_4(Y, X),$$

$L_1 = 0$ (irreducibility of V_2) and $\alpha \neq 0$(non-singularity).
If $L_2 \neq 0$, G leaves $V(L_2)$ invariant, $\overline{G}$ the restriction of G in W, the

$$\overline{G} \leq D_2$$

(always need in these arguments of case 1.b. that $\overline{G}$ is a subgroup of PSL_2, but if $\overline{G}$ is commutative we have to impose that is not a subgroup of SL_2(otherwise G commutative and is case 1.a.)), then by a change of variables of V_2 that the action of $\overline{G}$ is $u_1 : (x, y) \mapsto (-y, x)$ and $u_2 : (x, y) \mapsto (ix, -iy)$, then G can be only an extension of the $C_2 \times C_2$ (this group is $C_2 \times C_2$ in PSL_2 not in SL_2) situation above.

We need now to construct of possible $G's$ which $\overline{G}$ has the above property. Because C_2 sure is in G (see the comment in last paragraph) and re-scaling we can use the equations of (i), moreover C_4 is in G if we do the good elections, because $u_i^2 = -id$ and extending by 1 the action over X of $(X : Y : Z)$ one obtains a morphism of this degree. One can check that there are no more situations to consider, then u_1 and u_2 can be extended to G by the following matrices with ξ_4, ξ_4' 4-th root of unity (not necessary primitive):

$$\tilde{u_1} = \begin{pmatrix} \xi_4 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \ \tilde{u_2} = \begin{pmatrix} \xi_4' & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We need to study all the possibilities for ξ_4 and ξ_4' up to scaling, we are in $PGL_3(K)$.

Let us make explicit two situations, the others with similar techniques are studied.

Observe for our first election in PGL_3 we choose $\varphi \in PGL_3$ with determinant 1, we obtain that $\overline{G}$ is generated in $PGL_3(K)$ by

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \ \tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Observe that in PGL_3 we have $\tau_i^4 = id$ and one can check that this group is Q_8. As τ_2 defines an automorphism of order 4, from the equation (v) (need re-scaling and changing the variables because now the action of this cyclic group is $diag[i, -1, 1]$) we obtain that the equation is $X^4+Y^4+Z^4+\delta Z^2Y^2+\gamma X^2YZ$. Impose now that τ_1 and τ_2 are automorphism of this equation, τ_1 implies that $\gamma = 0$, and τ_2 gives invariant the curve

$$X^4 + Y^4 + Z^4 + \delta Z^2Y^2,$$

Since X only appears raised to the 4th power, this equation has automorphism group bigger of index 2 with respect to Q_8 with the automorphism acting only on X (we notice when $\delta = 0, \pm 6$ is isomorphic to $X^4+Y^4+Z^4$ which it will has bigger automorphism group, when $\delta = \pm 2$ is singular, and when $\delta = \pm 2\sqrt{-3}$ is isomorphic to $X^4 + Y^4 + Z^3X$, which one obtains a bigger automorphism group).

Let us now take another election in PGL_3 for $\overline{G}$, which we choose generated by:

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix};\ \tau_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

now $\tau_2^2 = 1$ and $\tau_1^4 = 1$ and moreover $\tau_1\tau_2 = \tau_2\tau_1^3$. Moreover we have that they have a subgroup isomorphic to $C_2 \times C_2 =< \tau_1^2 \times \tau_2 >$ where $\tau_1^2 = diag[1, -1, -1]$ and $\tau_2 = diag[1, 1, -1]$, therefore to construct the equation in this situation we can use the equation obtained in case 1.a, with $C_2 \times C_2$ group of automorphism (here we do not need any change variable, 1 acts on X). Let impose that the equation $Z^4+X^4+Y^4+Z^2(\alpha Y^2+\beta X^2)+\gamma Y^2X^2$ is invariant by τ_1 and τ_2. We obtain then for τ_1 that $\alpha = \beta$. Observe that τ_1, τ_2 generates H_8, therefore the curve

$$Z^4 + X^4 + Y^4 + \alpha Z^2(Y^2 + X^2) + \gamma Y^2X^2$$

has a subgroup of automorphism H_8, (moreover let us observe that when $\alpha = 0$ has also as subgroup Q_8, therefore has a bigger group because $C_2 \times C_2 \leq H_8$ but $C_2 \times C_2 \not\leq Q_8$).

If $L_2 = 0$ but $L_3 \neq 0$, here $\overline{G} \leq D_3$ obtains that with the invariants of this elements one obtains a singular curve.
If $L_2 = L_3 = 0$ but $L_4 \neq 0$, $\overline{G}$ leave $V(L_4)$ invariant. One knows

$$\overline{G} \leq A_4$$

of order 12. One should study all these subgroups; for $\mathbb{Z}/2 \times \mathbb{Z}/2$ we can restrict to the equation given by step 1a and one obtains the group of 16 elements and the group of 48 elements of the table.

Case 2: G has a normal transitive imprimitive subgroup H.

H is a subgroup given above and permutates cyclically coordinates, therefore the only situations possible are (here we need to check from the list of all G finite in PGL_3 with this property has normal subgroup which appears in some situation in case 1 as a possible group of automorphism, and for every one of these possible G's, take all the equations (here we have more than the 10 situations that we began in case 1.a. because we need to joint the situations with bigger group appearing in case 1) and check which has this permutation of the variables)(here we eliminate by isomorphism some equations, for example the ones coming from (v) with $\gamma \neq 0$ with a change of variables are isomorphic to the ones happening in case 1.b with a subgroup in the automorphic group equal to D_4, similarly some situations which permutations of variables give new automorphisms are already studied in case 1.b and then are not consider now in the following list):

$$Z^4 + \alpha Z^2 YX + Z(Y^3 + X^3) + \beta Y^2 X^2$$
$$Z^3 Y + Y^3 X + X^3 Z$$
$$Z^4 + Y^3 X + X^3 Y$$
$$Z^4 + Y^4 + Z^4 + 3a(Z^2 Y^2 + Z^2 X^2 + Y^2 X^2)$$

In the first one of these equations, we see that the automorphism group is S_3 with the restrictions appearing above in the argument (the group is C_6 in some situations already studied in case 1a).
The second curve appearing is the Klein quartic, whose automorp- hism group is $PSL_2(\mathbb{F}_7)$.
The third equation is isomorphic to Fermat's quartic $X^4 + Y^4 + Z^4$. It has as a subgroup of automorphism $C_4^2 \rtimes S_3$ (S_3 from permutation of three variables and $C_4 \times C_4$ from automorphism coming from making a scale of the variables by a 4-th root of unity) of order 96, therefore it cannot be bigger by the Hurwitz bound.
The fourth equation, if $a = 0$ is the Fermat's curve (isomorphic to the third equation), or $a = \frac{1}{2}(-1 \pm \sqrt{-7})$ is isomorphic to Klein curve. If a does not take these values, clearly a subgroup of the $Aut(C)$ which consists with change the sign of the variable with permutations of the variables. This subgroup has order 24, and is isomorphic to S_4. To obtain that this is the full group of automorphism, we need a more careful study of the action of the automorphism group on Weierstrass points.

Case 3: G is a simple group.

There are only two transitive primitive groups of $PGL_3(K)$, one is $PGL_3(\mathbb{F}_2)$ given a quartic (taking the $(X : Y : Z)$ invariants by this group) isomorphic to the Klein quartic model which we obtained in case 2 (see next talk in the seminar, [2]).
The other has order bigger than 168, therefore can not be $Aut(C)$ of any genus 3 curve (by Hurwitz theorem 6).

$\square$

2.2 Determination of $Aut(C)$ by cyclic covers

In this subsection we follow the proof which was printed firstly in an international accessible book (as far as I know). This is the work of Komiya and Kuribayashi [8]. We only write down some concrete situations of the proofs of the general statements, we refer to the original paper [8] for the interested reader.

Suppose that C is a non-hyperelliptic non-singular projective ge- nus 3 curve, and suppose that C has a non-trivial automorphism σ. Clearly by Hurwitz's formula $C/ < \sigma >$ has genus 0, 1, or 2. If it is 2, we have then $\sigma^2 = id$, thus by corollary 10 C is hyperelliptic, in contradiction with our hypothesis. Therefore $C/ < \sigma >$ has genus 0 or 1, i.e. C has a Galois cyclic cover to a projective line or to an elliptic curve (as K is algebraically closed, any genus 1 curve has points).

If $Aut(C)$ has an element of order > 4 then $C/ < \sigma >$ has genus 0 (use Hurwitz formula, proposition 7), therefore the Galois cyclic cover $\pi : C \to C/ < \sigma >$ is a cyclic cover of the projective line. We study the question about which groups are $Aut(C)$ for a genus 3 non-hyperelliptic curve C in two situations:

1. C curves which are a Galois cyclic cover of a projective line.
2. C curves which are a Galois cyclic cover of an elliptic curve but not of a projective line.

1. **Cyclic covers of a projective line**.
   Suppose that C has a Galois cyclic cover of order m then the extension of fields $K(C)/K(x)$ is a cyclic Galois extension with group C_m, then $K(C) = K(x, y)$ with $y^m \in K(x)$, therefore we can obtain an equation for our curve as follows:

$$y^m = (x - a_1)^{n_1} \cdot \ldots \cdot (x - a_r)^{n_r} \tag{1}$$

with $1 \leq n_i < m$ and $\sum_{i=1}^{r} n_i$ is divided by m where $a_1, \ldots, a_r$ are the points of the projective line over which the ramification occurs in the cyclic cover.
Apply now Hurwitz's proposition 7 with $g = 3$ and $\tilde{g} = 0$, we obtain that $m \leq 20$. In the original work [8] the situations C hyperelliptic and non-hyperelliptic genus 3 curve are deal together, but here we only do the non-hyperelliptic situation, (the results for hyperelliptic situation are stated in [8] and we refer to the interested reader there).

Theorem 19 (Theorem 1[8])**.** *The projective, non-singular, non-hyperelliptic genus 3 curves C which are a cyclic cover of order m (can have also a cyclic cover of order a multiple of m) of a projective line are listed below (up to isomorphism):*

m	*Equation*
3	$y^3 = x(x-1)(x-\alpha)(x-\beta)$
4	$y^4 = x(x-1)(x-\alpha)$
6	$y^3 = x(x-1)(x-\alpha)(x-(1-\alpha))$
7	$y^3 + yx^3 + x = 0$
8	$y^4 = x(x^2-1)$
9	$y^3 = x(x^3-1)$
12	$y^4 = x^3 - 1$

,

Observe that each equation above in $\mathbb{P}^2$ becomes a non-singular quartic.

Let us here only reproduce how runs the proof of the above theorem in some concrete situation, the general proof is a study case by case with similar techniques. We know that $m \leq 20$. By Hurwitz's formula the cover $C \to C/C_m$ is not possible for $m = 5, 11, 13, 17$ and 19. From the conditions of the equation 1, about the ramification r and the conditions on n_i, we have that $m = 15, 16, 18$ and 20 are not possible either. Let us fix a concrete remaining m, take $m = 8$. The values of v_i can be only divisors of 8, then 2,4,8, therefore all the possibilities for the index of ramification satisfying $n_i \leq m$ and the divisibility condition are the following three:

	v_1	v_2	v_3	v_4	v_5
(i)	2	2	2	2	2
(ii)	2	2	4	4	
(iii)	4	8	8		

In the situations $(i), (ii)$ the equation becomes reducible, these situations can not occur. In the situation (iii) there are three possible different equations:
(1) $y^8 = (x-a_1)^2(x-a_2)^3(x-a_3)^3$
(2) $y^8 = (x-a_1)(x-a_2)(x-a_3)^6$
(3) $y^8 = (x-a_1)^2(x-a_2)(x-a_3)^5$
by a birational transformation $x = X$ and $y = (X-a_1)^{-2}(X-a_2)^{-1}(X-a_3)^{-1}Y$, one obtains that (2) is birational equivalent to (1), and one observes that (2) is an hyperelliptic curve, situation that we do not work here in this talk.
Let us normalize the equation (3) as $y^8 = x^2(x-1)$. One computes a basis of differentials of the first kind $w_1 = y^{-3}dx$, $w_2 = y^{-6}xdx$, $w_3 = y^{-7}xdx$, and writing $x = -X^{-1}Y^4$, $y = Y$ one obtains a canonical model equation:

$$X^3Z + XZ^3 + Y^4 = 0$$

(and one observes that this quartic is isomorphic to Fermat's quartic $X^4 + Y^4 + Z^4 = 0$).

How can we obtain from theorem 19 the full automorphism group?
We use the equations in the projective model and case by case we study the group of elements of $PGL_3(K)$ that fix the quartic, we use here the result proposition 4, this is a work that you can find in [8, §2,§3] with the useful knowledge of lemma 2. More precisely, they distinguish different situations depending from the model equation, up to concrete missing situations they separate this study basically into two situations:
1) one with the affine model: $y^3 = x(x-1)(x-t)(x-s)$, and
2) second with the affine model: $y^4 = x(x-1)(x-t)$.

To obtain the exact group of automorphism (for 1) and 2)), one could study which $G \subseteq PGL_3(K)$ fixes the projective model, and this is the searching G, this is basically made in Komiya-Kuribayashi.

Let $F(X, Y, Z)$ be the equation of the quartic whose automorphism group we want to study (given by theorem 19). Solve the system of 15 equations (of degree 4 in the variables) from the equality

$$F(X', Y', Z') = kF(X, Y, Z)$$

with $k \neq 0$ where $\sigma(X, Y, Z) = (X', Y', Z')$ and $\sigma \in PGL_3(K)$.
This computation is so big, therefore to make this calculation one needs to use some more information. Komiya and Kuribayashi use the fact that WP maps by σ to WP (our lemma 2) to simplify the 15 equations to some managable systems with few equations.

They observe in the case 1) (which corresponds basically to Picard curves) that any automorphism σ fixes the point $P_\infty = (0:1:0)$, except for the equation $y^3 = x^4 - 1$. This simplify enormously the calculation of the automorphism group of the equation as a subgroup of PGL_3. In the case 2), they compute the Hessian, and observes that its Hessian has a good factorization given hight restrictions on Weierstrass points that lyes on a line of multiplicity two which appears in the factorization of the Hessian, simplifying the calculation of the automorphism group inside PGL_3 (remember that $(F \cdot Hessian(F)) =$ Weierstrass points (each one with its weight multiplying)), see [8, pp.68-74].

Then one obtains,

Theorem 20 (Komiya-Kuribayashi)**.** *The smooth, projective, non-hyperelliptic genus 3 curves which are a cyclic cover of order m of a projective line are isomorphic to one of the following equations and has the automorphism group associated to it:*

$Equation\{F(X,Y,Z)=0\}$	$Aut(C=V(F))$	m	$some\ P.R.$
$Y^3Z+XZ^3+X^3Y=0$	$PGL_2(\mathbb{F}_7)$	7	
$Y^4-X^3Z-XZ^3=0$	$(C_4\times C_4)\rtimes S_3$	8	
$Y^3Z-X^4+XZ^3=0$	C_9	9	
$Y^4-X^3Z+Z^4=0$	$C_4\circledcirc A_4$	12	
$Y^4-X^3Z+(\alpha-1)X^2Z^2$ $-\alpha XZ^3=0$	$C_4\circledcirc(C_2\times C_2)$	4	$\alpha\neq 1,\neq 0,...$
$X(X-Z)(X-\alpha Z)(X-(1-\alpha)Z)$ $-Y^3Z=0$	C_6	6	$\alpha\neq 0$
$-X(X-Z)(X-\alpha Z)(X-\beta Z)$ $+Y^3Z=0$	C_3	3	$\beta\neq 1-\alpha,$ $(x-\alpha)(x-\beta)$ $\neq x^2+x+1$

2. **Cyclic cover of a torus.**

We remember that the automorphism group has a cyclic element σ of order $m>4$ then the genus of $C/<\sigma>$ is zero and therefore a cyclic cover of a projective line, and we did it above.
Let us impose that $m=2,3$ or 4. Write n the size of the whole automorphism group associated to the genus 3 curve C. Let us impose that $n>4$ firstly, and we only make here a concrete proof in this situation to see now the key ingredients (as usual, for the general treatment, see [8] where work with C a general genus 3 curve which can also be an hyperelliptic curve).
For $n>4$ we have that $C/Aut(C)$ has genus 0 from Hurwitz formula and one can see that $r\geq 3$ in this formula.
In such a situation one deduces from Hurwitz's formula that the Galois cover $\pi : C\to C/Aut(C)$ verifies the following:

(a) If $r\geq 5$, then $n\leq 8$ and:
(1) $n=8$, $v_1=v_2=v_3=v_4=v_5=2$;
(2) $n=6$, $v_1=v_2=v_3=v_4=2$, $v_5=3$.

(b) If $r=4$ then $n\leq 24$ and:
(1) $n=24$, $v_1=v_2=v_3=2$, $v_4=3$
(2) $n=16$, $v_1=v_2=v_3=2$, $v_4=4$
(3) $n=12$, $v_1=v_2=2$, $v_3=v_4=3$
(4) $n=8$, $v_1=v_2=2$, $v_3=v_4=4$
(5) $n=6$, $v_1=v_2=v_3=v_4=3$.

(c) If $r=3$, then $n\leq 48$ and:
(1) $n=48$, $v_1=v_2=3$,$v_3=4$
(2) $n=24$, $v_1=3$, $v_2=v_3=4$
(3) $n=16$, $v_1=v_2=v_3=4$.

We need a study case by case. To show the ideas that they let us take the situation with $r \geq 5$ and $n = 6$. (There are situations that no such curve exists, another will obtain curves already studied above as cyclic cover of a projective line therefore we discard them).
Let us take $n = 6$ with ramification 2, 2, 2, 2, 3 and C be non-hyperelliptic. Because the automorphism group has order 6, we have an involution σ such that is bielliptic (see corollary 10). Let P_1 and P_2 be branch points with multiplicity 3 and τ the automorphism of order 3 by which P_1 and P_2 are fixed. We have that $\tau\sigma = \sigma\tau^2$ (is not cyclic here, otherwise we have already studied the situation by cyclic cover of projective line) and $C/ < \tau >$ is an elliptic curve (we can suppose is not a projective line because we suppose is not a cyclic cover of the projective line, and from Hurwitz's formula C has not genus 2).

We need some lemmas on divisors to help us:

Lemma 21. *Let C be a projective non-singular curve of genus g (≥ 3) and let ι an automorphism of C such that $C/ < \iota >$ is an elliptic curve. Denote by v_P the ramification multiplicity of a branch point of the covering $\pi : C \to C/ < \iota >$. Then the divisor $\sum(v_P - 1)P$ is canonical.*

Proof. Let w be a differential of first kind of the elliptic curve, think as differential of C by pull back we obtain

$$div_C(w) = \pi^{-1} div_{C/<\iota>}(w) + \sum(v_P - 1)P = \sum(v_P - 1)P.$$

□

The following lemma is not useful in our concrete situation $n = 6$ but it is useful in others. Let us write it here.

Lemma 22. *Let C be a projective, non-singular, non-hyperelliptic genus 3 curve. Assume that C has an automorphism ι of order 4 and ι has fixed points on C. Then the $v(\iota) = 4$, denote by P_1, P_2, P_3 and P_4 this four fixed points. Moreover we have that $\sum_{i=1}^{4} P_i$ and $4P_i$ $1 \leq i \leq 4$ are canonical divisors.*

Let us follow our concrete situation with $n = 6$. We obtain from lemma 21 that $2(P_1 + P_2)$ is canonical divisor.

Let also write the group $G = \{1, \tau, \tau^2, \sigma = \sigma_1, \sigma_2 = \tau\sigma_1, \sigma_3 = \tau^2\sigma_1\}$, where σ_i are involutions (all bielliptic).
Let $\{Q_i^{(1)}\}$, $\{Q_i^{(2)}\}$, $\{Q_i^{(3)}\}$ be the set of 4 fixed points by $\sigma_1, \sigma_2, \sigma_3$ respectively. By lemma 21 we know $\sum_{i=1}^{4}\{Q_i^{(1)}\}$, $\sum_{i=1}^{4}\{Q_i^{(2)}\}$ and $\sum_{i=1}^{4}\{Q_i^{(3)}\}$ are canonical divisors.

From the relation $\sigma_1\sigma_2\sigma_1 = \sigma_3$ we have hat $\sigma_1(\sum_{i=1}^4 \{Q_i^{(2)}\}) = \sum_{i=1}^4 \{Q_i^{(3)}\}$ and one checks that $\sigma_1 P_1 = P_2$. Let us define the meromorphic functions

$$div(x) = \sum_{i=1}^{4} \{Q_i^{(2)}\} - 2(P_1 + P_2)$$

$$div(y) = \sum_{i=1}^{4} \{Q_i^{(3)}\} - 2(P_1 + P_2)$$

we have $\sigma_1(x) = \alpha y$ and because σ_1 is an involution $\sigma_1(y) = \beta x$ with $\alpha\beta = 1$, rewrite y instead of αy.
Now one checks that $1, x, y$ are a basis for $L(2P_1 + 2P_2)$ with $\tau(x) = -y$ and $\tau(y) = x - y$.
Make now the following change

$$x_1 = \frac{x - 2y + 1}{x + y + 1},\ y_1 = \frac{-2x + y + 1}{x + y + 1},$$

where now the action of σ_1 and τ are given by

$$\sigma_1 : (x_1, y_1) \mapsto (y_1, x_1),\ \tau : (x_1, y_1) \mapsto (y_1/x_1, 1/x_1),$$

and one has $1, x_1, y_1$ are a basis for $L(K)$ where K means the canonical divisor. Because $dimL(4K) = 11$ we obtain an equation $f(x_1, y_1) = \sum a_{i,j} x_1^i y_1^j$ with $i + j \leq 4$, $i, j \geq 0$ and with homogenous coordinates the group acts by

$$\sigma_1 \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\tau \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

then the equation is invariant for the group S_3 and therefore the equation is

$$A(X^4 + Y^4 + Z^4) + B(X^3Y + Y^3X + Z^3X + X^3Z + Z^3Y + Y^3Z)$$

$$+C(X^2Y^2 + Y^2Z^2 + X^2Z^2) = 0$$

for some A, B, C. If $B = C = 0$ and $A \neq 0$ is isomorphic to $y^4 = x(x^2 - 1)$ which has cyclic cover of a projective line, this is already studied. If $B = 0$ and $AC \neq 0$ has a group of order 24, except when $C/A = 3\mu$

with$\mu \in \{\frac{-1\pm\sqrt{-7}}{2}\}$ where for this concrete situation is isomorphic to the Klein quartic which automorphism group is isomorphic to a group of 168 elements and is already studied in the cyclic cover of a projective line. For $ABC \neq 0$ we obtain that the full group of automorphism is G (this result is obtained by using proposition 4).

Working situation by situation Komiya and Kuribayashi (with similar techniques and some results on genus 3 curves with fix number of Weierstrass points from the article [7]) obtain the following statement:

Theorem 23 (Komiya-Kuribayashi)**.** *A smooth, projective, non-hyperelliptic genus 3 curves which is a cyclic cover of an elliptic curve and not of a projective line, is isomorphic to one of the following equations and has the indicated automorphism group:*

$Equation\{F(X,Y,Z)=0\}$	$Aut(C=V(F))$	$Some P.R.$
$X^4+Y^4+Z^4+3a(X^2Y^2+X^2Z^2+Z^2Y^2)=0$	S_4	$a\neq 0, \frac{-1\pm\sqrt{-7}}{2}$
$X^4+Y^4+aX^2Y^2+b(X^2Z^2+Y^2Z^2)+Z^4=0$	$H_8=D_4$	$a\neq b$
$(X^4+Y^4+Z^4)+c(X^2Y^2+Y^2Z^2+X^2Z^2)+$ $b(X^3Y+Y^3X+Z^3X+X^3Z+Z^3Y+Y^3Z)=0$	S_3	$bc\neq 0$
$X^4+Y^4+Z^4+2aX^2Y^2+2bX^2Z^2+2cY^2Z^2=0$	$C_2\times C_2$	
$a(X^4+Y^4+Z^4)+b(X^3Y-Y^3X)+cX^2Y^2$ $+d(X^2Z^2+Y^2Z^2)=0$	$C_2\times C_2\leq$	
$a(X^4+Y^4+Z^4)+b(X^3Y+Y^3Z+XZ^3)$ $+c(Y^3X+X^3Z+Y^3Z)+$ $d(X^2Y^2+X^2Z^2+Y^2Z^2)=0$	$C_3\leq$	
$(X^4+Y^4+Z^4)+Y^2(a_0X^2+a_1XZ+bZ^2)+$ $(a_2X^3Z+a_3X^2Z+a_4XZ^3)=0$	C_2	

Remark 24. In the column of $Aut(C)$ of the above table $\leq$ means that the group written is a subgroup of the whole automorphism group (check the appendix of [8] and also §III.6 [9]). These situations are listed above and then we can eliminate them from the table.

2.3 Final remarks

The approach of Komiya-Kuribayashi consists in listing all the group signature pairs, and for ach one obtain the exact automorphism group which occurs if such a situation is admissible, (i.e. if its possible). Let us introduce this language a little bit.
Let C be a curve of genus ≥ 2 (in this subsection). Let H be a subgroup of $Aut(C)$ then we can consider the cover $\pi : C \to C/H$ and denote by $g_0 = genus(C/H)$ and Hurwitz's formula reads:

$$2(g-1)/|H| = 2(g_0-1) + \sum_{i=1}^{r}(1-\frac{1}{m_i}),$$

then we define the signature associate to this cover is $(g_0; m_1, \ldots, m_r)$ where we have exactly r ramification points.

For Riemann surfaces of genus ≥ 2 we have a Fuchsian group K such that $C = \mathbb{H}/K$ and $Aut(C) = Norm(K)/K$ where $Norm(K)$ is the normalization inside $PSL_2(\mathbb{R})$ of K, these Fuchsian groups are added with a signature, and

π relates the Fuchsian group K as a normal subgroup of a concrete Fuchsian group of signature $(g_0; m_1, \ldots, m_r)$ (see [1] for an extended explanation).

Basically Hurwitz's formula gives restriction to the possible signatures for a subgroup H. One needs to obtain results in the direction: Is this group the exact group of automorphism or not?. Breuer [1] lists the possible signatures and subgroups H that could be subgroups of the automorphism group for curves of genus ≤ 48, but it remains to discard a lot of signatures which does not give the exact group of automorphism (as did originally Komiya-Kuribayashi in [8] for genus 3 curves), in this direction the work [9] reobtains Komiya-Kuribayashi's result using more the approach on Fuchsian groups. Next, we only list, for every group which is $Aut(C)$ for a genus 3 non-hyperelliptic curve, the signature that has for the covering $\pi : C \to C/Aut(C)$ from §2.2:

$Aut(C)$	signature
$PSL_2(\mathbb{F}_7)$	$(0; 2, 3, 7)$
S_3	$(0; 2, 2, 2, 2, 3)$
C_2	(1;2,2,2,2)
$C_2 \times C_2$	$(0; 2, 2, 2, 2, 2, 2)$
D_4	$(0; 2, 2, 2, 2, 2)$
S_4	(0;2,2,2,3)
$C_4^2 \rtimes S_3$	(0;2,3,8)
$C_4 \circledcirc (C_2)^2$	(0;2,2,2,4)
$C_4 \circledcirc A_4$	(0;2,3,12)
C_3	$(0; 3, 3, 3, 3, 3)$
C_6	(0;2,3,3,6)
C_9	(0;3,9,9)

Let us recall some facts presented in the seminar on "dessins d'enfants" [13] (genus of C is always is bigger than or equal to 2).

Let us denote by $\mathcal{M}_g$ the moduli space of genus g curves. Let us denote by $\mathcal{M}_{g,r}$ the moduli space of genus g curves with r different marked points where we view the marked points as unordered. It is known that the dimension of these moduli spaces ($genus \geq 2$) are given by

$$dim(\mathcal{M}_{g,r}) = 3g - 3 + r.$$

Remark 25. *From the above classification of curves with automorphism and joining the classification for hyperelliptic genus 3 curves, and because $dim(M_3) = 6$, we obtain that there a lot of non-hyperelliptic genus 3 curves that has no automorphism, in particular the generic curve for $\mathcal{M}_3$ has no automorphism. (See [11] for an equation of the generic genus 3 curve).*

A curve C is said to have a large automorphism group if its point in $\mathcal{M}_g$ has a neighborhood (in the complex topology) such that any other curve in this neighborhood has an automorphism group a group with strictly less elements than the automorphism group that has the curve C.

Theorem 26 (P.B.Cohen, J.Wolfart). *Let C be a curve over $\mathbb{C}$ with a large automorphism group ($g \geq 2$). Then $C/Aut(C)$ is the projective line and moreover the Galois cover $\pi : C \to C/Aut(C)$ is a Belyî morphism.*

We have by the general theory of "dessins d'enfants",

Corollary 27. *Any curve C with a large automorphism group is defined over $\overline{\mathbb{Q}}$ and therefore over a number field.*

Corollary 28. *Let C be a curve defined over $\mathbb{C}$ ($g \geq 2$). Then: C has a large automorphism group if and only if exists a Belyî function defining a normal covering $\pi : C \to \mathbb{P}^1$.*

If we center now in our tables for non-hyperelliptic genus 3 curves, observe from the signatures that the curves, which ramify in exactly three points and the genus of $C/Aut(C)$ is zero, are exactly the curves having a large automorphism group:

List of all non-hyperelliptic genus 3 curves
with large automorphism group (up to isomorphism):

C, curve	$Aut(C)$
$Z^3Y + Y^3X + X^3Z$	$PSL_2(\mathbb{F}_7)$
$Z^4 + X^4 + Y^4$	$C_4^2 \rtimes S_3$
$X^4 + Y^4 + XZ^3$	$C_4 \circledcirc A_4$
$Z^4 + ZY^3 + YX^3$	C_9

In [9] it is said that C has a large automorphism group if

$$|Aut(C)| > 4(g - 1).$$

According to this terminology the genus 3 curves with $|Aut(C)| > 8$ "have large automorphism group". This other terminology does not relate well with "desinn d'enfants" theory; see the situation for the curve with automorphism group S_4 and/or $C_4 \circledcirc C_2^2$ in the tables. Nevertheless is a general fact that with this second notion of "having a large automorphism group" one can prove that the curves C which satisfy this second notion have $C/Aut(C)$ of genus 0 and the cover $\pi : C \to C/Aut(C)$ ramifies at 3 or 4 points (pp. 258-260 [5]).

2.4 Henn's table

We reproduce Henn's table [6] which can be found in [12, p.62].

Let G be a finite group and let $\beta : G \to PGL_3(\mathbb{C})$ be a projective representation of G. Let $S(\beta) \subseteq \mathcal{M}_3 \setminus \mathcal{H}_3$ be the locus of moduli points of non-hyperelliptic curves containing $\beta(G)$ in their automorphism group. In §2.3 we compute $s_\beta := dim(S(\beta))$, which corresponds to the number of free parameters in the equation of genus 3 curves corresponding to the points of $S(\beta)$.

Theorem 29 (Henn)**.** *The following table classifies smooth plane quartics with non-trivial automorphisms. For each G in this table, there exists a smooth quartic C with $\beta(G) = Aut(C)$ and the locus $S(\beta)$ is an irreducible subvariety of $\mathcal{M}_3 \setminus \mathcal{H}_3$.*

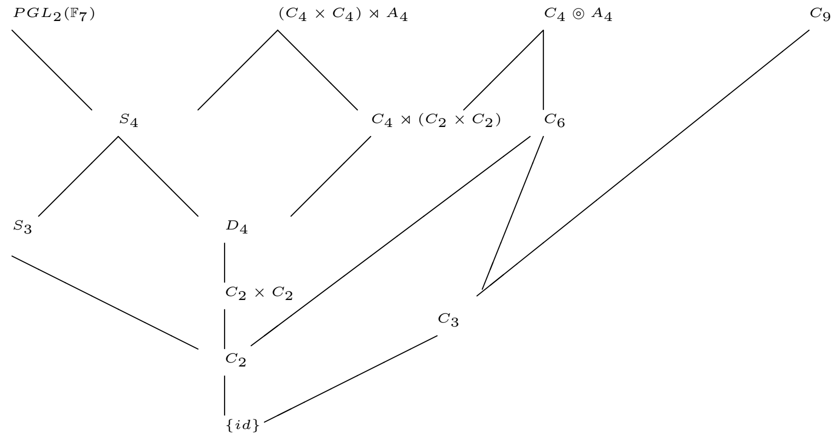
G	$Equation = \{F(X,Y,Z)\}$, $up\ K - isomorphism$	s_β	$generators\ of\ \beta(G)$
C_2	$X^4 + X^2 L_2(Y,Z) + L_4(Y,Z)$	4	$diag[-1,1,1]$
$C_2 \times C_2$	$X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2$	3	$diag[-1,1,1]$, $diag[1,-1,1]$
C_3	$Z^3Y + X(X-Y)(X-aY)(X-bY)$	2	$diag[1,1,\rho]$
C_6	$Z^3Y + X^4 + aX^2Y^2 + Y^4$	1	$diag[-1,1,\rho]$
S_3	$X^3Z + Y^3Z + X^2Y^2 + aXYZ^2 + bZ^4$	2	$diag[\rho,\rho^2,1]$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
D_4	$X^4 + Y^4 + Z^4 + aX^2Y^2 + bXYZ^2$	2	$diag[i,-i,1]$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
C_9	$X^4 + XY^3 + YZ^3$	0	$diag[\rho,1,\omega]$

$C_4 \circledcirc (C_2 \times C_2)$	$X^4 + Y^4 + Z^4 + aX^2Y^2$	1	$diag[-1,1,1]$, $diag[i,-i,1]$, $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
S_4	$X^4 + Y^4 + Z^4 +$ $a(X^2Y^2 + Y^2Z^2 + Z^2X^2)$	1	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$C_4 \circledcirc A_4$	$X^4 + Y^4 + Z^4 +$ $(4\rho + 2)X^2Y^2$	0	$\begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & \rho \end{pmatrix}$ $\begin{pmatrix} \frac{1+i}{2} & \frac{-1-i}{2} & 0 \\ \frac{-1+i}{2} & \frac{-1+i}{2} & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}$
$(C_4 \times C_4) \rtimes S_3$	$X^4 + Y^4 + Z^4$	0	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i & 0 \end{pmatrix}$
$PSL_2(\mathbb{F}_7)$	$X^3Y + Y^3Z + Z^3X$	0	$known(see$ [4] or [2]$)$

where ρ is a primitive 3-rd root of unity, $\omega^3 = \rho$ and:

We give here a kind of algorithm that we run only in a particular situation. We want to check when the model equation in the table has exact group of automorphism the one that is writed in the same line. For example: which models of type $X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cX^2Z^2$ of Henn's table have exact automorphism group $C_2 \times C_2$ and not a bigger automorphism group?

The algorithm uses the matrix presentation of the automorphism group for the models, for which we use Henn's table. The other tables help in this process too. Let us write down the scheme diagram of groups in the table ordered by inclusion (see [12, p.64]):



†

Let us describe an algorithm to check which of the equation models of type $X^4+Y^4+Z^4+aX^2Y^2+bY^2Z^2+cX^2Z^2$ of Henn's table has exact automorphism group $C_2 \times C_2$ and not a bigger automorphism group.

For the given scheme of groups, it is enough to prove that the model equation has no D_4 as a subgroup of automorphism. Let us modify the realization of D_4 in $PGL_3(\mathbb{C})$ given in Henn's table in order that the two generators of $C_2 \times C_2$ are given by $diag[-1, 1, 1]$ and $diag[1, 1, -1]$ (is the same group as Henn's gives, but we choose other generators for the group $C_2 \times C_2$). We write now the realization of D_4 in $PGL_3(\mathbb{C})$ in such a way that $C_2 \times C_2$ as a subgroup of D_4 is given by $diag[-1, 1, 1]$ and $diag[1, 1, -1]$. Henn's table shows us a realization of D_4 in $PGL_3(\mathbb{C})$, we need to do a conjugation by a matrix A of this realization

†Added February 2015: there is no line as a groups inside $PGL_3(K)$ for the given curves with maps C_3 inside S_3, because the subgroup of order 3 in S_3 of the genus 3 curves non-singular is not conjugate in $PGL_3(K)$ the cyclic group of ordre 3 given by C_3 of the curves that appears as full automorphism group. See details of this phenomena in Badr-Bars: "On smooth plane curves with a fixed automorphism group". (2015)

in order to obtain the one interested for us, in our concrete situation we need A such that:

$$A\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = diag[-1,1,1]A,$$

$$Adiag[-1,-1,1] = diag[1,1,-1]A,$$

where $diag[-1,-1,1] = (diag[i,\,-i,1])^2$ (here we fix some election in choosing the variables).

Imposing this conditions we obtain that we can choose A an invertible matrix of the form

$$A = \begin{pmatrix} 1 & -1 & 0 \\ r & r & s \\ t & t & u \end{pmatrix},$$

observe $det(A) = 2(ru-ts) \neq 0$.

Let us consider the automorphism ¥ of D_4 given by

$$\frac{1}{2(ru-ts)}\begin{pmatrix} 0 & 2ui & -2si \\ ir2(ru-ts) & -2st & 2rs \\ it2(ru-ts) & -2ut & 2ru \end{pmatrix} = Adiag[i,-i,1]A^{-1}.$$

In order that our model equation for $C_2 \times C_2$ has no bigger automorphism group is enough that ¥ is not automorphism of the model equation for $C_2 \times C_2$ in Henn's table. We compute which conditions a, b and c (in the model equation for $C_2 \times C_2$) should satisfies in order to have this ¥ as automorphism (we impose $F(¥(X,Y,Z)) = kF(X,Y,Z)$ with F the model for $C_2 \times C_2$ and $k \neq 0$ and/or that the model F_{D_4} of D_4 by the change of A become a multiple of the model for $C_2 \times C_2$ given by Henn's table; we do this last approach for the calculations). One obtains that all the possible solutions in which the model for $C_2 \times C_2$ comes from the model of D_4 are the following: when $a = b$ or $b = c$ or $a = c$ or $a = -b$ or $a = -c$ or $b = -c$. Observe moreover that if the model of equation of $C_2 \times C_2$ of Henn's table

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cX^2Z^2$$

satisfies $a = b$ or $b = c$ or $a = c$ or $a = -b$ or $a = -c$ or $b = -c$, we have seen in the table of theorem 16 (for the equation with D_4 as automorphism group) that has bigger automorphism group that $C_2 \times C_2$ (straightforward for the situations $a = b$ or $b = c$ or $a = c$, and for the situations with $-$ do the change of variables $X \longleftrightarrow iX$ or $Y \longleftrightarrow iY$ or $Z \longleftrightarrow iZ$ to conclude, compare then with the result in theorem 16).

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Francesc Bars Cortina
Depart. Matemàtiques, Edifici C,
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona,
Catalunya. Spain.
e-mail: francesc@mat.uab.es