ON FAKE ES-IRREDUCIBILE COMPONENTS OF CERTAIN STRATA OF SMOOTH PLANE SEXTICS

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Abstract. We construct the first examples of what we call *fake ES-irreducible components*; Definition 2.8. In our way to do so, we classify the automorphism groups of smooth plane sextics that only have automorphisms of order ≤ 3; Theorems 2.1, 2.4 and 2.5, Corollaries 2.9 and 2.11.

1. Introduction

Let $\mathcal{M}_g^{\mathrm{Pl}}$ be the set of K-isomorphism classes of smooth plane curves C of a fixed degree $d \geq 4$. Here K is an algebraically closed field of characterisitc $p = 0$ or $p > 2g + 1$, where $g = (d-1)(d-2)/2 \geq 3$ is the geometric genus of C.

We can associate to any $[C] \in \mathcal{M}_g^{\mathrm{Pl}}$ infinitely many non-singular plane models, each of them is given by a homogeneous polynomial equation $C : F(X, Y, Z) = 0$ of degree d in $\mathbb{P}^2(K)$. Moreover, two such plane models for C are K-isomorphic and their automorphism groups are $\mathrm{PGL}_3(K)$-conjugated via a projective change of variables $\phi \in \mathrm{PGL}_3(K)$.

Now, suppose that G is a finite non-trivial group that can be embedded into $\mathrm{PGL}_3(K)$. We write $[C] \in \mathcal{M}_g^{\mathrm{Pl}}(G)$ when there exists an injective representation $\varrho : G \hookrightarrow \mathrm{PGL}_3(K)$ such that $\varrho(G)$ is a subgroup of $\mathrm{Aut}(C)$; the automorphism group of $C : F(X, Y, Z) = 0$ inside $\mathrm{PGL}_3(K)$. Similarly, we write $[C] \in \widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G)$ when $\varrho(G) = \mathrm{Aut}(C)$, moreover, in this situation, we say that $[C]$ belongs to the component $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho(G))$ of $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G)$.

Clearly, if $\varrho_i : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$, for $i = 1, 2$, are $\mathrm{PGL}_3(\overline{k})$-conjugated, then $\mathcal{M}_g^{\mathrm{Pl}}(\varrho_1(G)) = \mathcal{M}_g^{\mathrm{Pl}}(\varrho_2(G))$ and $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho_1(G)) = \widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho_2(G))$. Accordingly,

$$\mathcal{M}_g^{\mathrm{Pl}}(G) = \bigcup_{[\varrho] \in R_G} \mathcal{M}_g^{\mathrm{Pl}}(\varrho(G)) \text{ and } \widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G) = \bigsqcup_{[\varrho] \in R_G} \widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho(G)).$$

Here $R_G := \{\varrho : G \hookrightarrow \mathrm{PGL}_3(K)\} / \sim$, where $\varrho_1 \sim \varrho_2$ if and only if $\varrho_1(G)$ and $\varrho_2(G)$ are $\mathrm{PGL}_3(K)$-conjugated.

Definition 1.1 (ES-irreducibility [3])**.** Each $[\varrho] \in R_G$ such that $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho(G)) \neq \emptyset$ is called an *ES-irreducible component* for $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G)$. We call $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G)$ *ES-irreducible* if it has exactly one ES-irreducible component.

Clearly, if a non-empty $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G)$ is not ES-irreducible, then it is not irreducible and the number of its ES-irreducible components is a lower bound for the number of its irreducible components inside the coarse moduli space $\mathcal{M}_g$ of K-isomorphism classes of smooth curves of genus g.

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Now, in the language of ES-irreducibility, one can interpret the results of Henn [9] and Komiya-Kuribayashi [10] for smooth plane quartic curves, which are genus $g = 3$ curves, as follows: the strata $\widetilde{\mathcal{M}_3^{\text{Pl}}}(G)$ are either empty or ES-irreducible. Thus each non-empty $\widetilde{\mathcal{M}_3^{\text{Pl}}}(G)$ is described by a single *normal form*; a homogenous polynomial equation $F(X,Y,Z) = 0$ in $\mathbb{P}^2(K)$ equipped with parameters as its coefficients such that any $[C] \in \widetilde{\mathcal{M}_3^{\text{Pl}}}(G)$ can be described by a smooth plane model through a specialization of those parameters.

Notations. Throughout the paper, $L_{i,B}$ denotes the generic homogeneous polynomial of degree i in the variables $\{X,Y,Z\} - \{B\}$.

By ζ_n we mean a fixed primitive nth root of unity in K.

A projective linear transformation $A = (a_{i,j}) \in \text{PGL}_3(K)$ is sometimes written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z].$$

For example, $[X : Z : Y]$ represents the projective change of variables $X \mapsto X$, $Y \mapsto Z$, $Z \mapsto Y$, and $\text{diag}(1, a, b)$ represents $X \mapsto X$, $Y \mapsto aY$, $Z \mapsto bZ$ with $a, b \in K^*$.

We use the formal GAP library notations "GAP(n, m)" to refer the finite group of order n that appears in the m-th position of the atlas for small finite groups [7]. See also GroupNames.

Fix the following subgroups in $\text{PGL}_3(K)$:

- $\varrho_1(\mathbb{Z}/2\mathbb{Z}) := \langle \text{diag}(1, 1, -1)\rangle$ and $\varrho_1((\mathbb{Z}/2\mathbb{Z})^2) := \langle \varrho_1(\mathbb{Z}/2\mathbb{Z}),\ \text{diag}(1, -1, 1)\rangle$,
- $\varrho_1(\mathbb{Z}/3\mathbb{Z}) := \langle \text{diag}(1, 1, \zeta_3)\rangle$ and $\varrho_1((\mathbb{Z}/3\mathbb{Z})^2) := \langle \varrho_1(\mathbb{Z}/3\mathbb{Z}),\ \text{diag}(1, \zeta_3, 1)\rangle$,
- $\varrho_2(\mathbb{Z}/3\mathbb{Z}) := \langle \text{diag}(1, \zeta_3, \zeta_3^{-1})\rangle$ and $\varrho_2((\mathbb{Z}/3\mathbb{Z})^2) := \langle \varrho_2(\mathbb{Z}/3\mathbb{Z}), [Y : Z : X]\rangle$,
- $\varrho_1(\text{S}_3) := \langle [Y : Z : X],\ [X : Z : Y]\rangle$ and $\varrho_2(\text{S}_3) := \langle \varrho_2(\mathbb{Z}/3\mathbb{Z}),\ [X : Z : Y]\rangle$,
- $\varrho_1(\mathbb{Z}/3\mathbb{Z} \rtimes \text{S}_3) := \langle \varrho_1(\text{S}_3),\ \varrho_2(\mathbb{Z}/3\mathbb{Z})\rangle$,
- $\varrho_1(\text{A}_4) := \langle \varrho_1((\mathbb{Z}/2\mathbb{Z})^2),\ [Y : Z : X]\rangle$ and $\varrho_2(\text{A}_4) := \langle \varrho_1((\mathbb{Z}/2\mathbb{Z})^2),\ [\zeta_6^{-1}Y : Z : X]\rangle$.

Remark 1.2. *P. Henn observed that $\mathcal{M}_3^{\text{Pl}}(\mathbb{Z}/3\mathbb{Z})$ admits two ES-components. One component corresponds to $\varrho_1(\mathbb{Z}/3\mathbb{Z})$ where any $[C] \in \mathcal{M}_3^{\text{Pl}}(\varrho_1(\mathbb{Z}/3\mathbb{Z}))$ is given by an equation of the form $Z^3Y + L_{4,Z} = 0$. The second component corresponds to $\varrho_2(\mathbb{Z}/3\mathbb{Z})$ such that any $[C'] \in \mathcal{M}_3^{\text{Pl}}(\varrho_2(\mathbb{Z}/3\mathbb{Z}))$ is given by an equation of the form $X^4 + X(Y^3 + Z^3) + \alpha_{2,1}X^2YZ + \alpha_{1,2}X(YZ)^2 = 0$ for some $\alpha_{2,1}, \alpha_{1,2} \in K$. In particular, C' has $[X : Z : Y]$ as an extra involution, thus C' always has the symmetry group S_3 as a subgroup of automorphisms. Therefore, $\widetilde{\mathcal{M}_3^{\text{Pl}}}(\varrho_2(\mathbb{Z}/3\mathbb{Z})) = \emptyset$ and $\mathcal{M}_3^{\text{Pl}}(\varrho_2(\mathbb{Z}/3\mathbb{Z})) \subseteq \mathcal{M}_3^{\text{Pl}}(\text{S}_3)$.*

Concerning smooth plane quintic curves, which are genus $g = 6$ curves, Badr-Bars [1] showed that all the strata $\widetilde{\mathcal{M}_6^{\text{Pl}}}(G)$ are either empty or ES-irreducible except when $G = \mathbb{Z}/4\mathbb{Z}$. In this case, $\widetilde{\mathcal{M}_6^{\text{Pl}}}(\mathbb{Z}/4\mathbb{Z})$ has exactly two ES-irreducible components. Moreover, we generalized this result in [3] for any odd degree $d \geq 5$. More precisely, we proved that $\widetilde{\mathcal{M}_g^{\text{Pl}}}(\mathbb{Z}/(d-1)\mathbb{Z})$ has at least two ES-irreducible components for any $g = (d-1)(d-2)/2$ with $d \geq 5$ odd. However, each of the strata $\widetilde{\mathcal{M}_6^{\text{Pl}}}(\varrho(G))$ is described again by a single normal form.

Accordingly, we were wondering if this is the situation in general. That is to say, there always exists a single normal form describing the elements of $\widetilde{\mathcal{M}_g^{\text{Pl}}}(\varrho(G))$ for each $\varrho \in R_G$. In this article, we will show that this impression is not true at least for smooth plane sextic curves, which are genus $g = 10$ curves. We establish two counter examples corresponding to $G = \mathbb{Z}/3\mathbb{Z}$ and A_4 respectively.

On the other hand, classifying automorphism groups of smooth curves is a long standing problem that receives interest by many people. In the case of hyperelliptic curve, the structure of the automorphism group is quite explicit, see [5, 6, 15, 16]. For non-hyperelliptic curves, we still have a lack of knowledge about the structure, except for some special cases. For example, the cases of low genus and also Hurwitz curves, see [4, 9, 11, 12, 13]. This lack motivates us to do more investigation in this direction, especially for the case of smooth plane curves of degree $d \geq 4$. In this paper, we classify the automorphism groups of smooth plane curves C of degree 6 such that 2 and 3 are the only divisors of $|\operatorname{Aut}(C)|$.

2. Main Results

Theorem 2.1. *Let C be a smooth plane sextic curve that admit an automorphism of maximal order 2. Up to K-isomorphism, C is defined by an equation of the form:*

$$C : Z^6 + Z^4 L_{2,Z} + Z^2 L_{4,Z} + L_{6,Z} = 0$$

such that $L_{6,Z}$ is of degree ≥ 5 in both X and Y, and at least one of the binary forms $L_{2,Z}$ and $L_{4,Z}$ is non-zero. Moreover, $\operatorname{Aut}(C) = \varrho_1(\mathbb{Z}/2\mathbb{Z})$ unless $L_{2,Z}$, $L_{4,Z}$ and $L_{6,Z}$ belong to the ring $K[X^2, Y^2]$. In the latter case, $\operatorname{Aut}(C) = \varrho_1((\mathbb{Z}/2\mathbb{Z})^2)$.

Corollary 2.2. *The strata $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/2\mathbb{Z})$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}((\mathbb{Z}/2\mathbb{Z})^2)$ are ES-irreducible.*

Definition 2.3 ([14])**.** An homology of period n is a projective linear transformation of the plane $\mathbb{P}^2(K)$, which is $\mathrm{PGL}_3(K)$-conjugate to $\operatorname{diag}(1, 1, \zeta_n)$. Such a transformation fixes pointwise a line $\mathcal{L}$ (its axis) and a point P off this line (its center). In its canonical form, $\mathcal{L} : Z = 0$ and center $P = (0 : 0 : 1)$.

Otherwise, it is called a *non-homology*.

Theorem 2.4. *Let C be a smooth plane sextic curve that admits an homology of period 3 as an automorphism of maximal order. Up to K-isomorphism, C is defined by an equation of the form $Z^6 + Z^3 L_{3,Z} + L_{6,Z} = 0$ where neither $L_{3,Z}$ nor $L_{6,Z}$ equals 0. Moreover, $\operatorname{Aut}(C)$ is always $\varrho_1(\mathbb{Z}/3\mathbb{Z})\rangle$ except when C is K-isomorphic to C' of the form $C' : X^6 + Y^6 + Z^6 + Z^3\left(\alpha_{3,0}X^3 + \alpha_{0,3}Y^3\right) + \alpha_{3,3}X^3Y^3 = 0$, such that $\alpha_{3,0}, \alpha_{0,3}, \alpha_{3,3}$ are pair-wise distinct modulo $\{\pm 1\}$. In this case, $\operatorname{Aut}(C') = \varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)$.*

Theorem 2.5. *Let C be a smooth plane sextic curve that admits a non-homology of period 3 as an automorphism of maximal order. Up to K-isomorphism, C is a member of one of the following families:*

$$\begin{aligned}
\mathcal{C}_1 \quad &: \quad X^6 + Y^6 + Z^6 + XYZ\left(\alpha_{4,1}X^3 + \alpha_{1,4}Y^3 + \alpha_{1,2}Z^3\right) + \alpha_{2,2}X^2Y^2Z^2 \\
&+ \quad \alpha_{3,3}X^3Y^3 + \alpha_{3,0}X^3Z^3 + \alpha_{0,3}Y^3Z^3 = 0 \\
\mathcal{C}_2 \quad &: \quad X^5Y + Y^5Z + XZ^5 + XYZ\left(\alpha_{3,2}X^2Y + \alpha_{1,3}Y^2Z + \alpha_{2,1}XZ^2\right) \\
&+ \quad \alpha_{2,4}X^2Y^4 + \alpha_{0,2}Y^2Z^4 + \alpha_{4,0}X^4Z^2 = 0.
\end{aligned}$$

In either way, $\sigma = \operatorname{diag}(1, \zeta_3, \zeta_3^{-1})$ is an automorphism of maximal order 3.

(1) The automorphism group $\operatorname{Aut}(\mathcal{C}_1) = \varrho_2(\mathbb{Z}/3\mathbb{Z})$ except when one of the following conditions hold.

(i) If $\alpha_{4,1} = \alpha_{1,4} = \alpha_{1,2} = \alpha_{2,2} = 0$, then $\mathcal{C}_1$ reduces to

$$X^6 + Y^6 + Z^6 + X^3\left(\alpha_{3,3}Y^3 + \alpha_{3,0}Z^3\right) + \alpha_{0,3}Y^3Z^3 = 0,$$

where $\operatorname{Aut}(\mathcal{C}_1) = \varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$.

(ii) *If* ***(a)*** $\alpha_{4,1} = \pm\alpha_{1,4}$ *and* $\alpha_{3,0} = \pm\alpha_{0,3}$, ***(b)*** $\alpha_{1,4} = \pm\alpha_{1,2}$ *and* $\alpha_{3,3} = \pm\alpha_{3,0}$, *or* ***(c)*** $\alpha_{4,1} = \pm\alpha_{1,2}$ *and* $\alpha_{3,3} = \pm\alpha_{0,3}$, *then* $\mathcal{C}_1$ *is* K*-isomorphic to*

$$\begin{aligned} \mathcal{C}'_1 \quad &: \quad X^6+Y^6+Z^6+\alpha'_{4,1}X^4YZ+\alpha'_{3,3}X^3(Y^3+Z^3)+\alpha'_{2,2}X^2Y^2Z^2 \\ &+ \quad \alpha'_{1,2}XYZ(Y^3+Z^3)+\alpha'_{0,3}Y^3Z^3=0, \end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}'_1) = \varrho_2(\mathrm{S}_3)$ *if* $\alpha'_{4,1} \neq \alpha'_{1,2}$ *or* $\alpha'_{3,3} \neq \alpha'_{0,3}$, *and* $\mathrm{Aut}(C') = \varrho_1(\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3)$ *otherwise.*

Remark 2.6. $(\alpha'_{3,3}, \alpha'_{1,2}) \neq (0,0)$ *or* $\mathrm{diag}(1, \zeta_6, \zeta_6^{-1})$ *will be an automorphism of order* $6 > 3$.

(iii) *If* ***(a)*** $(\alpha_{4,1}, \alpha_{1,2}, \alpha_{1,4})$, $(\alpha_{1,4}, \alpha_{4,1}, \alpha_{1,2})$ *or* $(\alpha_{1,2}, \alpha_{1,4}, \alpha_{4,1})$ *equals*

$$\left(\frac{2\left(29-54\lambda^6-54\mu^6\right)}{27\lambda\mu}, \frac{2\left(27\mu^6-54\lambda^6-52\right)}{27\lambda\mu^4}, \frac{2\left(27\lambda^6-54\mu^6-52\right)}{27\lambda^4\mu}\right),$$

(b) $(\alpha_{3,0}, \alpha_{3,3}, \alpha_{0,3})$, $(\alpha_{3,3}, \alpha_{0,3}, \alpha_{3,0})$ *or* $(\alpha_{0,3}, \alpha_{3,0}, \alpha_{3,3})$ *equals*

$$\left(\frac{2\left(81\lambda^6-27\mu^6-26\right)}{27\mu^3}, \frac{2\left(81\mu^6-27\lambda^6-26\right)}{27\lambda^3}, \frac{2\left(82-27\lambda^6-27\mu^6\right)}{27\lambda^3\mu^3}\right),$$

and ***(c)*** $\alpha_{2,2} = \dfrac{9\lambda^6+9\mu^6+10}{3\lambda^2\mu^2}$ *for some* $\lambda, \mu \in K^*$, *then* $\mathcal{C}_1$ *is* K*-isomorphic to*

$$\begin{aligned} \mathcal{C}_{1,\lambda,\mu} : X^6+Y^6+Z^6 \quad &+ \quad f_1(\lambda,\mu)X^2Y^2Z^2 + f_2(\lambda,\mu)(X^4Y^2+X^2Z^4+Y^4Z^2) \\ &+ \quad f_2(\mu,\lambda)(X^4Z^2+X^2Y^4+Y^2Z^4)=0, \end{aligned}$$

where

$$\begin{aligned} f_1(\lambda,\mu) \quad &:= \quad 3(80+81\lambda^6+81\mu^6), \\ f_2(\lambda,\mu) \quad &:= \quad 81\left(1+\zeta_3\lambda^6+\zeta_3^{-1}\mu^6\right). \end{aligned}$$

In this case, $\mathrm{Aut}(\mathcal{C}_{1,\lambda,\mu}) = \varrho_1(\mathrm{A}_4)$.

(2) *The automorphism group* $\mathrm{Aut}(\mathcal{C}_2) = \langle\sigma\rangle = \varrho_2(\mathbb{Z}/3\mathbb{Z})$ *except when one of the following conditions hold.*

(i) *If* $\alpha_{0,2} = \zeta_{21}^{-12r}\alpha_{4,0}$, $\alpha_{2,4} = \zeta_{21}^{3r}\alpha_{4,0}$, $\alpha_{1,3} = \zeta_{21}^{-6r}\alpha_{3,2}$, $\alpha_{2,1} = \zeta_{21}^{3r}\alpha_{3,2}$, *then* $\mathcal{C}_2$ *is* K*-isomorphic to*

$$\begin{aligned} \mathcal{C}'_2 \quad &: \quad X^5Y+Y^5Z+XZ^5+\alpha_{4,0}\zeta_{21}^{4r}\left(X^4Z^2+X^2Y^4+Y^2Z^4\right) \\ &+ \quad \alpha_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y+XZ^2+Y^2Z\right)=0, \end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}'_2) = \varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)$.

Remark 2.7. $(\alpha_{2,4}, \alpha_{1,3}) \neq (0,0)$ *or* $\mathrm{diag}(1, \zeta_{21}, \zeta_{21}^{-4})$ *will be an automorphism of order* $21 > 3$.

(ii) *If* ***(a)*** $(\alpha_{2,4}, \alpha_{4,0}, \alpha_{0,2})$, $(\alpha_{0,2}, \alpha_{2,4}, \alpha_{4,0})$ *or* $(\alpha_{4,0}, \alpha_{0,2}, \alpha_{2,4})$ *equals*

$$\left(\frac{\lambda^5\mu+4\mu^5}{2\lambda^4}, \frac{\lambda+4\lambda^5\mu}{2\mu^2}, \frac{4\lambda+\mu^5}{2\lambda^2\mu^4}\right)$$

and ***(b)*** $(\alpha_{1,3}, \alpha_{3,2}, \alpha_{2,1})$, $(\alpha_{2,1}, \alpha_{1,3}, \alpha_{3,2})$ *or* $(\alpha_{3,2}, \alpha_{2,1}, \alpha_{1,3})$ *equals*

$$\left(\frac{2\left(2\lambda^5\mu+2\lambda+\mu^5\right)}{\lambda^3\mu^2}, \frac{2\lambda^5\mu+4\lambda+4\mu^5}{\lambda^2\mu}, \frac{2\left(2\lambda^5\mu+\lambda+2\mu^5\right)}{\lambda\mu^3}\right),$$

then $\mathcal{C}_2$ is K-isomorphic to

$$\begin{aligned}\mathcal{C}_{2,\lambda,\mu}: X^6+Y^6+Z^6 &+ g_1(\lambda,\mu)(\zeta_3^{-1}X^4Y^2+X^2Z^4+Y^4Z^2)\\ &+ g_2(\lambda,\mu)(X^4Z^2+\zeta_3X^2Y^4+Y^2Z^4)=0,\end{aligned}$$

where

$$\begin{aligned} g_1(\lambda,\mu) &:= \frac{\sqrt{3}\zeta_9\left(\zeta_4\lambda^5\mu+\zeta_{12}\lambda+\zeta_{12}^5\mu^5\right)}{\lambda^5\mu+\lambda+\mu^5},\\ g_2(\lambda,\mu) &:= \frac{\sqrt{3}\zeta_{18}\left(\zeta_{12}^5\lambda^5\mu+\zeta_{12}\lambda+\zeta_4\mu^5\right)}{\lambda^5\mu+\lambda+\mu^5}.\end{aligned}$$

In this case, $\mathrm{Aut}(\mathcal{C}_{2,\lambda,\mu})=\varrho_2(\mathrm{A}_4)$.

We now introduce the notion of *fake ES-irreducible components.*

Definition 2.8. An ES-irreducible component $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho(G))$ is *fake* if it is not defined by a single normal form.

As a consequence of Theorems 2.4 and 2.5:

Corollary 2.9. *The strata $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z})$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}((\mathbb{Z}/3\mathbb{Z})^2)$ are not ES-irreducible and each of them has exactly two ES-irreducible components namely, $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_i\,(\mathbb{Z}/3\mathbb{Z}))$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_i\left((\mathbb{Z}/3\mathbb{Z})^2\right))$ respectively with $i=1$ and 2.*

On the other hand, $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2\,(\mathbb{Z}/3\mathbb{Z}))$ is the first example of fake ES-irreducible components. Any $[C]\in\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2\,(\mathbb{Z}/3\mathbb{Z}))$ in the family $\mathcal{C}_2$ has the property that its automorphism group $\mathrm{Aut}(C)=\varrho_2\,(\mathbb{Z}/3\mathbb{Z})$ fixes point-wise the three reference points $P_1=(1:0:0)$, $P_2=(0:1:0)$ and $P_2=(0:0:1)$ that all lie on C. This does not hold if C is in the family $\mathcal{C}_1$ in the sense that $\mathrm{Aut}(C)=\varrho_2\,(\mathbb{Z}/3\mathbb{Z})$ does not fix any points on C.

Corollary 2.10. *The strata $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{S}_3)$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)$ are ES-irreducible. More precisely, $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{S}_3)=\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2(\mathrm{S}_3))$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)=\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_1(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3))$.*

Corollary 2.11. *The stratum $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{A}_4)$ is ES-irreducible determined by $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_1(\mathrm{A}_4))$. It represents the second example of fake ES-irreducible components. Indeed, $\mathcal{C}_{2,\lambda,\mu}$ is K-isomorphic, via a change of variables $\phi=\mathrm{diag}(1,s,t)$ such that $s=t^2$ and $t^3=\zeta_6$, to ${}^{\phi}\mathcal{C}_{2,\lambda,\mu}: X^6+\zeta_3^{-1}Y^6+\zeta_3Z^6+$ lower order terms, where $\mathrm{Aut}({}^{\phi}\mathcal{C}_{2,\lambda,\mu})=\varrho_1(\mathrm{A}_4)$. Moreover, any $[C]\in\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_1(\mathrm{A}_4))$ in the family $\mathcal{C}_{1,\lambda,\mu}$ is a descendant of the Fermat curve $\mathcal{F}_6$ in the sense of Theorem 3.1 via a change of variables in the normalizer of $\varrho_1(\mathrm{A}_4)$ in $\mathrm{PGL}_3(K)$. This does not hold if $[C]$ is in the family ${}^{\phi}\mathcal{C}_{2,\lambda,\mu}$.*

3. Preliminaries about automorphism groups

Based entirely on geometrical methods, H. Mitchell [14, §1-10] proved that if G is a finite subgroups of $\mathrm{PGL}_3(K)$, then it fixes a point, a line or a triangle unless it is primitive and conjugate to some group in a specific list. However, as a consequence of Maschke's theorem in group representation theory, the first two cases are equivalent, in the sense that if G fixes a point (respectively a line), then it also fixes a line not passing through the point (respectively a point not lying the line).

Notations. For a non-zero monomial $cX^{i_1}Y^{i_2}Z^{i_3}$ with $c\in K^*$, its exponent is defined to be $\max\{i_1,i_2,i_3\}$. For a homogenous polynomial $F(X,Y,Z)$, the core of it is defined to be the sum of all terms of F with the greatest exponent. Now, let C_0 be a non-singular plane curve over K, a pair (C, G) with $G \leq \mathrm{Aut}(C)$ is said to be a descendant of C_0 if C is defined by a homogenous polynomial whose core is a defining polynomial of C_0 and G acts on C_0 under a suitable change of the coordinates system, i.e. G is $\mathrm{PGL}_3(K)$-conjugate to a subgroup of $\mathrm{Aut}(C_0)$.

An element of $\mathrm{PGL}_3(K)$ is called *intransitive* if it has the matrix shape

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

The subgroup of $\mathrm{PGL}_3(K)$ of all intransitive elements is denoted by $\mathrm{PBD}(2,1)$. Obviously, there is a natural map $\Lambda : \mathrm{PBD}(2,1) \to \mathrm{PGL}_2(K)$ given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{PBD}(2,1) \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \mathrm{PGL}_2(K).$$

Theorem 3.1 below is very helpful for determining the full automorphism groups of smooth plane curves. For more details, we refer to the work of T. Harui [8, Theroem 2.1].

Theorem 3.1. *Let C be a non-singular plane curve of degree $d \geq 4$ defined over an algebraically closed field K of characteristic 0. Then, one of the following situations holds:*

1. $\mathrm{Aut}(C)$ *fixes a point on C and then it is cyclic.*
2. $\mathrm{Aut}(C)$ *fixes a point not lying on C where we can think about $\mathrm{Aut}(C)$ in the following commutative diagram, with exact rows and vertical injective morphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^* & \longrightarrow & \mathrm{PBD}(2,1) & \xrightarrow{\Lambda} & \mathrm{PGL}_2(K) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & N & \longrightarrow & \mathrm{Aut}(C) & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

Here, N is a cyclic group of order dividing the degree d and G' is a subgroup of $\mathrm{PGL}_2(K)$, which is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m with $m \leq d-1$, a Dihedral group D_{2m} of order $2m$ with $|N| = 1$ or $m|(d-2)$, one of the alternating groups A_4, A_5, or the symmetry group S_4.

Remark 3.2. *We note that N is viewed as the part of $\mathrm{Aut}(C)$ acting on the variable $B \in \{X, Y, Z\}$ and fixing the other two variables, while G' is the part acting on $\{X, Y, Z\} - \{B\}$ and fixing B. For example, if $B = X$, then every automorphism in N has the shape $\mathrm{diag}(\zeta_n, 1, 1)$ for some nth root of unity ζ_n.*

3. $\mathrm{Aut}(C)$ *is conjugate to a subgroup G of $\mathrm{Aut}(\mathcal{F}_d)$, where $\mathcal{F}_d$ is the Fermat curve $X^d + Y^d + Z^d = 0$. In particular, $|G|$ divides $|\mathrm{Aut}(\mathcal{F}_d)| = 6d^2$, and (C, G) is a descendant of $\mathcal{F}_d$.*
4. $\mathrm{Aut}(C)$ *is conjugate to a subgroup G of $\mathrm{Aut}(\mathcal{K}_d)$, where $\mathcal{K}_d$ is the Klein curve curve $X^{d-1}Y + Y^{d-1}Z + XZ^{d-1} = 0$. In this case, $|\mathrm{Aut}(C)|$ divides $|\mathrm{Aut}(\mathcal{K}_d)| = 3(d^2 - 3d + 3)$, and (C, G) is a descendant of $\mathcal{K}_d$.*
5. $\mathrm{Aut}(C)$ *is conjugate to one of the finite primitive subgroup of $\mathrm{PGL}_3(K)$ namely, the Klein group $\mathrm{PSL}(2,7)$, the icosahedral group A_5, the alternating group A_6, or to one of the Hessian groups Hess_* with $* \in \{36, 72, 216\}$.*

Finally, we have:

Proposition 3.3. *The automorphism groups of the Fermat sextic curve $\mathcal{F}_6$ generated by $[X : Z : Y]$, $[Y : Z : X]$, $\operatorname{diag}(\zeta_6, 1, 1)$ and $\operatorname{diag}(1, \zeta_6, 1)$ of orders $2, 3, 6$ and 6 respectively is isomorphic to $\mathrm{GAP}(216, 92) = (\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$. On the other hand, the automorphism group of the Klein sextic curve $\mathcal{K}_6$ generated by $\operatorname{diag}(1, \zeta_{21}, \zeta_{21}^{-4})$ and $[Y : Z : X]$ of orders 21 and 3 respectively is isomorphic to $\mathrm{GAP}(63, 3) = \mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$.*

Proof. Regarding the generators of $\operatorname{Aut}(\mathcal{F}_6)$ and $\operatorname{Aut}(\mathcal{K}_6)$, we refer the reader to [8, Propositions 3.3, 3.5]. Now, for the Fermat curve $\mathcal{F}_6$, take $a = [X : Z : Y]$, $b = [Y : Z : X]$, $c = \operatorname{diag}(\zeta_6, 1, 1)$ and $d = \operatorname{diag}(1, \zeta_6, 1)$. One verifies that

$$(ab)^2 = (ac)(ca)^{-1} = (cd)(dc)^{-1} = ada(cd)^{-5} = bcb^{-1}(cd)^{-5} = 1.$$

These relations give us the 4th semidirect product of $(\mathbb{Z}/6\mathbb{Z})^2$ and S_3 acting faithfully, see semidirect products of $(\mathbb{Z}/6\mathbb{Z})^2$ and S_3 for more details.

For the Klein curve $\mathcal{K}_6$, the two generators $a = \operatorname{diag}(1, \zeta_{21}, \zeta_{21}^{-4})$ and $b = [Y : Z : X]$ of orders 21 and 3 respectively produce $\mathrm{GAP}(63, 3) = \mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ as $ba = (ab)^{-5}$. □

4. Proof of Theorem 2.4

In this case, $C : F(X, Y, Z) = 0$ has an homology σ of period 3 in its automorphism group. The results in [2] allows us to assume that σ acts as

$$(X : Y : Z) \mapsto (X : Y : \zeta_3 Z)$$

up to K-isomorphism, where ζ_3 is a fixed primitive 3rd root of unity in K. In particular, C is defined over K by a non-singular plane equation of the form:

$$C : Z^6 + Z^3 L_{3,Z} + L_{6,Z} = 0,$$

where $\sigma = \operatorname{diag}(1, 1, \zeta_3)$ is an automorphism of maximal order 3. By non-singularity, $L_{6,Z}$ should be of degree at least 5 in both variables X and Y. Also, $L_{3,Z} \neq 0$ or $\operatorname{diag}(1, 1, \zeta_6)$ would be an automorphism of order $6 > 3$.

In the sense of Theorem 3.1, we have the following:

- First, $\operatorname{Aut}(C)$ is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(K)$ since each of them contains elements of order > 3. Also, C is not a descendant of the Klein sextic curve $\mathcal{K}_6$ because $\operatorname{Aut}(\mathcal{K}_6)$ by Proposition 3.3 equals $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ and it does not contains homologies of order 3 similar to σ.
- Secondly, suppose that C is a descendant of the Fermat curve $\mathcal{F}_6$. So there is a $\phi \in \mathrm{PGL}_3(K)$ such that $\phi^{-1} \operatorname{Aut}(C)\phi \leq \operatorname{Aut}(\mathcal{F}_6)$ and the transformed equation ${}^{\phi}C$ is $X^6 + Y^6 + Z^6 +$ lower order terms in $X, Y, Z = 0$. There is no loss of generality to impose $\phi^{-1}\langle\sigma\rangle\phi = \langle\sigma\rangle$ since homologies of period 3 inside $\operatorname{Aut}(\mathcal{F}_6)$ form two conjugacy classes represented by σ and σ^{-1}. Hence ${}^{\phi}C$ reduces to

  $${}^{\phi}C : X^6 + Y^6 + Z^6 + Z^3 L_{3,Z} + \text{lower order terms in } X, Y = 0$$

  Furthermore, by assumption, the automorphisms of C have orders ≤ 3, then the group structure of $\operatorname{Aut}(\mathcal{F}_6) = (\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$ assures that $\operatorname{Aut}({}^{\phi}C)$ would be one of the following groups inside $\operatorname{Aut}(\mathcal{F}_6)$:

  $$\mathbb{Z}/3\mathbb{Z},\ (\mathbb{Z}/3\mathbb{Z})^2,\ \mathrm{S}_3,\ \mathrm{A}_4,\ \mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3,\ \mathrm{He}_3\,.$$

  For more details, check the subgroups lattice of $\operatorname{Aut}(\mathcal{F}_6)$.

  Now we tackle each of the above situations.

  - Any copy of S_3 (respectively A_4) inside $\operatorname{Aut}(\mathcal{F}_6)$ is $\operatorname{Aut}(\mathcal{F}_6)$-conjugate to either $\varrho_i(\mathrm{S}_3)$ (respectively $\varrho_i(\mathrm{A}_4)$) with $i = 1$ or 2. But non of these subgroups has homologies of period 3 similar to σ. So $\mathrm{Aut}(^{\phi}C)$ can not be an S_3 or A_4 inside $\mathrm{Aut}(\mathcal{F}_6)$.

- If $\mathrm{Aut}(^{\phi}C)$ equals a $(\mathbb{Z}/3\mathbb{Z})^2$, $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$ or He_3 in $\mathrm{Aut}(\mathcal{F}_6)$, then there must be $\sigma'\in\mathrm{Aut}(\mathcal{F}_6)\cap\mathrm{Aut}(^{\phi}C)$ of order 3 that commutes with σ as in any of these groups $\mathbb{Z}/3\mathbb{Z}$ is always contained in a $(\mathbb{Z}/3\mathbb{Z})^2$. By Proposition 3.3, the elements of order 3 in $\mathrm{Aut}(\mathcal{F}_6)$ are $\mathrm{diag}(1,s,t)$ with $s^3=t^3=1,[sY:tZ:X]$ and $[tZ:X:sY]$ with $s^6=t^6=1$. One easily verifies that only the diagonal shapes satisfies the description, equivalently, $\sigma'\in\langle\sigma,\mathrm{diag}(1,\zeta_3,1)\rangle$. In any case, we can reduce C up to K-isomorphism to

$$^{\phi}C: X^6+Y^6+Z^6+Z^3\left(\alpha_{3,0}X^3+\alpha_{0,3}Y^3\right)+\alpha_{3,3}X^3Y^3=0,$$

where $\varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)\leq\mathrm{Aut}(^{\phi}C)$.

Remark 4.1. *In this scenario, the parameters $\alpha_{3,0},\alpha_{0,3},\alpha_{3,3}$ must be pair-wise distinct modulo $\{\pm1\}$ or $^{\phi}C$ will admit automorphisms of order >3. For example, $[\zeta_3Y:X:Z]\in\mathrm{Aut}(^{\phi}C)$ has order 6 if $\alpha_{3,0}=\alpha_{0,3}$ and $[\zeta_3Y:X:-Z]\in\mathrm{Aut}(^{\phi}C)$ has order 6 if $\alpha_{3,0}=-\alpha_{0,3}$.*

A similar discussion shows that any $\sigma''\in\mathrm{Aut}(\mathcal{F}_6)$ that commutes with σ or σ' belongs to $\langle\sigma,\sigma'\rangle$. Therefore, $\mathrm{Aut}(^{\phi}C)$ can not be the Heisenberg group He_3 because this requires another automorphism $\sigma''\notin\langle\sigma,\sigma'\rangle$ that commutes with either σ or σ'.

Finally, for $\mathrm{Aut}(^{\phi}C)$ to be $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$, it is necessary that $\mathrm{Aut}(\mathcal{F}_6)\cap\mathrm{Aut}(^{\phi}C)$ has involutions in it. Proposition 3.3 tells us that the involutions of $\mathcal{F}_6$ are $\mathrm{diag}(-1,1,1)$, $\mathrm{diag}(1,-1,1)$, $\mathrm{diag}(1,1,-1)$, $[X:sZ:s^{-1}Y]$, $[s^{-1}Y:sX:Z]$ and $[sZ:Y:s^{-1}X]$ with $s^6=1$. If any of these involutions lies in $\mathrm{Aut}(^{\phi}C)$, then two of the parameters are equal modulo $\{\pm1\}$, which is absurd by Remark 4.1. For example, $\mathrm{diag}(-1,1,1)\in\mathrm{Aut}(^{\phi}C)$ only if $\alpha_{3,0}=\alpha_{3,3}=0,[sY:s^{-1}X:Z]\in\mathrm{Aut}(^{\phi}C)$ only if $\alpha_{3,0}=\pm\alpha_{0,3}$, and so on.

- Third, if $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ and a point P not lying on $\mathcal{L}$, then by Theorem 3.1 we can think about $\mathrm{Aut}(C)$ in a short exact sequence

$$1\to N=\langle\sigma\rangle\to\mathrm{Aut}(C)\to\Lambda(\mathrm{Aut}(C))\to1,$$

where $\Lambda(\mathrm{Aut}(C))\simeq\mathbb{Z}/3\mathbb{Z}$, D_4 or A_4.

- Any group of order 36 (respectively 12) that has a normal subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$ contains elements of order $6>3$, see Groups of order 12 and Groups of order 36 for more details. This allows us to exclude that $\Lambda(\mathrm{Aut}(C))$ equals A_4 or D_4.

- On the other hand, if $\Lambda(\mathrm{Aut})(C)$ equals $\mathbb{Z}/3\mathbb{Z}$ in $\mathrm{PGL}_2(K)$, then $\mathrm{Aut}(C)$ equals $(\mathbb{Z}/3\mathbb{Z})^2$ in $\mathrm{PBD}(2,1)$. In particular, $C:Z^6+Z^3L_{3,Z}+L_{6,Z}=0$ admits an automorphism $\sigma'\in\mathrm{PBD}(2,1)-\langle\sigma\rangle$ of order 3 that commutes with σ. Depending on whether σ' is an homology or a non-homology, it is conjugate via a change of variables $\phi\in\mathrm{PBD}(2,1)$, the normalizer of $\langle\sigma\rangle$, to $\mathrm{diag}(1,\zeta_3,1)$ or $\mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ respectively. In either way, $\mathrm{Aut}(^{\phi}C)=\varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ which appeared earlier.

Summing up, we deduce that $\mathrm{Aut}(C)$ is always cyclic of order 3 generated by σ except when C is projectively equivalent to C' of the form

$$C': X^6+Y^6+Z^6+Z^3\left(\alpha_{3,0}X^3+\alpha_{0,3}Y^3\right)+\alpha_{3,3}X^3Y^3=0,$$

such that $\alpha_{3,0},\alpha_{0,3},\alpha_{3,3}$ are pair-wise distinct modulo $\{\pm1\}$. In this case, $\mathrm{Aut}(C)$ is conjugate to $(\mathbb{Z}/3\mathbb{Z})^2$ generated by $\mathrm{diag}(1,\zeta_3,1)$ and $\mathrm{diag}(1,\zeta_3,1)$.

This proves Theorem 2.4.

5. Proof of Theorem 2.1

In this case, $C : F(X,Y,Z) = 0$ has an homology σ of period 2 in its automorphism group. By [2], there is no loss of generality to assume that σ acts as

$$(X : Y : Z) \mapsto (X : Y : -Z)$$

up to K-isomorphism. In particular, C is defined over K by a non-singular plane equation of the form:

$$C : Z^6 + Z^4 L_{2,Z} + Z^2 L_{4,Z} + L_{6,Z} = 0$$

where $\sigma = \mathrm{diag}(1,1,-1)$ is an automorphism of maximal order 2. Again $L_{6,Z}$ is of degree ≥ 5 in X and Y by non-singularity. Also, $L_{3,Z}$ or $L_{4,Z}$ does not vanish or $\mathrm{diag}(1,1,\zeta_4)$ will be an automorphism of order $4 > 3$ otherwise.

- Obviously, $\mathrm{Aut}(C)$ is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(K)$ as each of them contains elements of order > 2. Also, C can not be a descendant of the Klein sextic curve $\mathcal{K}_6$ since $2 \nmid |\,\mathrm{Aut}(\mathcal{K}_6)|$, recall that $|\,\mathrm{Aut}(\mathcal{K}_6)| = 63$ by Proposition 3.3.
- Secondly, if $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ and a point P off $\mathcal{L}$, then, by Theorem 3.1, $\mathrm{Aut}(C)$ is inside $\mathrm{PBD}(2,1)$ and satisfies a short exact sequence

  $$1 \to N = \langle\sigma\rangle \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1.$$

  Our assumptions that any automorphism of C has order ≤ 2 implies that $\Lambda(\mathrm{Aut}(C))$ is either $\mathbb{Z}/2\mathbb{Z}$ or D_4 inside $\mathrm{PGL}_2(K)$, so $\mathrm{Aut}(C)$ is conjugate to either $(\mathbb{Z}/2\mathbb{Z})^2$ or $(\mathbb{Z}/2\mathbb{Z})^3$. In both situations $\mathrm{Aut}(C)$ has another involution σ' that commutes with σ. Up to projective equivalence via a change of variables $\phi \in \mathrm{PBD}(2,1)$, the normalizer of $\langle\sigma\rangle$ in $\mathrm{PGL}_3(K)$, we can assume that $\sigma' = \mathrm{diag}(1,-1,1)$. Consequently, C is K-isomorphic to $C' : Z^6 + Z^4 L_{2,Z} + Z^2 L_{4,Z} + L_{6,Z} = 0$ for some $L_{i,Z} \in K[X^2, Y^2]$. Moreover, $\mathrm{Aut}(C)$ equals $(\mathbb{Z}/2\mathbb{Z})^3$ only if there is an involution $\sigma'' \notin \mathrm{PBD}(2,1) - \langle\sigma,\sigma'\rangle$ that commutes with both σ and σ'. It is straightforward to check that such σ'' does not exist, hence $\mathrm{Aut}(C)$ is not $(\mathbb{Z}/2\mathbb{Z})^3$ in this case.
- If C is a descendant of the Fermat curve $\mathcal{F}_6$ via a change of variables $\phi \in \mathrm{PGL}_3(K)$ with bigger automorphism group than $\langle\sigma\rangle$, then $\mathrm{Aut}(^{\phi}C)$ is a copy of $(\mathbb{Z}/2\mathbb{Z})^2$ inside $\mathrm{Aut}(\mathcal{F}_6)$. Indeed any other subgroup of $\mathrm{Aut}(\mathcal{F}_6)$ has elements of order > 2, see subgroups lattice of $\mathrm{Aut}(\mathcal{F}_6)$.

  Up to $\mathrm{Aut}(\mathcal{F}_6)$-conjugation, there are two copies of $(\mathbb{Z}/2\mathbb{Z})^2$ inside $\mathrm{Aut}(\mathcal{F}_6)$ namely, $\langle\sigma,\sigma'\rangle$ and $\langle\sigma,\tau\rangle$ with $\sigma' = \mathrm{diag}(1,-1,1)$ and $\tau = [Y : X : Z]$. However, both groups are $\mathrm{PGL}_3(K)$-conjugated via a transformation in $\mathrm{PBD}(2,1)$, the normalizer of $\langle\sigma\rangle$ in $\mathrm{PGL}_3(K)$. Thus there is no loss of generality to assume that $\mathrm{Aut}(C)$ is conjugate to $\varrho_1\left((\mathbb{Z}/2\mathbb{Z})^2\right)$, which was treated earlier.

Summing up, we deduce that $\mathrm{Aut}(C)$ is always cyclic of order 2 generated by σ except when $L_{i,Z} \in K[X^2, Y^2]$ for $i = 2,4,6$. In the latter case, $\mathrm{Aut}(C)$ equals $\varrho_1\left((\mathbb{Z}/2\mathbb{Z})^2\right)$, which shows Theorem 2.1.

6. Proof of Theorem 2.5

In this case, $C : F(X,Y,Z) = 0$ has a non-homology σ of period 3 in its automorphism group. By [2], one can assume that σ acts as

$$(X : Y : Z) \mapsto (X : \zeta_3 Y : \zeta_3^{-1} Z)$$

up to K-isomorphism, where ζ_3 is a fixed primitive 3rd root of unity in K. In particular, C is a K-isomorphic to a non-singular plane modelin one of the following families:

$$\begin{aligned}\mathcal{C}_1 \quad &: \quad X^6+Y^6+Z^6+XYZ\left(\alpha_{4,1}X^3+\alpha_{1,4}Y^3+\alpha_{1,2}Z^3\right)+\alpha_{2,2}X^2Y^2Z^2\\ &+ \quad \alpha_{3,3}X^3Y^3+\alpha_{3,0}X^3Z^3+\alpha_{0,3}Y^3Z^3=0\\ \mathcal{C}_2 \quad &: \quad X^5Y+Y^5Z+XZ^5+XYZ\left(\alpha_{3,2}X^2Y+\alpha_{1,3}Y^2Z+\alpha_{2,1}XZ^2\right)\\ &+ \quad \alpha_{2,4}X^2Y^4+\alpha_{0,2}Y^2Z^4+\alpha_{4,0}X^4Z^2=0.\end{aligned}$$

where $\sigma := \mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ is an automorphism of maximal order 3.

- Again $\mathrm{Aut}(\mathcal{C}_i)$ for $i=1$ and 2 is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(K)$.
- Suppose that $\mathrm{Aut}(\mathcal{C}_i)$ fixes a line $\mathcal{L}$ and a point P not lying on this line. Since σ is a non-homology inside $\mathrm{Aut}(\mathcal{C}_i)$ in its canonical form, $\mathcal{L}$ must be one of the reference lines; $B=0$ with $B=X,Y$ or Z and P is the reference point $(1:0:0)$, $(0:1:0)$ or $(0:0:1)$ respectively.

  - For $\mathcal{C}_2$, the point P belongs to $C: F(X,Y,Z)=0$. Hence $\mathrm{Aut}(\mathcal{C}_2)$ is cyclic, generated by $\langle\sigma\rangle$.

  - For $\mathcal{C}_1$, we can further impose $\mathcal{L}: X=0$ and $P=(1:0:0)$ (in the worst case scenario, one just needs to permute two of the variables and to fix the third one, which preserves the property that σ remains an automorphism). In particular, by Theorem 3.1, $\mathrm{Aut}(\mathcal{C}_1)\subseteq \mathrm{PBD}(2,1)$ and lives in a short exact sequence: $1\to N\to \mathrm{Aut}(\mathcal{C}_1)\to \Lambda(\mathrm{Aut}(\mathcal{C}_1))\to 1$, where $N=\langle\tau\rangle$ has order $1,2$ or 3 and $\Lambda(\mathrm{Aut}(C))$ is either $\mathbb{Z}/3\mathbb{Z}$, S_3 with $|N|=1$ or A_4 in $\mathrm{PGL}_2(K)$. First, we easily exclude the case when τ has order 2 because $\sigma\tau$ would be an automorphism of order $6>3$, a contradiction.

  Secondly, we handle each of the remaining cases:

  (i) If $\Lambda(\mathrm{Aut}(\mathcal{C}_1))=\mathbb{Z}/3\mathbb{Z}$ and $N=1$, then $\mathrm{Aut}(\mathcal{C}_1)=\mathbb{Z}/3\mathbb{Z}$ generated by σ.

  (ii) If $\Lambda(\mathrm{Aut}(\mathcal{C}_1))=\mathbb{Z}/3\mathbb{Z}$ and $N=\mathbb{Z}/3\mathbb{Z}$, then $\mathrm{Aut}(\mathcal{C}_1)=\varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$ generated by σ and $\tau=\mathrm{diag}(\zeta_3,1,1)$. In particular, $\alpha_{4,1}=\alpha_{2,2}=\alpha_{1,2}=\alpha_{1,4}=0$, and $\mathcal{C}_1$ reduces to

  $$X^6+Y^6+Z^6+Z^3\left(\alpha_{3,0}X^3+\alpha_{0,3}Y^3\right)+\ \alpha_{3,3}X^3Y^3=0,$$

  which happened before in Theorem 2.4.
  This shows Theorem 2.5, (1)-(i).

  (iii) If $\Lambda(\mathrm{Aut}(\mathcal{C}_1))=\mathrm{S}_3$ and $N=1$, then C should have an involution τ such that $\tau\sigma\tau=\sigma^{-1}$. So $\tau=[X:sZ:s^{-1}Y]$, $[sY:s^{-1}X:Z]$ or $[sZ:Y:s^{-1}X]$ with $s^6=1$. This holds if we are in one of the situations: $\alpha_{3,3}=\pm\alpha_{3,0}$ and $\alpha_{1,2}=\pm\alpha_{1,4},\alpha_{0,3}=\pm\alpha_{3,0}$ and $\alpha_{4,1}=\pm\alpha_{1,4}$, or $\alpha_{3,3}=\pm\alpha_{0,3}$ and $\alpha_{1,2}=\pm\alpha_{4,1}$. Moreover, in all scenarios we can reduce to $\tau=[X:Z:Y]$ via a change of variables ϕ in the normalizer of $\langle\sigma\rangle$, more precisely, via $\phi=\mathrm{diag}(1,\lambda,s\lambda)$ modulo $\langle[X:Z:Y],[Y:Z:X]\rangle$ with $\lambda^6=1$. That is, $\mathcal{C}_1$ is K-isomorphic to

$$\begin{aligned}\mathcal{C}_1' \quad &: \quad X^6+Y^6+Z^6+\alpha_{4,1}'X^4YZ+\alpha_{3,3}'X^3(Y^3+Z^3)+\alpha_{2,2}'X^2Y^2Z^2\\ &+ \quad \alpha_{1,2}'XYZ(Y^3+Z^3)+\alpha_{0,3}'Y^3Z^3=0.\end{aligned}$$

  Here $\mathrm{Aut}(\mathcal{C}_1')=\langle\sigma,\,\tau\rangle=\varrho_1\left(\mathrm{S}_3\right)$. In particular, we should impose $\alpha_{4,1}'\neq\alpha_{1,2}'$ or $\alpha_{3,3}'\neq\alpha_{0,3}'$ to avoid having $[Y:Z:X]$ as an extra automorphism. Also, $\left(\alpha_{3,3}',\alpha_{1,2}'\right)\neq(0,0)$ to avoid having $\mathrm{diag}(1,\zeta_6,\zeta_6^{-1})$ as an extra automorphism of order $6>3$.
  This shows part of Theorem 2.5, (1)-(ii).

(iv) If $\Lambda(\mathrm{Aut}(C)) = \mathrm{A}_4$, then the Group Structure of A_4 assures that $\Lambda(\mathrm{Aut}(C))$ contains $\Lambda(\tau)$ and $\Lambda(\tau')$ both of order 2 such that

$$\Lambda(\sigma)\Lambda(\tau)\Lambda(\sigma)^{-1} = \Lambda(\tau'),\ \Lambda(\sigma)\Lambda(\tau')\Lambda(\sigma)^{-1} = \Lambda(\tau')\Lambda(\tau) = \Lambda(\tau)\Lambda(\tau').$$

We aim to show that such τ and τ' do not exist. Write $\Lambda(\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then being of order 2 yields $(a+d)b = (a+d)c = 0$ and $a = \pm d$. So $\Lambda(\tau) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

- If $\Lambda(\tau) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, then

$$\Lambda(\tau') = \Lambda(\sigma)\Lambda(\tau)\Lambda(\sigma)^{-1} = \begin{pmatrix} 0 & \zeta_3^{-1}b \\ \zeta_3^{-1}c & 0 \end{pmatrix} = \Lambda(\tau) \text{ in } \mathrm{PGL}_2(K),$$

a contradiction.

- If $\Lambda(\tau) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then $\Lambda(\tau') = \Lambda(\sigma)\Lambda(\tau)\Lambda(\sigma)^{-1} = \begin{pmatrix} a & \zeta_3^{-1}b \\ \zeta_3^{-1}c & -a \end{pmatrix}$ such that $\Lambda(\tau)\Lambda(\tau') = \Lambda(\tau')\Lambda(\tau)$. That is,

$$\begin{pmatrix} a^2+\zeta_3 bc & (\zeta_3^{-1}-1)ab \\ (1-\zeta_3)ac & a^2+\zeta_3^{-1}bc \end{pmatrix} = \begin{pmatrix} a^2+\zeta_3^{-1}bc & -(\zeta_3^{-1}-1)ab \\ -(1-\zeta_3)ac & a^2+\zeta_3 bc \end{pmatrix} \text{ in } \mathrm{PGL}_2(K).$$

For this to be true, either $ab = ac = 0$ or $a^2+\zeta_3 bc = -(a^2+\zeta_3^{-1}bc)$. Assuming $ab = ac = 0$ yields $\Lambda(\tau') = \begin{pmatrix} 0 & \zeta_3^{-1}b \\ \zeta_3^{-1}c & 0 \end{pmatrix} = \Lambda(\tau)$ in $\mathrm{PGL}_2(K)$ or $\Lambda(\tau') = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = \Lambda(\tau)$ in $\mathrm{PGL}_2(K)$, which is again a contradiction. Assuming $a^2+\zeta_3 bc = -(a^2+\zeta_3^{-1}bc)$ yields $c = 2a^2/b$ with $ab \neq 0$. Moreover, $\Lambda(\sigma)\Lambda(\tau')\Lambda(\sigma)^{-1} = \Lambda(\tau)\Lambda(\tau')$, hence

$$\begin{pmatrix} a & \zeta_3 b \\ 2a^2/b & -a \end{pmatrix} = \begin{pmatrix} a(\zeta_3-\zeta_3^{-1}) & (\zeta_3^{-1}-1)b \\ 2a^2(1-\zeta_3)/b & -a(\zeta_3-\zeta_3^{-1}) \end{pmatrix} \text{ in } \mathrm{PGL}_2(K).$$

This is valid only if $(\zeta_3-\zeta_3^{-1})\zeta_3 = (\zeta_3^{-1}-1)$ and $(\zeta_3-\zeta_3^{-1}) = (1-\zeta_3)$, however, the second equation is never valid. This means that $\Lambda(\mathrm{Aut}(C)) \neq \mathrm{A}_4$.

- Thirdly, assume that $\mathcal{C}_i$ is a descendant of the Klein sextic curve $\mathcal{K}_6$.

Claim 1. *For $\mathcal{C}_1$, $\mathrm{Aut}(\mathcal{C}_1) = \varrho_2(\mathbb{Z}/3\mathbb{Z})$.*

Proof. (of Claim 1) If $\mathcal{C}_1$ is a descendant of $\mathcal{K}_6$ with bigger automorphism group than $\langle\sigma\rangle$, then, from the Group Structure of $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ and since the automorphisms of C have orders ≤ 3, $\mathrm{Aut}(\mathcal{C}_1)$ should be conjugate to a $(\mathbb{Z}/3\mathbb{Z})^2$ in $\mathrm{Aut}(\mathcal{K}_6)$. Thus $\mathcal{C}_1$ has another automorphism $\sigma' \notin \langle\sigma\rangle$ of order 3 that commutes with σ. Direct calculations show that we can take $\sigma' = \mathrm{diag}(1, s, t)$ with $s^3 = t^3 = 1$ or $[sY : tZ : X]$ with $s, t \in K^*$.

In the first case, σ' reduces to an homology as $\sigma' \notin \langle\sigma\rangle$. This is absurd because $\mathrm{Aut}(\mathcal{K}_6)$ does not contain any homologies of period 3. Regarding the second case, any descendant $\mathcal{C}'$ of the Klein curve $\mathcal{C}' : X^5Y + Y^5Z + Z^5X +$ lower terms in X, Y, Z satisfies the property that its automorphism group fixes the triangle Δ whose vertices are the three reference points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$, moreover, those points all lie on $\mathcal{C}'$. Because Δ is the only triangle fixed by $\langle\sigma, [sY : tZ : X]\rangle$ for any s, t and because non of its vertices lies on $\mathcal{C}_1$, we conclude that $\mathrm{Aut}(\mathcal{C}_1)$ can not equal $\langle\sigma,[sY:tZ:X]\rangle$. This proves the claim for $\mathcal{C}_1$. □

Claim 2. *For* $\mathcal{C}_2$, $\mathrm{Aut}(\mathcal{C}_2)$ *is either conjugate to* $\varrho_2(\mathbb{Z}/3\mathbb{Z})$ *or* $\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)$.

Proof. (of Claim 2) Similarly if $\mathcal{C}_2$ is a descendant of $\mathcal{K}_6$ with bigger automorphism group than $\langle\sigma\rangle$, then $\mathrm{Aut}(\mathcal{C}_2)=\langle\sigma,[sY:tZ:X]\rangle$ for some $s,t\in K^*$. For $\sigma'\in\mathrm{Aut}(\mathcal{C}_2)$, $s=\zeta_{21}^r$, $t=\zeta_{21}^{-4r}$, $\alpha_{0,2}=\zeta_{21}^{-12r}\alpha_{4,0}$, $\alpha_{2,4}=\zeta_{21}^{3r}\alpha_{4,0}$, $\alpha_{1,3}=\zeta_{21}^{-6r}\alpha_{3,2}$, $\alpha_{2,1}=\zeta_{21}^{3r}\alpha_{3,2}$, and $\mathcal{C}_2$ reduces to

$$\begin{aligned}X^5Y &+ Y^5Z+XZ^5+\alpha_{4,0}\left(X^4Z^2+\zeta_{21}^{3r}X^2Y^4+\zeta_{21}^{-12r}Y^2Z^4\right)\\ &+ \alpha_{3,2}XYZ\left(X^2Y+\zeta_{21}^{3r}XZ^2+\zeta_{21}^{-6r}Y^2Z\right)=0.\end{aligned}$$

In any situation, there exists a change of variables $\phi=\mathrm{diag}(1,\zeta_{21}^{r'},\zeta_{21}^{17r'})\in\mathrm{Aut}(\mathcal{K}_6)$ such that $21\,|\,18r'+r,\ 12r'-4r$ for some $r'\in\{0,1,...,20\}$ that transforms $\mathcal{C}_2$ up to K-isomorphism to

$$\begin{aligned}\mathcal{C}_2':X^5Y &+ Y^5Z+XZ^5+\alpha_{4,0}\zeta_{21}^{4r}\left(X^4Z^2+X^2Y^4+Y^2Z^4\right)\\ &+ \alpha_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y+XZ^2+Y^2Z\right)=0,\end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}_2')=\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)=\langle\sigma,[Y:Z:X]\rangle$. In particular, we must have $(\alpha_{2,4},\alpha_{1,3})\neq(0,0)$ or $\mathrm{diag}(1,\zeta_{21},\zeta_{21}^{-4})\in\mathrm{Aut}(\mathcal{C}_2')$ of order $21>3$.

This completes the proof, which in turns shows Theorem 2.5, (2)-(i). □

- Now, assume that $\mathcal{C}_i$ is a descendant of the Fermat curve $\mathcal{F}_6$. From the Group structure of $\mathrm{Aut}(\mathcal{F}_6)$, one sees that if $\mathcal{C}_i$ is a descendant of $\mathcal{F}_6$ with bigger automorphism group than $\langle\sigma\rangle$, then $\mathrm{Aut}(\mathcal{C}_i)$ is conjugate to one of the following groups inside $\mathrm{Aut}(\mathcal{F}_6)$:

$$(\mathbb{Z}/3\mathbb{Z})^2,\ \mathrm{S}_3,\ \mathrm{A}_4,\ \mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3,\ \mathrm{He}_3\,.$$

In what follows, we treat each of these cases for $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, more precisely, Claim 3 and Claim 4 below.

Claim 3. *For* $\mathcal{C}_1$, $\mathrm{Aut}(\mathcal{C}_1)$ *is conjugate to* $\varrho_2(\mathbb{Z}/3\mathbb{Z})$, $\varrho_2(\mathrm{S}_3)$, $\varrho_1(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)$, $\varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ *or* $\varrho_1(\mathrm{A}_4)$.

Claim 4. *For* $\mathcal{C}_2$, $\mathrm{Aut}(\mathcal{C}_2)$ *is conjugate to* $\varrho_2(\mathbb{Z}/3\mathbb{Z})$, $\varrho_2((\mathbb{Z}/3\mathbb{Z})^2)$ *or* $\varrho_2(\mathrm{A}_4)$.

Proof. (of Claim 3) - If $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to S_3 or $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$ inside $\mathrm{Aut}(\mathcal{F}_6)$, then $\mathcal{C}_1$ has an involution τ such that $\tau\sigma\tau=\sigma^{-1}$. Similarly as before, this holds only if $\alpha_{3,3}=\pm\alpha_{3,0}$ and $\alpha_{1,2}=\pm\alpha_{1,4},\alpha_{0,3}=\pm\alpha_{3,0}$ and $\alpha_{4,1}=\pm\alpha_{1,4}$, or $\alpha_{3,3}=\pm\alpha_{0,3}$ and $\alpha_{1,2}=\pm\alpha_{4,1}$. In this scenario, $\mathcal{C}_1$ is K-isomorphic to

$$\begin{aligned}\mathcal{C}_1' \quad:\quad & X^6+Y^6+Z^6+\alpha_{4,1}'X^4YZ+\alpha_{3,3}'X^3(Y^3+Z^3)+\alpha_{2,2}'X^2Y^2Z^2\\ + \quad & \alpha_{1,2}'XYZ(Y^3+Z^3)+\alpha_{0,3}'Y^3Z^3=0,\end{aligned}$$

where $\varrho_2(\mathrm{S}_3)$ generated by $\sigma=\mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ and $\tau=[X:Z:Y]$ is a subgroup of $\mathrm{Aut}(\mathcal{C}_1')$. Furthermore, if $\mathrm{Aut}(\mathcal{C}_1')$ equals $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$, then it must contain another automorphism $\sigma'\notin\langle\sigma,\tau\rangle$ of order 3 that commutes with σ and satisfies $\tau\sigma'\tau=\sigma'^{-1}$. Thus $\sigma'=[s'Y:s'^{-1}Z:X]$ and the invariance of the defining equation for $\mathcal{C}_1'$ under the action of σ' yields $s'^3=1,\alpha_{4,1}'=\alpha_{1,2}'$ and $\alpha_{3,3}'=\alpha_{0,3}'$. Hence $\mathcal{C}_1'$ becomes

$$\begin{aligned}X^6 &+ Y^6+Z^6+\alpha_{1,2}'XYZ(X^3+Y^3+Z^3)+\alpha_{3,3}'(X^3Y^3+Y^3Z^3+Z^3X^3)\\ &+ \alpha_{2,2}'X^2Y^2Z^2=0\end{aligned}$$

with $\mathrm{Aut}(\mathcal{C}_1')=\varrho_1(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)$. This shows the rest of Theorem 2.5, (1)-(ii).

- If $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to $(\mathbb{Z}/3\mathbb{Z})^2$ or He_3 inside $\mathrm{Aut}(\mathcal{F}_6)$, then $\mathcal{C}_1$ would have an automorphism $\sigma'\notin\langle\sigma\rangle$ of order 3 that commutes with σ since every copy of

$\mathbb{Z}/3\mathbb{Z}$ in any of these groups is contained in a $(\mathbb{Z}/3\mathbb{Z})^2$. Moreover, we guarantee that $\alpha_{2,2} \neq 0$ or $[X : \zeta_3 Z : \zeta_3^{-1} Y]$ would be an involution in $\mathrm{Aut}(\mathcal{C}_1)$, which is not accepted in this case.

Similarly as before, we can take $\sigma' = \mathrm{diag}(1, s', t')$ with $s'^3 = t'^3 = 1$ or $[s'Y : t'Z : X]$ with $s', t' \in K^*$.

(i) Suppose that $\sigma' = \mathrm{diag}(1, s', t') \in \mathrm{Aut}(\mathcal{C}_1)$. Because $\sigma' \notin \langle\sigma\rangle$, we have $\sigma' = \mathrm{diag}(1, 1, \zeta_3)$, $\mathrm{diag}(\zeta_3, 1, 1)$ or $\mathrm{diag}(1, \zeta_3, 1)$. Consequently, $\alpha_{4,1} = \alpha_{2,2} = \alpha_{1,2} = \alpha_{1,4} = 0$ and $\mathcal{C}_1$ reduces to

$$X^6 + Y^6 + Z^6 + \alpha_{3,3} X^3 Y^3 + \alpha_{3,0} X^3 Z^3 + \alpha_{0,3} Y^3 Z^3 = 0,$$

with $\varrho_1((\mathbb{Z}/3\mathbb{Z})^2) \subseteq \mathrm{Aut}(\mathcal{C}_1)$. On the other hand, $\mathrm{Aut}(\mathcal{C}_1)$ equals He_3 only if it contains an extra automorphism $\sigma'' \notin \langle\sigma, \sigma'\rangle$ of order 3 that commutes with σ and satisfies $\sigma''\sigma'\sigma''^{-1} = \sigma'\sigma^{-1}$. This gives us $\sigma'' = [s''Y : t''Z : X]$ for some $s'', t'' \in K^*$. Hence $s''^6 = t''^6 = 1$, $\alpha_{3,3} = s''^3\alpha_{3,0}$, $\alpha_{0,3} = t''^3\alpha_{3,0}$, and $\mathcal{C}_1$ becomes of the form:

$$X^6 + Y^6 + Z^6 + \alpha_{3,0}\left(\pm X^3Y^3 + X^3Z^3 + t''^3 Y^3 Z^3\right) = 0.$$

In particular, $[Y : X : t''Z]$ is an automorphism for $\mathcal{C}_1$ of order divisible by 2. This is a contradiction as $2 \nmid |\mathrm{He}_3|(= 27)$.

(ii) Suppose that $\sigma' = [s'Y : t'Z : X] \in \mathrm{Aut}(\mathcal{C}_1)$. Thus $s'^3 = t'^6 = (s't')^2 = 1$, $\alpha_{1,4} = \pm\alpha_{4,1}$, $\alpha_{1,2} = \pm\alpha_{4,1}$, $\alpha_{3,0} = \alpha_{3,3}$, $\alpha_{0,3} = \pm\alpha_{3,3}$, and $\mathcal{C}_1$ is defined by

$$\begin{aligned} X^6 &+ Y^6 + Z^6 + \alpha_{4,1} XYZ(X^3 \pm Y^3 \pm Z^3) + \alpha_{2,2} X^2Y^2Z^2 \\ &+ \alpha_{3,3}(X^3Y^3 + X^3Z^3 \pm Y^3Z^3) = 0, \end{aligned}$$

Hence $[X : Z : Y]$ is an involution for $\mathcal{C}_1$, which is not true if $|\mathrm{Aut}(\mathcal{C}_1)| = 9$ or 27.

- If $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to an A_4 inside $\mathrm{Aut}(\mathcal{F}_6)$, then it should be $\varrho_i(\mathrm{A}_4)$ with $i = 1$ or 2.

(i) First, suppose that $\phi^{-1}\,\mathrm{Aut}(\mathcal{C}_1)\phi = \varrho_1(\mathrm{A}_4)$. As all subgroups of A_4 of order 3 are A_4-conjugated, there is no loss of generality to take $\phi^{-1}\sigma\phi = [Y : Z : X]$ or $[Z : X : Y]$. In particular, ϕ has one of the following shapes:

$$\phi_1 := \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \zeta_3^{-1}\lambda & \zeta_3\lambda \\ \mu & \zeta_3\mu & \zeta_3^{-1}\mu \end{pmatrix},\ \phi_2 := \begin{pmatrix} \mu & \zeta_3\mu & \zeta_3^{-1}\mu \\ 1 & 1 & 1 \\ \lambda & \zeta_3^{-1}\lambda & \zeta_3\lambda \end{pmatrix},\ \phi_3 := \begin{pmatrix} \lambda & \zeta_3^{-1}\lambda & \zeta_3\lambda \\ \mu & \zeta_3\mu & \zeta_3^{-1}\mu \\ 1 & 1 & 1 \end{pmatrix},$$

$$\phi_4 := \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \zeta_3\lambda & \zeta_3^{-1}\lambda \\ \mu & \zeta_3^{-1}\mu & \zeta_3\mu \end{pmatrix},\ \phi_5 := \begin{pmatrix} \mu & \zeta_3^{-1}\mu & \zeta_3\mu \\ 1 & 1 & 1 \\ \lambda & \zeta_3\lambda & \zeta_3^{-1}\lambda \end{pmatrix},\ \phi_6 := \begin{pmatrix} \lambda & \zeta_3\lambda & \zeta_3^{-1}\lambda \\ \mu & \zeta_3^{-1}\mu & \zeta_3\mu \\ 1 & 1 & 1 \end{pmatrix},$$

for some $\lambda, \mu \in K^*$.

Now, we handle each of these situations to determine the restrictions on the defining equation of $\mathcal{C}_1$ for which this holds.

- For $\phi_1\,\mathrm{diag}(1, 1, -1)\phi_1^{-1}$ (respectively $\phi_4\,\mathrm{diag}(1, 1, -1)\phi_4^{-1}$) to be in $\mathrm{Aut}(\mathcal{C}_1)$, we must eliminate the coefficients of X^5Z, X^5Y, Y^5Z, XZ^5, YZ^5, X^4Y^2, X^4Z^2 from the transformed equation ${}^{\phi_i\,\mathrm{diag}(1,1,-1)\phi_i^{-1}}\mathcal{C}_1 =$

$\mathcal{C}_1$ with $i=1$ and 4 respectively. In this way, we obtain:

$$\begin{aligned}\alpha_{4,1} &= \frac{2\left(29-54\lambda^6-54\mu^6\right)}{27\lambda\mu}, \alpha_{3,3}=\frac{2\left(81\mu^6-27\lambda^6-26\right)}{27\lambda^3},\\ \alpha_{3,0} &= \frac{2\left(81\lambda^6-27\mu^6-26\right)}{27\mu^3}, \alpha_{1,4}=\frac{2\left(27\lambda^6-54\mu^6-52\right)}{27\lambda^4\mu},\\ \alpha_{1,2} &= \frac{2\left(27\mu^6-54\lambda^6-52\right)}{27\lambda\mu^4}, \alpha_{0,3}=\frac{2\left(82-27\lambda^6-27\mu^6\right)}{27\lambda^3\mu^3},\\ \alpha_{2,2} &= \frac{9\lambda^6+9\mu^6+10}{3\lambda^2\mu^2}.\end{aligned}$$

In particular, $\mathcal{C}_1$ is K-isomorphic via ϕ_1 (respectively ϕ_4 followed by $Y\leftrightarrow Z$) to $\mathcal{C}_{1,\lambda,\mu}$ described in Theorem 2.5, (1)-(iii).

- For $\phi_2\,\mathrm{diag}(1,1,-1)\phi_2^{-1}$ (respectively $\phi_5\,\mathrm{diag}(1,1,-1)\phi_5^{-1}$) to be in $\mathrm{Aut}(\mathcal{C}_1)$, one notices that $\phi_2=[Z:X:Y]\,\phi_1=\phi_1\circ[Z:X:Y]$ (respectively $\phi_5=[Z:X:Y]\,\phi_4=\phi_4\circ[Z:X:Y]$). This means that we get the same conclusion as above up to a permutation of the parameters, more precisely, after

$$\begin{aligned}(\alpha_{4,1},\alpha_{1,2},\alpha_{1,4}) &\mapsto (\alpha_{1,2},\alpha_{1,4},\alpha_{4,1}),\\ (\alpha_{0,3},\alpha_{3,3},\alpha_{3,0}) &\mapsto (\alpha_{3,3},\alpha_{3,0},\alpha_{0,3}).\end{aligned}$$

In other words, we have $\phi_i\,\mathrm{diag}(1,1,-1)\phi_i^{-1}$ with $i=2$ or 5 inside $\mathrm{Aut}(\mathcal{C}_1)$ only if

$$\begin{aligned}\alpha_{1,4} &= \frac{2\left(29-54\lambda^6-54\mu^6\right)}{27\lambda\mu}, \alpha_{0,3}=\frac{2\left(81\mu^6-27\lambda^6-26\right)}{27\lambda^3},\\ \alpha_{3,3} &= \frac{2\left(81\lambda^6-27\mu^6-26\right)}{27\mu^3}, \alpha_{1,2}=\frac{2\left(27\lambda^6-54\mu^6-52\right)}{27\lambda^4\mu},\\ \alpha_{4,1} &= \frac{2\left(27\mu^6-54\lambda^6-52\right)}{27\lambda\mu^4}, \alpha_{3,0}=\frac{2\left(82-27\lambda^6-27\mu^6\right)}{27\lambda^3\mu^3},\\ \alpha_{2,2} &= \frac{9\lambda^6+9\mu^6+10}{3\lambda^2\mu^2}.\end{aligned}$$

Once more $\mathcal{C}_1$ reduces to $\mathcal{C}_{1,\lambda,\mu}$ described in Theorem 2.5, (1)-(iii).
Similarly, $\phi_3=\phi_1\circ[Y:Z:X]$ and $\phi_6=\phi_4\circ[Y:Z:X]$. So $\phi_i\,\mathrm{diag}(1,1,-1)\phi_i^{-1}$ with $i=3$ or 6 is an automorphism for $\mathcal{C}_1$ only if

$$\begin{aligned}\alpha_{1,2} &= \frac{2\left(29-54\lambda^6-54\mu^6\right)}{27\lambda\mu}, \alpha_{3,0}=\frac{2\left(81\mu^6-27\lambda^6-26\right)}{27\lambda^3},\\ \alpha_{0,3} &= \frac{2\left(81\lambda^6-27\mu^6-26\right)}{27\mu^3}, \alpha_{4,1}=\frac{2\left(27\lambda^6-54\mu^6-52\right)}{27\lambda^4\mu},\\ \alpha_{1,4} &= \frac{2\left(27\mu^6-54\lambda^6-52\right)}{27\lambda\mu^4}, \alpha_{3,3}=\frac{2\left(82-27\lambda^6-27\mu^6\right)}{27\lambda^3\mu^3},\\ \alpha_{2,2} &= \frac{9\lambda^6+9\mu^6+10}{3\lambda^2\mu^2},\end{aligned}$$

where $\mathcal{C}_1$ becomes K-isomorphism to $C_{1,\lambda,\mu}$.
This shows Theorem 2.5, (1)-(iii).

(ii) Second, suppose that $\psi^{-1}\,\mathrm{Aut}(\mathcal{C}_1)\psi=\varrho_2(\mathrm{A}_4)$. Again, we can impose $\psi^{-1}\sigma\psi=[\zeta_6^{-1}Y:Z:X]$ or $[Z:\zeta_6X:Y]$, in particular, ψ has the shape of ψ_i below.

$$\psi_1 := \begin{pmatrix} 1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \\ \lambda & \zeta_{18}^{-8}\lambda & \zeta_{18}^{5}\lambda \\ \mu & \zeta_{18}^{4}\mu & \zeta_{18}^{-7}\mu \end{pmatrix}, \psi_2 := \begin{pmatrix} \mu & \zeta_{18}^{4}\mu & \zeta_{18}^{-7}\mu \\ 1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \\ \lambda & \zeta_{18}^{-8}\lambda & \zeta_{18}^{5}\lambda \end{pmatrix}, \psi_3 := \begin{pmatrix} \lambda & \zeta_{18}^{-8}\lambda & \zeta_{18}^{5}\lambda \\ \mu & \zeta_{18}^{4}\mu & \zeta_{18}^{-7}\mu \\ 1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \end{pmatrix},$$

$$\psi_4 := \begin{pmatrix} 1 & \zeta_{18}^{2} & \zeta_{18} \\ \lambda & \zeta_{18}^{-4}\lambda & \zeta_{18}^{7}\lambda \\ \mu & \zeta_{18}^{8}\mu & \zeta_{18}^{-5}\mu \end{pmatrix}, \psi_5 := \begin{pmatrix} \mu & \zeta_{18}^{8}\mu & \zeta_{18}^{-5}\mu \\ 1 & \zeta_{18}^{2} & \zeta_{18} \\ \lambda & \zeta_{18}^{-4}\lambda & \zeta_{18}^{7}\lambda \end{pmatrix}, \psi_6 := \begin{pmatrix} \lambda & \zeta_{18}^{-4}\lambda & \zeta_{18}^{7}\lambda \\ \mu & \zeta_{18}^{8}\mu & \zeta_{18}^{-5}\mu \\ 1 & \zeta_{18}^{2} & \zeta_{18} \end{pmatrix},$$

for some $\lambda, \mu \in K^*$. However, it is straightforward to check that non of these transformation transforms $\mathcal{C}_1$ to $\mathcal{C}'$ whose core is $X^6+Y^6+Z^6$. Consequently, $\mathcal{C}_1$ is never a descendant of the Fermat curve $\mathcal{F}_6$ with $\mathrm{Aut}(\mathcal{C}_1)$ conjugate to $\varrho_2(\mathrm{A}_4)$.

This proves Claim 3. □

It remains to prove Claim 4 for $\mathcal{C}_2$ that is a descendant of the Fermat curve $\mathcal{F}_6$.

Proof. (of Claim 4) - We easily discard the cases when $\mathrm{Aut}(\mathcal{C}_2)$ equals an S_3 or $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3$ inside $\mathrm{Aut}(\mathcal{F}_6)$ as non of the involutions $[X : sZ : s^{-1}Y]$, $[sY : s^{-1}X : Z]$ and $[sZ : Y : s^{-1}X]$ preserves the core $X^5Y + Y^5Z + Z^5X$ of $\mathcal{C}_2$.

- On the other hand, if $\mathrm{Aut}(\mathcal{C}_2)$ equals $(\mathbb{Z}/3\mathbb{Z})^2$ or He_3, then the discussion we had to show Claim 2 applies to conclude that $\mathcal{C}_2$ is K-isomorphic to

$$\begin{aligned} \mathcal{C}' : X^5Y &+ Y^5Z + XZ^5 + \alpha_{4,0}\zeta_{21}^{4r}\left(X^4Z^2 + X^2Y^4 + Y^2Z^4\right) \\ &+ \alpha_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y + XZ^2 + Y^2Z\right) = 0, \end{aligned}$$

where $\varrho_2((\mathbb{Z}/3\mathbb{Z})^2) \subseteq \mathrm{Aut}(\mathcal{C}')$. Next, if $\mathrm{Aut}(\mathcal{C}')$ is He_3, then there must be another automorphism $\sigma' \notin \varrho_2((\mathbb{Z}/3\mathbb{Z})^2)$ of order 3 that commutes with σ such that $\sigma'\,[Y : Z : X]\,\sigma'^{-1} = [Y : Z : X]\,\sigma^{-1}$. Straightforward calculations show that $\sigma' = [s'Y : t'Z : X]$ or $[s'Z : t'X : Y]$ with $s't' = \zeta_3$ and $s'^2t'^{-1} = \zeta_3^{-1}$. So σ' belongs to $\varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$ modulo $\langle [Y : Z : X]\rangle$. Obviously, none of these transformations leaves invariant the core of $\mathcal{C}'$. Therefore, $\mathrm{Aut}(\mathcal{C}_2)$ is never conjugate to He_3 inside $\mathcal{F}_6$.

- Thirdly, following the notations of Claim 3, a change of variables of the form $\phi = \phi_i$ for $i = 1, 2, ..., 6$ does not transform $\mathcal{C}_2$ to $\mathcal{C}_2' : X^6 + Y^6 + Z^6+$ lower order terms in X, Y, Z. Thus $\mathcal{C}_2$ is not a descendant of $\mathcal{F}_6$ such that $\phi^{-1}\,\mathrm{Aut}(\mathcal{C}_2)\phi = \varrho_1(\mathrm{A}_4)$. On the other hand, $\psi_i\,\mathrm{diag}(1, 1, -1)\psi_i^{-1} \in \mathrm{Aut}(\mathcal{C}_2)$ with $i = 1$ or 4 only if

$$\begin{aligned} \alpha_{2,4} &= \frac{\lambda^5\mu + 4\mu^5}{2\lambda^4}, \alpha_{4,0} = \frac{\lambda + 4\lambda^5\mu}{2\mu^2}, \alpha_{0,2} = \frac{4\lambda + \mu^5}{2\lambda^2\mu^4} \\ \alpha_{1,3} &= \frac{2\left(2\lambda^5\mu + 2\lambda + \mu^5\right)}{\lambda^3\mu^2}, \alpha_{3,2} = \frac{2\lambda^5\mu + 4\lambda + 4\mu^5}{\lambda^2\mu}, \alpha_{2,1} = \frac{2\left(2\lambda^5\mu + \lambda + 2\mu^5\right)}{\lambda\mu^3}. \end{aligned}$$

The above restrictions are consequences of eliminating the coefficients of X^6, Y^6, Z^6, X^5Z, Y^4Z^2, X^4Y^2, X^4Z^2 from the transformed equation ${}^{\psi_i\,\mathrm{diag}(1,1,-1)\psi_i^{-1}}\mathcal{C}_2 = \mathcal{C}_2$. Moreover, $\mathcal{C}_2$ is K-isomorphic via ψ_1 (respectively ψ_4 followed by $Y \leftrightarrow Z$) to $\mathcal{C}_{2,\lambda,\mu}$ described in Theorem 2.5, (2)-(ii). The rest is obvious by noticing that $\psi_2 = \psi_1 \circ [Z : X : Y]$, $\psi_5 = \phi_4 \circ [Z : X : Y]$, $\psi_3 = \psi_1 \circ [Y : Z : X]$ and $\psi_6 = \psi_4 \circ [Y : Z : X]$.

This proves Claim 4. □

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