ON TWISTS OF SMOOTH PLANE CURVES

ESLAM BADR, FRANCESC BARS, AND ELISA LORENZO

To Pilar Bayer for her 70^{th} birthday

Abstract. Given a smooth curve defined over a field k that admits a non-singular plane model over $\overline{k}$, a fixed separable closure of k, it does not necessarily have a non-singular plane model defined over the field k. We determine under which conditions this happens and we show an example of such phenomenon. Now, even assuming that such a smooth plane model exists, we wonder about the existence of non-singular plane models over k for its twists. We characterize twists possessing such models and use such characterization to improve, for the particular case of smooth plane curves, the algorithm to compute twists of non-hyperelliptic curves wrote recently down by the third author. We also show an example of a twist not admitting such non-singular plane model. As a consequence, we get explicit equations for a non-trivial Brauer-Severi surface. Finally, we obtain a theoretical result to compute all the twists of smooth plane curves with cyclic automorphism group having a k-model whose automorphism group is generated by a diagonal matrix. Some examples are also provided.

1. Introduction

Let C be a smooth curve over k, i.e. C a projective non-singular and geometrically irreducible curve defined over a field k. Denote by $\overline{C}$ the curve $C \times_k \overline{k}$ where $\overline{k}$ is a fixed separable closure of k, and by $\mathrm{Aut}(\overline{C})$ its automorphism group. We assume, once and for all, that $\overline{C}$ is non-hyperelliptic of genus $g \geq 3$. With the method exposed in [16] we can compute the twists of C; i.e. a smooth curve C' over k with a $\overline{k}$-isomorphism $\phi : C \to C'$. The set of twists of C modulo k-isomorphism, denoted by $\mathrm{Twist}_k(C)$, is in one to one correspondence with the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(\overline{C}))$. Given a cocycle $\xi \in \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(\overline{C}))$, the idea behind computing equations for the twist, is finding a $\mathrm{Gal}(\overline{k}/k)$-modulo isomorphism between the subgroup generated by the image of ξ in $Aut(\overline{C})$ with a subgroup of a general linear group $\mathrm{GL}_n(\overline{k})$. After that, by making explicit Hilbert's 90 Theorem, we can compute an isomorphism $\phi : C \to C'$ over $\overline{k}$ such that $\xi_\tau = \phi^{-1} \cdot {}^\tau \phi$ for all τ of the Galois group $\mathrm{Gal}(\overline{k}/k)$, and hence, obtain equations for the twist. For non-hyperelliptic curves, see a description in [15], the canonical model gives a natural inclusion $Aut(\overline{C}) \hookrightarrow \mathrm{PGL}_g(\overline{k})$ and with the natural representation of $Aut(\overline{C})$ on the regular differential forms, $Aut(\overline{C})$ will live inside a general linear group (then so does the subgroup generated by the image of a 1-cocycle in $\mathrm{PGL}_g(\overline{k})$ provided by the canonical model),

Now consider a smooth plane curve C over k, i.e. C is a smooth curve over k that admits a non-singular plane model over $\overline{k}$. Therefore, C has a g^2_d complete linear series which defines a map $\Upsilon : \overline{C} \hookrightarrow \mathbb{P}^2_{\overline{k}}$, where $\mathbb{P}^2_{\overline{k}}$ is the 2-th projective space over $\overline{k}$, and moreover $Image(\Upsilon)$ is defined by the zeroes of a degree d polynomial in X, Y, Z with coefficients in $\overline{k}$. Denote such a model by $F_{\overline{C}}(X, Y, Z) = 0$, in particular $g = \frac{1}{2}(d-1)(d-2)$. It is well-known that the complete linear series g^2_d is unique up to conjugation in $PGL_3(\overline{k})$, the automorphism group of $\mathbb{P}^2_{\overline{k}}$, see [10, Lemma 11.28]. Therefore, any $\overline{k}$-model of C is defined by $F_{P\overline{C}}(X, Y, Z) := F(P(X, Y, Z)) = 0$ for some $P \in \mathrm{PGL}_3(\overline{k})$. Furthermore, a plane model of C is defined over k if there exists a $Q \in \mathrm{PGL}_3(\overline{k})$ such that $F_{Q\overline{C}}(X, Y, Z) \in k[X, Y, Z]$. The group $Aut(\overline{C})$ is isomorphic via Υ to the automorphism group $Aut(F_{P\overline{C}})$ of any of its $\overline{k}$-models, and all these groups are conjugate in $\mathrm{PGL}_3(\overline{k})$. We say that C admits a smooth plane model over k if it is k-isomorphic to a smooth plane model defined over k.

The aim of this paper is to make a study of the twists for smooth plane curves by considering the embedding $Aut(\overline{C}) \hookrightarrow \mathrm{PGL}_3(\overline{k})$ instead the one given by the canonical model. The embedding is of $Gal(\overline{k}/k)$-groups if C admits a smooth plane model over k.

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This approach leads also to two natural questions: the first one, given C a smooth plane curve defined over a field k, does it admit a smooth plane model defined over the base field k; and secondly, if the answer is yes, does every twist of C over k also have a smooth plane model defined over k?

For both questions the answer is no in general, it does not. We obtain general results for the curves and the twists for which the above questions always have an affirmative answer, and also we show different examples concerning the negative general answer. Interestingly, in the way to get these examples, we need to handle with non-trivial Brauer-Severi surfaces, and we are able to compute explicit equations of a non-trivial one.

Moreover, for smooth plane curves defined over k with a cyclic automorphism group generated by a diagonal matrix, we obtain a general theoretical result to obtain all its twists. Families of such smooth plane curves have already been studied by the first two authors in [2,3]. These families have genus arbitrarily big, so the method in [16] does not work for them.

1.1. **Outline.** The structure of this paper is as follows. In Section 2, we introduce the background and basic definition about Galois cohomology and Brauer-Severi varieties needed in the sequel. We devote Section 3 to deal with the question on the field where there exists a non-singular model for a smooth plane curve C defined over k. We prove that if the degree of a non-singular plane model of C is coprime with 3 or C has a k-point or the 3-torsion of the Brauer group of k is trivial (in particular if k is a finite field), then the curve C admits a smooth plane model over k: Theorem 3.6 and Corollaries 3.2, 3.3. Moreover, we prove that a smooth plane model of C always exists in a finite field extension of k of degree dividing 3, see Theorem 3.5. We end Section 3 with an explicit example of a smooth plane curve defined over $\mathbb{Q}$ and not admitting a smooth plane model over $\mathbb{Q}$; however, we construct a smooth plane model over a degree 3 extension of $\mathbb{Q}$.

In Section 4, we assume that C is a smooth plane curve defined over k having a smooth plane model over k. We obtain Theorem 4.1 characterizing the twists of C having also a smooth plane model over k. Moreover, we construct a family of examples over $k=\mathbb{Q}$ where a twist of C over $\mathbb{Q}$ does not admit a non-singular plane model over $\mathbb{Q}$ (this construction is not explicit because we do not provide equations of such twists).

We dedicate Section 5 to detail an explicit example of a smooth plane curve defined over the field $\mathbb{Q}(\zeta_3)$ having a twist that does not possess such a model in the field $\mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive 3rd root of unity. Interestingly, we find the already mentioned explicit equations for a non-trivial Brauer-Severi variety.

In Section 6, we study the twists for smooth plane curve C over k with a non-singular plane model over k, $F_{P\overline{C}}\in k[X,Y,Z]$ such that $Aut(\overline{C})$ is a cyclic group. We prove that if $Aut(F_{P\overline{C}})$ is represented in $\mathrm{PGL}_3(\overline{k})$ by a diagonal matrix, then all the twists are diagonal, i.e. of the form $F_{PD\overline{C}}(X,Y,Z)=0$ with D a diagonal matrix, Theorem 6.2. We apply such result to a family of curves (one for each degree) where the techniques of [16] does not apply if the degree is too big, see Theorems 6.4, 6.5. In the case that C does not admit such a model where $Aut(F_{P\overline{C}})$ is cyclic diagonal in $\mathrm{PGL}_3(\overline{k})$, we also construct an example where not all the twists are given by a diagonal twist.

Finally, we apply the algorithm in [16] to the simplest degree 5 example in section 6, this shows the improvements of Theorem 6.4 and Theorem 6.2 to compute the twists.

1.2. **Notation and conventions.** We set the following notations, to be used throughout.

By k we denote a field, $\overline{k}$ is a separable closure of k and L is an extension of k inside $\overline{k}$. By ζ_n we always mean a fixed primitive n-th root of unity inside $\overline{k}$ when the characteristic of k is coprime with n. We write $\mathrm{Gal}(L/k)$ for the Galois group of L/k. All the Galois groups in this paper, when acting on sets, we denote it by left exponentiation. We write $H^i(\mathrm{Gal}(L/k),N)$ with $i\in\{0,1\}$ for the Galois cohomology set of a $\mathrm{Gal}(L/k)$-group N. In the particular case $L=\overline{k}$, we also denote G_k instead of $\mathrm{Gal}(\overline{k}/k)$ and $H^1(k,N)$ instead of $H^1(G_k,N)$. $Br(k)$ denotes the Brauer group of k consisting of the central simple algebras over k modulo k-algebras isomorphism.

When we work with groups, we use the SmallGroup Library-GAP [7]. Where the group $<N,r>$ or $GAP(N,r)$ denotes the group of order N that appears in the r-th position in such library. By $ID(G)$, we mean the corresponding GAP notation for the group G. For cyclic groups, we use the standard notation $\mathbb{Z}/n\mathbb{Z}$.

By smooth curve over k we mean a projective, non-singular and geometrically irreducible curve defined over k, and usually we denote it by C or C_k. As usual $\overline{C}$ corresponds to $C\times_k\overline{k}$, (the curve C over $\overline{k}$), $Aut(\overline{C})$ the automorphism group, and $g(\overline{C})$ denotes the genus of the curve, and once and for all we assume that $g(\overline{C})\geq 2$. We denote by $\mathbb{P}^r_k$ the r-th projective space over the field k.

By a smooth plane curve C over k we mean a smooth curve over k, which admits a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ of degree $d\geq 4$. We denote by $F_{P\overline{C}}(X,Y,Z)=0$ with $P\in\mathrm{PGL}_3(\overline{k})$ another non-singular model, where $F_{P\overline{C}}(X,Y,Z):=F_{\overline{C}}(P(X,Y,Z))$. By $Aut(F_{P\overline{C}})$ we mean the automorphism group of the curve $F_{P\overline{C}}(X,Y,Z)=0$ in $\mathbb{P}^2_{\overline{k}}$ which is a finite subgroup of $\mathrm{PGL}_3(\overline{k})$. We have that $Aut(F_{P\overline{C}})=P^{-1}Aut(F_{\overline{C}})P$ as subgroups of $\mathrm{PGL}_3(\overline{k})$.

A linear transformation $A=(a_{i,j})$ of $\mathbb{P}^2$ is also written as $[a_{1,1}X+a_{1,2}Y+a_{1,3}Z:a_{2,1}X+a_{2,2}Y+a_{2,3}Z:a_{3,1}X+a_{3,2}Y+a_{3,3}Z]$.

Given C a smooth plane curve over k, we say that C admits a non-singular plane model over L if there exists $P\in\mathrm{PGL}_3(\overline{k})$ such that $F_{P\overline{C}}(X,Y,Z)\in L[X,Y,Z]$, and C and $F_{P\overline{C}}(X,Y,Z)=0$ are isomorphic over L.

By an abuse of language, if a smooth plane curve C over k admits a non-singular plane model over k given by $F_{P\overline{C}}=0$, we identify C with the plane model $F_{P\overline{C}}=0$ and we identify $Aut(C)$ with $Aut(F_{P\overline{C}})$ as a fixed finite subgroup of $\mathrm{PGL}_3(\overline{k})$.

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2. Galois cohomology and Brauer-Severi varieties

In this section, we state different results about Galois cohomology to be used through the paper (see a general approach in [20, Chapter III]).

Definition 2.1. Given C a smooth curve over k. A twist of C over k is a smooth curve C' defined over k such that $\overline{C}\cong_{\overline{k}}\overline{C'}$. Given two twists of C over k, namely C_1,C_2, we say that they are equivalent if there exist an isomorphism $C_1\cong C_2$ defined over k. The set of twists of C over k modulo the above equivalence is denoted by $\mathrm{Twist}_k(C)$.

The following result is well-known.

Theorem 2.2. *There exists a bijection between the sets $Twist_k(C)$ and $H^1(k,Aut(\overline{C}))$.*

Recall that the above bijection $[C']\mapsto[\xi]$ sends a twist $\varphi:\overline{C}\to\overline{C'}$ to the cocycle $\xi:\ \tau\mapsto\xi_\tau:=\varphi^{-1}\cdot{}^{\tau}\varphi\in Aut(\overline{C})$ where $\tau\in G_k$.

Lemma 2.3. *Let C be a curve over k admitting a plane model $F_{\overline{C}}=0$ over $\overline{k}$. Let us assume that there exists a matrix $P\in PGL_3(\overline{k})$ with $F_{P\overline{C}}(X,Y,Z)\in k[X,Y,Z]$. Then, there exists a twist C' of C over k given by the non-singular plane model $F_{P\overline{C}}=0$ over k. Furthermore, we have a map $\Sigma':Twist_k(C)\to H^1(k,PGL_3(\overline{k}))$.*

Proof. Let $h:C\to F_{P\overline{C}}$ be a $\overline{k}$- isomorphism, and consider the 1-cocycle $\sigma\mapsto h^{-1}\cdot^{\sigma}(h)\in Aut(\overline{C})$ in $H^1(k,Aut(\overline{C}))=Twist_k(C)$. Then there exists a twist $\lambda:C\to C'$ of C over k such that $h^{-1}\cdot{}^{\sigma}h=\lambda^{-1}\cdot{}^{\sigma}\lambda$ for all $\sigma\in G_k$. Therefore $\lambda\circ h^{-1}:F_{P\overline{C}}\to C'$ is an isomorphism defined over k.

Now we can identify $Aut(\overline{C'})$ with $Aut(F_{P\overline{C}})$ as G_k-groups. Since $Aut(F_{P\overline{C}})$ has an injective representation inside $Aut(\mathbb{P}^2_{\overline{k}})=\mathrm{PGL}_3(\overline{k})$ as a G_k-group, we get a natural map in Galois cohomology

$$\Sigma':Twist_k(C)=H^1(k,Aut(\overline{C'}))\to H^1(k,PGL_3(\overline{k})).$$

□

Definition 2.4. A Brauer-Severi variety D over k of dimension r is a smooth projective variety such that the variety $D\otimes_k\overline{k}$ over $\overline{k}$ is isomorphic to the projective space $\mathbb{P}^r_{\overline{k}}$ of dimension r over $\overline{k}$.

The following result is also well-known [12, Corollary 4.7]:

Lemma 2.5. *The Brauer-Severi varieties over k of dimension r, up to k-isomorphism, are in bijection with $H^1(Gal(\overline{k}/k), PGL_{r+1}(\overline{k})) = H^1(Gal(\overline{k}/k), Aut_{\overline{k}}(\mathbb{P}^r_{\overline{k}}))$.*

Denote by Az_n^k the set of all isomorphic classes of central simple algebras A of dimension n^2 over k (they split in a separable extension of degree n of k).

The next result is Corollary 3.8 in [12].

Theorem 2.6. *Let k be a field, then there exists a bijection between the following sets*

$$Az_n^k \longleftrightarrow H^1(G_k, PGL_n(\overline{k})).$$

For completeness, we recall the map defining the above bijection: given $A \in \mathrm{Az}_n^k$, we always get a Galois extension L/k of degree n and an isomorphism $f : A \times_k L \to M_n(L)$ such that there exists a matrix $A_\tau \in Aut(M_n(L)) = PGL_n(L)$ for every $\tau \in \mathrm{Gal}(L/k)$ that makes the following diagram commutative:

$$\begin{array}{ccc} A \times_k L & \to^f & M_n(L) \\ \tau \uparrow & & \uparrow A_\tau \circ \tau \\ A \times_k L & \to^f & M_n(L) \end{array}$$

The map $\tau \mapsto A_\tau$ defines an element of $H^1(\mathrm{Gal}(L/k), \mathrm{PGL}_n(L))$, and we obtain an element in $H^1(k, \mathrm{PGL}_n(\overline{k}))$ through the inflation map.

Definition 2.7. Let L/k be a cyclic extension of degree n with $\mathrm{Gal}(L/k) = \langle\sigma\rangle$, and fix an isomorphism $\chi : \mathrm{Gal}(L/k) \to \mathbb{Z}/n\mathbb{Z}$. Given $a \in k^*$, we consider (a, χ), the n-dimensional vector space over L with basis $1, e, \ldots, e^{n-1}$, i.e.

$$(\chi, a) := \oplus_{1 \leq i \leq n-1} Le^i,$$

where the multiplication rules are given by $e\lambda = \sigma(\lambda)e$ for $\lambda \in L$ and $e^n = a$. Such (χ, a) becomes a central simple algebra of dimension n^2 over k which splits in L (see [23, §2]), and is called a cyclic algebra of k.

Theorem 2.8. *All the elements of Az_3^k are cyclic algebras of the form (χ, a) as in Definition 2.7 with $n = 3$. In particular, modulo isomorphism of k-algebras, $(\chi, a) \in Az_3^k$ is the trivial k-algebra if and only if a is a norm of L/k of an element of L. Moreover, the assignment*

$$(\chi, a) \in Az_3^k \mapsto inf(\{A_\tau\}_{\tau \in Gal(L/k)}) \in H^1(k, PGL_3(\overline{k}))$$

is given by an A_σ of the shape $\begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here inf denotes the inflation map in Galois cohomology.

Proof. By definition, any central cyclic simple algebra of Az_3^k can be expressed as described. Recall also that the map $f : A \otimes_k L \to M_3(L)$ is given by $f(\lambda \otimes 1) = diag(\lambda, \sigma(\lambda), \sigma^2(\lambda))$ and $f(e \otimes 1) = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. It is a result of Wedderbum [25], that all elements of Az_3^k corresponds to cyclic algebras. Moreover, it is well-known that a cyclic algebra (χ, a) is trivial if and only if a is a norm of its splitting field.

The last statement concerning the 1-cocycle assignment follows by [23, Example 5.5], after defining first the cocycle element in $H^1(\mathrm{Gal}(L/k), \mathrm{PGL}_3(L))$ and using [12, Lemma 3.7] for the inflation map in Galois cohomology groups. □

Because the Brauer group of a finite field is trivial, and taking cohomology of the short exact sequence $1 \to \overline{k}^* \to GL_n(\overline{k}) \to PGL_n(\overline{k}) \to 1$, we mention:

Lemma 2.9. *Let k be a finite field, then $H^1(G_k, PGL_n(\overline{k})) = 1$.*

3. The field of definition of a non-singular plane model

In this section, we prove that if a smooth curve defined over k admits a non-singular plane model over $\overline{k}$, then it is always possible to find a non-singular plane model defined over an extension L/k of degree dividing 3. Moreover, for a particular field k or if the smooth plane model has degree coprime with 3, we prove that we can always find a non-singular plane model defined over the base field k. We provide an example of a curve defined over $\mathbb{Q}$ that does not admit a smooth plane model over $\mathbb{Q}$, but that it does over a Galois extension of $\mathbb{Q}$ of degree 3.

Roé and Xarles prove the following result in [19, Corollary 6].

Theorem 3.1 (Roé-Xarles)**.** *Let C be a smooth projective curve defined over k such that $\overline{C}$ has a non-singular plane model. Let $\Upsilon : \overline{C} \hookrightarrow \mathbb{P}^2_{\overline{k}}$ be a morphism given by the uniqueness of the g^2_d-linear system over $\overline{k}$, then there exists a Brauer-Severi variety D (of dimension two) defined over k, together with a k-morphism $g : C \hookrightarrow D$ such that $g \otimes_k \overline{k} : \overline{C} \to \mathbb{P}^2_{\overline{k}}$ is equal to Υ.*

From the above result, one obtain remarkable consequences.

Corollary 3.2. *Let C be a smooth curve over k that admits a non-singular plane model. Assume that C has a k-point, i.e. $C(k)$ is not-empty. Then C admits a non-singular plane model over k.*

Proof. By a Severi result, see [12, Prop.4.8], a Brauer-Severi variety over k of dimension n with a k-point is isomorphic over k to $\mathbb{P}^n_k$. By Theorem 3.1, the map $g : C_k \to D \cong \mathbb{P}^2_k$ defined over k defines the non-singular plane model of C over k. □

Corollary 3.3. *Consider a field k such that $Br(k)[3]$ is trivial, where $Br(k)[3]$ denotes the 3-torsion of $Br(k)$. Then any smooth plane curve C over k, admits a non-singular plane model over k, and in particular any twist of C over k admits also a non-singular plane model over k.*

Proof. A non-trivial Brauer-Severi surface over k corresponds to a non-trivial 3-torsion element of $Br(k)$, therefore is such group is empty, by Theorem 3.1 the g^d_2-system factors through $g : C_k \hookrightarrow \mathbb{P}^2_k$ and all of them are defined over k, so, they define a plane model of C over k. □

Remark 3.4. *For a finite field k (see Lemma 2.9) or $k = \mathbb{R}$, it is known that $Br(k)[3]$ is trivial, therefore any smooth plane curve over such fields admits always a non-singular plane model over k.*

Theorem 3.5. *Let C be a smooth plane curve defined over k, then it admits a non-singular plane model over L such that $[L : k] \mid 3$, i.e. $\exists P \in PGL_3(\overline{k})$ such that $F_{P\overline{C}} \in L[X, Y, Z]$ and such that C and $F_{P\overline{C}} = 0$ are L-isomorphic.*

Proof. From Theorem 3.1, we have a k-morphism of C to a Brauer-Severi surface D over k. By Theorem 2.6, D corresponds to a central simple algebra over k of dimension 9 which splits (if is not the trivial algebra) in at degree 3 Galois extension L of k, therefore $D \otimes_k L$ corresponds to the trivial element in $H^1(\mathrm{Gal}(\overline{k}/L), \mathrm{PGL}_3(\overline{k}))$, by theorem 2.6. Thus, $D \otimes_k L \cong \mathbb{P}^2_L$ over L. We then obtain that

$$g \otimes_k L : C \otimes_k L \hookrightarrow \mathbb{P}^2_L$$

are all defined over L, thus we have a non-singular plane model of C over L. Lastly, because all the non-singular plane models of C over $\overline{k}$ are of the form $F_{P\overline{C}}(X, Y, Z) = 0$ for some $P \in \mathrm{PGL}_3(\overline{k})$, we deduce the result. □

The next result is a particular case of an argument by Roé and Xarles in [19] following Châtelet [6].

Theorem 3.6. *Let C be a smooth curve defined over k, such that admits a non-singular plane model of degree d with d coprime with 3. Then C admits a non-singular plane model over k.*

Proof. By the results of the previous section, Brauer-Severi surfaces over k corresponds to elements of $H^1(k, \mathrm{PGL}_3(\overline{k}))$, hence to the set of equivalence classes of central simple algebras of dimension 9 with a splitting field of degree 3 over k (thus, they are elements of the Brauer group $Br(k)$ of the field k of order dividing 3).

Moreover, if D is a Brauer-Severi surface over a field k, then its class $[D]$ in the Brauer group $Br(k)$ verifies that in the exact sequence

$$Pic(D) \to Pic(D \otimes_k \overline{k}) \cong \mathbb{Z} \to Br(k),$$

the last map send 1 to $[D]$, and hence the image of some generator of $Pic(D)$ is equal to m, where m is the order of $[D]$. Consequently, m divides 3, as the order of $[D]$ does. Now, if C is a curve over k in $Pic(D)$ such that $\overline{C}$ has a non-singular plane model of degree d, then the image of C in $Pic(B \otimes_k \overline{k}) \cong \mathbb{Z}$ is equal to the degree d. Therefore, if d is coprime with 3, we thus get $m = 1$, and D is the projective plane $\mathbb{P}^2_k$ (see [19, Theorem 13] for a more general statement on hypersurfaces in Brauer-Severi varieties).

□

Corollary 3.7. *Let C be a smooth curve defined over k which admits a non-singular plane model over $\overline{k}$ of degree d, coprime with 3. Then, every twist $C' \in Twist_k(C)$ admits a non-singular plane model over k.*

Proof. It follows, by our assumption, that every twist of C over k admits a non-singular plane model over $\overline{k}$ of degree d, coprime with 3. Hence, non-singular plane models over k exist for twists of C over k, by Theorem 3.6.

□

3.1. **An example of a smooth plane curve over $\mathbb{Q}$ without a non-singular plane model over $\mathbb{Q}$.** Let us consider $\mathbb{Q}_f$ the splitting field of the polynomial $f(x) = x^3+12x^2-64$. It is an irreducible polynomial and the discriminant of f is $(2^6 3^2)^2$, then $\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$, moreover by a computation in SAGE, the discriminant of the field $\mathbb{Q}_f$ is a power of 3, and the prime 2 becomes inert in $\mathbb{Q}_f$.

Let us denote the roots of f by a, b, c in a fixed algebraic closure of $\mathbb{Q}$, and let us call σ the element in the Galois group that acts by sending $a \to b \to c$.

Proposition 3.8. *The smooth plane curve over $\mathbb{Q}_f$*

$$C:\, 64x^6 + aby^6 + az^6 + 8x^3y^3 + \frac{ab}{8}y^3z^3 + az^3x^3 = 0,$$

has $\mathbb{Q}$ as a field of definition, but it does not admit a plane non-singular model over $\mathbb{Q}$.

Proof. The matrix

$$\phi = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

defines an isomorphism $\phi : {}^{\sigma}C \to C$. This isomorphism ϕ satisfies the Weil cocycle condition [26] ($\phi_{\sigma^3} = \phi_\sigma^3 = 1$), we therefore obtain that the curve is defined over $\mathbb{Q}$, and that there exists an isomorphism $\varphi_0 : C_{\mathbb{Q}} \to C$ where $C_{\mathbb{Q}}$ is a rational model such that $\phi = \varphi_0\, {}^{\sigma}\varphi_0^{-1} \in \mathrm{PGL}_3(\mathbb{Q})$. The assignation $\phi_\tau := \varphi_0\, {}^{\tau}\varphi_0^{-1}$ defines an element of $H^1(\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}), \mathrm{PGL}_3(\mathbb{Q}_f))$, by Theorem 2.8, this cohomology element is non-trivial because 2 is not a norm of an element of $\mathbb{Q}_f$ (since 2 is inert in $\mathbb{Q}_f$). Therefore φ_0 is not given by an element of $\mathrm{PGL}_3(\mathbb{Q}_f)$, or of $\mathrm{PGL}_3(\overline{\mathbb{Q}})$ because the cohomology class by the inflation map is neither trivial. Therefore the curve C over $\mathbb{Q}$ does not admits a non-singular plane model over $\mathbb{Q}$ (because if admitted a non-singular plane model over $\mathbb{Q}$, such model would be of the form $F_{PQ\overline{C}}(X,Y,Z) = 0$ for some $P \in \mathrm{PGL}_3(\overline{k})$ where $F_{Q\overline{C}}(X,Y,Z) = 0$ a non-singular model over $\mathbb{Q}_f$, therefore φ_0 would be representative by $P \in \mathrm{PGL}_3(\overline{k})$ which is not). □

Remark 3.9. *We have just seen an example of a curve defined over a field k not admitting a particular model (a plane one) over the same field. For hyperelliptic models, we find such examples after Proposition* 4.14 *in [13]. In [11, chp. 5,7], there are also examples of hyperelliptic curves and smooth plane curves where the field of moduli is not a field of definition, so, in particular, there are not such models defined over the fields of moduli.*

4. On twists of plane models defined over k

In this section, we assume, once and for all, that C is a smooth curve defined over k with a non-singular plane model also defined over k, i.e. we can assume that C is given by an equation $F_{\overline{C}} = 0$ with $F_{\overline{C}} \in k[X,Y,Z]$. We provide results characterizing when all the twists of C admit a non-singular plane model over k, and we give a (non-explicit) example of a family of such curves C having twists not admitting a plane model over k.

Theorem 4.1. *Let C be a curve defined over a field k with a plane non-singular model $F_{\overline{C}}(X,Y,Z)=0$ defined over k. Then there exists a natural map*

$$\Sigma : H^1(k, Aut(F_{\overline{C}})) \to H^1(k, PGL_3(\overline{k})),$$

defined by the inclusion $Aut(F_{\overline{C}}) \subseteq PGL_3(\overline{k})$ as G_k-groups. The kernel of Σ is the set of all twists of C that admit a non-singular plane model over k. Moreover, any such a plane model is obtained through an automorphism of $\mathbb{P}^2$, that is, it is of the form $F_{M\overline{C}}(X,Y,Z) := F_{\overline{C}}(M(X,Y,Z)) \in k[X,Y,Z]$ for some $M \in PGL_3(\overline{k})$.

Proof. The map is clearly well-defined. If a twist C' admits a non-singular plane model $F_{\overline{C'}}$ over k, the isomorphism from $F_{\overline{C'}}$ to $F_{\overline{C}}$ is then given by an element $M \in \mathrm{PGL}_3(\overline{k})$ (as any isomorphism between two non-singular plane curves of degrees > 3 is given by a linear transformation in $\mathbb{P}^2$, [5]). Hence, the corresponding 1-cocycle $\sigma \mapsto M\,{}^{\sigma}M^{-1} \in Aut(F_{\overline{C}})$ becomes trivial in $H^1(k, \mathrm{PGL}_3(\overline{k}))$. Conversely, if a twist C' is mapped by Σ to the trivial element in $H^1(k, \mathrm{PGL}_3(\overline{k}))$, then this twist is given by a $\overline{k}$-isomorphism $\varphi : F_{\overline{C}} \to C'$ defined by a matrix $M \in \mathrm{PGL}_3(\overline{k})$ that trivializes the cocycle and such an M produces a non-singular plane model defined over k. □

Remark 4.2. *We can reinterpret the map Σ in Theorem 4.1 as the map that sends a twist C' to the Brauer-Severi variety D in Theorem 3.1.*

Remark 4.3. *Consider a smooth plane curve C defined over k, and assume that it has degree d coprime with 3 or that $Br(k)[3]$ is trivial, then Σ in Theorem 4.1 is the trivial map by Corollaries 3.7 and 3.3.*

Remark 4.4. *Theorem 4.1 can be used to improve the algorithm for computing twists for non-hyperelliptic curves, see [16] or [15, Chp.1], for the special case of non-singular plane curves. The algorithm requires to compute a canonical model in $\mathbb{P}^{g-1}$, solutions to Galois embedding problems, and constructing equations for the twists. The last step needs to see $Aut(C) \subseteq GL_g(\bar{k})$ via the canonical embedding and the action of the automorphism group on the vector space of regular differentials, so, we can use an explicit version of Hilbert 90 Theorem.*

Now, if Σ is trivial in Theorem 4.1, then we can work on $\mathbb{P}^2$ instead of on $\mathbb{P}^g$. The exact sequence $1 \to \bar{k} \to GL_3(\bar{k}) \to PGL_3(\bar{k})$ gives the exact, well-defined sequence $1 \to H^1(\bar{k}, GL_3(\bar{k})) \to H^1(\bar{k}, PGL_3(\bar{k}))$. Hence, the trivial element of $H^1(\bar{k}, PGL_3(\bar{k}))$ corresponds to an element in $H^1(\bar{k}, GL_3(\bar{k}))$ and we can proceed again by Hilbert 90 Theorem.

In the appendix, we use this improvement to compute the twists of some particular families of plane curves over k having a cyclic diagonal automorphism group.

To finish we construct a family of smooth curves defined over $\mathbb{Q}$ that admits a non-singular plane model over $\mathbb{Q}$ but some of its twists do not admit a non-singular plane model over $\mathbb{Q}$. This construction is not explicit in the sense that we do not construct the equations of the twist and the Brauer-Severi surface, see next section for an explicit construction giving defining equations.

Theorem 4.5. *Let $p \equiv 3, 5 \ mod\ 7$ be a prime number. Take $a \in \mathbb{Q}$ with $a \neq -10, \pm 2, -1, 0$. Consider the family $C_{p,a}$ of smooth plane curves over $\mathbb{Q}$ given by*

$$C_{p,a} : X^6 + \frac{1}{p^2}Y^6 + \frac{1}{p^4}Z^6 + \frac{a}{p^3}(p^2X^3Y^3 + pX^3Z^3 + Y^3Z^3) = 0.$$

Then, there exists a twist $C' \in Twist_{\mathbb{Q}}(C_{p,a})$ which does not admit a non-singular plane model over $\mathbb{Q}$.

We need some lemmas before proving Theorem 4.5.

Lemma 4.6. *Consider the family of non-singular plane curves over $\mathbb{Q}$ defined by the equation*

$$C_a : X^6 + Y^6 + Z^6 + a(X^3Y^3 + X^3Z^3 + Y^3Z^3) = 0,$$

with $a \neq -10, \pm 2, -1, 0$. The full automorphism group $Aut(C_a)$ is generated by

$$S := [X; \zeta_3 Y; \zeta_3^2 Z],\ U := [X; Y; \zeta_3 Z],\ T := [Z; X; Y],\ \text{and}\ R := [Y; X; Z].$$

In particular, it is isomorphic to $GAP(54, 5)$.

Proof. First, it is necessary that $a \neq \pm 2, -1$ for non-singularity of C_a. Second, we use similar techniques and notation to the ones used and developed in [2], in particular $Aut(C_a)$ should be one of the groups in [2, Theorem 2]. We have $\langle R, S, T, U\rangle \leq \mathrm{Aut}(C_a)$, hence $Aut(C_a)$ is not conjugate to a cyclic group, the Klein group $PSL(2,7)$, the icosahedral group A_5, the alternating group A_6 (i.e. C_a is not K-equivalent to the Wiman sextic curve), or the Hessian groups $Hess_*$ with $* \in \{36, 72\}$. On the other hand, C_a is not a descendant of the degree 6 Klein curve K_6, since $54 \nmid |Aut(K_6)| (= 63)$, also $Aut(C_a)$ fixes no points in the projective plane $\mathbb{P}^2$, thus from Mitchel's classification that is explicitly explained in [2], $Aut(C_a)$ fixes a triangle. Consequently, we need only to worry about the following situations: $Aut(C_a)$ is conjugate to the Hessian group $Hess_{216}$ or to a subgroup of $Aut(F_6)$ (i.e. C_a is a descendant of the Fermat curve of degree 6).

We note that there is a unique representation of the Hessian group $Hess_{216}$ inside $\mathrm{PGL}_3(K)$, up to conjugation. Thus, without loss of generality, we consider $Hess_{216} = \langle S, T, U, V\rangle$ where

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}.$$

Now, if $\mathrm{Aut}(C_a) = P^{-1} \cdot \langle S, T, U, V\rangle \cdot P$ for some $P \in \mathrm{PGL}_3(\overline{k})$, then it is easy to check that it must be $\mathrm{Aut}(C_a) = \langle S, T, U, V\rangle$. Moreover, if $V \in \mathrm{Aut}(C_a)$ then $a = -10$ (see the coefficient of X^4YZ), which is not our case.

Let us now inspect the automorphism group of the Fermat curve of degree 6.

$$Aut(F_6) = \langle [\zeta_6 X; Y; Z],\ [X; \zeta_6 Y; Z],\ [Y; Z; X],\ [X; Z; Y]\,\rangle \simeq GAP(216, 92),$$

where ζ_6 is a primitive 6-th root of unity.

This group contains a unique group of order 54, up to conjugation. Hence, we may assume that C_a is a descendant of the Fermat curve $F_6 : X^6 + Y^6 + Z^6 = 0$ through a projective transformation $P \in \mathrm{PGL}_3(K)$ such that $P^{-1}\langle S, U, T, R\rangle P = \langle S, U, T, R\rangle$. Consequently, the transformed equation should be again of the form $C_P : X^6 + Y^6 + Z^6 + a'(X^3Y^3 + X^3Z^3 + Y^3Z^3)$ for some $a' \in \mathbb{Q}$. Finally, elements of $Aut(F_6)$ are of the forms

$$[X; \zeta_6^r Y; \zeta_6^{r'} Z],\ [X; \zeta_6^{r'} Z; \zeta_6^r Y],\ [\zeta_6^r Y; X; \zeta_6^r Z],\ [\zeta_6^{r'} Z; \zeta_6^r Y; X],\ [\zeta_6^r Y; \zeta_6^{r'} Z; X],\ [\zeta_6^{r'} Z; X; \zeta_6^r Y]$$

with $r, r' \in \mathbb{Z}$. Because $a \neq 0$, then any of these forms belongs to $Aut(C_P)$ only if $2|r, r'$. This gives exactly $6(3 \times 3) = 54$ automorphisms inside $Aut(F_6)$.

As a conclusion, the full automorphism group of C_a with $a \neq -10, \pm 2, -1, 0$ is of order 54 and is isomorphic to $GAP(54, 5)$. □

Remark 4.7. *If the automorphism group of a smooth plane curve, as a subgroup of $PGL_3(\overline{k})$ contains a subgroup conjugate to $\langle S, U, T, R\rangle$, then the degree d of the plane model is divisible by 3. This follows because the non-singularity plus having the automorphism T implies that we must have one of the following cores* [1]*:*

(1) $X^d + Y^d + Z^d$,
(2) $X^{d-1}Y + XZ^{d-1} + Y^{d-1}Z$,
(3) $X^{d-1}Z + YZ^{d-1} + XY^{d-1}$

But also we have U as an automorphism of the plane model, therefore the plane model could only have the core $X^d + Y^d + Z^d$ with $3|d$.

Lemma 4.8. *Given $a \neq -10, \pm 2, -1, 0$ and $\alpha_0 \in \overline{\mathbb{Q}}$, the family of curves*

$$C_{\alpha_0, a} : X^6 + \frac{1}{\alpha_0^2}Y^6 + \frac{1}{\alpha_0^4}Z^6 + \frac{a}{\alpha_0^3}(\alpha_0^2X^3Y^3 + \alpha_0X^3Z^3 + Y^3Z^3) = 0,$$

has automorphism group isomorphic to $GAP(54, 5)$ with $[Y, Z, \alpha_0 X] \in Aut(C_{\alpha_0,a})$, and the elements of $Aut(C_{\alpha_0,a})$ are defined over the field $\mathbb{Q}(\zeta_3, \sqrt[3]{\alpha_0})$, where ζ_3 is a primitive 3-rd root of unity.

[1]For a non-zero monomial $cX^iY^jZ^k$, its exponent is defined to be $max\{i, j, k\}$. For a homogeneous polynomial F, the core of F is the sum of all terms of F with the greatest exponent.

Proof. The curve $C_{\alpha_0,a}$ and C_a of Lemma 4.6 are $\overline{k}$-isomorphic through the projective transformation $P = diag(1;\beta;\mu)$ where $\beta^3 = \frac{1}{\alpha_0}$, and $\mu^3 = \frac{1}{\alpha_0^2}$. $\square$

Proof. (of Theorem 4.5)
Consider the Galois extension $M/\mathbb{Q}$ with $M = \mathbb{Q}(cos(2\pi/7), \zeta_3, \sqrt[3]{p})$ where all the elements of $Aut(C_{p,a})$ are defined. Let σ be a generator of the cyclic Galois group $\mathrm{Gal}(\mathbb{Q}(cos(2\pi/7))/\mathbb{Q})$. We define a 1-cocycle in $\mathrm{Gal}(M/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{p})/\mathbb{Q})$ to $Aut(C_{p,a})$ by mapping $(\sigma, id) \mapsto [Y, Z, pX]$ and $(id, \tau) \mapsto id$. This defines an element of $H^1(M/\mathbb{Q}, Aut(C_{p,a}))$.

Consider its image by Σ inside $H^1(M/\mathbb{Q}, \mathrm{PGL}_3(M))$. We need to check that its image is not the trivial element, and then the result is an immediate consequence by Theorem 4.1.

By Theorem 2.6, $H^1(M/\mathbb{Q}, \mathrm{PGL}_3(M))$ is the set of central simple algebras over $\mathbb{Q}$ of dimension 9 which splits in a degree 3 field inside M. If we consider the image in $H^1(\mathrm{Gal}(\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}), \mathrm{PGL}_3(\mathbb{Q}(cos(2\pi/7))))$ then it is non-trivial if and only if p is not a norm of the field extension $\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}$.

By [24, Theorem 2.13], the ideal (p) is prime in $\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}$, therefore p is not a norm of an element of $\mathbb{Q}(\cos(2\pi/7))$. Now $H^1(M/\mathbb{Q}, \mathrm{PGL}_3(M))$ is the union of the above central simple algebras over $\mathbb{Q}$ running through the subfields $F \subset M$ of degree 3 over $\mathbb{Q}$, see [12]. Thus the element is not trivial, which was to be shown. $\square$

5. Explicit non-plane model twists over k of a plane model defined over k

In this section, we explicitly construct a twist of a curve C_a over $\mathbb{Q}(\zeta_3)$ which does not admit a plane model over $\mathbb{Q}(\zeta_3)$. In particular, we construct a non-trivial Brauer-Severi surface over $\mathbb{Q}(\zeta_3)$ giving its equations inside $\mathbb{P}^9_{\mathbb{Q}(\zeta_3)}$.

Let us consider the curve $C_a : X^6 + Y^6 + Z^6 + a(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$ defined over a number field $k \supseteq \mathbb{Q}(\zeta_3)$ where ζ_3 is a primitive third root of unity and $a \in k$. For $a \neq -10, -2, -1, 0, 2$, it is a non-hyperelliptic, non-singular plane curve of genus $g = 10$ and its automorphism group is the group of order 54 determined in the previous section.

The algorithm in [16], allows us to compute all the twists of C_a, previous computation of its canonical model in $\mathbb{P}^9$. We follow such algorithm, since this time we will see that Σ is not trivial, so we cannot use the improvements in Remark 4.4.

5.1. **A canonical model of C_a in $\mathbb{P}^9$.** Let us denote by α_i the six different root of the polynomial $T^6+aT^3+1 = 0$, and define the points on C_a: $P_i = (0 : \alpha_i : 1)$, $Q_i = (\alpha_i : 0 : 1)$ and $\infty_i = (\alpha_i : 1 : 0)$. The divisor of the function $x = X/Z$ is $\mathrm{div}(x) = P_i - \infty$. Let $P = (X_0 : Y_0 : 1) \in C_a$, the function x is a uniformizer at P if the polynomial $T^6 + a(X_0^3 + 1)T^3 + X_0^6 + aX_0^3 + 1 = 0$ does not have double roots. That is, if $X_0^6 + aX_0^3 + 1 \neq 0$ or $4(X_0^6 + aX_0^3 + 1) \neq a^2(X_0^3 + 1)^2$. Let us denote by β_i the six different roots of the polynomial $T^6 + \frac{2a}{a+2}T^3 + 1 = 0$ and denote by $V_{ij} = (\beta_i : \zeta_3^{j+1}\sqrt[3]{-\frac{a}{2}(\beta_i^3 + 1)} : 1)$ where $j \in \{1, 2, 3\}$. In order to compute $\mathrm{ord}_P(dx)$ we need to use the expression

$$dx = -\frac{y^2}{x^2}\frac{2y^3 + a(x^3 + 1)}{2x^3 + a(y^3 + 1)}dy$$

for the points Q_i and $V_{i,j}$. Notice that $\mathrm{div}(2y^3 + a(x^3 + 1)) = V_{i,j} - 3\infty$. For the points at infinity, we use that the degree of a differential is $2g - 2 = 18$. We finally get

$$\mathrm{div}(dx) = 2Q_i + V_{i,j} - 2\infty's.$$

Hence, a basis of regular differentials is given by

$$\omega_1 = \omega = \frac{xdx}{y(2y^3 + a(x^3 + 1))}, \omega_2 = \frac{x^2}{y}\omega, \omega_3 = \frac{y^2}{x}\omega, \omega_4 = \frac{1}{xy}\omega$$

$$\omega_5 = x\omega, \omega_6 = \frac{y}{x}\omega, \omega_7 = \frac{1}{y}\omega, \omega_8 = y\omega, \omega_9 = \frac{x}{y}\omega, \omega_{10} = \frac{1}{x}\omega.$$

We list the divisors of these differentials below.

$$\mathrm{div}(\omega_1) = P_i + Q_i + \infty, \mathrm{div}(\omega_2) = 3P_i, \mathrm{div}(\omega_3) = 3Q_i, \mathrm{div}(\omega_4) = 3\infty$$

$$\mathrm{div}(\omega_5)=2P_i+Q_i,\ \mathrm{div}(\omega_6)=2Q_i+\infty,\ \mathrm{div}(\omega_7)=P_i+2\infty,$$
$$\mathrm{div}(\omega_8)=P_i+2Q_i,\ \mathrm{div}(\omega_9)=2P_i+\infty,\ \mathrm{div}(\omega_{10})=Q_i+2\infty.$$

Lemma 5.1. *The ideal of the canonical model of C_a in $\mathbb{P}^9[\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8,\omega_9,\omega_{10}]$ is generated by the polynomials*

$$\omega_4\omega_9=\omega_7^2,\ \omega_4\omega_6=\omega_{10}^2,\ \omega_4\omega_1=\omega_7\omega_{10},\ \omega_4\omega_5=\omega_9\omega_{10},\ \omega_4\omega_8=\omega_6\omega_7,\ \omega_4\omega_2=\omega_7\omega_9,\ \omega_4\omega_3=\omega_6\omega_{10},$$
$$\omega_3\omega_{10}=\omega_6^2,\ \omega_2\omega_7=\omega_9^2,\ \omega_6\omega_9=\omega_1^2,\ \omega_3\omega_5=\omega_8^2,\ \omega_2\omega_3=\omega_5\omega_8,\ \omega_2\omega_8=\omega_5^2,$$
$$\omega_2^2+\omega_3^2+\omega_4^2+a(\omega_5\omega_8+\omega_6\omega_{10}+\omega_7\omega_9)=0.$$

We denote by $\mathcal{C}_a$ this canonical model.

Proof. If $\omega_4\neq 0$, then the des-homogenization of this ideal with respect to ω_4 gives the affine curve C_a for $Z=1$. If $\omega_4=0$, then $\omega_7=\omega_{10}=0$, so $\omega_6=\omega_9=0$ and $\omega_1=0$, so if $\omega_3\neq 0$ we recover the part at infinity $(Z=0)$ of C_a. If $\omega_4=\omega_3=0$, then all the variables are equal to zero which produces a contradiction.

To check that it is non-singular, we need to see if the rank of the matrix of partial derivatives of the previous generating functions has rank equal to $8=\dim(\mathbb{P}^9)-\dim(C)$ at every point. If $\omega_4\neq 0$, then the partial derivatives of the first seven equation plus the last one produce linearly independent vectors If $\omega_4=0$, we have already seen that $\omega_3\neq 0$ and by equivalent arguments, neither it is ω_2. Then the $6th$, $7th$, $8th$, $9th$ equations plus the last four equations produce the linearly independent vectors. $\square$

Remark 5.2. *The canonical embedding of C_a in $\mathbb{P}^{g-1}=\mathbb{P}^9$ coincides with the composition of the g_2^d-linear system of C_a with the Veronese embedding given by:*

$$\mathbb{P}^2\hookrightarrow\mathbb{P}^9:\ (x:y:z)\to(xyz:x^3:y^3:z^3:x^2y:y^2z:z^2x:xy^2:x^2z:yz^2).$$

In particular, we get that the ideal defining the projective space $\mathbb{P}^2$ in $\mathbb{P}^9$ by the Veronese embedding is generated by the polynomials defined in Lemma 5.1 after removing the last one.

5.2. **The automorphism group of C_a in $\mathbb{P}^9$.** Let us consider the automorphisms of the curve C_a given by $R=[y;x;z]$, $T=[z;x;y]$ and $U=[x;y;\zeta_3 z]$. We easily check that $<R,T,U>\subseteq\mathrm{Aut}(C_a)$ and by Lemma 4.6, we obtain that $\mathrm{Aut}(C_a)=<R,T,U>$.

Notice that the pullbacks $R^*(\omega)=-\omega$, $T^*(\omega)=\omega$ and $U^*(\omega)=\zeta_3^2\omega$. So, in the canonical model, these automorphisms look like

$$R\to-\mathcal{R}=-\left(\begin{array}{c|ccc|ccc|ccc}1&0&0&0&0&0&0&0&0&0\\\hline 0&0&1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\\\hline 0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\end{array}\right),T\to\mathcal{T}=\left(\begin{array}{c|ccc|ccc|ccc}1&0&0&0&0&0&0&0&0&0\\\hline 0&0&0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&1&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\\hline 0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0&0\end{array}\right)$$

and $U\to\zeta_3^2\mathrm{Diag}(1,\zeta_3^2,\zeta_3^2,\zeta_3^2,\zeta_3^2,1,\zeta_3,\zeta_3^2,1,\zeta_3)=\zeta_3^2\mathcal{U}$. We define the faithful linear representation $\mathrm{Aut}(C_a)\hookrightarrow\mathrm{GL}_{10}(\overline{k})$ by sending $R,T,U\to\mathcal{R},\mathcal{T},\mathcal{U}$. Moreover, it preserves the action of the Galois group G_k.

5.3. **A explicit twist over $k=\mathbb{Q}(\zeta_3)$ of C_a without a non-singular plane model over k.** Let us consider the subgroup N of $\mathrm{Aut}(C_a)$ generated by $N:=<TU>\simeq\mathbb{Z}/3\mathbb{Z}$.

Let us consider the curve C_a defined over $k=\mathbb{Q}(\zeta_3)$, and the field extension $L=k(\sqrt[3]{7})$ with Galois group $\mathrm{Gal}(L/k)=<\sigma>\simeq\mathbb{Z}/3\mathbb{Z}$, where $\sigma(\sqrt[3]{7})=\zeta_3\sqrt[3]{7}$. We define the cocycle $\xi\in\mathrm{Z}^1(G_k,\mathrm{Aut}(C_a))\hookrightarrow\mathrm{Z}^1(G_k,\mathrm{PGL}_{10}(\overline{k}))$ given by $\xi_\sigma=\mathcal{T}\mathcal{U}$.

Lemma 5.3. *The image of the cocycle ξ by the map $\Sigma:H^1(G_k,Aut(C_a))\to H^1(G_k,PGL_3(\overline{k}))$ is not trivial.*

Proof. By construction, the image of the cocycle ξ in $H^1(k, \mathrm{PGL}_3(\overline{k}))$ coincides with the inflation of the cocycle in $H^1(Gal(L/k), \mathrm{PGL}_3(L))$ where $\xi_\sigma = TU$. Now by Theorem 2.8 we conclude, since ζ_3 is not a norm in L/k (no new primitive root of unity appears in L than k and ζ_3 is not a norm of an element of L). $\square$

In order to compute equations defining the twist $\mathcal{C}'_a$ associated to the cocycle ξ (and the Brauer-Severi surface that contains such twist), we need to find a matrix $\phi \in \mathrm{PGL}_{10}(\overline{k})$ such that $\xi_\sigma = \phi \cdot^\sigma \phi^{-1}$.

We can then take

$$\phi = \left(\begin{array}{c|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \sqrt[3]{7} & \sqrt[3]{7^2} & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} & 7\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt[3]{7} & \zeta_3^2\sqrt[3]{7^2} & 7\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \zeta_3\sqrt[3]{7} & \sqrt[3]{7^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta_3 & \sqrt[3]{7} & \sqrt[3]{7^2} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3\sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & \zeta_3\sqrt[3]{7} & \sqrt[3]{7^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2}
\end{array}\right)$$

If we simply substitute this isomorphism ϕ in the equations of $\mathcal{C}_a$, we will get equations for $\mathcal{C}'_a$. However, even defining a curve over k, this equations are defined over $L = k(\sqrt[3]{7})$. In order to get generators of the ideal defined over k, we use next lemma.

Lemma 5.4. *Let $f_0, f_1, f_2 \in k[x_1, ..., x_n]$, and define $g_0 = f_0 + \sqrt[3]{7}f_1 + \sqrt[3]{7^2}f_2$, $g_1 = f_0 + \zeta_3\sqrt[3]{7}f_1 + \zeta_3^2\sqrt[3]{7^2}f_2$, $g_2 = f_0 + \zeta_3^2\sqrt[3]{7}f_1 + \zeta_3\sqrt[3]{7^2}f_2$. Then the ideals in $L[x_1, ..., x_n]$ generated by $< g_0, g_1, g_2 >$ and $< f_0, f_1, f_2 >$ are equal.*

Proof. Clearly, we have the inclusion $< g_0, g_1, g_2 > \subseteq < f_0, f_1, f_2 >$. The reverse inclusion can be checked by writing $3f_0 = g_0 + g_1 + g_2$, $(\zeta_3 - 1)\sqrt[3]{7}f_1 = g_1 - \zeta_3 g_2 + (\zeta_3 - 1)f_0$ and $\sqrt[3]{7^2}f_2 = g_0 - f_0 - \sqrt[3]{7}f_1$. $\square$

Proposition 5.5. *The equations in $\mathbb{P}^9$ of the non-trivial Brauer-Severi surface B over k constructed as in Theorem 4.1 from the cocycle ξ above are*

$$\begin{array}{ll}
\omega_1\omega_2 = \zeta_3\omega_5\omega_9 + \zeta_3\omega_6\omega_8 + 7\zeta_3\omega_7\omega_{10}, & \omega_2^2 - 7\omega_3\omega_4 = \zeta_3\omega_5\omega_{10} + \zeta_3\omega_7\omega_8 + \zeta_3\omega_6\omega_9, \\
\omega_1\omega_3 = \omega_5\omega_{10} + \zeta^2\omega_7\omega_8 + \zeta_3\omega_6\omega_9, & 7\omega_3^2 - 7\zeta_3\omega_2\omega_4 = \omega_5\omega_9 + \zeta_3^2\omega_6\omega_8 + 7\zeta_3\omega_7\omega_{10}, \\
7\omega_1\omega_4 = \zeta_3\omega_5\omega_8 + 7\omega_6\omega_{10} + 7\zeta_3^2\omega_7\omega_9, & 49\omega_4^2 - 7\zeta_3^2\omega_2\omega_3 = \omega_5\omega_8 + 7\zeta_3\omega_6\omega_{10} + 7\zeta_3^2\omega_7\omega_9,
\end{array}$$

$$\begin{array}{ll}
\omega_5^2 + 14\zeta_3\omega_6\omega_7 = 7\zeta_3\omega_2\omega_{10} + 7\omega_4\omega_8 + 7\zeta_3\omega_3\omega_9, & \omega_5^2 - 7\zeta_3\omega_6\omega_7 = 7\omega_2\omega_{10} + 7\omega_4\omega_8 + 7\zeta_3^2\omega_3\omega_9, \\
\omega_6^2 + 2\zeta_3\omega_5\omega_7 = \zeta_3\omega_2\omega_9 + \omega_3\omega_8 + 7\zeta_3\omega_4\omega_{10}, & \omega_6^2 - \zeta_3\omega_5\omega_7 = \omega_2\omega_9 + \zeta_3\omega_3\omega_8 + 7\zeta_3\omega_4\omega_{10}, \\
7\omega_7^2 + 2\zeta_3\omega_5\omega_6 = \zeta_3\omega_2\omega_8 + 7\zeta_3^2\omega_3\omega_{10} + 7\zeta_3^2\omega_4\omega_9, & 7\omega_7^2 - \zeta_3\omega_5\omega_6 = \omega_2\omega_8 + 7\omega_3\omega_{10} + 7\zeta_3^2\omega_4\omega_9,
\end{array}$$

$$\begin{array}{ll}
\omega_8^2 + 14\zeta_3\omega_9\omega_{10} = 7\zeta_3^2\omega_2\omega_7 + \omega_4\omega_5 + 7\zeta_3^2\omega_3\omega_6, & \omega_8^2 - 7\zeta_3^2\omega_9\omega_{10} = 7\zeta_3^2\omega_2\omega_7 + 7\zeta_3^2\omega_4\omega_5 + 7\omega_3\omega_6, \\
\omega_9^2 + 14\zeta_3^2\omega_8\omega_{10} = \zeta_3\omega_2\omega_6 + \zeta_3^2\omega_3\omega_5 + 7\zeta_3\omega_4\omega_7, & \omega_9^2 - 7\zeta_3^2\omega_8\omega_{10} = \zeta_3^2\omega_2\omega_6 + \zeta_3\omega_3\omega_5 + 7\zeta_3\omega_4\omega_7, \\
7\omega_{10}^2 + 2\zeta_3^2\omega_8\omega_9 = \zeta_3^2\omega_2\omega_5 + 7\omega_3\omega_7 + 7\omega_4\omega_6, & 7\omega_{10}^2 - \zeta_3^2\omega_8\omega - 9 = \zeta_3^2\omega_2\omega_5 + 7\zeta_3^2\omega_3\omega_7 + 7\omega_4\omega_6,
\end{array}$$

Proof. We only need to plug the equations of the isomorphism ϕ into the equations defining $\mathcal{C}_a$ and apply Lemma 5.4. $\square$

In order to get the equations of the twisted curve, we only need to add the equation that we get by plugging ϕ in $\omega_2^2 + \omega_3^2 + \omega_4^2 + a(\omega_5\omega_8 + \omega_6\omega_{10} + \omega_7\omega_9) = 0$, and apply Lemma 5.4 again.

Proposition 5.6. *The curve $\mathcal{C}'_a$ is a twist over k of the curve C_a for $a \neq -10, -2, -1, 0, 2$ which does not admits a non-singular plane model over k and the defining equations of $\mathcal{C}'_a$ in $\mathbb{P}^9$ are the ones given in Proposition 5.5 plus the extra equation:*

$$\omega_2^2 + 14\omega_3\omega_4 + a(\omega_2^2 - 7\omega_3\omega_4) = 0$$

5.4. **A non-singular plane model for the twist of $\mathcal{C}'_a$ over finite fields.** We consider the reductions $\tilde{C}_a$ and $\tilde{\mathcal{C}}_a$ at a prime $\mathfrak{p}$ of good reduction of the curve C_a/k and the twist $\mathcal{C}'_a/k$ computed in subsection 5.3. Since $k=\mathbb{Q}(\zeta_3)$, the resulting reductions curves are defined over a finite field $\mathbb{F}_q$ with $q\equiv 1 \bmod 3$, and $q=p^f$ for some $f\in\mathbb{N}$ and $\mathfrak{p}\mid p$. We also assume that $p>21=(6-1)(6-2)+1$ in order to ensure that $\mathrm{Aut}(\tilde{C}_a)\simeq <54,5>$, see [1, §6] and Lemma 4.6.

The natural map $G_k\to G_{\mathbb{F}_q}$ induces a map $\mathrm{H}^1(k,\mathrm{Aut}(C_a))\to\mathrm{H}^1(\mathbb{F}_q,\mathrm{Aut}(\tilde{C}_a))$. Since $\mathrm{Z}^1(\mathbb{F}_q,\mathrm{Aut}(\tilde{C}_a))\hookrightarrow \mathrm{Z}^1(G_{\mathbb{F}_q},\mathrm{PGL}_3(\overline{\mathbb{F}_q}))$ and $\mathrm{H}^1(G_{\mathbb{F}_q},\mathrm{PGL}_3(\overline{\mathbb{F}_q}))=1$, see lemma 2.9, the reduction twist has a non-singular plane models.

Clearly, if $7\in\mathbb{F}_q^3$, then the twist $\mathcal{C}'_a$ becomes trivial. Otherwise, we get that the reduction of the cocycle ξ is given by its image at π, the Frobenius endomorphism [17], and ξ_π can take the values

$$\begin{pmatrix} 0 & 0 & \zeta_3^e\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}^e,$$

where $e=0,1,2$ according to the splitting behaviour of the prime $\mathfrak{p}$ in $L=k(\sqrt[3]{7})$. In the first case, we get the trivial twist. In the later and the former, let assume $e=1$ (the other can be treat symmetrically) and $q\not\equiv 1 \bmod 9$, we can then take a generator η of $\mathbb{F}_{q^3}/\mathbb{F}_q$, such that $\eta^3=\zeta_3$. Then, the cocycle is given ($\xi_\sigma=\phi\,{}^\sigma\phi^{-1}$) by the isomorphism

$$\phi=\begin{pmatrix} 1 & \eta & \eta^2\\ \eta^2 & \zeta_3^2 & \eta\\ \eta & \zeta_3^2\eta^2 & \zeta_3^2\end{pmatrix}:\tilde{C}'_a\to\tilde{C}_a,$$

and the twist $\tilde{\mathcal{C}}'_a$ has a non-singular plane model

$$\tilde{C}'_a:\ 2(x^5z+z^5y+\zeta_3y^5x)-5(1+\zeta_3)(y^4z^2+x^2z^4)+9x^4y^2+20\zeta_3(x^3yz^2+x^2y^3z)-20(\zeta_3+1)xy^2z^3+$$

$$+a(-(x^5z+z^5y+\zeta_3y^5x)+2x^4y^2-2(\zeta_3+1)(x^2z^4+y^4z^2)-\zeta_3(x^3yz^2-x^2y^3z)+(\zeta_3+1)xy^2z^3).$$

If $q\equiv 1 \bmod 9$, the same ϕ works, but this time the cocycle becomes trivial since $\eta\in\mathbb{F}_q$.

6. Twists of smooth plane curves with diagonal cyclic automorphism group

We observed in Remark 4.4, that the algorithm for computing $Twist_k(C)$ described in [16] can be substantially improved if the smooth curve C over k admits a non-singular plane model and such that the morphism Σ in theorem 4.1 is trivial.

In this section, we apply this algorithm for curves C having an extra property: there exists a plane k-model $F_{\overline{C}}(X,Y,Z)=0$ having a diagonal cyclic automorphism group, i.e., we have that $Aut(F_{\overline{C}})=<\alpha>$ with α a diagonal matrix. In such case, we prove that all the elements of $Twist_k(F_{\overline{C}}=0)$ are given by non-singular plane models of the form $F_{D\overline{C}}=0$ with D a diagonal matrix. We apply this method to some particular families of smooth plane curves.

Definition 6.1. Consider a smooth plane curve C over k with a non-singular plane model over k given by $F_{\overline{C}}(X,Y,Z)=0$. We say that $C'\in Twist_k(C)$ is a diagonal twist of C if there exist $M\in\mathrm{PGL}_3(k)$ and D a diagonal matrix in $\mathrm{PGL}_3(\overline{k})$ such that C' is k-isomorphic to $F_{MD\overline{C}}(X,Y,Z)=0$.

The condition of having cyclic automorphism group is not enough to ensure that all the twists having plane non-singular models, are diagonal twists. We will show an example.

6.1. **Diagonal cyclic automorphism group: all twists are diagonal.**

Motivated by the results in Section 4 and following the philosophy of the third author's thesis in [15], we prove the next result.

Theorem 6.2. *Let $C:\, F_{\overline{C}}(X,Y,Z)=0$ be a non-singular plane curve defined over k. Assume that $Aut(F_{\overline{C}})\subseteq PGL_3(\overline{k})$ is a non-trivial cyclic group of exact order n (prime with the characteristic of k) generated by an element $\alpha=diag(1,\zeta_n^a,\zeta_n^b)$ where $a,b\in\mathbb{N}$.*

Then all the twists $Twist_k(C)$ are given by plane equations of the form $F_{D\overline{C}}(X,Y,Z)=0$ with $F_{D\overline{C}}(X,Y,Z)\in k[X,Y,Z]$ and D is a diagonal matrix. In particular, the map Σ is trivial.

Proof. We just need to notice that the map Σ in Theorem 4.1 factors as follows:

$$\Sigma:\ \mathrm{H}^1(k,\mathrm{Aut}(F_{P\overline{C}}))\to(\mathrm{H}^1(k,\mathrm{GL}_1(\overline{k})))^3\to\mathrm{H}^1(k,\mathrm{GL}_3(\overline{k}))\to\mathrm{H}^1(k,\mathrm{PGL}_3(\overline{k})).$$

Hence, Σ is trivial and all the cocycles are given by diagonal matrices. □

Remark 6.3. *A general statement of the above result is as follows: Assume that C is a curve defined over k with a plane non-singular model over k say, $F_{\overline{C}}(X,Y,Z)=0$, and that there exists $[C']\in Twist_k(C)$ such that C' admits a non-singular plane model over k given by $F_{Q\overline{C}}(X,Y,Z)=0$ with $Aut(F_{Q\overline{C}})=\langle diag(1,\xi_n^a,\xi_n^b)\rangle$. Then, any $[C'']\in Twist_k(C)$ has a representative C'' given by a non-singular plane model over k of the form $F_{QD\overline{C}}=0$ with D diagonal. In particular, $Aut(F_{QD\overline{C}})$ is diagonal, and the twists of C over k are diagonal.* [2]

We apply now Theorem 6.2 to some particular smooth plane curves with cyclic automorphism group. These twists are also computed in the appendix by the algorithm in [16] with the improvement of Theorem 4.1 in order to run the algorithm in [16] in PGL_3 instead of PGL_g where g the genus of the smooth plane curve.

For a finite group G, we denote by $\widetilde{M_g^{Pl}(G)}$ the elements, in the moduli space M_g of smooth, genus g curves over $\overline{k}$, that admit a non-singular plane model, and their full automorphism group is isomorphic to G. The strata $\widetilde{M_g^{Pl}(G)}$ is the disjoint union of the different components $\rho(\widetilde{M_g^{Pl}(G)})$ where ρ denote different non-conjugate injective representations of G inside $\mathrm{PGL}_3(\overline{k})$, we refer to [1] for complete details.

Let k be a field of characteristic 0 or $p>(d-1)(d-2)+1$ (see the last section in [1] and [3]). Then we have:

$$M_g^{Pl}(\widetilde{\mathbb{Z}/d(d-1)}\mathbb{Z})=\{X^d+Y^d+XZ^{d-1}=0\},$$
$$M_g^{Pl}(\widetilde{\mathbb{Z}/(d-1)}^2\mathbb{Z})=\{X^d+Y^{d-1}Z+XZ^{d-1}=0\}.$$

Both curves are defined over k and they have a non-singular plane model over k whose automorphism groups are cyclic diagonal of orders $d(d-1)$ and $(d-1)^2$ respectively. These groups are generated by

$$diag(1,\zeta_{d(d-1)}^{d-1},\zeta_{d(d-1)}^{d}),\ \text{and}\ diag(1,\zeta_{(d-1)^2},\zeta_{(d-1)^2}^{(d-1)(d-2)}),$$

respectively, see Theorem 1 in [3].

Theorem 6.4. *Let k be a field of characteristic zero or $p>(d-1)(d-2)+1$. Consider the curve $C:\ X^d+Y^d+XZ^{d-1}=0$ with $d\geq5$. Then, the set $Twist_k(C)$ is parameterized by $\mathfrak{A}_1:=(k^*\backslash {k^*}^d)\times(k^*\backslash {k^*}^{d-1})/\sim$, where $(M,N)\sim(M',N')$ if and only if $M'=n^dM$, $N'=nm^{d-1}N$ for some $n,m\in k$. More precisely, a pair*

[2] Here we present a different proof of Theorem 6.2 assuming the general statement:

Consider the natural map $\Sigma:H^1(k,Aut(F_{PQ\overline{C}}))\to H^1(k,\mathrm{PGL}_3(\overline{k}))$, then by Theorem 4.1, any twist that has a non-singular plane model over k is given by some $Q'\in\mathrm{PGL}_3(\overline{k})$, i.e. it has a non-singular plane model over k of the form $F_{PQQ'\overline{C}}=0$. This, in particular, defines the cocycle $\sigma\mapsto Q'\cdot^{\sigma}(Q')^{-1}$, which is trivial in $H^1(k,\mathrm{PGL}_3(\overline{k}))$ because $Q'\in\mathrm{PGL}_3(\overline{k})$.

Now, $Q'\cdot^{\sigma}Q'^{-1}\in Aut(F_{PQ\overline{C}})=\langle diag(1,\xi_n^a,\xi_n^b)\rangle$ (observe that $Q'^{-1}:F_{PQ\overline{C}}\to F_{PQQ'\overline{C}}$), thus ${}^{\sigma}(Q'^{-1})=Q'^{-1}diag(1,\xi_n^{a'},\xi_n b')$ for some integers a',b'. Writing $Q'^{-1}=(a_{i,j})$, one easily deduces that $\sigma(a_{i,j})=u_ja_{i,j}$ with $u_1=1,u_2=\xi_n^{a'},u_3=\xi_n^{b'}$; therefore fixing j, we get $\sigma(a_{i,j})a_{i,j}^{-1}=\sigma(a_{i',j})a_{i',j}^{-1}$. In particular, $a_{i,j}a_{i',j}^{-1}$ is invariant under all $\sigma\in G_k$, and thus $a_{i,j}=q_ia_{i',j}$ for some $q_i\in k$. Then, we can consider $Q'=DM$ with D diagonal over $\overline{k}$ and $M\in\mathrm{PGL}_3(k)$.

Now, the k-model $F_{PQDM\overline{C}}(X,Y,Z)=0$ is k-isomorphic to $F_{PQD\overline{C}}(X,Y,Z)=0$, which is a diagonal twist of $F_{PQ\overline{C}}(X,Y,Z)=0$. Thus, all the twists of C over k admitting a non-singular plane model over k are diagonal. But, also the morphism $Aut(F_{PQ\overline{C}})=<diag(1,\xi_n^a,\xi_n^b)>\hookrightarrow\mathrm{PGL}_3(\overline{k})$ factors through $GL_3(\overline{k})$, and $H^1(k,\mathrm{GL}_3(\overline{k}))$ is trivial. Hence Σ is the zero map, and all the twists of C over k admit also a non-singular plane model over k.

This completes the proof.

$(M,N) \in \mathfrak{A}_1$ corresponds to a twist of the form $MX^d + Y^d + NXZ^{d-1} = 0$ and vice versa, i.e, every twist corresponds to some pair $(M,N) \in \mathfrak{A}_1$.

Proof. The description of the twists is an easy consequence of Theorem 4.1. Finally, the equivalence of twists comes from the fact that two such cohomologous cycles are related by a diagonal matrix. □

Similarly, we have:

Theorem 6.5. *Let k be a field of characteristic zero or positive characteristic $> (d-1)(d-2)+1$. Consider the curve $X^d + Y^{d-1}Z + XZ^{d-1} = 0$ with $d \geq 5$. Then, the set $Twist_k(C)$ is parameterized by $\mathfrak{A}_2 := (k^* \setminus {k^*}^{d-1}) \times (k^* \setminus {k^*}^{d-1})/\sim$, where $(M,N) \sim (M',N')$ if and only if $M' = n^{d-1}mM$, $N' = m^{d-1}N$ for some $n,m \in k$. That is, we associate to an $(M,N) \in \mathfrak{A}_2$ the twist $X^d + MY^{d-1}Z + NXZ^{d-1} = 0$, and vice versa.*

Remark 6.6. *In the work in [3] (or in [2], for curves of genus 6), the first two authors give different families of curves depending on certain parameters for $\rho(\widetilde{M_g^{Pl}}(\mathbb{Z}/n\mathbb{Z}))$ with $\rho(\mathbb{Z}/n\mathbb{Z})$ generated by a diagonal matrix. So, it is possible to apply Theorem 6.2 to compute the twists of curves in these families. In order to get a precise parametrization of the twists, we need to deal with representative families for such strata $\rho(\widetilde{M_g^{Pl}}(\mathbb{Z}/n\mathbb{Z}))$ (see the ideas of Lercier, Ritzenthaler, Rovetta and Sisjling in [14]). In the upcomming work [4], we study representative families for the different stratas of $\rho(\widetilde{M_6^{Pl}}(G))$.*

6.2. **Aut(C) cyclic does not imply diagonal twists.** The hypothesis that the automorphism group of C is generated by a diagonal matrix on Theorem 6.2 cannot be removed.

Example 6.7. Consider the non-singular plane curve C_0 defined over $\mathbb{Q}$ by the equation

$$X^4Y + Y^4Z + XZ^4 + (X^3Y^2 + Y^3Z^2 + X^2Z^3) = 0$$

and denote such model by $F_{\overline{C_0}}(X,Y,Z) = 0$.

Lemma 6.8. *The group $Aut(F_{\overline{C_0}})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ (then so does $Aut(\overline{C_0})$), and it is generated by $[Y;Z;X]$ as a subgroup of $PGL_3(\overline{\mathbb{Q}})$.*

Proof. We have $s := [Y;Z;X] \in Aut(F_{\overline{C_0}})$ is of order 3, therefore the group $Aut(F_{\overline{C_0}})$ should be one of the groups in [3, Table 2] with 3 dividing the order. First, we note that there are no elements $t \in Aut(F_{C_0})$ of order 2 such that $tst = s^{-1}$: To show this we consider, for simplicity, the $\overline{\mathbb{Q}}$-equivalent model $F_{P\overline{C_0}}$ of the form

$$4X^5 + 20X^3YZ + \left((-5-9i\sqrt{3})Y^3 + (-5+9i\sqrt{3})Z^3\right)X^2 - 6XY^2Z^2 - 4YZ(Y^3+Z^3),$$

through the transformation P of the shape

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi_3 & \xi_3^2 \\ 1 & \xi_3^2 & \xi_3 \end{pmatrix}.$$

Recall that $Aut(F_{P\overline{C_0}}) = P^{-1}Aut(F_{\overline{C_0}})P$, in particular $s' := P^{-1}sP = diag(1;\xi_3;\xi_3^2) \in Aut(F_{P\overline{C_0}})$. Now, if $t' \in PGL_3(\overline{\mathbb{Q}})$ is of order 2 such that $t's't' = s'^{-1}$, then t' should be of the shapes $[X;aZ;a^{-1}Y]$, $[aZ;Y;a^{-1}X]$ or $[aY;a^{-1}Y;Z]$ for some $a \in \overline{\mathbb{Q}}$. But non of these transformations retains the defining equation $F_{P\overline{C_0}} = 0$, hence S_3 does not occur as a bigger group of automorphisms. Then so are the groups $GAP(30,1)$ and $GAP(150,5)$, as both groups contain an S_3 and also there exists a single conjugacy class of elements of order 3 inside these groups. In particular, we get the same conclusion for $Aut(F_{\overline{C_0}})$, which was to be shown.

Second, assume that $Aut(F_{\overline{C_0}})$ is conjugate, through some $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ to $GAP(39,1)$ such that $F_{P\overline{C_0}}(X,Y,Z) = X^4Y + Y^4Z + XZ^4$. Now, any element of order 3 in $GAP(39,1)$, with respect to the given representation in [2, Table 2], is conjugate to s or s^{-1}. Then we may impose, without loss of generality, that $P^{-1}sP = s$, since s is not conjugate to s^{-1} in $\mathrm{PGL}_3(\overline{\mathbb{Q}})$. In particular, P has the shape

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \zeta_3^r\alpha_3 & \zeta_3^r\lambda\alpha_1 & \zeta_3^r\alpha_2 \\ \zeta_3^{-r}\alpha_2 & \zeta_3^{-r}\alpha_3 & \zeta_3^{-r}\alpha_1 \end{pmatrix} \in \mathrm{PGL}_3(\overline{\mathbb{Q}}),$$

When $r = 0$, we need to cancel the coefficients of $Y^5, Y^4X, Y^3X^2, Y^3Z^2$, and Y^3XZ in $F_{P\overline{C_0}}$, which is impossible due to invertibility of P. Furthermore, if $r = 1$ or 2, we also force the coefficients of X^5, Y^5 and Z^5 in $F_{P\overline{C_0}}$ to be zeros, and then we obtain a diagonal transformation P. In particular, the defining equation $F_{P\overline{C_0}} = 0$ is not the claimed one. That is $Aut(F_{C_0})$ can not be conjugate to $GAP(39, 1)$. Hence, $Aut(F_{C_0})$ is cyclic of order 3, and we are done.

□

Lemma 6.9. *There exists a non-diagonal twist C_0 over $\mathbb{Q}$.*

Proof. The defining equation $F_{\overline{C_0}} = 0$ has degree 5, thus any twist of $\overline{C_0}$ admits also a non-singular plane model over $\mathbb{Q}$ defined by $F_{P\overline{C_0}}(X, Y, Z) = 0$ for some $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$.

We construct the twist following the classical algorithm in [16] and Theorem 4.1 because Σ is trivial.

Firstly, the twisted product $\Gamma := Aut(F_{\overline{C_0}}) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ (recall that the field K, where all automorphisms of $F_{\overline{C_0}}$ are defined, is $\mathbb{Q}$). The group $Aut(F_{\overline{C_0}})$ is identified with $lat[2]$ (see notation in the code) inside the lattice of subgroups of Γ. Then, if we modify the MAGMA code in [15, Table 5.5] to fit our case, then we deduce that $F_{\overline{C_0}}$ has exactly two non-trivial twists for each cyclic cubic field extension $L/\mathbb{Q}$. Since the set of such extensions is not empty, the curve $\overline{C_0}$ has a non-trivial twist.

Secondly, a twist of $F_{\overline{C_0}}$ through a diagonal isomorphism $D \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ is always trivial (then so are all the twists of the form $F_{DM\overline{C_0}}$ with $M \in \mathrm{PGL}_3(\mathbb{Q})$). On the other hand, to obtain a non-trivial twist of the form $F_{MD\overline{C_0}}$ with $D = diag(1; a; b) \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ and $M = (a_{ij}) \in \mathrm{PGL}_3(\mathbb{Q})$, we must satisfy the 1-cocycle condition $(MD).^h(MD)^{-1} = s$ (recall that $\phi = (MD)^{-1} : F_{\overline{C_0}} \to F_{MD\overline{C_0}}$), where h is a generator of $\mathrm{Gal}(L/\mathbb{Q})$. In particular,

$$\begin{pmatrix} a_{11} & aa_{12} & ba_{13} \\ a_{21} & aa_{22} & ba_{23} \\ a_{31} & aa_{32} & ba_{33} \end{pmatrix} = \lambda \begin{pmatrix} a_{21} & h(a)a_{22} & h(b)a_{23} \\ a_{31} & h(a)a_{32} & h(b)a_{33} \\ a_{11} & h(a)a_{12} & h(b)a_{13} \end{pmatrix}$$

for some $\lambda \in \overline{\mathbb{Q}}$. From the 1^{st} column, we get $\lambda a_{21} = a_{11}, \lambda a_{31} = a_{21}$, and $\lambda a_{11} = a_{31}$. Hence $\lambda^3 - 1 = 0$, but also $a_{ij} \in \mathbb{Q}$, then $\lambda = 1$. Consequently, $h(a^3) = a^3$, and $h(b^3) = b^3$. That is $a^3, b^3 \in L^{\langle h \rangle}(= \mathbb{Q})$, thus $a = \sqrt[3]{N}$, and $b = \sqrt[3]{N'}$ for some $N, N' \in \mathbb{Q}$. In particular, the twist ϕ has $\mathbb{Q}(\sqrt[3]{N}, \sqrt[3]{N'})$ as its splitting field, which is not Galois if a or b does not belong to $\mathbb{Q}$, a contradiction.

Consequently, C_0 has a non-trivial twist, which can not be obtained through any diagonal isomorphism modulo $\mathrm{PGL}_3(\mathbb{Q})$.

□

Therefore we obtain,

Proposition 6.10. *Let C be a non-singular plane curve over k, a field of characteristic zero, admitting a non-singular plane model $F_{\overline{C}}(X, Y, Z) = 0$ over k such that $Aut(F_{\overline{C}}) \subseteq PGL_3(\overline{k})$ is a cyclic group of order n generated by a matrix α, and no element in the conjugacy class of α in $PGL_3(k)$ is neither a diagonal matrix. Then the twists mapping to zero by Σ (i.e., those ones admitting a plane non-singular model over k), are not necessarily diagonal twist.*

Remark 6.11. *The above example in §6.2 extends to positive characteristic p, for $p > (d-1)(d-2)+1$ with $d = 5$, (Lemma 6.8 remains true by the arguments in [1, §6]), and $\zeta_3 \notin k$ in order to construct the non-trivial diagonal twist in the proof of Lemma 6.9.*

Remark 6.12. *Degree* 5 *is the smallest degree for which such an example exists, see the third author thesis [15] to discard degree* 4 *exceptions.*

Appendix A. The classical algorithm on twists for non-hyperelliptic curves

The algorithm for computing $Twist_k(C)$ of a non-hyperelliptic curve C of genus $g \geq 3$ developed in [15, Chp.1] and [16] has three main steps: (1) canonical model of C, (2) Solutions of the Galois embedding problem, (3) Explicit equation of Twists.

Assume that C is a smooth curve over k with a plane non-singular model over k such that Σ is trivial, in such case all the twist admits a plane non-singular model over k, see Theorem 4.1. Now, instead of computing a canonical model of C, we can consider a plane model over k associated to C, modifying point (1) of the algorithm. The point (2) is independent of the embedding of C inside a projective space. In point (3), the algorithm of [16] requires to investigate the solutions in $\mathrm{GL}_g(\overline{k})$ using [15, Lemma 1.1.3]. Now, in the modified algorithm, it is enough to look for solutions in $\mathrm{GL}_3(\overline{k})$. As in [15, 16], the isomorphisms covering the solutions of the Galois embedding problems in $\mathrm{GL}_g(\overline{k})$ or $\mathrm{GL}_3(\overline{k})$ is quite hard except that we have a control of the matrix shape that could appear.

We use the above modified algorithm to compute the twist over k for the curve $C:\ X^5+Y^5+XZ^4=0$ where k is a field of zero characteristic or positive characteristic $>(5-1)(5-2)+1=13$. Here we recover the result obtained for such curve in Theorem 6.4.

The automorphism group $Aut(C)\cong GAP(20,2)$ is generated by the elements $s:=diag(1;1;i)$, and $t:=diag(1;\zeta_5;1)$, so it is defined over $K=k(i,\zeta_5)$ (we assume the generic case in which $i,\zeta_5\notin k$).

Then, $[K:k]=8$ and the Galois group $\mathrm{Gal}(K/k)$ is generated by $\tau_1: i\mapsto -i$, $\zeta_5\mapsto\zeta_5$ and $\tau_2: i\mapsto i$, $\zeta_5\mapsto\xi_5^2$ of order 2 and 4 respectively with $\tau_2\tau_1=\tau_1\tau_2$ (in particular, $\mathrm{Gal}(K/k)\cong GAP(8,2)$).

The group $\mathrm{Gal}(K/k)$ acts naturally on $Aut(C)$ as: $\tau_1: s\mapsto s^3$, $t\mapsto t$ and $\tau_2: s\mapsto s$, $t\mapsto t^2$.

The twisted product $\Gamma:=Aut(C)\rtimes\mathrm{Gal}(K/k)$ is isomorphic to $GAP(160,207)$, and generated by the elements $x:=(st,1), y:=(1,\tau_1)$ and $z:=(1,\tau_2)$, where $x^{20}=y^2=z^4=1$, $yxy=x^{11}$, $xz=zx^{13}$ and $yzy=z$.

The degree of the defining equation of C is coprime with 3, thus, by Corollary 3.7, every twist of $\overline{C}$ has a non-singular plane model over k. Consequently, by Theorem 4.1, the map Σ is trivial. In particular, we only look for solutions of the Galois embedding problems, generated by a similar MAGMA code as in [15, Table 5.5], inside $GL_3(\overline{k})$, not in $GL_6(\overline{k})$. One finds that all the twists of C over k are covered by diagonal matrices, and they are of the form $\alpha X^d+Y^d+\beta XZ^{d-1}$ for some $\alpha,\beta\in k$ through an isomorphism of the shape $diag(a;1;c)$ in $\mathrm{GL}_3(\overline{k})$.

We thus collect the computations into the following result:

Theorem A.1. *(Galois embedding problems) The set $Twist_k(C)$ is completely determined by table 1.*

Remark A.2. *The Galois embedding problems for C are given by the pairs (G,H) appearing in the second and the third columns in GAP notations. This means that a twist $\phi: C\to C'$ of C over k has a splitting field L such that $Gal(L/k)\cong G$ and $Gal(L/K)\cong H$ for some pair (G,H) in the list. By the aid of Proposition 4.1 in [15], we find solutions to these Galois embedding problems as described in the 8^{th} column, where $N\in k^*\setminus k^{*2}$ and $M\in k^*\setminus k^{*5}$. We also provide generators of G and H in the 4^{th}, 5^{th}, and 6^{th} columns: G is generated by the elements $h\rtimes 1\in H\rtimes 1$ and $g_i\rtimes\tau_i$ with $i=1,2$. The integer $n_{(G,H)}$ that appears in the 7^{th} column is the number of non-equivalent twists of C with the same splitting field L. In the remaining part of the table, we give the associated set of non-equivalent twists which are defined by equations of the form $\alpha X^d+Y^d+\beta XZ^{d-1}=0$ through an isomorphism of the form $diag(a;1;c)$ with $a=\sqrt[5]{\alpha}$, $r\in\{1,25\}$, $\ell\in\{1,2,3,4\}$, $j_i\in\{0,1\}$.*

TABLE 1. The pairs (G, H), and Twists

	$ID(G)$	$ID(H)$	$gen\ (H)$	g_1	g_2	$n_{(G,H)}$	L	α	β	φ
1	$< 8, 2 >$	$< 1, 1 >$	1	s	1	1	K	1	-4	$c = \sqrt[4]{\beta}$
2				s	s^2				-100	
3				1	s^2				25	
4				1	1				1	
5	$< 16, 10 >$	$< 2, 1 >$	s^2	s	1	2	$K(\sqrt{N})$	1	$-4rN^2$	
6	$< 16, 3 >$			s	s		$K(\sqrt[4]{5N^2})$		$-20rN^2$	
7	$< 16, 3 >$			1	s				$5rN^2$	
8	$< 16, 10 >$			1	1		$K(\sqrt{N})$		rN^2	
9	$< 40, 12 >$	$< 5, 1 >$	t	1	s^2	4	$K(\sqrt[5]{M})$	M^ℓ	$25M^\ell$	$c = \sqrt[4]{\frac{\beta}{M^\ell}}\sqrt[5]{M^\ell}$
10				1	1				M^ℓ	
11				s	s^2				$-100M^\ell$	
12				s	1				$-4M^\ell$	
13	$< 32, 25 >$	$< 4, 1 >$	s	1	1	8	$K(\sqrt[4]{N})$	1	$(-4)^{j_1}(25)^{j_2}N^{2j_3+1}$	$c = \sqrt[4]{\beta}\sqrt[5]{M^\ell}$
14	$< 80, 50 >$	$< 10, 2 >$	$s^2,\ t$	s	1	8	$K(\sqrt{N},\ \sqrt[5]{M})$	M^ℓ	$-4rN^2M^\ell$	
15	$< 80, 34 >$			1	s		$K(\sqrt[4]{5N^2},\ \sqrt[5]{M})$		$125rN^2M^\ell$	
16	$< 80, 34 >$			s	s				$-500rN^2M^\ell$	
17	$< 80, 50 >$			1	1		$K(\sqrt{N},\ \sqrt[5]{M})$		rN^2M^ℓ	
18	$< 160, 207 >$	$< 20, 2 >$	$s,\ t$	1	1	32	$K(\sqrt[4]{N},\ \sqrt[5]{M})$	M^ℓ	$(-4)^{j_1}(25)^{j_2}N^{2j_3+1}M^\ell$	

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• Eslam Essam Ebrahim Farag Badr

Departament Matemàtiques, Edif. C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia
E-mail address: eslam@mat.uab.cat

Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt
E-mail address: eslam@sci.cu.edu.eg

• Francesc Bars Cortina

Departament Matemàtiques, Edif. C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia
E-mail address: francesc@mat.uab.cat

• Elisa Lorenzo

Universiteit Leiden, 2300 Leiden, The Nederlands
E-mail address: e.lorenzo.garcia@math.leidenuniv.nl