Joyal's CatLab

Distributors and barrels

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Barrels

Recall that a set of objects S in category A is said to be a *sieve* if the implication

$$\mathrm{target}(f) \in S \;\Rightarrow\; \mathrm{source}(f) \in S$$

is true for every arrow $f \in A$. Dually, a set of objects $S \subseteq A$ is said to be a *cosieve* if the implication

$$\mathrm{source}(f) \in S \;\Rightarrow\; \mathrm{target}(f) \in S$$

is true for every arrow $f \in A$. Notice that a subset $S \subseteq \mathrm{Ob}(A)$ is a sieve iff its complement is a cosieve. We shall often identify a sieve $S \subseteq \mathrm{Ob}(A)$ with the full subcategory of A spanned by the objects in S. If $S \subseteq A$ is a sieve (resp. cosieve) then there exists a unique functor $p : A \to [1]$ such that $S = p^{-1}(0)$ (resp. $S = p^{-1}(1)$). This defines a bijection between the set of sieves (resp. cosieves) in A and the set of functors $A \to [1]$.

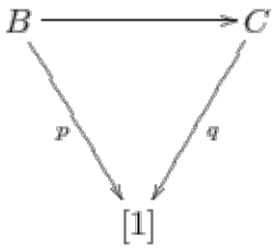
Definition 1. We shall say that an object of the category $\mathbf{Cat}/[1]$ is a *barrel*. The *bottom* of a barrel $B = (B, p)$ is the sieve $B(0) = p^{-1}(0)$ and its *top* is the cosieve $B(1) = p^{-1}(1)$.

The opposite of a barrel $p:B \to [1]$ is a barrel $p^o:B^o \to [1]^o \simeq [1]$. Notice that $B^o(0) = B(1)^o$ and that $B^o(1) = B(0)^o$. The opposition functor $B \mapsto B^o$ is an automorphism of the category $\mathbf{Cat}/[1]$.

Proposition 2. The category $\mathbf{Cat}/[1]$ is cartesian closed.

Proof. We have to show that every barrel (C, p) is exponentiable. For this, it suffices to verify that the functor $p:C \to [1]$ is a Conduché fibration?. But the Conduché condition? is trivially satisfied, since the arrow $0 \to 1$ in $[1]$ has no nontrivial factorisation. Hence the category $\mathbf{Cat}/[1]$ is cartesian closed. ▮

The internal hom $\mathrm{Hom}[B, C]$ between two barrels $B = (B, p)$ and $C = (C, q)$ can be described as follows.



We have $\mathrm{Hom}[B, C](0) = [B(0), C(0)]$ and $\mathrm{Hom}[B, C](1) = [B(1), C(1)]$. Thus, an object of the category $\mathrm{Hom}[B, C]$ is either a functor $B(0) \to C(0)$ or a functor $B(1) \to C(1)$. The morphisms between two functors in $\mathrm{Hom}[B, C](0)$ is a natural transformation, and similarly for a morphism between two functors in $\mathrm{Hom}[B, C](1)$. If $f:B(0) \to C(0)$ and $g:C(1) \to D(1)$, then morphism $f \to g$ in the category $\mathrm{Hom}[B, C]$ is a barrel map $h:C \to D$ which is extending the functor $f \sqcup g: C(0) \sqcup C(1) \to D(0) \sqcup D(1)$. We leave to the reader the description of the composition law between the different kind of morphisms.

Consider the functor

$$(\mathrm{bot}, \mathrm{top}): \mathbf{Cat}/[1] \to \mathbf{Cat} \times \mathbf{Cat}$$

which associates to a barrel (C, p) the pair of categories $(C(0), C(1))$. The functor has a left adjoint and a right adjoint. The left adjoint associates to a pair of categories (A, B) the category $A \sqcup B$ equipped with the canonical functor $A \sqcup B \to 1 \sqcup 1 \to [1]$. The right adjoint associates to a pair of categories (A, B) their join? $A \star B$ equipped with the canonical functor $A \star B \to 1 \star 1 = [1]$.

Distributors

Recall that a *distributor* $D:A \nrightarrow B$ between two categories A and B is defined to be a functor $D:A^o \times B \to \mathbf{Set}$. We shall regard the set $D(a, b)$ as functor of two variables, contravariant in $a \in A$ and covariant in $b \in B$. For example, the functor $\mathrm{Hom}_A : A^o \times A \to \mathbf{Set}$ is defining the *unit distributor* $I_A : A \nrightarrow A$. The distributors $A \nrightarrow B$ form a category

$$\mathbf{D}(A, B) = [A^o \times B, \mathbf{Set}] .$$

The opposite of a distributor $D : A \nrightarrow B$ is the distributor $D^o : B^o \nrightarrow A^o$ obtained by putting $D^o(b^o, a^o) = D(a, b)$. We shall denote by $f^o \in D^o(b^o, a^o)$ the element corresponding to an alement $f \in D(a, b)$, so that

$$D^o(b^o, a^o) = \{f^o : f \in D(a, b)\} = \{f : f \in D(a, b)\}^o = D(a, b)^o .$$

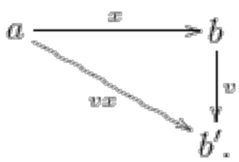
Example 3. To every functor $f : A \to B$ we can associate two distributors $D(f) : A \nrightarrow B$ and $D^o(f) : B \nrightarrow A$ by putting

$$D(f)(a, b) = B(f(a), b) \qquad D^o(f)(b, a) = B(b, f(a))$$

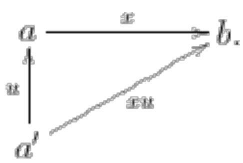
for every object $a \in A$ and every object $b \in B$. Notice that $D^o(f) = D(f^o)^o$. An adjunction $\theta : f \vdash g$ between two functors $f : A \to B$ and $g : B \to A$ is exactly an isomorphism of distributors $\theta : D(f) \simeq D^o(g)$.

Notation 4. it is often convenient to represent an element $x \in D(a, b)$ of a distributor $D : A \nrightarrow B$ by a dotted arrow or a wavy arrow

and more simply by a plain arrow $x : a \to b$ if the context is clear. If $x \in D(a, b)$ and $v \in B(b, b')$, we shall write the element $D(a, v)(x) \in D(a, b')$ as a composite $vx : a \to b'$,



Dually, if $u \in A(a', a)$, we shall write the element $D(u, b)(x) \in D(a', b)$ as a composite xu,



Definition 5. To every distributor $D : A \nrightarrow B$ we can associate a category $C = \mathrm{Col}(D) = A \star_D B$ called the *collage* of D constructed as follows:

$\mathrm{Ob}(C) = \mathrm{Ob}(A) \sqcup \mathrm{Ob}(B)$ and for $x, y \in \mathrm{Ob}(C)$, we put

$$
\begin{aligned}
C(x,y) &= D(x,y) \quad \text{if} \quad x \in A \quad \text{and}\, y \in B \\
&= A(x,y) \quad \text{if} \quad x \in A \quad \text{and}\, y \in A \\
&= B(x,y) \quad \text{if} \quad x \in B \quad \text{and}\, y \in B \\
&= \varnothing \quad \text{if} \quad x \in B \quad \text{and}\, y \in A
\end{aligned}
$$

The composition of arrows in $A \star_D B$ is defined as above. The associativity of composition is equivalent to the functorialty of D.

There is a unique functor $p: A \star_D B \to [1]$ such that $p^{-1}(0) = A$ and $p^{-1}(1) = B$. This shows that the collage category $\mathrm{Col}(D)$ carries the structure of a barrel. Notice the isomorphism $\mathrm{Col}(D)^o = \mathrm{Col}(D^o)$. We would like to say that collage operation is a functor

$$\mathrm{Col}: \mathbf{Dist} \to \mathbf{Cat}/[1],$$

but we have not yet defined the category of distributors $\mathbf{Dist}$. A *map* $D \to D'$ between two distributors $D: A \nrightarrow B$ and $D': A' \nrightarrow B'$ is defined to be triple (f, α, g), where $f: A \to A'$ and $g: B \to B'$ are two functors and $\alpha: D \to (f^o \times g)^*(D')$ is a natural transformation,

$$\alpha(a,b): D(a,b) \to D'(fa, gb)\,.$$

Maps of distributors can be composed and this defines the category $\mathbf{Dist}$. It is obvious from this construction that the functor

$$(\text{source}, \text{target}): \mathbf{Dist} \to \mathbf{Cat} \times \mathbf{Cat}$$

which associates to a distributor $D: A \nrightarrow B$ the pair of categories (A, B) is a Grothendieck fibration. A map of distributors $(f, g, \alpha): D \to D'$ induces a functor between the collage categories,

$$f \star_\alpha g = \mathrm{Col}(f, \alpha, g): \mathrm{Col}(D) \to \mathrm{Col}(D')\,.$$

This defines the *collage functor*

$$\mathrm{Col}: \mathbf{Dist} \to \mathbf{Cat}/[1]\,.$$

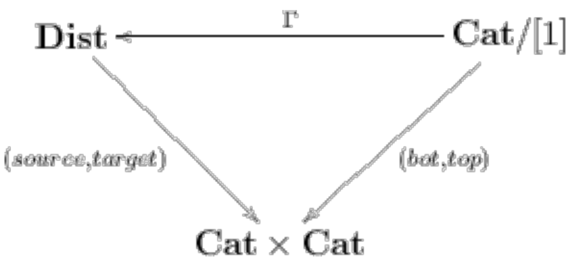
Proposition 6. The collage functor is an equivalence of categories.

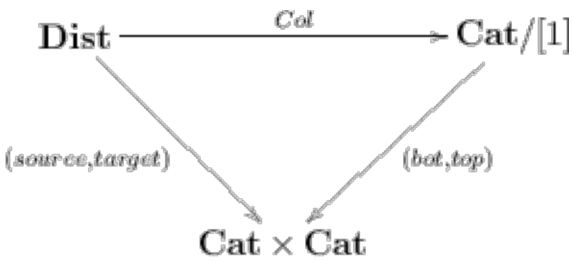
Proof. Let us define the inverse functor in the other direction

$$\Gamma: \mathbf{Cat}/[1] \to \mathbf{Dist}\,.$$

It associates to a barrel $C = (C, p)$ the distributor $\Gamma(C): C(0) \nrightarrow C(1)$ defined by putting $\Gamma(C)(a, b) = C(a, b)$ for every pair of objects $(a, b) \in C(0) \times C(1)$. It is easy to see that there is a natural isomorphism $C \simeq \mathrm{Col}(\Gamma(C))$ for every barrel C and a natural isomorphism $D \simeq \Gamma(\mathrm{Col}(D))$ for every distributor D. ▮

Remark 7. The first of the following two triangles commutes strictly, whilst the second commutes only up to a natural isomorphism.

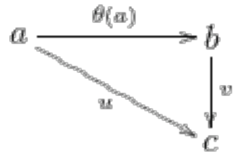




Definition 8. We shall say that a distributor $D: A \nrightarrow B$ is *locally representable* if the functor $D(a, -): B \to \mathbf{Set}$ is representable for every object $a \in A$ and say that D is is *representable* if it is isomorphic to a distributor $D(f)$ for a functor $f: A \to B$. Dually, we shall say that D is *locally corepresentable* if the functor $D(-, b): A^o \to \mathbf{Set}$ is representable for every object $b \in B$ and say that D is *corepresentable* if it is isomorphic to a distributor $D^o(g)$ for a functor $g: B \to A$.

Proposition 9. A distributor $D: A \nrightarrow B$ is representable iff it is locally representable iff the full subcategory $B \subseteq A \star_D B$ is reflexive. Dually, D is corepresentable iff it is locally corepresentable iff the full subcategory $A \subseteq A \star_D B$ is coreflexive.

Proof. ($1 \Rightarrow 2$) Let us suppose that D is represented by a functor $f: A \to B$ together with a natural isomorphism $\theta: D(f) \simeq D$. Then the map $\theta(a, -): A(fa, -) \simeq D(a, -)$ is a natural isomorphism
for every object $a \in A$ and this shows that D is locally representable. ($2 \Rightarrow 3$) If D is locally representable, then for each object $a \in A$, there exists an object $b \in B$ together with an element $\theta(a) \in D(a, b)$ which represents the functor $D(a, -)$. By definition, for every object $c \in B$ and every element $u \in D(a, c)$, there exists a unique morphism $v: b \to c$ such that $v\theta(a) = u$.



This means that the morphism $\theta(a): a \to b$ of the category $A \star_D B$ is reflecting the object a into the subcategory B. Thus, B is a reflective subcategory of $A \star_D B$.

(3 ⇒ 1) If B is a reflective subcategory of $A \star_D B$, let us show that the distributor D is representable. For each object $a \in A$, let us choose an object $fa \in B$ together with a morphism $\theta(a): a \to fa$ which reflects the object a into B (we are using the axiom of choice here). Then for every morphism $x: a \to a'$ in A, there exists a unique morphism $y: fa \to fa$ such that $\theta(a')x = y\theta(a)$,

$$\begin{array}{ccc} a & \xrightarrow{\theta(a)} & fa \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ a' & \xrightarrow{\theta(a')} & fa'. \end{array}$$

Let us put $y = fx$. This defines a functor $f: A \to B$ which represents the distributor D. ∎

Remark 10. The proof that a locally representable distributor is representable depends on the axiom of choice. A locally representable distributor $A \nrightarrow B$ is called an *anafunctor* $A \to B$ by Makkai here.

Remark 11. The functor $D: [A, B] \to \mathbf{D}(A, B)$ which associates to a functor $f: A \to B$ the distributor $D(f): A \nrightarrow B$ induces an equivalence between the category $[A, B]$ and the full subcategory of $\mathbf{D}(A, B)$ spanned by the representable distributors.

Remark 12. A functor $f: A \to B$ has a right adjoint iff the distributor $D(f): A \nrightarrow B$ is corepresentable, and it has a left adjoint iff the distributor $D^o(f):: B \nrightarrow A$ is representable.

The distributors also form a bicategory? $\mathcal{Dist}$ whose objects are the small categories. The *composite* of a distributor $X: A \nrightarrow B$ with a distributor $Y: B \nrightarrow C$ is the distributor $Y \circ X: A \nrightarrow C$ defined by putting

$$(Y \circ X)(a, c) = Y(-, c) \otimes_B X(a, -) = \int^{b \in B} Y(b, c) \times X(a, b)$$

where the tensor product? between the contravariant functor $Y(-, c)$ and the covariant functor $X(a, -)$ is used.

Proposition 13. The composition law

$$\mathbf{D}(B, C) \times \mathbf{D}(A, B) \to \mathbf{D}(A, C)$$

is coherently associative and the distributor $\mathrm{Hom}_A: A^o \times A \to \mathbf{Set}$ is a unit $I_A: A \nrightarrow A$ for this composition.

Idea of proof. This follows from the properties of the tensor product $Y \otimes_B X$ for Y a right B-"module" and X a left B-"modules". By definition, the tensor product is a quotient of the matrix-product

$$(Y \times_B X)(a,c) = \bigsqcup_{b \in \mathrm{Ob}(B)} Y(b,c) \times X(a,b)$$

and the canonical map

$$\phi : Y \times_B X \to Y \otimes_B X$$

is universal among the maps satisfying the compatibility condition $\phi(y, fx) = \phi(yf, x)$ for $(y,f,x) \in Y(b',c) \times B(b,b') \times X(a,b)$. If Y is a (B,C)-"bimodule" and Z is a right C-"module", then the associativity isomorphism

$$Z \otimes_C (Y \otimes_B X) \simeq (Z \otimes_C Y) \otimes_B X$$

can be obtained by showing that the two sides are actually isomorphic to the triple-tensor product $Z \otimes_C Y \otimes_B X$. By definition, the triple-tensor product of (Z,Y,X) is a quotient of the triple matrix-product $Z \times_C Y \times_B X$, and the canonical map

$$\phi : Z \times_C Y \times_B X \to Z \otimes_C Y \otimes_B X$$

is universal among the maps satisfying the "trilinearity" conditions $\phi(zh, y, x) = \phi(z, hy, x)$ and $\phi(z, yf, x) = \phi(z, y, fx)$. The unit isomorphism $X \otimes_A I_A \simeq X$ is obtained by showing that the right action $X(a',b) \times \mathrm{Hom}(a,a') \to X(a,b)$ is a universal "bilinear" map. ▮

The composition functor $(E,D) \mapsto E \circ D$ is divisible? on both sides. Hence the functor $E \circ (-) : \mathbf{D}(A,B) \to \mathbf{D}(A,C)$ has a right adjoint $S \mapsto E \backslash S$ for every distributor $E : B \nrightarrow C$. By construction, we have

$$(E \backslash S)(a,b) = hom(E(b,-), S(a,-)) = \int_{c \in C} S(a,c)^{E(b,c)},$$

where the hom set of maps $E(b,-) \to T(a,-)$ in the category $[C, \mathbf{Set}]$ is used. Dually the functor $(-) \circ D : \mathbf{D}(B,C) \to \mathbf{D}(A,C)$ has a right adjoint $S \mapsto S/D$ for every distributor $D : A \nrightarrow B$. By construction, we have

$$(S/D)(b,c) = hom(D(-,b), S(-,c)) = \int_{a \in A} S(a,c)^{D(a,b)},$$

where the hom set of maps $D(-,b) \to S(-,c)$ in the category $[A^o, \mathbf{Set}]$ is used.

Let us denote by $\mathbf{CCAT}$ the category whose objects are the cocomplete locally small categories and whose morphisms are the cocontinuous functors. For any small category A, we have

$$\mathbf{D}(1, A) = [A, \mathbf{Set}] = \mathbf{Set}^A \quad \text{and} \quad \mathbf{D}(A,\ 1) = [A^o, \mathbf{Set}] = \mathbf{Set}^{A^o}$$

For a fixed $D: A \nrightarrow B$, the composition functor

$$D_! = D \circ (-): \mathbf{D}(1, A) \to \mathbf{D}(1, B)$$

is cocontinuous, since it has a right adjoint. This defines a functor $D \mapsto D_!$,

$$(\ -\ )_!: \mathbf{D}(A, B) \to \mathbf{CCAT}\left(\mathbf{Set}^A, \mathbf{Set}^B\right).$$

Dually, the composition functor

$$D^! = (-) \circ D: \mathbf{D}(B,\ 1) \to \mathbf{D}(A,\ 1)$$

is cocontinuous, since it has a right adjoint. This defines a functor $D \mapsto D^!$,

$$(\ -\ )^!: \mathbf{D}(A, B) \to \mathbf{CCAT}\left(\mathbf{Set}^{B^o}, \mathbf{Set}^{A^o}\right).$$

Notice the canonical isomorphisms $(E \circ D)_! = E_! D_!$ and $(E \circ D)^! = D^! E^!$.

Theorem 14. (Morita-Watts-Lawvere-Benabou) The functors $(\ -\ )_!$ and $(\ -\ )^!$ defined above are equivalence of categories.

To every functor $f: A \to B$ in $\mathbf{Cat}$ is associated a pair of adjoint cocontinuous functors

$$f_!: [A^o, \mathbf{Set}] \leftrightarrow [B^o, \mathbf{Set}]: f^*$$

where $f^*(G) = G \circ f$ for a functor $G: B^o \to \mathbf{Set}$, and where $f_!(F)$ is the left Kan extension? along f of a functor $F: A^o \to \mathbf{Set}$. The two functors can represented by distributors. For every $G: B \nrightarrow 1$ we have

$$f^*(G)(a) = G(fa) = \int^{b \in B} B(fa, b) \times G(b)$$

and this means that we have a canonical isomorphism $f^* = D(f)^!$, where $D(f)$ is the distributor $A \nrightarrow B$ defined by putting $D(f)(a, b) = B(fa, b)$. For every $F: A \nrightarrow 1$ we have

$$f_!(F)(b) = \int^{a \in A} B(b, fa) \times F(a)$$

and this means that we have a canonical isomorphism $f_! = D^o(f)^!$, where $D^o(f)$

is the distributor $B \nrightarrow A$ obtained by putting $D^o(f)(b,a) = B(b, fa)$. From the adjunction $f_! \vdash f^*$, we obtain an adjunction between distributors $D(f) \vdash D^o(f)$ (beware that $(E \circ D)^! = D^! E^!$). The unit of this adjunction is a map $\eta : I_A \to D^o(f) \circ D(f)$ in $\mathbf{D}(A,A)$ and the counit is a map $\epsilon : D(f) \circ D^o(f) \to I_B$ in $\mathbf{D}(B,B)$. The counit ϵ is the obvious map

$$\int^{a \in A} B(fa, b') \times B(b, fa) \to B(b, b')$$

defined by composing the pairs of arrows $b \to fa \to b'$. The composite $D^o(f) \circ D(f)$ is the distributor

$$B(fa, fa') = \int^{b \in B} B(b, fa') \times B(fa, b).$$

The unit η is the map $A(a,a') \to B(fa, fa')$ induced by the functor f.

Duality

The bicategory of distributors $\mathcal{Dist}$ is symmetric monoidal. The *tensor product* of a distributor $X : A \nrightarrow A'$ with a distributor $Y : B \nrightarrow B'$ is the distributor $X \times Y : A \times B \nrightarrow A' \times B'$ defined by putting

$$(X \times Y)(a, b; a', b') = X(a, a') \times Y(b, b').$$

The tensor product functor

$$\times : \mathbf{D}(A, A') \times \mathbf{D}(B, B') \to \mathbf{D}(A \times B, A' \times B')$$

is really a cartesian product in the category $\mathbf{Dist}$. More precisely, we have a canonical isomorphism

$$\mathrm{Col}(X \times Y) = \mathrm{Col}(X) \times_{[1]} \mathrm{Col}(Y)$$

in the category of cylinders $\mathbf{Cat}/[1]$, where $\mathrm{Col}(X)$ is the collage barrel of a distributor X.

If A, B and C are small categories, then the categories $\mathbf{D}(A \times B, C)$ and $\mathbf{D}(B, A^o \times C)$ are equivalent, since they are both isomorphic to the the category

$$[A^o \times B^o \times C, \mathbf{Set}].$$

The equivalence

$$\Theta : \mathbf{D}(A \times B, C) \simeq \mathbf{D}(B, A^o \times C)$$

is actually *natural* when B and C are varying in the bicategory of distributors. In other words, the endo-functor $A \times (-)$ of the bicategory $\mathcal{Dist}$ is left adjoint to the endo-functor $A^o \times (-)$. It follows that the objects A and A^o are mutually dual in the symmetric monoidal bicategory $\mathcal{Dist}$. Hence the bicategory $\mathcal{Dist}$ is **compact closed**. Let us examine this duality explicitly. If $B = 1$ and $C = A$, then $\Theta(1_A)$ is a distributor $\eta_A : 1 \nrightarrow A^o \times A$. If $B = A^o$ and $C = 1$, then $\Theta^{-1}(1_{A^o})$ is a distributor $\epsilon_A : A \times A^o \nrightarrow 1$. The distributor η_A is given by the functor $\mathrm{Hom}_A : 1^o \times (A^o \times A) \to \mathbf{Set}$ and the distributor ϵ_A by the same functor $\mathrm{Hom}_A : (A^o \times A) \times 1 \to \mathbf{Set}$. Notice that $\eta_A(x_1, x_2) = A(x_1^o, x_2)$ is a covariant functor of $(x_1, x_2) \in A^o \times A$, whilst $\epsilon_A(x_1, x_2) = A(x_1, x_2^o)$ is a contravariant functor of $(x_1, x_2) \in A \times A^o$. Notice also that $\epsilon = \eta^{\mathrm{o}}$.

A monoidal bicategory is a tri-category. In this context, the adjunction identities are taking the form of a pair of isomorphisms between distributors,

$$\alpha_A : (\epsilon_A \times A) \circ (A \times \eta_A) \simeq I_A \quad \text{and} \quad \beta_A : (A^o \times \epsilon_A) \circ (\eta_A \times A^o) \simeq I_{A^o}.$$

The domain of α_A is the composite D of the distributors,

$$A \xrightarrow{A \times \eta_A} A \times A^o \times A \xrightarrow{\epsilon_A \times A} A.$$

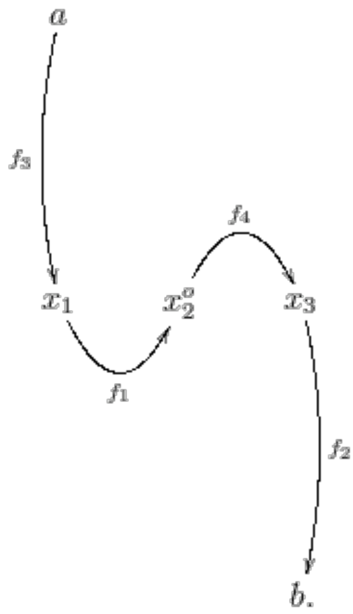
Thus, $D(a, b)$ is a coend,

$$\int^{x_1 \in A} \int^{x_2 \in A^o} \int^{x_3 \in A} \big(A(x_1, x_2^o) \times A(x_3, b)\big) \times \big(A(a, x_1) \times A(x_2^o, x_3)\big).$$

Notice that the product $A(a, x_1) \times A(x_2^o, x_3)$ in the integrant is covariant in $(x_1, x_2, x_3) \in A \times A^o \times A$, whilst the product $A(x_1, x_2^o) \times A(x_3, b)$ is contravariant. The isomorphism $\alpha_A : D(a, b) \simeq A(a, b)$ is then induced by the map

$$A(x_1, x_2^o) \times A(x_3, b) \times A(a, x_1) \times A(x_2^o, x_3) \to A(a, b)$$

which takes a quadruple (f_1, f_2, f_3, f_4) to their composite $f_2 f_4 f_1 f_3 : a \to b$ in A,



Dually,the domain of β_A is defined to be the composite

$$A^o \xrightarrow{\eta_A \times A^o} A^o \times A \times A^o \xrightarrow{A^o \times \epsilon_A} A^o.$$

This domain is isomorphic to D^o, since $\eta_A = \epsilon_A^o$ and $\epsilon_A = \eta_A^o$. The isomorphism $\beta_A : D^o \to I_{A^o}$ is induced by the isomorphism $\alpha_A : D \to I_A$.

Exercises

Exercise. Show that the functor

$$(\mathrm{bot}, \mathrm{top}) : \mathbf{Cat}/[1] \to \mathbf{Cat} \times \mathbf{Cat}$$

is essentially? a Grothendieck bifibration.

Exercise. In a compact monoidal category, every map $f : X \to Y$ has a transpose ${}^t f : Y^* \to X^*$. In the monoidal bicategory of distributors, show that the transpose of a distributor $D : A \nrightarrow B$ is its opposite $D^o : B^o \nrightarrow A^o$.

References

Historical notes.

The notion of distributor was first introduced by Lawvere in a talk that he gave in 1966 at a meeting in Oberwolfach. They were used to represent cocontinuous functors between presheaf categories. The theory of distributors was later developed extensively by Bénabou who introduced also the notion of bicategory. The collage category is also due to him, but the terminology “collage category” was introduced by

Street in his Rendiconti paper.

In a letter adressed to me (April 11,2010), Anders Kock has expressed his view on the invention of distributors. I find his opinion worth to be made public (with his permission):

The notion of profunctor/bimodule was not a creation of 1966, but is a result of an evolution, which includes for instance the section (p.22-23) in Cartan-Eilenberg on (the bicategory of) bimodules, and their tensor product (= composition in the bicategory), and the tensor product of a covariant and a contravariant functor on a category, cf. Watts' contribution in the LaJolla volume. The important thing about the evolution of these notions was that they led to the notion of bicategory - likewise an evolution having many stages and inputs, including 2-categories, and monoidal categories. In this evolution, Benabou is a main actor. The identification of profunctors with cocontinuous functors between categories of presheaves is likewise part of an evolution, where ancestors are Morita's characterization of equivalences between module categories, and Watts' characterization of cocontinuous functors between module categories (Proc.Amer.Math.Soc, 1960).

I can agree with that. Most mathematical ideas are the result of an evolution. Mutations are playing an important role in biological evolution. What about mutations of mathematical ideas? What could be the role of mutations in the evolution of mathematical ideas?

The notion of symmetric monoidal bicategory has a complicated history. Let me quote Street:

An important point here was the realization that there are really three kinds of commutativity for monoidal bicategories and then it was important to find the correct definition of braided monoidal bicategory. Kapranov and Voevodsky made a definition of braiding on (I think) a slightly restricted notion of monoidal bicategory: and they left out an axiom. The missing axiom was corrected by Larry Breen and Baez-Neuchl. In my paper with Day we made up the term "syllepsis" for the extra notion of commutativity between "braiding" and "symmetry".

Let me quote Breen:

Actually, I did not "correct" the Kapranov -Voevodsky definition, in fact I was aware of the full definition at least as far back as 1988. I wrote about this in a letter to Deligne, dated 4 Feb. 1988, (now on the website). I did not write this up as a paper at the time, but returned to this topic (in a more general topos context) in the last chapter of my Asterisque monograph (vol. 225, 1994), where I discuss these questions and in particular explicitly mention on page 148 the *necessary correction to Kapranov-Voevodsky. In my letter the associativities are strict, so that only the commutativity conditions are considered. I had played around with the more complete diagrams involving both associativity and commutativity but I hadn't at the time worked out the lax definition of the 2-categorical braiding axioms as Kapranov-Voevodsky did in their paper.*

Many people have discovered independently that the symmetric monoidal bicategory $\mathcal{Dist}$ is compact closed. It was known to the Australian school for several decades. The general case of the symmetric monoidal bicategory of V-distributors, for a closed symmetric monoidal category V, is treated in the paper of Day and Street referred to below.

Books:

F. Borceux: *Handbook of Categorical Algebra* in 3 volumes, CUP, (1994)

Papers:

J. Baez, M. Neuchl *Braided Monoidal 2-categories*, Advances in Math, 121 (1996) 196-244

J. Bénabou: *Introduction to bicategories*, Springer Lecture Notes in Math **47**, 1-17, (1967)

J. Bénabou: *2-Dimensional limits and colimits of distributors*, Mathematisches Forschungsinstitut Oberwolfach, Tagungsbericht 30 (1972) 6–7.

J. Bénabou: *Les Distributeurs*, Univ. Catholique de Louvain, Séminaire de Mathématiques Pures, Rapport 33 (1973)

J. Bénabou: *Distributors at work*, (2000), pdf

B. Day, R. Street: *Monoidal bicategories and Hopf algebroids*, Advances in Math. 129 (1997) 99-157; MR99f:18013.

M. Kapranov and V.Voevodsky: *2-categories and Zamolodchikov tetrahedra equations*, Proc. Symp. Pure Math. 56 (1994), part 2, p. 177-259.

M. Kapranov and V.Voevodsky: *Braided monoidal 2-categories and Manin-Schechtman higher braid groups*, J. Pure Appl. Algebra, 92 (1994), 241-267.

R. Street: *Cauchy characterization of enriched categories*, Rendiconti del Seminario Matematico e Fisico di Milano 51 (1981) 217-233; MR85e:18006. Reprints in Theory and Applications of Categories 4 (2004) 1-16, link

R. Street: *Two constructions on lax functors*, Cahiers topologie et géométrie différentielle 13 (1972) 217-264; MR50#436; numdam

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