Formal Homotopy Quantum Field Theories and 2-groups.

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Definition

A *crossed module*, $\mathcal{C} = (C, P, \partial)$, consists of

groups C, P,

a (left) action of P on C (written $(p, c) \to {}^{p}c$) and

a homomorphism

$$\partial : C \to P.$$

These are to satisfy:

CM1 $\partial({}^{p}c) = p \cdot \partial c \cdot p^{-1}$ for all $p \in P$, $c \in C$,

and

CM2 ${}^{\partial c}c' = c \cdot c' \cdot c^{-1}$ for all $c, c' \in C$.

Examples:

- If N is a normal subgroup of a group P, then P acts by conjugation on N, ${}^{p}n = pnp^{-1}$, and the inclusion $\iota : N \to P$ is a crossed module.
- If M is a left P-module and we define $0 : M \to P$ to be the trivial homomorphism, $0(m) = 1_P$, for all $m \in M$, then $(M, P, 0)$ is crossed module.
- If G is any group, $\alpha : G \to Aut(G)$, the canonical map sending $g \in G$ to the inner automorphism determined by g, is a crossed module for the obvious action of $Aut(G)$ on G.
- Also for an algebra, L, $(U(L), Aut(L), \delta)$ is a crossed module, where $U(L)$ is the group of units of L and δ maps a unit e to the automorphism given by conjugation by e.

Examples continued

- Central extension: $1 \to A \to E \xrightarrow{\partial} P \to 1$,
  $E \xrightarrow{\partial} P$ is a crossed module.
- Fibration:
  $$F \to E \to B \quad \text{pointed spaces}$$
  $\pi_1(F) \to \pi_1(E)$ is a crossed module.
- Special case: (X, A) pointed pair of spaces,
  $\pi_2(X, A) \to \pi_1(A)$ is a crossed module.
- Extra special case: X, CW-complex, $A = X_1$, 1-skeleton of X,
  $$\pi_2(X, X_1) \xrightarrow{\partial} \pi_1(X_1)$$
  determines 2-type of X. NB. $\ker \partial = \pi_2(X), \mathrm{coker}\partial = \pi_1(X)$.

TQFTs (over $\mathbb{C}$)

TQFT = Topological Quantum Field Theory

- $\{$Oriented d-manifolds, $X\} \to \{$ f.d. Vector spaces, $T(X)\}$
- $\{$cobordisms, $M : X \to Y\} \to$
  $\{$lin. trans., $T(M) : T(X) \to T(Y)\}$,

'Tensor' of manifolds is disjoint union, $X \sqcup Y$
$\to$ usual tensor, $T(X) \otimes T(Y)$

$\emptyset$ is the unit for the monoidal structure on d-cobord

Compose cobordisms :

Usual picture with usual provisos on associativity etc.

- Get that a TQFT is simply a monoidal functor:

$$\begin{array}{c}(d-\text{cobord}, \sqcup)\\ \big\downarrow T\\ (\text{Vect}, \otimes)\end{array}$$

- $\emptyset$ is a d-manifold so
- a closed $d+1$ manifold M is a cobordism from $\emptyset$ to $\emptyset$ and hence $T(M) : T(\emptyset) \to T(\emptyset)$.
- T is monoidal, thus $T(\emptyset) \cong \mathbb{C}$, so
- $T(M)$ is a linear map from $\mathbb{C}$ to itself, i.e., is specified by a single complex number, a numerical invariant of M

HQFT = Homotopy Quantum Field Theory : Turaev 1999.

- Problem : would like to have a theory with manifolds *with extra structure*, e.g. a given structural G-bundle, or metric, etc.
- Partial solution by Turaev (1999): Replace X by characteristic structure map, $g : X \to B$, where B is a 'background' space, e.g. BG.
- For the cobordisms, want $F : M \to B$ agreeing with the structure maps on the ends, but F will only be given *up to homotopy relative to that boundary*.

- Get a monoidal category $d{-}Hocobord(B)$: (Rodrigues, 2000)
- Definition: A HQFT is a monoidal functor

$$d{-}Hocobord(B) \to Vect.$$

Classification results for cases where: (i) $B = K(\pi, 1)$, (ii) $B = K(A, 2)$.

General result (Rodrigues) : d-dimensional HQFTs over B depend on the $(d+1)$-type of B only, so can assume $\pi_n(B)$ is trivial for $n > d+1$.
Clearest classification results are in dimensions $d = 1$ and $d = 2$. (Will look at $d = 1$ only.)

- d-manifold = disjoint union of oriented circles,
- cobordism = oriented surface possibly with boundary,
- can restrict to B being a 2-type.

$B = K(\pi, 1)$

Here

- $\pi_1(B) = \pi$, $\pi_i(B) = 1$ for $i > 1$, and 'extra structure' = 'isomorphism class of principal π-bundles'.
- (Turaev, 1999) HQFTs over $K(\pi, 1)$ $\longleftrightarrow$ π-graded algebras with inner product and 'extra structure' = *crossed π-algebras.*

Crossed π-algebras.

$L = \bigoplus_{g\in\pi} L_g$, a π-graded algebra , so

- if ℓ_1 is graded g, and ℓ_2 is graded h, then $\ell_1\ell_2$ is graded gh;
- L has a unit $1 = 1_L \in L_1$ for 1, the identity element of π;
- there is a symmetric K-bilinear form

$$\rho : L \otimes L \longrightarrow K$$

such that

(i) $\rho(L_g \otimes L_h) = 0$ if $h \neq g^{-1}$;

(ii) the restriction of ρ to $L_g \otimes L_{g^{-1}}$ is non-degenerate for each $g \in \pi$, (so $L_{g^{-1}} \cong L_g^*$, the dual of L_g); and

(iii) $\rho(ab, c) = \rho(a, bc)$ for any $a, b, c \in L$.

Crossed π-algebras, continued

- a group homomorphism $\phi : \pi \to Aut(L)$ satisfying:
  (i) if $g \in \pi$ and we write $\phi_g = \phi(g)$ for the corresponding automorphism of L, then ϕ_g preserves ρ, (i.e. $\rho(\phi_g a, \phi_g b) = \rho(a, b)$) and $\phi_g(L_h) \subseteq L_{ghg^{-1}}$ for all $h \in \pi$;
  (ii) $\phi_g|_{L_g} = id$ for all $g \in \pi$;
  (iii) for any $g, h \in \pi$, $a \in L_g$, $b \in L_h$, $\phi_h(a)b = ba$;
  (iv) (**Trace formula**) for any $g, h \in \pi$ and $c \in L_{ghg^{-1}h^{-1}}$,

$$Tr(c\phi_h : L_g \to L_g) = Tr(\phi_{g^{-1}}c : L_h \to L_h),$$

  where Tr denotes the K-valued trace of the endomorphism.

$B = K(A, 2)$

(Brightwell and Turner, 2000)

- HQFTs over $K(A, 2)$ $\longleftrightarrow$
  Frobenius algebras with an action of A
  $= A$-Frobenius algebras.

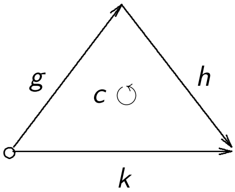
(V.Turaev and TP, 2003-2005)

- General 2-type B given by a crossed module $\mathcal{C} = (C, P, \partial)$. Can one find corresponding algebras and ways of studying leading to a formal classification theorem for HQFTs over $B\mathcal{C}$.

  Progress so far: Notion of Formal HQFT based on combinatorial models of structural maps. 'Crossed $\mathcal{C}$-algebras' giving classification for this formal model.

Formal $\mathcal{C}$- map on a manifold M

- decomposition of manifold M into cells, e.g. a triangulation.
- labelling of edges with elements of P;
- labelling of faces with elements of C;
- boundary condition:



  with $g, h, k \in P$ and $c \in C$, and where $\partial c = kh^{-1}g^{-1}$.
- cocycle condition from any 3-simplices.
  This corresponds combinatorially to the characteristic map $g : M \to B\mathcal{C}$.

Formal HQFTs - definition

A *formal HQFT* with background $\mathcal{C}$ assigns

- to each formal $\mathcal{C}$-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, a K-vector space $\tau(\mathbf{g})$, and by extension, to each formal $\mathcal{C}$-map on a 1-manifold S, given by a list, $\mathbf{g} = \{\mathbf{g}_i \mid i = 1, 2, \ldots, m\}$ of formal $\mathcal{C}$-circuits, a vector space $\tau(\mathbf{g})$ and an isomorphism,
$$\tau(\mathbf{g}) = \bigotimes_{i=1,\ldots,m} \tau(\mathbf{g}_i),$$
identifying $\tau(\mathbf{g})$ as a tensor product;
- to any formal $\mathcal{C}$-cobordism, $(M, \mathbf{F})$ between $(S_0, \mathbf{g}_0)$ and $(S_1, \mathbf{g}_1)$, a K-linear transformation
$$\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_1),$$

FHQFTs continued

These assignments are to satisfy the following axioms:

(i) Disjoint union of formal $\mathcal{C}$-maps corresponds to tensor product:

$$\tau(\mathbf{g} \sqcup \mathbf{h}) \xrightarrow{\cong} \tau(\mathbf{g}) \otimes \tau(\mathbf{h}), \qquad \tau(\emptyset) \xrightarrow{\cong} \mathbb{C}.$$

(ii) For formal $\mathcal{C}$-cobordisms

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2$$

with composite $\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}$, we have

$$\tau(\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}) = \tau(\mathbf{G})\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_2).$$

(iii) $\tau(1_{\mathbf{g}}) = 1_{\tau(\mathbf{g})}$.

(iv) Interaction of cobordisms and disjoint union is transformed correctly by τ.

Crossed $\mathcal{C}$-algebras - the definition

Let $\mathcal{C} = (C, P, \partial)$ be a crossed module. A *crossed $\mathcal{C}$-algebra* consists of a crossed P-algebra, $L = \bigoplus_{g \in P} L_g$, together with elements $\tilde{c} \in L_{\partial c}$, for $c \in C$, such that

(a) $\tilde{1} = 1 \in L_1$;

(b) for $c, c' \in C$, $\widetilde{(c'c)} = \tilde{c'} \cdot \tilde{c}$;

(c) for any $h \in P$, $\phi_h(\tilde{c}) = \widetilde{{}^h c}$.

We note that the first two conditions make 'tilderisation' into a group homomorphism $(\tilde{\ }) : C \to U(L)$, the group of units of the algebra, L.

In fact:

Suppose that L is a crossed $\mathcal{C}$-algebra. The diagram

$$\begin{array}{ccc} C & \xrightarrow{(\tilde{\ })} & U(L) \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \delta} \\ P & \xrightarrow[\phi]{} & Aut(L) \end{array} \qquad (1)$$

is a morphism of crossed modules from $\mathcal{C}$ to $\mathfrak{Aut}(L)$. Thus a crossed $\mathcal{C}$-algebra is a Frobenius algebra together with an action of $\mathcal{C}$ on it by automorphisms.

Classification Theorem for 2-D FHQFTS

Main theorem

Formal HQFTs over $\mathcal{C}$ correspond to crossed $\mathcal{C}$-algebras up to isomorphism.

Work in progress

1. Investigate change of algebras along morphisms of crossed modules, and the general link with methods of non-Abelian cohomology ('long' exact sequences, HQFTs as linearisation of non-Abelian cohomology ....);
2. Formal $\mathcal{C}$-maps specify $\mathcal{C}$-bundles, and the whole theory 'sheafifies' well, allowing coefficients in a sheaf on the gros topos, e.g. de Rham structures...do it!
3. Open-closed HQFTs (Moore-Segal) correspond to K-theoretic invariants of the crossed algebras for $B = K(G, 1)$. Generalise.

Work in progress cont'd

1. Extend the theory (i) to 2+1 dimensions and then to formal coefficients in a model for a 3-type ... and beyond!
2. Crossed modules *are* 2-groupoids. Extend the theory to handle coefficients in a general small 2-category and interpret.
3. Look for higher dimensional trace formulae which give more information on the objects.
4. ..... and a lot more!

The End

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