Algebraic K-theory for categorical groups

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Outline

The Whitehead group of a ring R

- For any ring R , if $GL_n(R)$ is the general linear group of invertible matrices $n \times n$ with entries in R, there is a sequence

$$GL_1(R) \subset GL_2(R) \subset GL_3(R) \subset \cdots$$

  whose direct limit is denoted $GL(R)$.
- The subgroup $E(R)$ of $GL(R)$ generated by the elementary matrices (e_{ij}^{λ}) is just the derived subgroup $[GL(R), GL(R)]$
- The quotient group, $GL(R)/E(R)$, which is an abelian group, is the Whitehead group of R and is denoted by $K_1 R$.
- Note that, K_1 is a covariant functor from the category of rings to the category of abelian groups.

Steiner groups.

- The Steiner groups $St_n(R)$ are groups given by generators x_{ij}^{λ} and relations encapsulating the key rules of the elementary matrices e_{ij}^{λ}.
- The canonical homomorphism $\Phi_n : St_n(R) \to E_n(R)$, $x_{ij}^{\lambda} \mapsto e_{ij}^{\lambda}$, induces a homomorphism in the corresponding direct limits

$$St(R) \xrightarrow{\Phi} GL(R) .$$

- $Im(\Phi) = E(R)$.
- $Ker(\Phi) = K_2(R)$, the 2-th group of algebraic K-theory.
- $St(R) \xrightarrow{\Phi} GL(R)$ is a crossed module of groups (we explain this fact soon)

Higher K-groups.

- Higher K-groups were defined by Quillen
- Given a ring R, K_iR, $i \geq 1$, is given by the composition of covariant functors,

$$K_i : R \mapsto GLR \mapsto BGLR \mapsto BGLR^+ \mapsto \pi_i BGLR^+$$

- $BGL(R)$ is the classifying space of the group $GL(R)$.
- $BGL(R)^+$ its Quillen plus-construction.

$$\pi_1 BGL(R)^+ \cong \frac{\pi_1 BGL(R)}{E(R)} = \frac{GL(R)}{E(R)} = K_1 R\,,$$

$$\pi_2 BGL(R)^+ \cong K_2 R\,.$$

- Quillen K-groups K_1R and K_2R are recognized, as the cokernel and the kernel of $St(R) \xrightarrow{\Phi} GL(R)$ (a crossed module of groups).

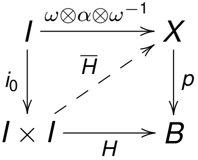
The fundamental crossed module of a fibration

For any fibration $p : (X, x_0) \to (B, b_0)$ with fiber $F = p^{-1}(b_0)$, the morphism

$$\pi_1(F, x_0) \xrightarrow{i} \pi_1(X, x_0),$$

induced by the inclusion $i : (F, x_0) \hookrightarrow (X, x_0)$, is a crossed module of groups, the fundamental crossed module of the fibration p.

If $[\alpha] \in \pi_1(F, x_0)$, and $[\omega] \in \pi_1(X, x_0)$, then $p(\omega \otimes \alpha \otimes \omega^{-1})$ is homotopic to the constant loop in B, through a homotopy of loops $H : I \times I \to X$.



Then ${}^{[\omega]}[\alpha] = [\overline{H}_1] \in \pi_1(X, x_0)$

The fundamental crossed module of a fibration

Standard procedure in homotopy theory of factoring a map of pointed spaces $f:(X,x_0)\to(Y,y_0)$:

- Homotopy equivalence $(X,x_0)\to(\overline{X},\overline{x_0})$ $(\overline{X}=\{(x,\omega)\in X\times Y^I/\omega(1)=f(x)\})$
- Fibration $\overline{f}:(\overline{X},\overline{x_0})\to(Y,y_0)$

gives a functor $f\mapsto\overline{f}$ from maps to fibrations.

$$f:(X,x_0)\to(Y,y_0)\rightsquigarrow \text{fundamental crossed module}$$

If Kf is the homotopy kernel of f (the fiber of $\overline{f}$)

$$\pi_1(Kf,x_0)\xrightarrow{\pi_1(kf)}\pi_1(X,x_0)$$

is called the fundamental crossed module of the fiber homotopy sequence $Kf\xrightarrow{kf}X\xrightarrow{f}Y$

$\Phi:St(R)\to GL(R)$ is a crossed module arising from this general procedure.

A basic structure for Algebraic K-theory

- The fiber homotopy sequence

$$F(R) \to BGL(R) \to BGL(R)^+$$

- The associated fundamental crossed module $\pi_1 F(R) \xrightarrow{\theta} \pi_1 BGL(R)$ is equivalent to $St(R) \xrightarrow{\Phi} GL(R)$.
- $Coker(\theta) = K_1$ and $Ker(\theta) = K_2$

Where we go!

We'll need:

1. Notion of homotopy categorical groups associated to any pointed space.
2. Existence of 2-exact sequences associated to any pair of pointed spaces and to any fibration.
3. Notion of crossed module in the 2-category of categorical groups.
4. Existence of such structure associated to any fibration of pointed spaces (the fundamental categorical crossed module of a fibration).

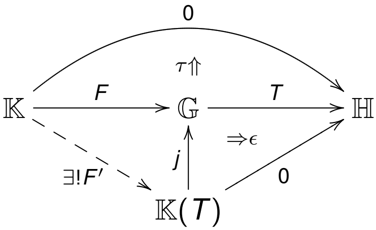
1) and 4) allows to define notions of K-theory categorical groups of a ring R, $\mathbb{K}_i R$, $i \geq 1$, and identify the K-categorical groups $\mathbb{K}_i R$, $i = 1, 2$, respectively as the homotopy cokernel and the homotopy kernel of the fundamental categorical crossed module associated to the fibre homotopy sequence $F(R) \to BGL(R) \to BGL(R)^+$.

Notation.

- We will denote by $\mathbb{G}$ a categorical group.
- We will denote by $\mathcal{CG}$ the 2-category of categorical groups and by $\mathcal{BCG}$ the 2-category of braided categorical groups.
- The set of connected components of $\mathbb{G}$, $\pi_0(\mathbb{G})$, has a group structure (which is abelian if $\mathbb{G} \in \mathcal{BCG}$) with operation $[X] \cdot [Y] = [X \otimes Y]$.
- $\pi_1(\mathbb{G}) = Aut_{\mathbb{G}}(I)$ is an abelian group.

2-exactness.

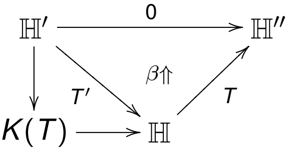
- The kernel of a homomorphism $\mathbf{T} = (T, \mu) : \mathbb{G} \to \mathbb{H}$ consists of a universal triplet $(K(\mathbf{T}), \mathbf{j}, \epsilon)$, where $K(\mathbf{T})$ is a categorical group, $\mathbf{j} : K(\mathbf{T}) \to \mathbb{G}$ is a homomorphism and $\epsilon : \mathbf{Tj} \to \mathbf{0}$ is a monoidal natural transformation.
- The categorical group $K(\mathbf{T})$ is also a standard homotopy kernel and is determined, up to isomorphism, by the following strict universal property:



such that $\mathbf{jF'} = \mathbf{F}$ and $\epsilon\mathbf{F'} = \tau$.

2-exactness.

- Given a diagram in $\mathcal{C}G$



  the triple (T', β, T) is said to be 2-exact if the factorization of T' through the homotopy kernel of T is a full and essentially surjective functor.

- If (T', β, T) is 2-exact, then $\pi_i(\mathbb{H}' \xrightarrow{T'} \mathbb{H} \xrightarrow{T} \mathbb{H}'')$, $i = 0, 1$, is an exact sequence of groups.

Homotopy categorical groups.

- We will denote by $\wp_1(Y)$ the fundamental groupoid of a topological space Y.
- If (X, x_0) is a pointed topological space with base point $x_0 \in X$, then $\wp_2(X, x_0) = \wp_1(\Omega(X, x_0))$, the fundamental groupoid of the loop space $\Omega(X, x_0)$, is enriched with a natural categorical group structure and refer to it as the *fundamental categorical group* of (X, x_0).
- If we define for all $n \geq 2$, $\wp_n(X, x_0) = \wp_1(\Omega^{n-1}(X, x_0))$, then $\wp_3(X, x_0)$ is a braided categorical group and $\wp_n(X, x_0)$, $n \geq 4$, are symmetric categorical groups.
- There is a categorical group action of $\wp_2(X, x_0)$ on $\wp_n(X, x_0)$.
- $\wp_n$, $n \geq 2$, define functors from the category of pointed topological spaces to the category of (braided or symmetric) categorical groups, with $\pi_0\wp_n(X, x_0) \cong \pi_{n-1}(X, x_0)$ and $\pi_1\wp_n(X, x_0) \cong \pi_n(X, x_0)$.

Relative Homotopy Categorical Groups.

- For any pointed topological pair (X, A, x_0), the homotopy kernel of the inclusion $i : (A, x_0) \hookrightarrow (X, x_0)$ is given by the subspace $Ki = \{(a, \omega) \in A \times X^I \,/\, \omega(0) = x_0,\ \omega(1) = a\}$ and the map $ki : (Ki, x_0) \to (A, x_0)$ is given by $ki(a, \omega) = a$.
- We define:

$$\wp_2(X, A, x_0) = \wp_1(Ki, (x_0, \omega_0))$$

and, for $n \geq 3$,

$$\wp_n(X, A, x_0) = \wp_1(\Omega^{n-2}(Ki, (x_0, \omega_0)))\ .$$

- Thus, $\wp_2(X, A, x_0)$ is a groupoid, $\wp_3(X, A, x_0)$ is a categorical group, $\wp_4(X, A, x_0)$ is a braided categorical group and $\wp_n(X, A, x_0)$, $n \geq 5$, is a symmetric categorical group. We refer to these categorical groups as the relative homotopy categorical groups of the pair (X, A, x_0).

2-exact sequences.

- For $n \geq 3$,
$$\pi_0 \wp_n(X, A, x_0) \cong \pi_{n-1}(X, A, x_0)$$
$$\pi_1 \wp_n(X, A, x_0) \cong \pi_n(X, A, x_0)$$
- For any pointed map $f : (X, x_0) \to (Y, y_0)$, the map $q : \Omega(Y, y_0) \to (Kkf, ((x_0, \omega_{y_0}), \omega_{x_0}))$, given by $q(\omega) = ((x_0, \omega), \omega_{x_0})$ is a homotopy equivalence. Then the sequence of iterated homotopy kernels
$$\cdots Kkkf \longrightarrow Kkf \xrightarrow{kkf} Kf \xrightarrow{kf} X \xrightarrow{f} Y$$
is homotopy equivalent to the sequence
$$\cdots \Omega Kf \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow Kf \xrightarrow{kf} X \xrightarrow{f} Y$$

2-exact sequences.

Proposition.*

For any pointed map $f : (X, x_0) \to (Y, y_0)$, there exists a long 2-exact sequence of categorical groups and pointed groupoids (in the last three terms)

$$\ldots \to \wp_n(Kf, (x_0, \omega_{y_0})) \to \wp_n(X, x_0) \to \wp_n(Y, y_0) \to \wp_{n-1}(Kf, (x_0, \omega_{y_0})) \to \ldots$$

$$\ldots \to \wp_2(Kf, (x_0, \omega_{y_0})) \to \wp_2(X, x_0) \to \wp_2(Y, y_0) \to \wp_1(Kf) \to \wp_1(X) \to \wp_1(Y).$$

* M. Grandis, E.M. Vitale, A higher dimensional homotopy sequence, Homology, Homotopy and Appl. 4 (1), 59-69, 2002

2-exact sequences.

Corollary.(The 2-exact homotopy sequence of a pair of spaces)

For any pointed topological pair (X, A, x_0) there exists a long 2-exact sequence of categorical groups and pointed groupoids (the last three terms)

$$\dots \to \wp_{n+1}(X, A, x_0) \to \wp_n(A, x_0) \to \wp_n(X, x_0) \to \wp_n(X, A, x_0) \to \dots$$

$$\dots \to \wp_3(X, A, x_0) \to \wp_2(A, x_0) \to \wp_2(X, x_0) \to \wp_2(X, A, x_0) \to \wp_1(A) \to \wp_1(X).$$

that is called the *2-exact homotopy sequence* of the pair (X, A, x_0).

- The exact homotopy sequence of the pair $(X, A, *)$ follows, from this 2-exact sequence, by taking π_0:

$$\pi_{n+1}(X, A, *) \to \pi_n(A, *) \to \pi_n(X, *) \to \pi_n(X, A, *) \to \dots$$

$$\to \pi_2(X, A, *) \to \pi_1(A, *) \to \pi_1(X, *) \to \pi_1(X, A, *) \to \pi_0(A, *) \to \pi_0(X, *)\,.$$

2-exact sequences.

Theorem.

Let $p : X \to B$ a fibration and suppose $b_0 \in B' \subset B$. Let $X' = p^{-1}(B')$ and let $x_0 \in p^{-1}(b_0)$. Then, p induces a functor $p : \wp_n(X, X', x_0) \longrightarrow \wp_n(B, B', b_0)$ which is a full and essentially surjective functor, for $n = 2$, and a monoidal equivalence for all $n \geq 3$.

Proof (for $n \geq 3$):
$\wp_n(X, X', x_0)$ and $\wp_n(B, B', b_0)$ are categorical groups and $p : \pi_q(X, X', x_0) \to \pi_q(B, B', b_0)$ is a bijection for every $q \geq 1$, then we have that

$$\pi_0\wp_n(X, X', x_0)) \cong \pi_{n-1}(X, X', x_0) \cong \pi_{n-1}(B, B', b_0) \cong \pi_0\wp_n(B, B', b_0))$$

and

$$\pi_1\wp_n(X, X', x_0)) \cong \pi_n(X, X', x_0) \cong \pi_n(B, B', b_0) \cong \pi_1\wp_n(B, B', b_0))$$

2-exact sequences.

Using a result given by Sinh * we conclude that

$$p : \wp_n(X, X', x_0) \longrightarrow \wp_n(B, B', b_0)$$

is a monoidal equivalence for $n \geq 3$.

Corollary.

Let $p : (X, x_0) \to (B, b_0)$ be a fibration with fibre $F = p^{-1}(b_0)$. Then, the induced functor $p : \wp_n(X, F, x_0) \to \wp_n(B, b_0)$, is a full and essentially surjective functor, for $n = 2$, and a monoidal equivalence for $n \geq 3$.

Now, combining the 2-exact sequence of the pair (X, F, x_0) with the equivalence of this corollary:

* H.X. Sinh, Gr-catégories, Université Paris 7, Thèse de doctorat,1975

2-exact sequences.

Corollary.(The 2-exact homotopy sequence of a fibration)

Let $p : (X, x_0) \to (B, b_0)$ be a fibration with fibre $F = p^{-1}(b_0)$. Then, there exists a long 2-exact sequence

$$... \to \wp_{n+1}(B, b_0) \xrightarrow{\partial} \wp_n(F, x_0) \xrightarrow{i} \wp_n(X, x_0) \xrightarrow{p} \wp_n(B, b_0) \xrightarrow{\partial} ...$$

$$\to \wp_3(B, b_0) \xrightarrow{\partial} \wp_2(F, x_0) \xrightarrow{i} \wp_2(X, x_0) \to \wp_2(X, F, x_0) \to \wp_1(F, x_0) \to \wp_1(X, x_0)$$

that is called the *2-exact homotopy sequence* of the fibration p.

2-exact sequences

- We remark that

$$\pi_0\wp_2(X,F,x_0)\cong\pi_1(X,F,x_0)\cong\pi_1(B,b_0)\cong\pi_0\wp_2(B,b_0)$$

and then, applying π_0 to previous sequence, we obtain the well-known group exact sequence of the fibration p:

$$\ldots\to\pi_{n+1}(B,b_0)\xrightarrow{\partial}\pi_n(F,x_0)\xrightarrow{i}\pi_n(X,x_0)\xrightarrow{p}\pi_n(B,b_0)\xrightarrow{\partial}\ldots$$

$$\to\pi_2(B,b_0)\xrightarrow{\partial}\pi_1(F,x_0)\xrightarrow{i}\pi_1(X,x_0)\to\pi_1(B,b_0)\to\pi_0(F,x_0)\to\pi_0(X,x_0).$$

Crossed modules of groups

- A crossed module of groups is a system $\mathcal{L} = (H, G, \varphi, \delta)$, where $\delta : H \to G$ is a group homomorphism and $\varphi : G \to Aut(H)$ is an action (so that H is a G-group) for which the following conditions are satisfied:

$$\delta({}^{x}h) = x\delta(h)x^{-1} \quad , \quad {}^{\delta(h)}h' = hh'h^{-1} \ .$$

- The category of crossed modules is equivalent to the following categories:
  - The category of cat^1-groups. (A cat^1-group consist of a group G with two endomorphisms $d_0, d_1 : G \to G$, such that

$$d_0 d_1 = d_1 \ , \ \ d_1 d_0 = d_0 \ , \ \ [Kerd_0, Kerd_1] = 0 \ .)$$

  - The category of internal groupoids in Groups.
  - The category of strict categorical groups.

Categorical crossed modules

- A categorical crossed module * $\langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle$, consists of a morphism of categorical groups $T = (T, \mu) : \mathbb{H} \to \mathbb{G}$ together with an action of $\mathbb{G}$ on $\mathbb{H}$, $\mathbb{G} \times \mathbb{H} \longrightarrow \mathbb{H}$, $(X, A) \mapsto {}^{X}A$, and two families of natural isomorphisms in $\mathbb{G}$ and $\mathbb{H}$, respectively

$$\nu = \left(\nu_{X,A} : T({}^{X}A) \otimes X \longrightarrow X \otimes T(A)\right)_{(X,A) \in \mathbb{G} \times \mathbb{H}}$$

$$\chi = \left(\chi_{A,B} : {}^{TA}B \otimes A \longrightarrow A \otimes B\right)_{(A,B) \in \mathbb{H}}$$

  such that the coherence conditions hold.
- Categorical crossed modules and morphisms between them form a 2-category.
- Categorical crossed modules are crossed modules of categorical groups.

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* P. Carrasco, A.R. Garzón, E.M. Vitale, On categorical crossed modules, TAC 16 (22), 585-618, 2006

Equivalent categories??

- Weak groupoids internal to the 2-category of categorical groups
- cat^1-categorical groups
- Certain monoidal bicategories
- Others?

1. Any crossed module of groups $H \stackrel{\delta}{\to} G$ is a categorical crossed module when H and G are seen has discrete categorical groups.
2. The zero-morphism $\mathbf{0} : \mathbb{A} \to \mathbf{0}$,with $\mathbb{A}$ braided, is a categorical crossed module where, for any $A, B \in \mathbb{A}$, $\chi_{A,B} :^{I} B \otimes A \to A \otimes B$ is given by the braiding $c_{A,B}$, up to composition with a canonical isomorphism.
3. Consider a morphism $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ of categorical groups and $p_2 : \mathbb{G} \times \mathbb{H} \to \mathbb{H}$ as action of $\mathbb{G}$ on $\mathbb{H}$. Then, if $\mathbb{G}$ is braided, $\nu_{X,A} = c^{-1}_{X,TA}$ and $\chi_{A,B} = c_{B,A}$ gives a categorical crossed module structure to $\mathbf{T} : \mathbb{H} \to \mathbb{G}$.

Advantages: Parallelism with the theory of groups

The category of groups (abelian) is a reflexive subcategory (coreflexive subcategory) of the category of crossed modules of groups.

Theorem *

i) The category of categorical groups is a reflexive subcategory of the category of categorical crossed modules. The left adjoint to the inclusion functor is given by the homotopy cokernel construction.

ii) The category of braided categorical groups is a coreflexive subcategory of the category of categorical crossed modules. The right adjoint to the inclusion functor is given by the homotopy kernel construction.

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* P. Carrasco, A.M. Cegarra, A.R. Garzón, The homotopy categorical crossed module of a CW-complex, Topology and its Applications 154, 834-847, 2007

The cokernel

The cokernel of a categorical crossed module $< \mathbb{H}, \mathbf{T} : \mathbb{H} \to \mathbb{G}, \nu, \chi >$, is define in the following way:

- Objects: those of $\mathbb{G}$
- Premorphisms pairs: $(A, f) : X \to Y$ with $A \in \mathbb{H}$ and $f : X \to T(A) \otimes Y$.
- Morphisms: classes of premorphisms $[A, f]$, where two pairs $[A, f]$ and $[A', f']$ are equivalent if there is $a : A \to A'$ in $\mathbb{H}$ such that

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ TA \otimes Y & \xrightarrow[T(a)\otimes 1_Y]{} & TA' \otimes Y \end{array}$$

- Tensor: given $[A, f] : X \multimap Y$ and $[B, g] : H \multimap K$, $[A, f] \otimes [B, g]$ is given by

$$[A \otimes {}^{Y}B, X \otimes H \overset{f\otimes g}{\to} T(A) \otimes Y \otimes T(B) \otimes K \xrightarrow{1\otimes\nu^{-1}\otimes 1} T(A) \otimes T({}^{Y}B) \otimes Y \otimes K \overset{\mathrm{can}}{\to} T(A \otimes {}^{Y}B) \otimes Y \otimes K$$

Homotopy types

- Carrasco, Garzón and Vitale, observed that if $\langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle$ is a categorical crossed module, then *KerT*, the homotopy kernel of T, is a braided categorical group and *CokerT* the homotopy cokernel of the categorical crossed module is a categorical group.
- $\pi_0 KerT \cong \pi_1 CokerT$,
- The homotopy groups of the categorical crossed module are defined by *:

$$\Pi_i \langle \mathbb{H}, \mathbb{G}, T, \nu, \chi \rangle = \begin{cases} \pi_0 CokerT & \text{for} \quad i = 1 \\ \pi_0 KerT \cong \pi_1 CokerT & \text{for} \quad i = 2 \\ \pi_1 KerT & \text{for} \quad i = 3 \,. \end{cases}$$

* P. Carrasco, A.M. Cegarra, A.R. Garzón, The homotopy categorical crossed module of a CW-complex, Topology and its Applications 154, 834-847, 2007

The linkage with algebraic 3-type

- A categorical crossed module is associated with any pointed pair of spaces*
- A categorical crossed module is associated with any pointed CW-complex

If $(X,*)$ is a pointed CW-space, the pointed topogical pair $(X, X^1, *)$ gives a categorical crossed module.
There is a functor $\mathcal{W} : \mathrm{CW\text{-}complexes}_* \to \mathrm{Categorical\ crossed\ module}$ and
$\Pi_1(\mathcal{W}(X,*)) \cong \pi_1(X,*)$
$\Pi_2(\mathcal{W}(X,*)) \cong \pi_2(X,*)$
$\Pi_3(\mathcal{W}(X,*)) \cong \pi_3(X,*)$
$\mathcal{W}(X,*)$ represents the homotopy 3-type of $(X,*)$.

* P. Carrasco, A.M. Cegarra, A.R. Garzón, The classifying space of a categorical crossed module to appear in Math. Nachr.

The linkage with algebraic 3-type

There is a functor

$$B : \text{Categorical crossed module} \longrightarrow \text{CW-complexes}$$

$$\mathcal{L} = < \mathbb{H}, \mathbb{G}, T, \nu, \chi > \longmapsto B(\mathcal{L}),$$

where $B\mathcal{L}$ is the classifying space of $\mathcal{L}$.
The only homotopy groups of this space are just $\pi_1 B\mathcal{L}$, $\pi_2 B\mathcal{L}$ and $\pi_3 B\mathcal{L}$.
Composing both functors, there is a continuos map $X \mapsto B\mathcal{W}X$, inducing and isomorphism of the homotopy groups

$$\pi_i X \cong \pi_i B\mathcal{W}X\,, \quad \text{for i} = 1, 2, 3.$$

The linkage with algebraic 3-type

- Crossed squares correspond, up to isomorphisms, to strict categorical crossed modules ($\mathbb{H}$ and $\mathbb{G}$ are strict categorical groups, the action of $\mathbb{G}$ on $\mathbb{H}$ is strict and T is strictly equivariant and χ is an identity)
- 2-crossed modules correspond, up to isomorphisms, to special semistrict categorical crossed modules ($\mathbb{H}$ is a strict categorical groups and $\mathbb{G}$ is a discrete categorical group acting strictly on $\mathbb{H}$)
- Every reduced Gray groupoid has associated a special semistrict categorical crossed module
- Associated to any semistrict categorical crossed module there is a reduced Gray groupoid ($\mathbb{H}$ and $\mathbb{G}$ are strict categorical groups, the action of $\mathbb{G}$ on $\mathbb{H}$ is strict and T is strictly equivariant)

The fundamental categorical crossed module of a fibration.

Theorem

Let $p : (X, *) \to (B, *)$ be a fibration with fibre $F = p^{-1}(*)$ and consider the induced categorical group homomorphism $\wp_2(F, *) \xrightarrow{i} \wp_2(X, *)$ given in the 2-exact homotopy sequence of the fibration p. Then the homotopy categorical group $\wp_2(F, *)$ is a $\wp_2(X, *)$-categorical group and, for any $\omega \in \wp_2(X, *)$ and $\alpha, \alpha' \in \wp_2(F, *)$, there are natural isomorphisms

$$\nu = \nu_{\omega,\alpha} : i({}^{\omega}\alpha) \otimes \omega \to \omega \otimes \alpha \quad , \quad \chi = \chi_{\alpha,\alpha'} : {}^{i(\alpha)}\alpha' \otimes \alpha \to \alpha \otimes \alpha'$$

such that $\langle \wp_2(F, *), \wp_2(X, *), i, \nu, \chi \rangle$ is a categorical crossed module which we call the fundamental categorical crossed module of the fibration p.

The fundamental categorical crossed module of a fibration

Proof:
To define a categorical group action of $\wp_2(X, x_0)$ on $\wp_2(F, x_0)$

$$\wp_2(X, x_0) \times \wp_2(F, x_0) \xrightarrow{ac} \wp_2(F, x_0)$$

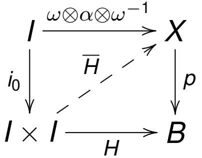
we consider the continuous map

$$\Omega(X, x_0) \times \Omega(F, x_0) \longrightarrow \Omega(F, x_0)\ , \quad (\omega, \alpha) \mapsto {}^{\omega}\alpha$$

where ${}^{\omega}\alpha$ is defined as follows.
Let $\omega \otimes \alpha \otimes \omega^{-1} \in \Omega(X, x_0)$ and consider the projection $p(\omega \otimes \alpha \otimes \omega^{-1}) \in \Omega(B, b_0)$ which is homotopic, to the constant loop in B at b_0 through a homotopy of loops $H : I \times I \to B$.
Then, $H_0(s) = H(s, 0) = p(\omega \otimes \alpha \otimes \omega^{-1})(s)$ and $H_1(s) = H(s, 1) = b_0$.

The fundamental categorical crossed module of a fibration

Since p is a fibration, using the homotopy lifting property in the diagram



$\overline{H}_{\alpha,\omega} = \overline{H} : I \times I \to X$ such that

$$\overline{H}_0(s) = \overline{H}(s,0) = \omega \otimes \alpha \otimes \omega^{-1}$$

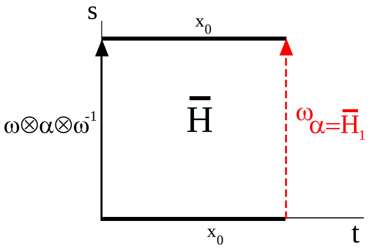
and

$$p\overline{H} = H$$

$$p\overline{H}_1(s) = p\overline{H}(s,1) = H(s,1) = b_0$$

$$Im\overline{H}_1 \subseteq F, \text{that is}, \overline{H}_1 \in \Omega(F, x_0)$$

The fundamental categorical crossed module of a fibration



The fundamental groupoid functor $\wp_1$ preserves products,

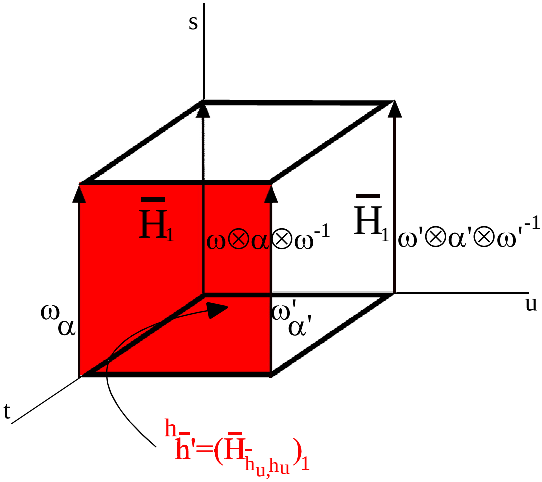
$$ac(\omega, \alpha) =^{\omega} \alpha$$

on arrows $(\omega, \alpha) \xrightarrow{([h],[\bar{h}])} (\omega', \alpha')$,

$$^{[h]}[\bar{h}] = [\,^{h}\bar{h}] : \,^{\omega}\alpha \to \,^{\omega'}\alpha'$$

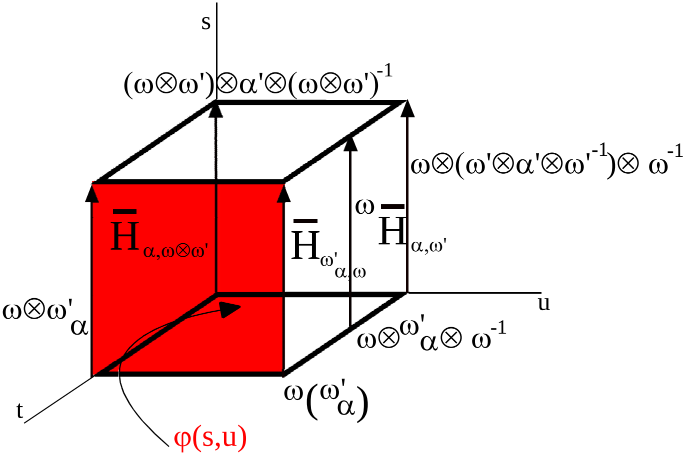
where $(\,^{h}\bar{h})(s,u) = (\overline{H}_{\bar{h}_u, h_u})_1(s) = \overline{H}_{\bar{h}_u, h_u}(s,1)$

The fundamental categorical crossed module of a fibration



The fundamental categorical crossed module of a fibration

For any $\omega, \omega' \in \wp_2(X, x_0)$ and $\alpha \in \wp_2(F, x_0)$, we define a natural isomorphism $\Phi = \Phi_{\omega,\omega',\alpha} : {}^{\omega\otimes\omega'}\alpha \longrightarrow {}^{\omega}({}^{\omega'}\alpha)$ .We define $\Phi_{\omega,\omega',\alpha} = [\varphi]$

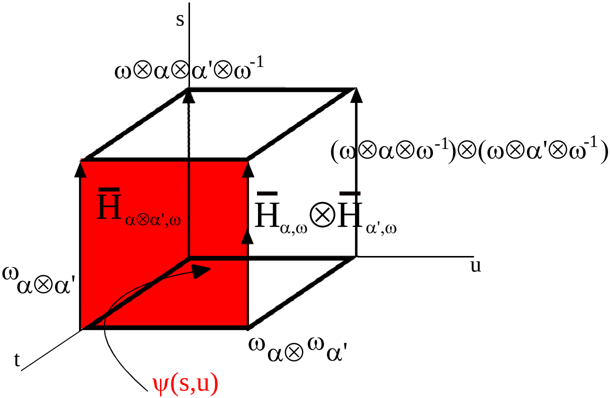


The fundamental categorical crossed module of a fibration

For any $\omega \in \Omega(X, *)$ and $\alpha, \alpha' \in \Omega(F, *)$ the natural isomorphism

$$\Psi = \Psi_{\omega,\alpha,\alpha'} : {}^{\omega}(\alpha \otimes \alpha') \longrightarrow {}^{\omega}\alpha \otimes {}^{\omega}\alpha'$$

is defined as the class of the front face of the following cube

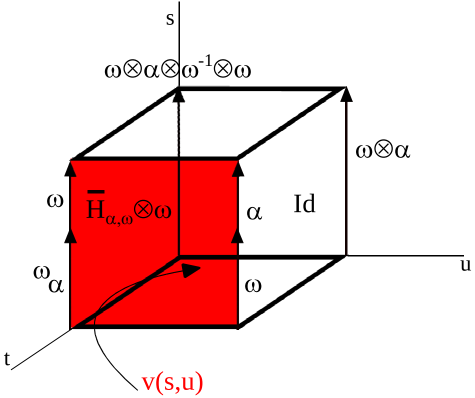


The fundamental categorical crossed module of a fibration

For any $\omega \in \wp_2(X, x_0)$ and $\alpha \in \wp_2(F, b_0)$, the natural isomorphism

$$\nu = \nu_{\omega,\alpha} :^{\omega} \alpha \otimes \omega \to \omega \otimes \alpha,$$

is the class of the front face of the cube



The fundamental categorical crossed module of a fibration

For any $\alpha, \alpha' \in \wp_2(F, b_0)$, the natural isomorphism

$$\chi = \chi_{\alpha,\alpha'} : {}^{\alpha}\alpha' \otimes \alpha \to \alpha \otimes \alpha',$$

we define

$$\chi_{\alpha,\alpha'} = \nu_{\alpha,\alpha'}$$

With all this natural isomorphisms we prove that
$< \wp_2(F, x_0), \wp_2(X, x_0), i, \nu, \chi >$ is a categorical crossed module.

The fundamental categorical crossed module of a fibration

The projection by π_0,

$$\pi_0(\wp_2(F, x_0) \xrightarrow{i} \wp_2(X, x_0))$$

gives the fundamental crossed module

$$\pi_1(F, x_0) \xrightarrow{i} \pi_1(X, x_0)$$

of the fibration p.

The fundamental categorical crossed module of a fibration

If $((Kf, (x_0, \omega_0), kf)$ is the homotopy kernel of f (the fiber of $\overline{f}$)

$$\cdots \to \pi_2(Y, y_0) \xrightarrow{\partial} \pi_1(Kf, (x_0, \omega_{y_0})) \xrightarrow{\pi_1(kf)} \pi_1(X, x_0) \to \pi_1(Y, y_0) \to \cdots$$

and

$$\pi_1(Kf, (x_0, \omega_{y_0})) \xrightarrow{\pi_1(kf)} \pi_1(X, x_0)$$

is called the fundamental crossed module of the fibre homotopy sequence

$$(Kf, (x_0, \omega_{y_0})) \xrightarrow{kf} (X, x_0) \xrightarrow{f} (Y, y_0)$$

There is also the 2-exact sequence

$$\to \wp_3(Y, y_0) \xrightarrow{\partial} \wp_2(Kf, (x_0, \omega_{y_0})) \xrightarrow{\wp_2(kf)} \wp_2(X, x_0) \to \wp_2(\overline{X}, F, x_0) \to \wp_1(F, x_0) \to \wp_1(X, x_0)$$

The fundamental categorical crossed module of a fibration

Definition

The fundamental categorical crossed module of a fibre homotopy sequence $(Kf,(x_0,\omega_{y_0})) \xrightarrow{kf} (X,x_0) \xrightarrow{f} (Y,y_0)$ is defined as the categorical crossed module

$$\wp_2(Kf,(x_0,\omega_{y_0})) \xrightarrow{\wp_2(kf)} \wp_2(X,x_0)$$

obtained from the fibration $\overline{f}:(\overline{X},\overline{x_0}) \to (Y,y_0)$ according to previous Theorem.

The fundamental categorical crossed module of a fibration

Note that in the particular case in which we consider a pair of pointed topological spaces (X, A, x_0), associated to the inclusion, there is the fibration $\overline{A} \to X$ where $\overline{A}$ is the space of paths in X ending at some point of A and the maps send each path to its starting point. The fibre of this fibration is given by the subspace $Ki = \{(a, \omega) \in A \times X^I \,/\, \omega(0) = x_0,\, \omega(1) = a\}$ whose homotopy categorical groups are $\wp_n(X, A, x_0) = \wp_1(\Omega^{n-2}(Ki, (x_0, \omega_{x_0}))$, $n \geq 3$. In this way, just we obtain the homotopy categorical crossed module

$$\partial : \wp_3(X, A, x_0) \longrightarrow \wp_2(A, x_0),$$

so that, as in the group case, the fundamental categorical crossed module of a pair of spaces can be deduced from the fundamental categorical crossed module of a fibration.

Recall that given a ring R, K_iR, $i \geq 1$, is given by the composition of

$$K_i : R \mapsto GLR \mapsto BGLR \mapsto BGLR^+ \mapsto \pi_i BGLR^+$$

$$\wp_i(X, x_0) = \wp_1(\Omega^{i-1}(X, x_0))$$

Definition

For any ring R we define K-categorical groups $\mathbb{K}_iR$, $i \geq 1$, as the composition of covariants functors

$$\mathbb{K}_i : R \mapsto GLR \mapsto BGLR \mapsto BGLR^+ \mapsto \wp_{i+1} BGLR^+ .$$

- $\pi_0 \mathbb{K}_i R = \pi_0 \wp_{i+1} BGLR^+ = \pi_i BGLR^+ = K_i R$
- $\pi_1 \mathbb{K}_i R = \pi_1 \wp_{i+1} BGLR^+ = \pi_{i+1} BGLR^+ = K_{i+1} R .$

$\mathbb{K}_iR$, $i \geq 2$, are completely determined, up to isomorphisms, by the K_iR and $K_{i+1}R$ and the quadratic map $K_iR \to K_{i+1}R$.

$$F(R) \stackrel{d_R}{\to} BGL(R) \stackrel{q_R}{\to} BGL(R)^+ .$$

This has associated the crossed module

$$\pi_1 F(R) \stackrel{\pi_1(d_R)}{\longrightarrow} \pi_1 BGL(R)$$

which is equivalent to

$$St(R) \stackrel{\Phi}{\longrightarrow} GL(R)$$

whose cokernel is $K_1 R$ and its kernel $K_2 R$. According to the previuos theorem, associated to the homotopy fibration there is also the categorical crossed module

$$\wp_2(F(R)) \stackrel{\wp_2 d_R}{\longrightarrow} \wp_2(BGL(R))$$

Theorem

For any ring R, $\mathbb{K}_1 R$ and $\mathbb{K}_2 R$ are, respectively, up to monoidal equivalence, the cokernel and the kernel of the categorical crossed module $\wp_2(d_R)$, that is:

$$\mathbb{K}_1 R \simeq Coker\, \wp_2(d_R) \;\; , \;\; \mathbb{K}_2(R) \simeq Ker\, \wp_2(d_R)\,.$$

Corollary

For any ring R,

$$\pi_0 Coker \wp_2(d_R) = K_1 R \; ,$$

$$\pi_1 Coker \wp_2(d_R) = K_2 R \; (\cong \pi_0 Ker \wp_2(d_R))$$

and

$$\pi_1 Ker \wp_2(d_R) \cong K_3 R\,.$$