# PERFECT POINTS OF ABELIAN VARIETIES

$K$ INFINITE FIN. GEN FIELD OF CAR $= p > 0$

(E.G. $\mathbb{F}_p(T)$, $\mathbb{F}_p[x,y] \big/ (y^2 - x(x-1)(x-2))$)

$A/K$ (SIMPLE) ABELIAN VARIETY

$A$ IS NOT DEFINED OVER $\mathbb{F}_q$

LANG-NÉRON:
$A(K)$ IS FINITELY GENERATED

## 1 MOTIVATION

$\Gamma \subseteq A(\bar{K})$ FINITE RANK

$(\dim(\Gamma \otimes \mathbb{Q}) < +\infty)$

$X \subseteq A$ IRREDUCIBLE SUB-VARIETY.

CONJECTURE (ML).

IF $\Gamma \cap X(K) \subseteq X$ IS ZARISKI DENSE THEN $X$ IS THE TRANSLATION OF AN ABELIAN SUBVARIETY OF $A$.

EX $\dim(X) = 1$, $X$ SMOOTH

ML $\Rightarrow$ MORDELL CONJECTURE

**RMQ** IF $\mathrm{CAR}(K)=0$

ML IS A THEOREM OF FALTINGS.

**THM** (HRUSHOVSKI)

ML HOLDS IF

$\Gamma$ IS FINITELY GENERATED

$K^{PERF}$ PERFECTION OF $K$

(E.G. $\mathbb{F}_p(T)^{PERF} = \mathbb{F}_p(T, T^{1/p}, T^{1/p^2}, \dots, T^{1/p^n}, \dots)$)

**THM** (GHIOCA-MOOSA).

IF ML HOLDS FOR $\forall\, \Gamma \subset A(K^{PERF})$

THEN ML // $\forall\, \Gamma$

(2) $A(K^{perf})$ ??

$\boxed{Q}$ IS $A(K^{perf})$ FIN. GEN?

IF THIS IS TRUE

GHIOCA-MOOSA + HRUSHOVSKI

$\Downarrow$

ML HOLDS FOR A

$\boxed{A}$ NO!!!!!!! $\dim(A) = g$

$$A[p](\bar{K}) = \left(\mathbb{Z}/p\mathbb{Z}\right)^e$$

$$0 \le e \le g$$

NOT ÉTALE

$$0 \to \underset{\text{CONNECTED}}{A[p]^0} \to A[p] \to \underset{\text{ÉTALE}}{A[p]^{ét}} \to 0$$

$e$ p-RANK OF A

EX IF $e=0$

THEN $A(K^{perf})$ IS NOT FINITELY GENERATED.

$$A(K^{perf}) \xrightarrow{p} A(K^{perf})$$

$A \xrightarrow{p} A$ INSEPARABLE

EX (HELM) $\exists$ ABELIAN VARIETY ORDINARY ($e=g$) SUCH THAT $A(K^{perf})$ IS NOT FIN. GEN.

EX (RÖSSLER)

IF $A$ IS ORDINARY AND $A(K^{perf})$ NOT FIN. GEN THEN $A$ HAS EVERYWHERE GOOD-REDUCTION.

$$0 \to A[p^n]^0 \to A[p^n] \to A[p^n]^{\text{ét}} \to 0$$

$$0 \to A[p^\infty]^0 \to A[p^\infty] \to A[p^\infty]^{\text{ét}} \to 0$$

$\mathrm{END}(A) \otimes \mathbb{Q}$ IS A DIVISION ALGEBRA.

$$\begin{tikzcd}
\mathrm{END}(A) \otimes \mathbb{Q} \arrow[r] \arrow[d] & \mathrm{END}(A[p^\infty]) \otimes \mathbb{Q} \arrow[d] \\
\mathrm{END}(A) \otimes \mathbb{Q}_p \arrow[r] & \mathrm{END}(A[p^\infty]^{\text{ÉT}}) \otimes \mathbb{Q}
\end{tikzcd}$$

$$f \longmapsto f[p^\infty]^{\text{ÉT}}$$

THM (A.)

① $A(K^{\text{PERF}})$ IS NOT FINITELY GEN
$\Updownarrow$
$\exists\, 0 \neq f \in \mathrm{END}(A) \otimes \mathbb{Q}_p$ SUCH THAT
- $f^2 = f$ (IDEMPOTENT)
- $f[p^\infty]^{\text{ÉT}} = 0$

② IF $e > 0$
$A(K^{\text{PERF}})$ CONTAINS

NO INFINITELY $p$-DIVISIBLE ELEMENT.

EX • IF $\mathrm{END}(A) = \mathbb{Z}$, $e > 0$

THEN $A(K^{\mathrm{perf}})$ IS FINITELY GENERATED.

• IF $e = 0$

$f = 1$

• IF $A$ IS DEFINED OVER $\mathbb{F}_q$

$$0 \to A[p^\infty]^0 \to A[p^\infty] \to A[p^\infty]^{\text{ét}} \to 0$$

SPLITS.

③ WARM-UP (TORSION).

LEMMA $|A(K^{\mathrm{perf}})_{\mathrm{tor}}| < +\infty$

$\forall \ell \neq p$ $\quad A[\ell^\infty](K^{\mathrm{perf}})$

ÉTALE $\quad \|$

$A[\ell^\infty](K)$

$$\mathrm{GAL}(K^{\mathrm{perf}}) = \mathrm{GAL}(K)$$

• $\ell = p$

$$|A(K^{\text{pos NF}})[p^\infty]| < \infty \, ?$$

$$\|$$

$$\mathrm{Hom}_{\pi_1 K^{\text{pos NF}}}\left(\mathbb{Q}_p/\mathbb{Z}_p,\ A[p^\infty]\right)$$

$$\|$$

$$\mathrm{Hom}_{K^{\text{pos NF}}}\Big(\mathbb{Q}_p/\mathbb{Z}_p,\ \overset{\text{ÉTALE}}{\widehat{A[p^\infty]^{\text{ét}}}}\Big)$$

$$0 \to A[p^\infty]^{\text{ét}} \to A[p^\infty] \to A[p^\infty]^{\circ} \to \cdots$$

$K^{\text{pos NF}}$

$$= \mathrm{Hom}_K\Big(\underbrace{\mathbb{Q}_p/\mathbb{Z}_p}_{\text{weight } 0},\ \underbrace{A[p^\infty]^{\text{ét}}}_{\text{weight } 1}\Big) = 0$$

③ ABEL-JACOBI MAPS.

$$A(K^{perf})\left[\tfrac{1}{p}\right] = A(K)\left[\tfrac{1}{p}\right]$$

• $\ell \neq p$ $\quad 0 \to A[\ell^n] \to A \xrightarrow{\ell^n} A \to 0$

$$\begin{tikzcd}
A(K)\otimes \mathbb{Z}/\ell \arrow[r] \arrow[d, equal] & H^1(K, A[\ell^n]) \arrow[d, equal] \\
A(K^{perf})\otimes \mathbb{Z}/\ell \arrow[r] & H^1(K^{perf}, A[\ell^n])
\end{tikzcd}$$

• $\ell = p$

$$0 \to A[p^n] \to \underbrace{A \to A}_{\text{FLAT}} \to 0$$

$$\begin{tikzcd}
A(K) \arrow[r, hook] \arrow[d] \arrow[dr, "AJ^{ét}"] & H^1(K, A[p^\infty]) \arrow[d] \\
A(K)\otimes \mathbb{Z}/p \arrow[r, "AJ_p^{ét}"'] & H^1(K, A[p^\infty])^{ét}
\end{tikzcd}$$

PROP

- $A(K^{\mathrm{PERF}})$ IS NOT FINITELY GENERATED

$\Updownarrow$

$AJ_p^{\text{ÉT}}$ IS NOT INJECTIVE

- $A(K^{\mathrm{PERF}})$ CONTAINS INFINITELY P-DIVISIBLE ELEMENTS $\Leftrightarrow$ $AJ^{\text{ÉT}}$ IS NOT INJECTIVE

§ AJ VS $AJ_p$

$$\begin{tikzcd} A(K) \arrow[r, "AJ"] \arrow[d] & H^1(K, A[p^\infty]) \\ A(K) \otimes \mathbb{Z}/p \arrow[ur, "AJ_p"'] & \end{tikzcd}$$

$x \in A(K) \longleftrightarrow$ 1-MOTIVE

$$M_x = \left[\begin{array}{c} \mathbb{Z} \to A \\ 1 \mapsto x \end{array}\right]$$

$AJ(x)$ IS THE P-ADIC

REALISATION OF $M_x$

$(1,y) \mapsto x + p^n y$

$$M_x[p^n] \simeq \frac{\operatorname{KER}(\mathbb{Z}\times A \to A)}{\operatorname{IM}(\mathbb{Z} \to \mathbb{Z}\times A)}$$

$$1 \mapsto (p^n, -x)$$

$$\begin{array}{c} 0 \\ \downarrow \\ A[p^n] \end{array} \to M_x[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

$$0 \to A[p^\infty] \to M_x[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

$$e \in H^1(K, A[p^\infty])$$

TATE CONJECTURE FOR 1-MOTIVES

$$\operatorname{END}(M_x[p^\infty]) = \operatorname{END}(M_x) \otimes \mathbb{Z}_p$$

**LEMMA** $x$ NOT TORSION

(A SIMPLE) $\Downarrow$

$$\mathrm{END}(M_x) = \mathbb{Z}$$

**PROVE:**

$$
\begin{array}{c}
1 \longmapsto x \\
M_x : \quad
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "n"] & A \arrow[d, "f"] \\
\mathbb{Z} \arrow[r] & A
\end{tikzcd} \\
1 \longmapsto x
\end{array}
$$

$$f : A \to A$$

$f(x) = nx$ FOR SOME $n$.

$$(f-n)(x) = 0$$

A SIMPLE $\Rightarrow$ $f-n$ TRIVIAL OR ISOMORPH.

$$x \in \ker(f-n)$$

$\Downarrow$

$f-n$ IS TRIVIAL. $\blacksquare$