The Gross–Kohnen–Zagier theorem via p-adic uniformization
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Overview

$E/\mathbb{Q}$ elliptic curve, $K/\mathbb{Q}$ imag. quad. extension

$$\rightsquigarrow P_K \in E(\mathbb{Q}) \quad \text{Heegner point}$$

- P_K constructed from CM points on Shimura curve

Question: Are there relations between different P_K?
Answer: Yes! More precisely, Gross–Kohnen–Zagier showed that the generating series of P_K is a modular form

Aim: new proof using p-adic uniformization of Shimura curves

1. Orthogonal Shimura varieties
2. p-adic uniformization
3. p-adic families of theta series

Orthogonal Shimura varieties

(V, Q) quadratic space$/\mathbb{Q}$, $\operatorname{sign}(V) = (n, 2)$
$\rightsquigarrow$ associated bilinear form $Q(x) = 1/2 \cdot \langle x, x \rangle$

$$D := \{ w \in V_{\mathbb{C}} \setminus \{0\} \mid Q(w) = 0,\ \langle w, \overline{w} \rangle < 0 \} / \mathbb{C}^{\times} \subseteq \mathbb{P}(V_{\mathbb{C}})$$

- D is a smooth complex manifold of dimension n
- D has two connected components
- both are contractible

Fix $\mathbb{Z}$-lattice $L \subseteq V$ with $Q(x) \in \mathbb{Z}$ for all $x \in L$ and $\Gamma \subseteq \mathrm{O}(L)$ congruence subgroup

$$\rightsquigarrow X_{\Gamma} := \Gamma \backslash D \quad \text{orthogonal Shimura variety}$$

- X_{Γ} is an algebraic variety$/\mathbb{C}$

Examples

$B/\mathbb{Q}$ quaternion algebra, e.g.: $B = \mathrm{M}_2(\mathbb{Q})$
$\rightsquigarrow$ $V = B^{\mathrm{trd}=0}$, $Q = \mathrm{nrd}$

$$\mathrm{sign}(V) = \begin{cases} (3,0) & \text{if } B_{\mathbb{R}} = \mathbb{H} \\ (1,2) & \text{if } B_{\mathbb{R}} = \mathrm{M}_2(\mathbb{R}) \end{cases}$$

$B^\times$ acts on V via conjugation: $b.x := b \cdot x \cdot b^{-1}$

$$\rightsquigarrow B^\times/\mathbb{Q}^\times \xrightarrow{\cong} \mathrm{SO}(V)$$

In particular, $\mathrm{SO}(V) = \mathrm{PGL}_2(\mathbb{Q})$ if $B = \mathrm{M}_2(\mathbb{Q})$

Examples cont.

Suppose that $B_{\mathbb{R}} = \mathrm{M}_2(\mathbb{R})$

$$D \subseteq \{A \in \mathrm{M}_2(\mathbb{C}) \smallsetminus \{0\} \mid \mathrm{trd}(A) = 0,\ \mathrm{nrd}(A) = 0\}/\mathbb{C}^\times$$

$$\begin{aligned} \{\text{nilpot. mat.}\}/\text{scaling} &\xrightarrow{\cong} \mathbb{P}^1 \\ A &\longmapsto \ker(A) \end{aligned}$$

$$\rightsquigarrow D \xrightarrow{\cong} \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$$

Let $R \subseteq B$ be a $\mathbb{Z}$-order and $\Gamma \subseteq R^\times_{\mathrm{nrd}=1}$

$$\rightsquigarrow X_\Gamma \text{ Shimura curve}$$

Special cycles

$x \in V$ with $Q(x) > 0$
$\Rightarrow$ $x^{\perp}$ has signature $(n-1, 2)$

$$D_x := \{[w] \in D \mid \langle w, x \rangle = 0\} \subseteq D \quad \text{codimension 1}$$
$$\rightsquigarrow Z(x) \subseteq X_\Gamma \quad \text{divisor}$$

$$Z(m) := \sum_{\substack{x \in \Gamma \backslash L \\ Q(x) = m}} Z(x), \quad m > 0$$

In example:

- $Z(x)$ consists of two (or no) CM points
- $Z(m) = \{\text{CM points of disc } m\}$

The GKZ theorem

Theorem (Borcherds 1999)

The generating series

$$[Z] = \sum_{m=0}^{\infty} [Z(m)] q^m \in \mathrm{Pic}(X_\Gamma)[\![q]\!]$$

is a modular form of weight $n/2$.

- $\varphi(Z)$ mod. form of weight $n/2$ for all $\varphi\colon \mathrm{Pic}(X_\Gamma) \to \mathbb{C}$
- n=2, X_Γ Hilbert modular surface: Hirzebruch–Zagier 1976
- n=1, X_Γ modular curve: Gross–Kohnen–Zagier 1987
- n=2, X_Γ admits p-adic uniformization: Darmon–G–Lipnowski 2023
- n=1, X_Γ admits p-adic uniformization: today

About Darmon–G–Lipnowski

$V/\mathbb{Q}$ definite quadratic space unramified at p,

$$D_p \subseteq \{w \in V_{\mathbb{C}_p} \smallsetminus \{0\} \mid Q(w) = 0\}/\mathbb{C}_p^\times$$

$\Gamma \subseteq O(V)$ p-arithmetic congruence subgroup
$\rightsquigarrow$ consider $X_\Gamma = \Gamma\backslash D_p$ and special cycles $Z(m)$

- Theorem of DGL: analogue of Borcherds for $\dim(V) \geq 4$
- If $\dim(V) = 3, 4$, p-adic uniformization implies new $Z(m)$ agrees with old $Z(m)$

p-adic uniformization

$B'/\mathbb{Q}$ quaternion algebra, split at ∞, ramified at p
$R' \subseteq B'$ Eichler order maximal at p, $\Gamma' = R'_{\mathrm{nrd}=1}$
$\rightsquigarrow X := X_{\Gamma'} = \Gamma' \backslash \mathcal{H}$ has a model over $\mathbb{Q}$

$$\text{Cerednik–Drinfeld} \quad \Rightarrow \quad X(\mathbb{C}_p) \cong \Gamma \backslash \mathcal{H}_p$$

- $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{P}^1(\mathbb{Q}_p)$ p-adic upper half plane
- $\Gamma = R_{\mathrm{nrd}=1}$ where $R \subseteq B$ is a $\mathbb{Z}[1/p]$-order
- $B/\mathbb{Q}$ quaternion algebra with invariants switched at ∞ and p

p-adic CM points

B split at p, $\mathbb{P}^1_{\mathbb{Q}_p} \cong \{\text{nilpot. mat.}\}/\text{scaling}$

$$\rightsquigarrow Z(m) = \sum_{\substack{x \in R \\ Q(x)=m}} Z(x) \in \mathrm{Div}(\Gamma \backslash \mathcal{H}_p) = \mathrm{Div}^{\dagger}(\mathcal{H}_p)^{\Gamma}$$

- new $Z(m)$ agrees with old $Z(m)$
- $Q(p \cdot x) = p^2 \cdot Q(x) \Rightarrow Z(p^2 m) = Z(m)$

$$\Rightarrow Z := \sum_{m=0}^{\infty} Z(m) q^m \text{ fulfils } U_{p^2} Z = Z$$

In particular, Z is ordinary

Jacobians of Mumford curves

$\mathcal{A}$ (rigid) analytic functions on $\mathcal{H}_p \rightsquigarrow \mathcal{A}^\Gamma$ rational functions on X_Γ
$\mathcal{M}$ (rigid) meromorphic functions on $\mathcal{H}_p$, ...

$$0 \longrightarrow \mathcal{A}^\times \longrightarrow \mathcal{M}^\times \longrightarrow \mathrm{Div}^\dagger(\mathcal{H}_p) \longrightarrow 0$$

$$(\mathcal{M}^\times)^\Gamma \longrightarrow \mathrm{Div}^\dagger(\mathcal{H}_p)^\Gamma \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{A}^\times)$$

$$\rightsquigarrow \mathrm{Pic}(X_\Gamma) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{A}^\times)$$

In fact:

$$\begin{array}{ccccc} \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{A}^\times) & \twoheadrightarrow & \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) \\ \downarrow\!\!\downarrow & & \downarrow \cong & & \downarrow \cong \\ \mathrm{Jac}(X_\Gamma) & \hookrightarrow & \mathrm{Pic}(X_\Gamma) & \xrightarrow{\deg} & \mathbb{Z} \end{array}$$

Hecke splitting

Hecke action on $\mathrm{Pic}(X_\Gamma)$

- degree part is Eisenstein
- Jacobian is cuspidal

$\rightsquigarrow$ splitting $\mathrm{Pic}(X_\Gamma) \approx \mathrm{Jac}(X_\Gamma) \oplus \mathbb{Z}$
$\rightsquigarrow$ can consider both summands separately (after hitting with Hecke)

easy calculation: $\deg(Z)$ is a Theta series and Eisenstein
$\rightsquigarrow$ can assume that $Z(m)$ has degree 0

can assume something stronger (degree 0 condition on preimages of simplices of BT tree)
$\rightsquigarrow$ canonically lift of $[Z(m)] \in \mathrm{Jac}(X_\Gamma)$ to $[Z(m)] \in \mathrm{H}^1(\Gamma, \mathcal{O}_{\mathbb{C}_p}^\times)$

A crucial formula

$[Z(m)] \in \mathrm{H}^1(\Gamma, \mathcal{O}_{\mathbb{C}_p}^\times) = \mathrm{Hom}(\Gamma, \mathcal{O}_{\mathbb{C}_p}^\times)$
to show: $\mathrm{ev}_\gamma([Z])$ is a modular form for all $\gamma \in \Gamma$ hyperbolic

γ hyperbolic $\Rightarrow$ $V_{\mathbb{Q}_p} = V_+ \oplus V_- \oplus V_0$
$\rightsquigarrow$ $\mathbb{Q}_p$-rational isotropic vectors ϖ_+ and ϖ_-

For all $\mathcal{D} \in \mathrm{Div}(\mathcal{H}_p)$ of strong degree 0:

$$\mathrm{ev}_\gamma([\Gamma.\mathcal{D}]) = \prod_{\alpha \in \gamma^{\mathbb{Z}} \backslash \Gamma} \prod_{x \in \mathcal{H}_p} \left(\frac{\langle w_+, x \rangle}{\langle w_-, x \rangle} \right)^{\mathrm{ord}_x(\alpha D)}$$

p-adic families of theta series

Given $k \geq 0$, $\varpi \in V_{\mathbb{C}}$ isotropic and Φ Schwartz function

$$\rightsquigarrow \vartheta_{\Phi,\varpi,k} := \sum_{v \in V} \Phi(v) \cdot \langle \varpi, v \rangle^k \cdot q^{\langle v,v \rangle}$$

Classical: $\vartheta_{\Phi,\varpi,k}$ is a modular form of weight $3/2 + k$

For appropiate choice of $\Phi^{\pm}$:

- $\Theta_k := \vartheta_{\Phi^+,\varpi_+,k} - \vartheta_{\Phi^-,\varpi_-,k}$ forms a p-adic family
- $\Theta_0 = 0$

$\Rightarrow \Theta' := \frac{d}{dk}\Theta_k|_{k=0}$ is a p-adic modular form
$\Rightarrow e_{\mathrm{ord}_p}(\Theta')$ is a classical ordinary modular form

Remark: $\frac{d}{dk}\langle w, v \rangle^k|_{k=0} = \log_p(\langle w, v \rangle)$

The main formula

$$\log_p(\mathrm{ev}_\gamma([Z])) = e_{\mathrm{ord}_p}(\Theta')$$

Fin

Thank you