Computing Schneider p-adic heights on hyperelliptic Mumford curves

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joint work in progress with Marc Masdeu,
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From Benasque...

Introduction

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For a prime number p, a p-adic height pairing is a function

$$A(F) \times \tilde{A}(F) \to \mathbb{Q}_p$$

which can be regarded as a p-adic analogue of the Néron–Tate height pairing.

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Goal

Present an algorithm to compute the Schneider p-adic height pairing on (Jacobians of) hyperelliptic Mumford curves.

Overview

1. p-adic numbers & rigid analytic geometry
2. Schneider p-adic heights
3. Mumford curves and their Jacobians
   - Mumford curves
   - Hyperelliptic Mumford curves
   - Jacobians of Mumford curves
4. Schneider heights on Mumford curves
   - Theta functions
   - Werner's formula
5. Computing Schneider heights on hyperelliptic Mumford curves
   - Setting
   - An algorithm for local components at p
6. Numerical example

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Warning: Formally, we need to use the language of rigid analytic spaces for what follows, but, for simplicity, we'll be sweeping things under the rug.

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Schneider's pairing decomposes into local factors. For a finite prime $\mathfrak{p}$ of F, set

$F_{\mathfrak{p}}$	-	the completion of F at $\mathfrak{p}$
$C_{\mathfrak{p}}$	-	$C \otimes F_{\mathfrak{p}}$
$\mathrm{Div}^0(C_{\mathfrak{p}})$	-	the group of divisors of degree 0 on $C_{\mathfrak{p}}$

Theorem (Schneider): For each finite prime $\mathfrak{p}$ of F, there exists a local pairing

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Local components away from p: If $\mathfrak{p}$ does not lie over p, then

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Local components at p: If $\mathfrak{p}$ lies over p, then a formula for $(D,E)_{\mathfrak{p}}$ was given by Werner in the case where $C_{\mathfrak{p}}$ is a *Mumford* curve.

§3.2. Mumford Curves

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$$\mathcal{L}_\Gamma := \text{the set of } \textbf{limit} \text{ points of } \Gamma,$$
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Remark: For any Schottky group Γ, X_Γ has *split degenerate* reduction: it has a semistable $\mathcal{O}_K$-model $\mathfrak{X}$ such that

- all irreducible components of $\mathfrak{X}_k$ are isomorphic to $\mathbb{P}^1_k$, and
- all double points are k-rational with two k-rational branches,

where k is the residue field.

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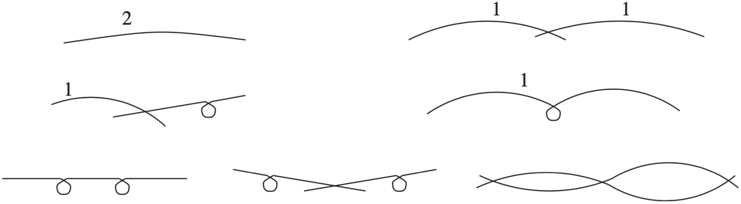

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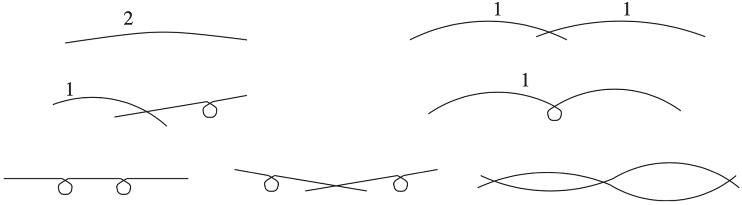

A genus 2 curve has split degenerate reduction precisely when the special fiber of its stable model is one of the three pictures at the bottom (picture taken from Liu's **Algebraic Geometry and Arithmetic Curves** book).

Theorem (Mumford): The map $\Gamma \mapsto X_\Gamma$ induces a bijection

$$\left\{\begin{matrix}\text{conjugacy classes of Schottky} \\ \text{groups in } \mathrm{PGL}_2(K)\end{matrix}\right\} \to \left\{\begin{matrix}\text{isomorphism classes of curves over} \\ K \text{ with split degenerate reduction}\end{matrix}\right\}.$$

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- is Γ'-invariant and induces an isomorphism $\Omega/\Gamma' \simeq \mathbb{P}^1$.

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- Every hyperelliptic Mumford curve can be parametrized by a Whittaker group in this way.

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Question: Which abelian varieties are uniformizable?

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$$A(K) \simeq (K^\times)^g/\Lambda$$

for some lattice Λ. Not every abelian variety is uniformizable.

Question: Which abelian varieties are uniformizable?

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Result: Not only Mumford curves, but also their Jacobians have nice reduction types.

§4.1. Theta functions

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X - $C \otimes K$; a curve over K of genus g

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Properties:

- It's a meromorphic function.
- It's an automorphic form with constant factors of automorphy: for all $\gamma \in \Gamma$ and all $z \in \Omega$,

  $$\Theta(a, b; z) = c(a, b, \gamma) \cdot \Theta(a, b; \gamma(z))$$

  for some $c(a, b, \gamma) \in K^{\times}$.

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§4.2. Werner's formula for $(D, E)_{\mathfrak{p}}$

Now let $\rho\colon K^\times \to \mathbb{Q}_p$ be a non-trivial continuous homomorphism. In practice, it will be $\log_p \circ N_{K/\mathbb{Q}_p}$ where $\log_p$ is the branch of the p-adic logarithm that sends p to 0.

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Definition: We say ρ is Λ-**invertible** if the matrix

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Proposition (Werner): ρ is Λ-invertible $\Longleftrightarrow$ Schneider condition is fulfilled.

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We then have

$$(D, E)_{\mathfrak{p}} = \rho\left(\frac{\Theta(x', y'; z')}{\Theta(x', y'; w')}\right) - \sum_{k=1}^{g} \chi_k(z', w')\rho(c(x', y', \gamma_k)).$$

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F	-	a number field
C	-	a hyperell. curve over F of genus $g \geq 1$ s.t. for every finite prime $\mathfrak{p}$ of F above p, $C \otimes F_{\mathfrak{p}}$ is a Mumford curve

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Local components at p: If $\mathfrak{p}$ lies over p, we'll use Werner's formula for $(D, E)_{\mathfrak{p}}$. There are three main steps:

- Θ: computing theta functions Θ,
- Γ: determining the Schottky group Γ,
- Ω: lifting points from the curve to Ω.

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a_i, b_i	-	fixed points of s_i

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Remark: Another method due to Masdeu–Xarles allows us to compute this function in a comparatively fast way.

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Answer: Of course not. But, thanks to Kadziela's approximation theorem, we can simultaneously approximate both S and F such that

$$F(S) = R.$$

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- $$F(z) \equiv \begin{cases} -4b_0 \prod_{i=1}^{g-1} \left(1 - \left(\frac{a_i - b_i}{a_i + b_i}\right)^2\right) \mod \pi^2 & \text{if } z = b_0, \\ -2z \prod_{i=1}^{g-1} \left(1 + \frac{(a_i - b_i)^2}{(a_i + b_i)(2z - a_i - b_i)}\right) \mod \pi^2 & \text{otherwise,} \end{cases}$$

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- $F(z) \mod \pi^t = \prod_{m=0}^{t-2} F_m(z \mod \pi^t)$ for $t \geq 3$,

where π is a uniformizer in K.

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Recall that R consists the roots of the defining polynomial of X. We may assume that

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If we know correctly the elements z in $S \setminus \{0, 1, \infty\}$ mod π^t, and use them to approximate

- the elliptic matrices s_i,
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$$R = \{0, r_0, r_1, \ldots, r_{2g-2}, 1, \infty\}.$$

If we know correctly the elements z in $S \setminus \{0, 1, \infty\}$ mod π^t, and use them to approximate

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In other words, we guess the elements in S digit by digit using the approximation theorem. This algorithm is a brute force algorithm but works quite well when g and p are small.

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\downarrow & \searrow & \\
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where the isomorphism $\Omega/\Gamma^{'} \simeq \mathbb{P}^1$ is induced by $F = F_{a,b} : \Omega \to \mathbb{P}^1$ for parameters $a, b \in \Omega$.

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Question: But... How do we distinguish?

Theorem (K.–Masdeu–Müller–van der Put)

Set $\gamma := \gamma_1 \cdots \gamma_g$, *and*

$$H(z) := \Theta(a, \gamma(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z), \quad z \in \Omega.$$

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- *The curve* $X = \Omega/\Gamma$ *is parametrized by* $z \in \Omega \mapsto (F(z), H(z))$*.*

Corollary: If $H(z) = y$, then z is a lift of P. Else $s_0(z)$ is a lift of P.

§6. Numerical example

Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^5 - 326x^4 + 1052 \cdot 5^2 x^3 - 5914 \cdot 5^2 x^2 + 39 \cdot 5^5 x.$$

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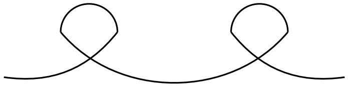
The prime $p = 5$ is a prime of bad reduction. Moreover, the corresponding (stable) reduction is a projective line with two ordinary double points:



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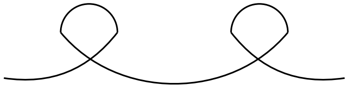

Set $D = (x) - (y)$ and $E = (z) - (w)$, where

$$x = (7,\ 1+3\cdot5+4\cdot5^2+5^5+5^6+O(5^7)), \qquad y = (12,\ 1+2\cdot5+3\cdot5^2+5^5+4\cdot5^6+O(5^7)),$$

$$z = (-3,\ 1+2\cdot5^2+4\cdot5^4+2\cdot5^5+5^6+O(5^7)), \qquad w = (-18,\ 1+3\cdot5+2\cdot5^3+5^4+5^5+2\cdot5^6+O(5^7)).$$

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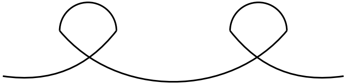

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Goal

Compute the local height $(D, E)_p$.

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We have:

$$a_0 = 0, \qquad b_0 = 3\cdot 5^3 + 3\cdot 5^4 + 3\cdot 5^5 + 3\cdot 5^6 + O(5^7),$$
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Question: How do we know that this is correct?

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The function $(\cdot,\cdot)_p$ is symmetric, and we have

$$(E,D)_5 = 3\cdot 5 + 2\cdot 5^2 + 4\cdot 5^3 + 2\cdot 5^5 + O(5^6). \quad :-)$$

Gràcies!

- *Basic Notions of Rigid Analytic Geometry* - Schneider
- *Non-archimedean Uniformization and Monodromy Pairing* - Papikian
- *Schottky Groups and Mumford Curves* - Gerritzen–van der Put
- *Rigid Geometry of Curves and Their Jacobians* - Lütkebbohmert

- *p-adic Height Pairings I* - Schneider
- *Local Heights on Mumford Curves* - Werner

- *Algorithms for Mumford Curves* - Morrison–Ren
- *Rigid Analytic Uniformization of Hyperelliptic Curves* - Kadziela
- *Algorithms for Heights On Mumford Curves* (to be modified) - Kaya–Masdeu–Müller–van der Put