Quadratic Chabauty for modular curves

Montserrat, 28 June 2023

Jan Vonk

Outline

The class number one problem

A **quadratic form** (*primitive* if $\gcd(a,b,c)=1$) is an element

$$\langle a,b,c\rangle := aX^2+bXY+cY^2 \quad \in \mathbf{Z}[X,Y]$$

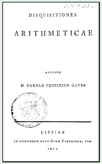
Have right $\mathrm{SL}_2(\mathbf{Z})$-action on $\mathbf{Z}[X,Y]$ by ring automorphisms, defined by

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} : \begin{cases} X \longmapsto pX+qY \\ Y \longmapsto rX+sY \end{cases}$$

This action preserves the set of quadratic forms, respects primitivity, and preserves the **discriminant** $\Delta := b^2-4ac$ of a quadratic form $\langle a,b,c\rangle$.

Let $\mathcal{F}_\Delta$ be the set of primitive forms of discriminant Δ (with $a>0$ if $\Delta<0$).
When Δ is a non-square discriminant of a quadratic order $\mathcal{O}$, have bijection

$$\mathcal{F}_\Delta/\,\mathrm{SL}_2(\mathbf{Z}) \longrightarrow \mathrm{Pic}^+(\mathcal{O})\ ;\ \langle a,b,c\rangle \longmapsto [(a,(-b+\sqrt{\Delta})/2)].$$



DISQVISITIONES

ARITHMETICAE

AVCTORE

D. CAROLO FRIDERICO GAVSS

LIPSIAE

IN COMMISSIS APVD GERH. FLEISCHER, IVN.

Gauß made computational study of number $h(\Delta)$ of equivalence classes.
He conjectured:

- For $\Delta<0$ we have $h(\Delta)=1$ for precisely 13 discriminants:

$$-\Delta \in \{3,4,7,8,11,12,16,19,27,28,43,67,163\}.$$

- For $\Delta>0$ we have $h(\Delta)=1$ for infinitely many discriminants Δ.

The class number one problem

Heegner (1952) resolved the case $\Delta < 0$ using modular functions. Most important: Klein j-function, defined in the variable $q = \exp(2\pi i \tau)$ on $|q| < 1$ by

$$\begin{aligned} j(q) &= \left(1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n}\right)^3 \div \left(q \prod_{n \geq 1} (1 - q^n)^{24}\right) \\ &= q^{-1} + 744 + 196884q + 21493760q^2 + \dots \end{aligned}$$

Values at roots τ of forms with $\Delta < 0$ (*singular moduli*) are algebraic integers of degree $h(\Delta)$:

$$j\left(\frac{1+\sqrt{-7}}{2}\right) = -3^3 \cdot 5^3 \qquad j\left(\frac{1+\sqrt{-163}}{2}\right) = -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$$

Not yet enough for class number one, since $\mathbf{Z}$ is infinite! Heegner uses two ingredients:

Special values of the cube root $\gamma_2 = \sqrt[3]{j}$

$$\gamma_2(q) = q^{-1/3} + 248q + 4124q^2 + \dots$$

Value at quadratic τ with $3 \nmid \Delta < 0$ is an *algebraic integer* of degree $h(\Delta)$.

Special values of the *Weber functions*

$$\begin{aligned} \mathfrak{f}(q) &= q^{-1/48} \prod (1 + q^{n-1/2}) \\ \mathfrak{f}_1(q) &= q^{-1/48} \prod (1 - q^{n-1/2}) \\ \mathfrak{f}_2(q) &= \sqrt{2} q^{1/24} \prod (1 + q^n) \end{aligned}$$

Reduces the problem to finding integral solutions of $2x(x^3 + 1) = y^2$.

Non-split Cartan modular curves

Geometric interpretation of Heegner/Stark argument for $\Delta < 0$:

- the function γ_2 is parameter on $X_{\mathrm{ns}}^{+}(3) \simeq \mathbf{P}^1$
- the equation $y^2 = 2x(x^3 + 1)$ is a model for $X_{\mathrm{ns}}^{+}(24)$.

Let p be prime, then points on $X_{\mathrm{ns}}^{+}(p)(\mathbf{Q})$ correspond to elliptic curves $E/\mathbf{Q}$ with image of

$$\rho_{E,p} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}(E[p]) \simeq \mathrm{GL}_2(\mathbf{F}_p)$$

contained in the normaliser of a non-split Cartan subgroup

$$\mathbf{F}_{p^2}^{\times} \subset \mathrm{GL}_2(\mathbf{F}_p).$$

For a CM curves E equivalent to p inert in $\mathcal{O} \simeq \mathrm{End}(E)$. When $h(\Delta) = 1$, this is implied by the condition $p \nmid \Delta < -4p$, giving for each such Δ an integral point on $X_{\mathrm{ns}}^{+}(p)$.

Q: (Mazur / Serre) Are all integral/rational points obtained in this way?

- Siegel (1968) parametrises $X_{\mathrm{ns}}^{+}(5) \simeq \mathbf{P}^1$ (two cusps) and finds points on $X_{\mathrm{ns}}^{+}(15)$.
- Kenku (1985) parametrises $X_{\mathrm{ns}}^{+}(7) \simeq \mathbf{P}^1$ (three cusps).
- Ligozat (1976) parametrises $X_{\mathrm{ns}}^{+}(11)$ genus 1, finds integral points.
- Baran (2014) parametrises $X_{\mathrm{ns}}^{+}(13)$ genus 3, rational points (BDMTV 2019)
- Mercuri–Schoof (2018) parametrise $X_{\mathrm{ns}}^{+}(17)$ genus 6, rational points (BDMTV 2023)

Outline

Let K be a number field, and X_K be a smooth projective curve, then we know:

If X_K is of genus $g \geq 2$, then $X(K)$ is finite.

(1) Proved **unconditionally** by Faltings (1983) and Lawrence–Venkatesh (2020). Consider a certain *Parshin family* $\mathcal{C} \longrightarrow X$, with $\mathcal{C}_{\overline{x}}$ a finite covering of X unramified away from x. One then associates a (very structured) Galois representation

$$\rho : \; x \longmapsto \mathrm{H}^1_{\text{ét}}(\mathcal{C}_{\overline{x}}, \mathbf{Q}_p).$$

Then show that the association is finite to one, and has finitely many images.

(2) Proved much earlier by Chabauty (1941) **conditionally** on $r < g$.

Method of Chabauty proceeds as follows: Choose $b \in X(K)$, get Abel–Jacobi map

$$\mathrm{AJ}_b : X_K \longrightarrow J_K.$$

For $\mathfrak{p}$ prime of good reduction, consider $\mathfrak{p}$-adic logarithm, get commutative diagram:

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X(K_{\mathfrak{p}}) \arrow[d] \arrow[dr] & \\
J(K) \arrow[r] & J(K_{\mathfrak{p}}) \arrow[r, "\log"] & \mathrm{H}^0(X_{K_{\mathfrak{p}}}, \Omega^1)^{\vee}
\end{tikzcd}$$

Chabauty proves two statements in $\mathrm{H}^0(X_{K_{\mathfrak{p}}}, \Omega^1)^{\vee}$:

- The closure of $J(K)$ is in a proper $K_{\mathfrak{p}}$-subspace.
- The intersection $X(K_{\mathfrak{p}}) \cap \log \overline{J(K)}$ is finite.

What if $r \geq g$? $\rightarrow$ Reinterpret Chabauty cohomologically.

Let $V := T_p(J) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ the p-adic Tate module, and define

$$
\begin{array}{llcccc}
\kappa : & J(K) & \longrightarrow & \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_n J(K)/p^n J(K) & \xrightarrow{\sim} & \mathrm{H}^1_f(G, V) \\
\kappa_{\mathfrak{p}} : & J(K_{\mathfrak{p}}) & \longrightarrow & \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_n J(K_{\mathfrak{p}})/p^n J(K_{\mathfrak{p}}) & \xrightarrow{\sim} & \mathrm{H}^1_f(G_{\mathfrak{p}}, V)
\end{array}
$$

where f = unramified outside bad reduction, crystalline at places above p.

Set $V_{\mathrm{dR}} := \mathrm{H}^1_{\mathrm{dR}}(X_{K_{\mathfrak{p}}})^\vee$, then

$$
\begin{aligned}
\mathrm{H}^1_{\mathrm{f}}(G_{\mathfrak{p}}, V) &\simeq V_{\mathrm{dR}}/\,\mathrm{Fil}^0 \\
&\simeq \mathrm{H}^0(X_{K_{\mathfrak{p}}}, \Omega^1)^\vee
\end{aligned}
$$

isomorphism, constructed by Bloch–Kato. Get a commutative diagram:

$$
\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X(K_{\mathfrak{p}}) \arrow[d] \arrow[dr] & \\
J(K) \arrow[r] \arrow[d, "\kappa"'] & J(K_{\mathfrak{p}}) \arrow[r, "\log"] \arrow[d, "\kappa_{\mathfrak{p}}"'] & \mathrm{H}^0(X_{K_{\mathfrak{p}}}, \Omega^1)^\vee \arrow[d, "\wr"'] \\
\mathrm{H}^1_{\mathrm{f}}(G, V) \arrow[r, "\mathrm{loc}_{\mathfrak{p}}"] & \mathrm{H}^1_{\mathrm{f}}(G_{\mathfrak{p}}, V) \arrow[r, "\simeq"] & V_{\mathrm{dR}}/\,\mathrm{Fil}^0
\end{tikzcd}
$$

Cutting the middle row, we find a diagram more amenable to generalisation! Note that

$$\pi_1^{\text{ét}}(\overline{X}, b) \longrightarrow \mathrm{H}^1_{\text{ét}}(\overline{X}, \widehat{\mathbf{Z}})^\vee \longrightarrow T_p(J).$$

Can we replace the bottom row with cohomology valued in larger quotient?

Grothendieck conjectured that

$$X(K) \longrightarrow \mathrm{H}^1\big(G_K, \pi_1^{\text{ét}}(\overline{X}, b)\big)\,; \qquad x \longmapsto \pi_1^{\text{ét}}(\overline{X}; b, x)$$

is an isomorphism. Unfortunately, this cohomology set has very little structure!

$$\begin{array}{lcll} X(K) & \longrightarrow & \mathrm{H}^1_{\mathrm{f}}\,(G_K, V) & (\mathrm{H}^1 \text{ has } \textbf{too much} \text{ structure if } r \geq g.) \\ X(K) & \longrightarrow & \mathrm{H}^1\,\big(G_K, \pi_1^{\text{ét}}(\overline{X}, b)\big) & (\mathrm{H}^1 \text{ has } \textbf{too little} \text{ structure.}) \end{array}$$

We instead work with a *unipotent* quotient U of $\pi_1^{\text{ét}}(\overline{X}, b)$, assuring that each group $\mathrm{H}^1_{\mathrm{f}}$ is the set of $\mathbf{Q}_p$-points of *algebraic* variety, and $\mathrm{loc}_{\mathfrak{p}}$ is algebraic (*Selmer varieties*).

Proposal of Minhyong Kim: Want a unipotent quotient U such that

1. we can prove that $\dim \mathrm{H}^1_{\mathrm{f}}(G, \mathrm{U}) < \dim \mathrm{H}^1_{\mathrm{f}}(G_{\mathfrak{p}}, \mathrm{U})$,
2. the quotient is "motivic", so that we get replacement of

$$\log : J(K_{\mathfrak{p}}) \longrightarrow \mathrm{H}^0(X_{K_{\mathfrak{p}}}, \Omega^1)^{\vee}$$

For such a U, we get the following commutative diagram:

$$\begin{tikzcd} X(K) \arrow[r] \arrow[d, "j^{\text{ét}}"'] & X(K_{\mathfrak{p}}) \arrow[d, "j_p^{\text{ét}}"'] \arrow[dr, "j^{\mathrm{dR}}"] & \\ \mathrm{H}^1_{\mathrm{f}}(G, \mathrm{U}) \arrow[r, "\mathrm{loc}_{\mathfrak{p}}"] & \mathrm{H}^1_{\mathrm{f}}(G_{\mathfrak{p}}, \mathrm{U}) \arrow[r, "\mathrm{D}_{\mathrm{cris}}"] & \mathrm{U}^{\mathrm{dR}} / \mathrm{Fil}^0. \end{tikzcd}$$

Kim: Such a quotient exists if we assume Bloch–Kato / Fontaine–Mazur.

Outline

Quadratic Chabauty–Coleman

In quadratic Chabauty–Coleman, we use a suitable algebraic correspondence $Z \subset X \times X$ to construct a unipotent quotient U such that

$$1 \longrightarrow \mathbf{Q}_p(1) \longrightarrow \mathrm{U} \longrightarrow V \longrightarrow 1.$$

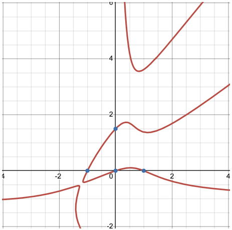
The example $X = X_{\mathrm{ns}}^{+}(13)$: Baran finds the model

$$(2y^2 + y)x^2 - (y^3 - y^2 + 2y - 1)x + (2y^2 - 3y) = x^3(y + 1)$$

Quadratic Chabauty–Kim associates representations to points:

$$\begin{array}{lll} x \in X(\mathbf{Q}) & \longmapsto & [\rho_x : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_8(\mathbf{Q}_p)] \\ x \in X(\mathbf{Q}_p) & \longmapsto & [\rho_x : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_8(\mathbf{Q}_p)] \end{array}$$

with a *computable* condition on global representations.
Quadratic Chabauty uses the bilinearity of p-adic height pairing.



Numerical method works from the equations, only arithmetic input:

- A correspondence $Z \subset X \times X$, and computation of its action on cohomology. This was the subject of Edixhoven's thesis (1989).
- Use Edixhoven (1990) to determine semi-stable model at $\ell = 13$. General: Edixhoven–Parent (2021), all max. subgroups $\mathrm{GL}_2(\mathbf{F}_p)$.
- Use Chen, Edixhoven–de Smit (1999) to determine rank $\mathrm{Jac}_X(\mathbf{Q})$.

Constructing the quotient U

Let U_n be the quotient by the n-th lower central series filtration on the $\mathbf{Q}_p$-unipotent completion of $\pi_1^{\text{ét}}(\overline{X}, b)$. We have $\mathrm{U}_1 = V$ and U_2 is an extension

$$1 \longrightarrow \mathrm{Coker}\left(\mathbf{Q}_p(1) \xrightarrow{\cup^*} \wedge^2 V\right) \longrightarrow \mathrm{U}_2 \longrightarrow V \longrightarrow 1. \tag{1}$$

Assume there exists a nonzero class $Z \in \mathrm{NS}(J)$ with trace zero. This element gives a cycle class $\xi_Z : \mathrm{Coker}(\cup^*) \longrightarrow \mathbf{Q}_p(1)$ along which we push out the extension (1) to obtain

$$1 \longrightarrow \mathbf{Q}_p(1) \longrightarrow \mathrm{U} \longrightarrow V \longrightarrow 1.$$

This quotient is very convenient for Chabauty–Kim, since it is motivic, and we have

$$\begin{array}{lclcl} \mathrm{H}^1_{\mathrm{f}}(G_{\mathbf{Q}}, \mathbf{Q}_p(1)) & = & \mathbf{Z}^\times \widehat{\otimes} \mathbf{Q}_p & = & 0, \\ \mathrm{H}^1_{\mathrm{f}}(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1)) & = & \mathbf{Z}_p^\times \widehat{\otimes} \mathbf{Q}_p & = & \mathbf{Q}_p . \end{array}$$

We end up with a commutative diagram with the required dimension inequality <u>when $r = g$</u>.

$$\begin{tikzcd} X(K) \arrow[r] \arrow[d, "j^{\text{ét}}"'] & X(K_{\mathfrak{p}}) \arrow[d, "j_p^{\text{ét}}"'] \arrow[dr, "j_p^{\mathrm{dR}}"] & \\ \mathrm{H}^1_{\mathrm{f}}(G, \mathrm{U}) \arrow[r, "\mathrm{loc}_{\mathfrak{p}}"] & \mathrm{H}^1_{\mathrm{f}}(G_{\mathfrak{p}}, \mathrm{U}) \arrow[r, "\mathrm{D_{cris}}"] & \mathrm{U}^{\mathrm{dR}} / \mathrm{Fil}^0. \end{tikzcd}$$

$\dim = r$ $\qquad$ $\dim = r + 1$

Computing the de Rham realisation

The map $j_p^{\text{ét}} : X(k) \longrightarrow \mathrm{H}^1_{\mathrm{f}}(G_k, \mathrm{U})$ associates to each rational point an *extension*

$$1 \to \mathrm{U} \to E \to \mathbf{Q}_p \to 1$$

of G_k-representations. Want to **compute** its image under $\mathrm{D}_{\mathrm{cris}}$, a filtered ϕ-module.

These filtered ϕ-modules are the fibres of a vector bundle with connection $\mathcal{V}_Z$ on X, with:

- A filtration $\mathcal{V}_Z \supset \mathrm{Fil}^0$
- A Frobenius structure $\phi^* \mathcal{V}_Z \simeq \mathcal{V}_Z$.

These structures on the bundle $\mathcal{V}_Z$ are rigid: we know them on graded pieces, and we know them on the fibre at $b \in X(\mathbf{Q})$. This determines them uniquely (and computably).

Condition on global points

There is a continuous <u>bilinear</u> height pairing due to Nekovář

$$h : \mathrm{H}^1_{\mathrm{f}}(G, \mathrm{U}) \longrightarrow \mathrm{H}^1_{\mathrm{f}}(G, V) \times \mathrm{H}^1_{\mathrm{f}}(G, V^*(1)) \longrightarrow \mathbf{Q}_p$$

which has a decomposition into local components

$$h = h_p + \sum_{v \neq p} h_v, \qquad h_v : \mathrm{H}^1_{\mathrm{f}}(G_v, \mathrm{U}) \longrightarrow \mathbf{Q}_p\,.$$

For the cursed curve, have $h_v = 0$. Compute h_p from filtered ϕ-module, and solve for $h = h_p$.

Outline

"Proving" theorems

From a recent talk of Kevin Buzzard (Jan 2020, Pittsburgh):

The future of mathematics?

Kevin Buzzard

Introduction.

Human proofs.

Computer proofs.

I want to move away from errors now and talk about other issues.

In 2019, Balakrishnan, Dogra, Mueller, Tuitman and Vonk found all the rational solutions to a certain important quartic curve in two variables (the modular curve $X_s(13)$, a.k.a. $y^4 + 5x^4 - 6x^2y^2 + 6x^3 + 26x^2y + 10xy^2 - 10y^3 - 32x^2 - 40xy + 24y^2 + 32x - 16y = 0$).

This calculation had important consequences in arithmetic (new proof of class number 1 problem etc).

The proof makes essential use of calculations in `magma`, an unverified closed-source system using fast unrefereed algorithms.

It would be difficult, but certainly not impossible, to port everything over to an unverified open source system such as `sage`.

Nobody has any plans to do this. Hence part of the proof remains secret (and may well remain secret forever). Is this science?

13 / 28

Quadratic Chabauty–Coleman (d'après Edixhoven-Lido)

Choose b in $X(\mathbf{Q})$, gives $X \to J$ and

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Pic}^0(J) \arrow[r] \arrow[d, equal] & \mathrm{Pic}(J) \arrow[r] \arrow[d] & \mathrm{NS}(J) \arrow[r] \arrow[d] & 0 \\ 1 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathbf{Z} \arrow[r] & 0. \end{tikzcd}$$

Take non-trivial Z in $\mathrm{NS}(J)$ which maps to zero in $\mathbf{Z} \simeq \mathrm{NS}(X)$. Get unique lift $\mathscr{L}_Z$ in $\mathrm{Pic}(J)$ that is trivial on X. Working over $\mathbf{Z}$, obtain $X \longrightarrow \mathscr{L}_Z^{\times}$, unique up to $\mathbf{Z}^{\times} = \{\pm 1\}$.

Fundamental group U of $\mathscr{L}_Z^{\times}$ is an extension (Bertrand–Edixhoven 2020):

$$1 \to \mathbf{Q}_p(1) \to \mathrm{U} \to \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbf{Q}_p)^{\vee} \to 1$$

Associating path torsors, get diagram:

$$\begin{tikzcd} X(\mathbf{Q}) \arrow[r] \arrow[d] & X(\mathbf{Q}_p) \arrow[d] \arrow[dr] & \\ \mathscr{L}_Z^{\times}(\mathbf{Q}) \arrow[r] \arrow[d] & \mathscr{L}_Z^{\times}(\mathbf{Q}_p) \arrow[r, "\log"] \arrow[d] & \mathrm{Lie}\, \mathscr{L}_Z^{\times}(\mathbf{Q}_p) \arrow[d, "\wr"'] \\ \mathrm{H}^1_{\mathrm{f}}(G, \mathrm{U}) \arrow[r, "\mathrm{loc}_p"] & \mathrm{H}^1_{\mathrm{f}}(G_p, \mathrm{U}) \arrow[r, "\simeq"] & \mathrm{U}_{\mathrm{dR}} / \mathrm{Fil}^0 \end{tikzcd}$$

Edixhoven–Lido "puts the middle row back in". Place geometry of Poincaré torsor central. These ideas are being developed by Duque-Rosero, Hashimoto, Spelier.

Real quadratic fields

Discriminants $\Delta < 0$

MULTITUDO MEDIOCRIS CLASSIUM. 367

I. 1 ... 1, 2, 3, 4, 7
I. 3 ... 11, 19, 23, 27, 31, 43, 67, 163
I. 5 ... 47, 79, 103. 127
I. 7 ... 71, 151, 223, 343, 463, 487
II. 1 ... 5, 6, 8, 9, 10, 12, 13, 15, 16, 18, 22, 25, 28, 37, 58

Finite list with $h(\Delta) = 1$.

Discriminants $\Delta > 0$

reliqui 145 unam classem in quovis genere. — Quaestio curiosa foret, nec geometrarum sagacitate indigna, secundum quam legem determinantes unam classem in quovis genere habentes continuo rariores fiant, investigare; hactenus nec per theoriam decidere possumus, nec per observationem satis certo coniectare, utrum tandem omnino abrumpantur (quod tamen parum probabile videtur), aut saltem *infinite rari* evadant, an ipsorum frequentia ad limitem fixum continuo magis acce-

Infinite list with $h(\Delta) = 1$?

Constructing singular moduli must be very different for real quadratic fields!

- **Objection 1:** There are finitely many rational points on every $X_{\mathrm{ns}}^{+}(p)$.
- **Objection 2:** Gross–Zagier showed that *differences* of singular moduli are smooth:

$$
\begin{aligned}
j\left(\frac{1+\sqrt{-67}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) &= -2^{15}\cdot 3^{3}\cdot 5^{3}\cdot 11^{3} + 2^{18}\cdot 3^{3}\cdot 5^{3}\cdot 23^{3}\cdot 29^{3} \\
&= 2^{15}\cdot 3^{7}\cdot 5^{3}\cdot 7^{2}\cdot 13\cdot 139\cdot 331
\end{aligned}
$$

A similar theory of singular moduli for $\Delta > 0$ would contradict the abc-conjecture.

Darmon–V. (2021) construct p-adic quantity $\Theta_p^{\times}[\tau_1, \tau_2] \in \mathbf{C}_p^{\times}$ for RM points. Very explicit. For example, when $p = 13$ and $(\tau_1, \tau_2) = (2\sqrt{2}, \sqrt{31})$ we find a root of

$$
1201712(x^4+1) - 3946488(x^3+x) + 5631681x^2 = 0 \pmod{13^{200}}, \qquad 1201712 = 2^4\cdot 19\cdot 59\cdot 67.
$$

- Mimics (conjecturally!) all properties of singular moduli discovered in Gross–Zagier.
- A *multiplicative* quantity, relates (conj!) to p-adic heights of points on modular *Jacobians*.

Thanks for having me!