Abel-Jacobi maps

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Abstract

This short note is a rough draft of the material presented in the student seminar in McGill, Fall 2009. The main references are the paper of D. Arapura and K. Oh ([AO94]) and the preprint of M. Bertolini, H. Darmon and K. Prasanna ([BDP09]). A good reference for generalities on Hodge Theory is the book of C. Voisin ([Voi02]).

1 The Abel-Jacobi map on curves

Let X be a compact and connected Riemann surface. Define $\mathrm{Div}(X)$ as the free abelian group on the points of X. That is, the elements of $\mathrm{Div}(X)$ are formal sums:

$$\sum_{P\in X} n_P P,$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for almost all P. Define the degree map:

$$\deg\colon \mathrm{Div}(X) \to \mathbb{Z}$$

which maps $\sum n_P P \mapsto \sum n_P$. This is a group homomorphism, and we set $\mathrm{Div}(X)_0 := \ker \deg$.

Let $D \in \mathrm{Div}(X)_0$. We can write then $D = \sum_{i=1}^n Q_i - P_i$. For each i, let γ_i be a continuous path in X joining $P_i \to Q_i$. Consider the 1-cycle $Z := \sum_{i=1}^n \gamma_i$. Then $\partial Z = D$. Consider a functional:

$$I_Z = \int_Z \colon H^0(X,\Omega^1) \to \mathbb{C},$$

which assigns to a holomorphic 1-form $\omega = f(z)dz$ the path integral

$$\sum_{i=1}^n \int_{\gamma_i} \omega = \sum_{i=1}^n \int_{\gamma_i} f(z)dz.$$

Of course, the choice of paths γ_i is not unique, and so the functional I_Z depends on that. But if $\partial Z = \partial Z'$, then $Z - Z'$ is a *closed* 1-cycle. There is an injection:

$$I\colon H_1(X,\mathbb{Z}) \to H^0(X,\Omega^1)^\vee,$$

which sends closed 1-cycles C to I_C, and which identifies $H_1(X,\mathbb{Z})$ as a lattice inside $H^0(X,\Omega^1)^\vee$ which is isomorphic by Poincaré duality to $H^1(X,\mathbb{Z})$. This defines a group homomorphism:

$$\mathrm{AJ}_{\mathbb{C}}\colon \operatorname{Div}(X)_0 \to \frac{H^0(X,\Omega^1)^\vee}{H^1(X,\mathbb{Z})}.$$

Define the subgroup of *principal divisors* as $P(X) := \{\operatorname{div}(f) \mid f \in \mathbb{C}(X)\}$, where given a meromorphic function $f \in \mathbb{C}(X)$ we define $\operatorname{div}(f) := \sum_{P\in X} \operatorname{ord}_P(f)P$. The fact that a meromorphic function has a finite number of zeroes and poles makes this into a divisor. It will be of degree zero because it is the pull-back of the degree-zero divisor $(0)-(\infty)$ of $\mathbb{P}^1_{\mathbb{C}}$ via the map $f\colon X \to \mathbb{P}^1_{\mathbb{C}}$.

Theorem 1.1 (Abel-Jacobi)**.** *The map* $\mathrm{AJ}_{\mathbb{C}}$ *is surjective and* $\ker \mathrm{AJ}_{\mathbb{C}}$ *is the group* $P(X)$ *of principal divisors.*

In this way, we get a canonical isomorphism

$$\operatorname{Div}(X)_0/P(X) \cong \frac{H^0(X,\Omega^1)^\vee}{H^1(X,\mathbb{Z})}.$$

The left-hand side is called $\operatorname{Pic}^0(X)$, the *degree*-0 *Picard group of* X, and the right-hand side is called the *Jacobian* of X. It has the structure of a complex Lie group. In fact, it is isomorphic to $\mathbb{C}^g/\Lambda$, where g is the genus of X, and $\Lambda = H^1(X,\mathbb{Z})$ is a rank-$2g$ $\mathbb{Z}$-lattice inside $\mathbb{C}^g$.

The goal of these lectures is to generalize the Abel-Jacobi map in many ways. First recall the Hodge filtration:

$$0 \to H^0(X,\Omega^1) \to H^1_{\mathrm{dR}}(X) \to H^1(X,\mathcal{O}_X) \to 0.$$

Serre duality gives a perfect pairing

$$H^1(X,\mathcal{O}_X) \times H^0(X,\Omega^1) \to \mathbb{C}$$

defined as $\langle f,\omega\rangle := \sum_{P\in X} \operatorname{res}_P(f g_{\omega,P})$, where $g_{\omega,P}$ is a local primitive of ω around P. This induces an isomorphism $H^0(X,\Omega^1)^\vee \cong H^1(X,\mathcal{O}_X) \cong H^1_{\mathrm{dR}}(X)/H^0(X,\Omega^1)$. Hence we can see $\mathrm{AJ}_{\mathbb{C}}$ as a map:

$$\mathrm{AJ}_{\mathbb{C}}\colon \operatorname{Div}(X)_0 \to J^0(X) := \frac{H^1_{\mathrm{dR}}(X)}{H^1(X,\mathbb{Z}) + H^0(X,\Omega^1)}.$$

In the next section we will see how to deal with objects like the one appearing in the right-hand side of the previous display.

2 Hodge Structures

We construct an abstract category that will be suitable to deal with the previous problem.

Definition 2.1. A *pure Hodge structure* of weight k is a pair $(H_{\mathbb{Z}}, F^\bullet)$, where:

1. $H_{\mathbb{Z}}$ is a finitely-generated abelian group,

2. $F^\bullet$ is an exhaustive and separated decreasing filtration on $H_\mathbb{C} := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$, satisfying for all $p \in \mathbb{Z}$:
$$H_\mathbb{C} = F^p H_\mathbb{C} \oplus \overline{F^{k+1-p} H_\mathbb{C}}.$$

Example. 1. The kth Betti cohomology $H := H^k_B(X)$ of a smooth proper variety over $\mathbb{C}$, together with the Hodge filtration given by the de Rham theorem, is a pure Hodge structure of weight k.

2. The Hodge structure of Tate $\mathbb{Z}(-1)$ is defined as follows: $H_\mathbb{Z} := \frac{1}{2\pi i}\mathbb{Z} \subset \mathbb{C}$, and $F^\bullet$ is:
$$F^p H_\mathbb{C} = \begin{cases} \mathbb{C} & p \leq 1, \\ 0 & p \geq 2. \end{cases}$$
Note that it is of weight $k = 2$. In general, $\mathbb{Z}(-t) := \mathbb{Z}(-1) \otimes \cdots \otimes \mathbb{Z}(-1)$ is a pure Hodge structure of weight $2t$.

Definition 2.2. A *mixed Hodge structure (MHS)* is a triple $(H_\mathbb{Z}, W_\bullet, F^\bullet)$, where:

1. $H_\mathbb{Z}$ is a finitely-generated abelian group,
2. $W_\bullet$ is an increasing filtration on $H_\mathbb{Q} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$, called the *weight filtration*, and
3. $F^\bullet$ is a decreasing filtration on $H_\mathbb{C}$, called the Hodge filtration,

such that $F^\bullet$ induces a pure Hodge structure of weight m on the graded piece $\mathrm{gr}^{W_\bullet}_m := W_m / W_{m-1}$.

A morphism of MHS is a group homomorphism of the underlying abelian groups which preserves the two filtrations.

In the category of MHS there is an internal hom: given two MHS's A and B, define H as follows: $H_\mathbb{Z} := \mathrm{Hom}(A_\mathbb{Z}, B_\mathbb{Z})$. The filtrations are defined as follows:
$$W_m H_\mathbb{Q} := \{\phi \mid \phi(W_r A_\mathbb{Q}) \subseteq W_{r+m} B_\mathbb{Q}, \forall r\},$$
$$F^p \,\mathrm{Hom}_\mathbb{C} := \{\phi \mid \phi(F^r A_\mathbb{C}) \subseteq F^{r+p} B_\mathbb{C}, \forall r\}.$$

Definition 2.3. A MHS B is said to be *larger than* a MHS A, written $B > A$, if there exists m_0 such that $W_m A_\mathbb{Q} = A_\mathbb{Q}$ for all $m \geq m_0$ and $W_m B_\mathbb{Q} = 0$ for all $m < m_0$.

Finally, let H be a MHS and suppose that $H_\mathbb{Z}$ is free. Define in this case the pth Jacobian of H as:
$$J^p H := \frac{H_\mathbb{C}}{H_\mathbb{Z} + F^p H_\mathbb{C}}.$$

Example. Let X be a compact and connected Riemann surface. Let A be the pure Hodge structure given by the 1st Betti cohomology of X, twisted by 1. That is, $A_\mathbb{Z} = H^1(X, \mathbb{Z})$. Also, $A_\mathbb{C} \cong H^1_{\mathrm{dR}}(X)$, and the filtration is:
$$F^p A_\mathbb{C} = \begin{cases} H^1_{\mathrm{dR}}(X) & p \leq 0, \\ H^0(X, \Omega^1) & p = 1, \\ 0 & p \geq 2. \end{cases}$$

Let's compute then $J^0 \operatorname{Hom}(\mathbb{Z}(-1), H^1(X,\mathbb{Z}))$: as an abelian group, it is $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, H^1(X,\mathbb{Z})) \cong H^1(X,\mathbb{Z})$. The 0th filtered subspace is:

$$F^0 \operatorname{Hom}(\mathbb{C}, H^1_{\mathrm{dR}}(X)) = \{\phi \mid \phi(F^r\mathbb{C}) \subseteq F^r H^1_{\mathrm{dR}}(X), \forall r\},$$

which are the homomorphisms ϕ such that $\phi(\mathbb{C}) \subseteq F^1 H^1_{\mathrm{dR}}(X) = H^0(X,\Omega^1)$. That is:

$$J^0 \operatorname{Hom}(\mathbb{Z}(-1), H^1(X,\mathbb{Z})) \cong J^0(\operatorname{Hom}(\mathbb{Z}, H^1(X,\mathbb{Z})(1)) \cong \frac{H^1_{\mathrm{dR}}(X)}{H^1(X,\mathbb{Z}) + H^0(X,\Omega^1)}. \tag{1}$$

We recover the space which is the target of the Abel-Jacobi map.

Consider the trivial Hodge structure $\mathbb{Z}$. Let H be a Hodge structure of negative pure weight. Then we have:

$$J^0 \operatorname{Hom}(\mathbb{Z}, H) \cong \frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^0 H_{\mathbb{C}}}.$$

Theorem 2.4 (Carlson)**.** *There is a natural isomorphism:*

$$\operatorname{Ext}^1_{MHS}(\mathbb{Z}, H) \cong \frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^0 H_{\mathbb{C}}}.$$

Proof. We just explain the maps. Consider an extension:

$$0 \to H \xrightarrow{i} E \xrightarrow{\pi} \mathbb{Z} \to 0.$$

Choose an element $s^{\mathrm{hol}} \in F^0 E_{\mathbb{C}}$ and an element $s^{\mathrm{int}} \in E_{\mathbb{Z}}$. The difference $s := s^{\mathrm{hol}} - s^{\mathrm{int}}$ lies in $\ker \pi$ and hence can be seen as an element of $H_{\mathbb{C}}$. This gives a well-defined map to the given jacobian, since the indeterminacy in the choices of s^{hol} and s^{int} makes s change by an element of $F^0 H_{\mathbb{C}}$ or $H_{\mathbb{Z}}$, respectively.

We now describe the map in the opposite direction. Suppose given an element $s \in H_{\mathbb{C}}$. Define a MHS E as follows: as an abelian group, $E_{\mathbb{Z}} := H_{\mathbb{Z}} \oplus \mathbb{Z}$. Also $W_m E := W_m H \oplus W_m \mathbb{Z}$. Lastly, define:

$$F^p E_{\mathbb{C}} := \{(h,z) \in H_{\mathbb{C}} \oplus \mathbb{C} \mid z \in F^p\mathbb{C} \text{ , and } h - zs \in F^p H_{\mathbb{C}}\}.$$

One can check that this is well defined and gives the inverse map. $\square$

The map $\mathrm{AJ}_{\mathbb{C}}$ can thus be described as a map with target $\operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^1_B(X,\mathbb{Z})(1))$, which we describe in the next section.

3 The Abel-Jacobi map revisited

Let U be the complement of a finite set $S \subset X$. One can define a cohomology group $H^1(U,\mathbb{Z})$ as cohomology with log-singularities. Its elements can be represented by differential one forms which are meromorphic on X, holomorphic on U and such that locally around each $P \in S$ they can be written as df/f for some $f \in \mathcal{O}_{X,P}$. This group can be given the structure of a mixed Hodge structure, as follows:

$$W_m H^1(U,\mathbb{Z}) = \begin{cases} 0 & m \leq 0, \\ \operatorname{img}\left(H^1(X,\mathbb{Z}) \to H^1(U,\mathbb{Z})\right) & m = 1, \\ H^1(U,\mathbb{Z}) & m \geq 2. \end{cases}$$

and

$$F^p H^1(U,\mathbb{C}) = \begin{cases} H^1(U,\mathbb{C}) & p \leq 0, \\ \operatorname{img}\left(H^0(X,\Omega^1(\log S)) \to H^1(U,\mathbb{C})\right) & p = 1, \\ 0 & p \geq 2. \end{cases}$$

This fits in an exact sequence of mixed Hodge structures:

$$0 \to H^1(X,\mathbb{Z})(1) \to H^1(U,\mathbb{Z})(1) \to \bigoplus_{P \in S} \mathbb{Z}P \stackrel{\deg}{\to} H^2(X,\mathbb{Z})(1) \to 0. \tag{2}$$

From this sequence we can define the Abel-Jacobi map. Let $D \in \operatorname{Div}(X)_0$ be a divisor of degree 0. Let $S := \{P \in X \mid n_P \neq 0\}$ be the *support* of D, and let $U := X \setminus S$. Define also K to be the kernel of the degree map in the sequence of Equation (2). Define a morphism of pure Hodge structures:

$$\phi_D : \mathbb{Z} \to \bigoplus_{P \in S} \mathbb{Z}P,$$

by mapping 1 to $\sum_{P \in S} n_P P$. Since D is of degree 0, the map ϕ_D factors through K. This defines by pull-back a new extension:

$$\begin{tikzcd} 0 \arrow[r] & H^1(X,\mathbb{Z})(1) \arrow[r] \arrow[d, equal] & E_D \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\phi_D"] & 0 \\ 0 \arrow[r] & H^1(X,\mathbb{Z})(1) \arrow[r] & H^1(U,\mathbb{Z})(1) \arrow[r] & K \arrow[r] & 0. \end{tikzcd}$$

The class of the extension E_D can be seen as an element of $\operatorname{Ext}^1(\mathbb{Z}, H^1(X,\mathbb{Z})(1))$, and this gives a map:

$$\mathrm{AJ}_{\mathbb{C}} \colon \operatorname{Div}(X)_0 \to \operatorname{Ext}^1(\mathbb{Z}, H^1(X,\mathbb{Z})(1)) \cong \frac{H^1_{\mathrm{dR}}(X)}{H^1(X,\mathbb{Z}) + H^0(X,\Omega^1)} \simeq \frac{H^0(X,\Omega^1)^\vee}{H^1(X,\mathring{A})},$$

where the isomorphisms follows from Equation (1) and the identification $H^1_{\mathrm{dR}}(X)/H^0(X,\Omega^1)$ with $H^0(X,\Omega^1)^\vee$.

Theorem 3.1. *The two descriptions of the Abel-Jacobi map agree.*

Proof. Write $D = \sum_{P \in S} n_P P$, and set $U = X \setminus S$. Note that:

$$F^0 E_D = \left\{(\eta,\lambda) \in H^0\left(X,\Omega^1(\log S)\right) \times \mathbb{C} \mid \operatorname{res}_P(s) = \lambda n_P\right\}.$$

Let then $s^{\mathrm{hol}} := (\eta_D, 1)$, where $\eta_D \in H^0(X,\Omega^1(\log S))$ is a holomorphic differential on U which is meromorphic on X and has simple log-singularities at $P \in S$, with $\operatorname{res}_P(\eta_D) = n_P$ for all $P \in S$.

Choose a basis $\{\xi_1, \dots, \xi_m\}$ for $H^1(X, \mathbb{Z})$ such that they vanish in a neighborhood $N(D)$ of S. Let $\{\xi^1, \dots, \xi^m\}$ be its dual basis. For each $P \in S$, let B_P be a small ball centered at P and whose closure is contained in $N(D)$. Set $B(D) := \cup_P B_P \subset N(D)$. Define η'_D as follows:

$$\eta'_D := \eta_D - \sum_{i=1}^{m} \langle \eta_D, \xi_i \rangle \xi^i.$$

Then one sees that η'_D is an element of $H^1(U, \mathbb{Z})$ which satisfies $\mathrm{res}_P(\eta'_D) = n_P$ for all $P \in S$, because the ξ_i vanish at S. Then the image of Abel-Jacobi, as defined using extensions of MHS, is such that it sends $\omega \in H^0(X, \Omega^1)$ to:

$$\mathrm{AJ}_{\mathbb{C}}(D)(\omega) = \langle \eta_D - \eta'_D, \omega \rangle = \frac{1}{2\pi i} \int_X (\eta_D - \eta'_D) \wedge \omega.$$

Let Δ be a fundamental domain in the universal covering space of X whose boundary doesn't intersect $|S|$. Let $\pi \colon \Delta \to X$ be the projection. Fix $P_0 \in X$. Identify P_0 with $z_0 := \pi^{-1}(P_0) \in \Delta$.Then we can integrate $\pi^*\omega$ on Δ, and we call $\int_{P_0} \omega$ the holomorphic function on Δ such that

- $d \int_{P_0} \omega = \pi^* \omega$,
- The value of $\int_{P_0} \omega$ at z_0 is 0.

Claim.

$$\frac{1}{2\pi i} \int_X (\eta_D - \eta'_D) \wedge \omega = \sum_P n_P \int_{P_0}^{P} \omega,$$

where $\int_{P_0}^{P} \omega$ means the value of $\int_{P_0} \omega$ at $\mathrm{pr}^{-1}(P)$. Note that the right-hand side is well defined, since D is of degree 0.

Proof. Write $\omega = \sum_i c_i \xi_i + df$, where f is a meromorphic form on X and the c_i are complex constants.

$$\begin{aligned}
\int_X (\eta_D - \eta'_D) \wedge \omega &= \int_X (\eta_D - \eta'_D) \wedge \left(\sum_i c_i \xi_i \right) \quad \text{(Stokes', since } X \text{ is compact)} \\
&= \int_X \eta_D \wedge \left(\sum c_i \xi_i \right) \quad (\langle \eta'_D, \xi_j \rangle = 0, \forall j) \\
&= \int_{X \setminus B(D)} \eta_D \wedge (\omega - df) \quad \text{(the } \xi_i \text{ vanish on } B(D)) \\
&= \int_{X \setminus B(D)} -\eta_D \wedge df \quad \text{(since } \eta_D, \omega \text{ are holomorphic on } X \setminus B(D)) \\
&= \int_{\partial B(D)} f \eta_D \quad \text{(Stokes' Theorem, again)} \\
&= \int_{\partial B(D)} \left(\int_{P_0} \omega \right) \eta_D \quad \text{(on } B(D), \omega = df) \\
&= 2\pi i \sum_{P \in S} (\mathrm{res}_P \, \eta_D) \int_{P_0}^{P} \omega = 2\pi i \sum_{P \in S} n_P \int_{P_0}^{P} \omega.
\end{aligned}$$

□

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4 Exact sequences in cohomology

The purpose of this and the following sections is to generalize all the previous concepts to the setting of algebraic varieties with semistable reduction. The price to pay is a considerably more complicated language.

Let X be a smooth projective variety over K, and let $i\colon Z \hookrightarrow X$ be a closed immersion. Let $j\colon U \hookrightarrow X$ be an open immersion such that X is the disjoint union of $i(Z)$ and $j(U)$. Let $\mathcal{F}$ be a sheaf on the étale site of X. Define $i^!$ to be the right adjoint of i_*. That is, for all sheaves $\mathcal{G}$ on Z, we have:

$$\mathrm{Hom}_Z(\mathcal{G}, i^!\mathcal{F}) = \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

One can actually construct $i^!\mathcal{F}$ as:

$$i^!\mathcal{F} := i^* \ker(\mathcal{F} \to j_*j^*\mathcal{F}).$$

One also checks that $i_*i^!\mathcal{F}$ is the largest subsheaf of $\mathcal{F}$ which is zero outside Z. Since $i^!$ has a left adjoint, it is left-exact. Also, since i_* is exact, it implies that $i^!$ preserves injectives.

Definition 4.1. The group

$$\Gamma(X, i_*i^!\mathcal{F}) = \Gamma(Z, i^!\mathcal{F}) = \ker\left(\mathcal{F}(X) \to \mathcal{F}(U)\right)$$

is called the group of **sections of $\mathcal{F}$ with support on** Z.

The functor

$$\mathcal{F} \mapsto \Gamma(Z, i^!\mathcal{F})$$

is left-exact, and so it makes sense to consider its right-derived functors.

Definition 4.2. The functors

$$H^k_{|Z|}(X, \mathcal{F}) := \mathcal{F} \mapsto R^k\Gamma(Z, i^!\mathcal{F})$$

are called the **(étale) cohomology groups of $\mathcal{F}$ with support on** Z.

Proposition 4.3. *Let $\mathcal{F}$ be a sheaf on the étale site of X. There is a long exact sequence*

$$0 \to (i^!\mathcal{F})(Z) \to F(X) \to F(U) \to \cdots \to H^k_{et}(X, \mathcal{F}) \to H^k_{et}(U, \mathcal{F}) \to H^{k+1}_{|Z|}(X, \mathcal{F}) \to \cdots$$

Proof. Follows from applying the cohomology functor to the exact sequence of sheaves on the étale site of X:

$$0 \to i_*i^*\mathcal{F} \to \mathcal{F} \to j_!j^*\mathcal{F} \to 0.$$

□

Assume now that K is algebraically closed, and that Z is smooth over K. Let c be the codimension of Z in X. That is, each of the connected components of Z is of codimension c inside the corresponding component of X.

Let $\mathcal{F}$ be a locally constant torsion sheaf on X, such that its torsion is coprime to $\mathrm{char}(K)$. As a special case of cohomological purity, (see [Mil80] VI.5.1), we have:

Theorem 4.4. *For every $k \in \mathbb{Z}$ there is a canonical isomorphism*

$$H^k_{|Z|}(X, \mathcal{F}) \simeq H^{k-2c}_{et}(Z, i^*\mathcal{F}(-c)).$$

Proof. Consider the commutative diagram of functors:

$$\begin{tikzcd} \mathbf{Sh}(X_{\mathrm{et}}) \arrow[r, "i^!"] \arrow[dr, "H^0_{|Z|}(X,-)"'] & \mathbf{Sh}(Z_{\mathrm{et}}) \arrow[d, "\Gamma(Z,-)"] \\ & \mathbf{Ab}. \end{tikzcd}$$

Both $i^!$ and $\Gamma(Z, -)$ are left-exact, and $i^!$ preserves injectives. Therefore there exists the Grothendieck spectral sequence:

$$E_2^{r,s} = H^r_{\mathrm{et}}(Z, R^s i^! \mathcal{F}) \implies H^{r+s}_{|Z|}(X, \mathcal{F}).$$

By cohomological purity, $E_2^{r,s} = 0$ unless $s = 2c$. Hence the spectral sequence degenerates at the E_2-term, and:

$$H^k_{|Z|}(X, \mathcal{F}) \cong H^{k-2c}_{\mathrm{et}}(Z, R^{2c} i^! \mathcal{F}) \cong H^{k-2c}_{\mathrm{et}}\left(Z, i^*\mathcal{F}(-c)\right).$$

$\square$

Corollary 4.5. *For $0 \leq k \leq 2c-2$, $H^k_{et}(X, \mathcal{F}) \simeq H^k_{et}(U, \mathcal{F})$. Moreover, if $d = \dim(X)$, there is a long exact sequence:*

$$\begin{aligned} 0 \to H^{2c-1}_{et}(X, \mathcal{F}) \to H^{2c-1}_{et}(U, \mathcal{F}) \to H^{2c}_{|Z|}(X, \mathcal{F}) \xrightarrow{i_*} H^{2c}_{et}(X, \mathcal{F}) \to \cdots \\ \cdots \to H^{2d}_{|Z|}(X, \mathcal{F}) \to H^{2d}_{et}(X, \mathcal{F}) \to H^{2d}_{et}(U, \mathcal{F}) \to 0. \end{aligned} \tag{3}$$

Remark. In the previous corollary we could replace the groups $H^{2c}_{|Z|}(X, \mathcal{F})$ and $H^{2d}_{|Z|}(X, \mathcal{F})$ with $H^0_{\mathrm{et}}(Z, i^*\mathcal{F}(-c))$ and $H^{2(d-c)}_{\mathrm{et}}(Z, i^*\mathcal{F}(-c))$, respectively.

5 The l-adic Abel-Jacobi Map

Let K be a field of characteristic 0, and let l be a prime number. Let X be a smooth projective variety over K. We want to generalize the divisor group to not just cycles of codimension 1.

Definition 5.1. The *cth Chow group* of X, written $\mathrm{CH}^c(X)$ is the group consisting of codimension-c cycles with *rational* coefficients, modulo rational equivalence. That is, if Z_1, Z_2 are two subvarieties of codimension c, we say $Z_1 \sim Z_2$ if there exists a flat family over $\mathbb{P}^1_K$ and contained in $X \times \mathbb{P}^1_K$ such that Z_1 and Z_2 are two of its fibers.

Consider the locally-constant sheaves $\mathcal{F}_n = \mathbb{Z}/l^n\mathbb{Z}(c)$ in the previous section, and take projective limits with respect to n, to get $\mathbb{Z}_l$-valued cohomology. Inverting p we get $\mathbb{Q}_l$-valued cohomology. So by definition:

$$H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l(c)) := \varprojlim_n H^i\left(\overline{X}_{\mathrm{et}}, \mathbb{Z}/l^n\mathbb{Z}(c)\right)(1/p).$$

The Gysin map i_* in Equation 3 induces by restriction to rational cycles the **cycle class map** (see [Mil80, Chapter VI.9]):

$$\mathrm{cl}\colon\ \mathrm{CH}^c(X) \to H^{2c}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_l(c)\right)^{G_K}.$$

Let $\mathrm{CH}^c(X)_0 := \ker \mathrm{cl}$. Given a class $[Z] \in \mathrm{CH}^c(X)_0$, represented by a cycle Z, consider the short exact sequence of G_K-modules:

$$0 \to H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_l(c)\right) \to H^{2c-1}_{\mathrm{et}}\left(\overline{X} \setminus |\overline{Z}|, \mathbb{Q}_l(c)\right) \to H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_l(c)\right)_0 \to 0, \tag{4}$$

where

$$H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_l(c)\right)_0 := \ker\left(H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_l(c)\right) \xrightarrow{i_*} H^{2c}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_l(c)\right)\right)$$

is the kernel of the *Gysin map*.

Consider the map $\alpha\colon \mathbb{Q}_l \mapsto H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_l(c)\right)_0$ which sends

$$1 \mapsto \mathrm{cl}^{\overline{X}}_{\overline{Z}}(\overline{Z}) \in H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_l(c)\right)_0.$$

Pulling back the exact sequence (4) by α we obtain an extension

$$\begin{tikzcd}
0 \arrow[r] & H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_l(c)\right) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \mathbb{Q}_l \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_l(c)\right) \arrow[r] & H^{2c-1}_{\mathrm{et}}\left(\overline{X} \setminus |\overline{Z}|, \mathbb{Q}_l(c)\right) \arrow[r] & H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_l(c)\right)_0 \arrow[r] & 0.
\end{tikzcd} \tag{5}$$

Definition 5.2. The l-**adic étale Abel-Jacobi map** is the map

$$\mathrm{AJ}^{\mathrm{et}}_l\colon\ \mathrm{CH}^c_0(X) \to \mathrm{Ext}^1\left(\mathbb{Q}_l, H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_l(c)\right)\right),$$

which assigns to a class $[Z]$ the class of the extension (5).

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