Algebraic Cycles & p-adic L-functions.

depends on Hodge conjecture.

2 themes
- Construction of rational points on elliptic curves. ← Alg cycles
- alg. cycles on higher-dimensional varieties. ← Alg cycles
  (non-torsion elements in the Griffiths group).

Plan of lectures.

1, 2 (K): Intro to alg. cycles.

3, 4 (H): Main theorems on rational points, Hodge conjecture.

5 (K): Intro to p-adic L-functions.

6, 7 (H): Ideas in proof: p-adic Abel-Jacobi, computing it.

8 (K): Applications to construction of non-torsion elements of the Griffiths gp.

Gross-Zagier: have a deg-4 L-function, which factors w/ sign $= -1$.

$$E(K) \nearrow E(\mathbb{Q}) \qquad E(K) \searrow E(K) \qquad \text{(depending on signs).}$$

In our setting:

$$\text{cycle} \nearrow \text{point on } E \qquad \text{cycle} \searrow \text{nontorsion in Griffiths gp.}$$

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1. Curves:

Let X = smooth complete curve $/\mathbb{C}$.

Define $Z^1(X)$ = free abelian gp on the points of $X = \{\sum_{P \in X(\mathbb{C})} n_P P \;:\; n_P = 0 \text{ a.a.}\}$.

There's the degree map:

$$Z^1(X) \xrightarrow{\deg} \mathbb{Z}$$
$$\sum n_P P \longmapsto \sum n_P$$

which gives an exact sequence:

$$0 \to Z^1(X)_0 \to Z^1(X) \to \mathbb{Z} \to 0$$

The subgroup $Z^1(X)_0$ contains $Z^1(X)_{rat} := \{\operatorname{div}(f) : f \in \mathbb{C}(X)\}$.

$$0 \to Z^1(X)_{rat} \to Z^1(X)_0 \to \underbrace{Z^1(X)_0 / Z^1(X)_{rat}}_{\text{II}} \to 0$$

Actually, we define:

$$CH^1(X) := \frac{Z^1(X)}{Z^1(X)_{rat}} \quad ; \quad CH^1(X)_0 = \frac{Z^1(X)_0}{Z^1(X)_{rat}}.$$

<u>Jacobian variety.</u>

Let $\Omega^1(X)$:= space of holomorphic 1-forms on X.

The Jacobian variety is defined as:

$$J(X) := \frac{\Omega^1(X)^\vee}{H_1(X,\mathbb{Z})} \quad \text{where} \quad H_1(X,\mathbb{Z}) \overset{I}{\hookrightarrow} \Omega^1(X)^\vee$$
$$\gamma \longmapsto \int_\gamma .$$

<u>Rmk</u>:

- I is an injection: if $\gamma \in \ker I$, then $\int_\gamma \omega = 0 \;\; \forall \omega \in \Omega^1(X) = H^{0,1}(X)$.

  But then $\int_\gamma \bar\omega = 0$ as well, so $\int_\gamma \omega = 0 \;\; \forall \omega \in H^1_{dR}(X) \Rightarrow \gamma = 0$.

- The same proof shows that the induced map

  $$H_1(X, \mathbb{R}) \to \Omega^1(X)^\vee \quad \text{is an } \mathbb{R}\text{-linear isomorphism}.$$

  Therefore $H_1(X,\mathbb{Z})$ is a <u>lattice</u> in $\Omega^1(X)^\vee$.

<u>Def</u>: $J(X)$ is a complex torus. In fact, a principally-polarized abelian variety.

<u>Recall</u>: the criterion for a complex torus to be an abelian variety:

<u>Prop</u>: Spz V is a $\mathbb{C}$-vectorspace of dim g, $\Lambda \subseteq V$ a lattice (a $\mathbb{Z}$-submodule of rk $2g$ s.t $\Lambda \otimes \mathbb{R} \cong V$).

Then V/Λ is an abelian variety

$\Leftrightarrow$ V admits a positive definite hermitian form $H: V \times V \to \mathbb{C}$ s.t

$E := \mathrm{Im}\, H$ is integrally-valued on Λ.

In this case, there exists a polarization on V/Λ of degree $\sqrt{\det E}$.

Poincaré duality on $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ is given by:

$$(\omega, \eta) \longmapsto \int_{X(\mathbb{C})} \omega \wedge \eta .$$

Then:

$$J(X) = \frac{H^{1,0}(X)^\vee}{H_1(X,\mathbb{Z})} \overset{PD}{\cong} \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})}$$

?

Define a pairing $H: H^{0,1}(X) \times H^{0,1}(X) \to \mathbb{C}$

$$(\omega, \eta) \longmapsto H(\omega,\eta) := \frac{i}{2}\int \omega \wedge \bar{\eta} = \frac{i}{2}\langle \omega, \bar{\eta}\rangle.$$

Ex: check that H is a positive definite, Hermitian, & $E = \mathrm{Im}\, H$ is $\mathbb{Z}$-valued on $H_1(X,\mathbb{Z})$.
Moreover, $\det E = 1 \Rightarrow$ principal polarization.

Back to cycles: define the Abel-Jacobi map: pick $P_0 \in X(\mathbb{C})$.

$$Z^1(X)_0 \xrightarrow{\Phi} J(X).$$

$$Z = \sum n_P P \longmapsto \Phi\left(\sum n_P P\right) : \omega \mapsto \sum_P n_P \int_{P_0}^{P} \omega \quad \text{, for some paths } P_0 \to P.$$

This is well-defined as an element of $J(X)$.

Since $\deg Z = 0$, this is independent also of P_0.

<u>Theorem</u> (Abel-Jacobi):

i) $\mathrm{Ker}\, \Phi = Z^1(X)_{rat}$.

ii) $\mathrm{Im}\, \Phi = J(X)$.

So Φ induces an isomorphism of groups $CH^1(X)_0 \cong J(X)$.

How to generalize this:

i) Points on a higher-dim variety.

ii) Divisors on a higher-dim variety $\leftarrow$ nicer theory.

<u>Set up</u>: Let X be a smooth, projective variety over $\mathbb{C}$.
Define $Z^1(X) =$ free abelian gp on codimension-1 irreducible subvarieties.

<u>Def</u>: $D \in Z^1(X)$ is said to be <u>rationally-equivalent to 0</u> if $D = \mathrm{div}\, f$, for some rational function f.

The analogue of the degree map on curves is the "cycle-class map": Let $n = \dim X$.
First let D be a smooth <u>irreducible</u> divisor, ($\dim_{\mathbb{C}} D = n-1$)

$$\begin{array}{ccccc} H_{2n-2}(D,\mathbb{Z}) & \cong & \underset{\substack{\uparrow \\ \text{fundamental class}}}{\mathbb{Z}} & \ni & 1 \\ \downarrow & & & & \downarrow \\ H_{2n-2}(X,\mathbb{Z}) & \overset{PD}{\cong} & H^2(X,\mathbb{Z}) & \ni & cl(D) \end{array}$$

For general D, use resolution of singularities, and then extend by linearity.

We obtain the class map:

$$cl : Z^1(X) \to H^2(X, \mathbb{Z})$$

(and if $\dim X = 1$, then $cl = \deg$ $\left(H^2(X,\mathbb{Z}) \simeq \mathbb{Z} \text{ canonically}\right)$.)

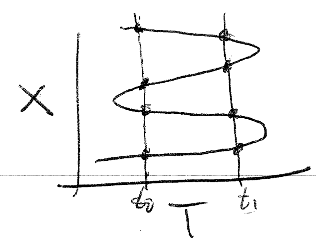
<u>Def</u>: $D \in Z^1(X)$ is said to be <u>homologically-trivial</u> if $cl(D) = 0$.

Two divisors D, D' are said to be <u>homologically equivalent</u> if $D - D'$ is hom. triv.

There are two other equivalence relations that are interesting: <u>algebraic</u> and <u>numerical</u>.

<u>Def</u>: D, D' are <u>algebraically equivalent</u> if $\exists$ a smooth connected variety T, and a divisor Y on $X \times T$ such that $D = Y_{(t_0)}$, $D' = Y_{(t_1)}$, for two points $t_0, t_1 \in T$.

$\left(Y_{(t_i)} := pr_1\left(Y \cap X \times \{t_i\}\right) \right)$.



Ex (i) if $D \underset{alg}{\sim} D'$ and $D' \underset{alg}{\sim} D''$, then $D \underset{alg}{\sim} D''$.

(ii) if $D \underset{alg}{\sim} 0$ and $D' \underset{alg}{\sim} 0$, then $D + D' \underset{alg}{\sim} 0$.

<u>Def</u>: D is said to be <u>numerically equivalent to 0</u> if for every smooth curve $C \subset X$, the intersection number $D \cdot C = 0$.

$$D \cdot C = \deg\left(\mathcal{L}(D)_C\right).$$

<u>Exercise</u>: show:

$$\begin{array}{ccccccccc} Z^1(X)_{rat} & \subseteq & Z^1(X)_{alg} & \subseteq & Z^1(X)_0 & \subseteq & Z^1(X)_{num} & \subseteq & Z^1(X) \\ & & & & \downarrow & & & & \downarrow \\ & & & & 0 & & \longrightarrow & & H^2(X,\mathbb{Z}) \end{array}$$

<u>Def</u>: $CH^1(X) := \dfrac{Z^1(X)}{Z^1(X)_{rat}}$

$$CH^1(X)_0 := \frac{Z^1(X)_0}{Z^1(X)_{rat}}.$$

Have an exact sequence:

$$\begin{array}{c} H^2(X,\mathbb{Z}). \\ \uparrow cl \\ Z^1(X) \end{array}$$

$0 \to CH^1(X)_0 \to CH^1(X)$

Questions:

A) What is the image of $cl : Z^1(X)/Z^1(X)_0 \hookrightarrow H^2(X,\mathbb{Z})$?

B) What can one say about the structure of $CH^1(X)_0$. For instance, is it naturally identified with the complex points of an abelian variety ?

<u>Key tool</u>: Use the relation between divisors and line bundles.

(Weil) divisors = Cartier divisors = global sections of $\mathcal{K}^\times/\mathcal{O}_X^\times$ $\mapsto$ line bundles $\mathcal{L}(D)$.

Principal W-divisors : principal Cartier divisors

$$\mathrm{Div}/\mathrm{PDiv} \cong \text{iso. classes of line bundles.}$$

Let $\mathrm{Pic}(X)$ = gp of iso classes of line bundles. Then $CH^1(X) = \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$.

Recall GAGA: X, X^{an} = complex-analytic space.

$\mathcal{L}, \mathcal{L}^{an}$ (coh-sheaf $\mathcal{F} \to$ coh sheaf $\mathcal{F}^{an}$).

<u>Thm</u> (GAGA): The assignment $\mathcal{F} \mapsto \mathcal{F}^{an}$ gives an equivalence of categories:

$$\mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}).$$

Furthermore, there are natural isos:

$$H^i(X,\mathcal{F}) \cong H^i(X^{an}, \mathcal{F}^{an}).$$

Consider the exponential sequence on X^{an}:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

giving rise to an exact sequence in cohomology:

$$\begin{tikzcd}
H^0(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\mathcal{O}_X^\times) \arrow[r, "c_1"] \arrow[d, equal] & H^2(X,\mathbb{Z}) \arrow[r, "\alpha"] \arrow[d] & H^2(X,\mathcal{O}_X) \arrow[d, equal, "\wr"] \\
 & & \mathrm{Pic}(X) \arrow[d, equal] & H^2(X,\mathbb{C}) \arrow[r, two heads, "\text{projection}"'] \arrow[d, equal] & H^{0,2}(X,\mathbb{C}) \\
 & & CH^1(X) \arrow[uur, "cl"] & H^{02} \oplus H^{11} \oplus H^{20} &
\end{tikzcd}$$

<u>Thm</u>: 1) $c_1 = cl$. That is:

$$c_1(\mathcal{L}(D)) = cl([D]).$$

2) α is the composite

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \xrightarrow{\text{proj}} H^{0,2}(X,\mathbb{C}) \xrightarrow{\simeq} H^2(X,\mathcal{O}_X)$$

This will answer our question:

$$\mathrm{Im}(cl) = \mathrm{Im}(c_1) = \mathrm{Ker}\,\alpha = \{\gamma \in H^2(X,\mathbb{Z}) \text{ whose image in } H^2(X,\mathbb{C}) \text{ is zero on the } (0,2)\text{-component}\} =$$

$$= \{\gamma \in H^2(X,\mathbb{Z}) \text{ whose image in } H^2(X,\mathbb{C}) \text{ lives in } H^{1,1}\} =: \text{"}H^2(X,\mathbb{Z}) \cap H^{1,1}(X)\text{"}$$

↑ b/c $H^{2,0} = \overline{H^{0,2}}$.

note that there can be torsion on $H^2(X,\mathbb{Z})$!

This is called the Lefschetz (1,1) theorem.

About question B:

$$CH^1(X)_0 = \ker cl = \mathrm{Pic}^0(X) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})}$$

The same argument as in the case of curves shows that the induced map:

$H^1(X,\mathbb{R}) \longrightarrow H^{0,1}(X)$ is an $\mathbb{R}$-linear isomorphism. $\Rightarrow H^1(X,\mathbb{Z})$ is a lattice in $H^{0,1}(X)$.

Pick an ample line bundle $\mathcal{L}$ on X, and let $\beta = c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$, which is an (1,1) form.

Let $H : H^{0,1}(X) \times H^{0,1}(X) \longrightarrow \mathbb{C}$

$$(\omega,\eta) \longmapsto \frac{i}{2}\int_{X(\mathbb{C})} \beta^{n-1}\wedge\omega\wedge\bar{\eta}.$$

Exercise: Show that H is pos. def. hermitian, and $E := \mathrm{Im}\,H$ is $\mathbb{Z}$-valued on $H^1(X,\mathbb{Z})$.

Hint: Reduce to the case of curves by using the fact that, for some N, $N\beta^{n-1}$ is Poincaré-dual to the class of an irreducible smooth curve in X.

So this complex torus $\mathrm{Pic}^0(X)$ is an abelian variety, but the choice of polarization depends on the choice of β. In particular, $\mathrm{Pic}^0(X)$ may not be principally polarized.

<u>The A-J map in this case:</u>

$D \in Z^1(X)_{hom}$; $\dim_{\mathbb{C}} D = n-1 \Rightarrow \dim_{\mathbb{R}} D = 2n-2$

Can find a $\mathbb{R}$(2n-1) chain Γ s.t. $\partial\Gamma = D$.

$$\Phi : CH^1(X)_0 \longrightarrow \frac{H^{n,n-1}(X)^\vee}{H_{2n-1}(X,\mathbb{Z})}$$

$$D \longmapsto \int_\Gamma$$

Lemma: Any element in $H^{n,n-1}(X)$ can be represented by a closed form of type $(n,n-1)$ that is well-defined up to a form of type $d\eta$, where η is of type $(n,n-2)$.

This makes Φ well-defined:

$$\Phi(D)([\omega]) = \int_\Gamma \omega \,, \quad \text{and} \quad \int_\Gamma d\eta = \int_{\partial\Gamma} \eta = \int_{\underset{\text{dim } n-1}{D}} \overset{\text{type } (n,n-2)}{\eta} = 0.$$

(well-def. in the quotient, since a change of Γ goes to $H_{2n-1}(X,\mathbb{Z})$).

By Poincaré duality, $\dfrac{H^{n,n-1}(X)^\vee}{H_{2n-1}(X)} = \dfrac{H^{0,1}(X)}{H^1(X,\mathbb{Z})} = \mathrm{Pic}^0(X)$.

Theorem: Via the previous identification, $\Phi =$ the isomorphism constructed earlier.
(Abel-Jacobi).

(see [Voisin], Prop 12.7).

Theorem: $\sim_{alg} = \sim_{hom}$.

This theorem follows from the following:

Thm: There exists a line bundle $\mathcal{P}$ (Poincaré bundle) on $X \times T$, $T := \dfrac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$ such that $\forall t \in T$, $\mathcal{P}|_{X\times\{t\}} \cong \mathcal{L}_t$.

$\underset{\mathcal{L}_t}{\overset{\shortparallel}{t}}$

Pf: $H^2(X\times T) \xleftarrow{\text{Künneth}} H^2(X)\otimes H^0(T) \oplus \underbrace{H^1(X)\otimes H^1(T)}_{\text{interested in this}} \oplus H^0(X)\otimes H^2(T)$.

$T = \dfrac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} \Rightarrow H_1(T,\mathbb{Z}) \simeq H^1(X,\mathbb{Z})$ so get a tautological class in:

$$H^1(X,\mathbb{Z})\otimes H^1(T,\mathbb{Z}) \ni e$$

Then consider the image of e in $H^2(X\times T)$.

Claim: This is a $(1,1)$-class.

Note that $H^1(T,\mathbb{C}) = H_1(T,\mathbb{C})^\vee = H^1(X,\mathbb{C})^\vee$

$\cup$

$H^{1,0}(T)$ = identified with linear forms on $H^1(X,\mathbb{C})$ that vanish on the $(1,0)$-comp.

(and $H^{0,1}(T)$ = forms on $H^1(X,\mathbb{C})$ that vanish on the $(0,1)$-component)

Let $\{\omega_i\}$ be a basis for $H^{1,0}(X)$, $\{\bar\omega_i\}$ a basis for $H^{0,1}(X)$.

Get a basis for $H^1(X, \mathbb{C})$. Let $\{\omega_i^*, \bar\omega_i^*\}$ be the dual basis.

Then $e = \sum_i \omega_i \otimes \bar\omega_i^* + \bar\omega_i \otimes \omega_i^*$ is of $(1,1)$-type.

$\therefore$ e is $= cl(D)$, for some D divisor on $X \times T$.

Fix $t_0 \in T$. Get a map

$$T \longrightarrow T$$
$$t \longmapsto [D_t - D_{t_0}]$$

Idea: analyse the induced map on $H_1(T, \mathbb{Z})$. It is:

$$\gamma \longmapsto P_{1,*}\left(P_2^{-1}(\gamma) \cap D\right)$$

$$H_1(T, \mathbb{Z}) \longrightarrow H_{2n-1}(X, \mathbb{Z}) = H_1(T, \mathbb{Z}).$$

...

$\Rightarrow \sim_{alg} = \sim_{hom}$.

As for numerical equivalence:

<u>Thm</u>: $Z^1(X)_{num} / Z^1(X)_0 = $ torsion $=$ torsion in $H^2(X, \mathbb{Z})$.

<u>The case of points</u>: (instead of divisors).

$$Z^n(X) \xrightarrow{deg} \mathbb{Z}$$

In general, rational equivalence is defined as follows:

$z \in Z^k(X)$ is $z \underset{rat}{\sim} 0$ if $\exists$ a finite collection of subvarieties V_i,
($Z^k(X)$: free abelian gp on cycles of codim k.)
of codimension $k-1$, and a collection of rational functions f_i on V_i s.t

$$z = \sum_i \operatorname{div} f_i.$$

Have $Z^n(X)_{rat} \subseteq Z^n(X)_0 \subseteq Z^n(X)$. Define $CH^n(X)_0 = Z^n(X)_0 / Z^n(X)_{rat}$.

Also, have

$$AJ : CH^n(X)_0 \longrightarrow Alb(X) = \frac{H^{0,1}(X)^\vee}{H_1(X, \mathbb{Z})}$$

Ex: 1) $Alb(X)$ is dual to $Pic^0(X)$. So $Alb(X)$ is an abelian variety.

In this case, it turns out that the kernel of AJ is quite mysterious.

Thm (Mumford): If X is a surface with $P_g(X) > 0$ $(H^{2,0}(X) \neq 0)$, then $\mathrm{Ker}(AJ)$ is not "finite-dimensional".

For example, for each d, could consider $X^{(d)} = X \times \overset{(d)}{\cdots} \times X$

$$\begin{tikzcd} X^{(d)} \times X^{(d)} \arrow[r] \arrow[d] & CH^2(X)_0 \\ S^dX \times S^dX \arrow[ur] & \end{tikzcd}$$

Then if $P_g(X) > 0$, $S^dX \times S^dX \to CH^2(X)_0$ is not surjective, for any d.

Lecture 2

Ref. Fulton "Intersection Theory".

Let X be a smooth projective variety of dim n.

Define $Z^k(X)$:= free ab. gp of closed irreducible subvarieties of codim k.

We already defined the notion of rational equivalence: to say $Z \sim_{rat} 0$ is equivalent to saying that $\exists$ on $X \times \mathbb{P}^1$ a collection of subvarieties W_i of codim k s.t.

$$Z = \sum_i \left(W_i(0) - W_i(\infty) \right).$$ (prove it as an exercise).

Homological equivalence.

Need the cycle class map: $cl : Z^k(X) \to H^{2k}(X, \mathbb{Z})$.

If $z \in Z^k(X)$ is irreducible + smooth, then:

$$\begin{tikzcd} H_{2n-2k}(X) \arrow[r, "PD"] & H^{2k}(X, \mathbb{Z}) \\ H_{2n-2k}(Z) \cong \mathbb{Z} \ni 1 \arrow[u] \arrow[r, mapsto] & cl(Z) \in H^{2k}(X,\mathbb{Z}) \end{tikzcd}$$

(+ resolution of singularities).

Remark: The ~~class of~~ image of $cl(z)$ in $H^{2k}(X, \mathbb{C})$ is easily checked to be the PD of $\gamma_Z \in H^{k,k}(X, \mathbb{C})^\vee$.

$$H_{2n-2k}(X) \longrightarrow (H^{n-k,n-k})^\vee$$

$$[Z] \longmapsto \left[\omega \mapsto \int_{\tilde{Z}} i^* \omega \right] =: \gamma_Z$$

$\Rightarrow$ image of cl $\subseteq$ "$H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X, \mathbb{C})$".

Def: $Z \in Z^k(X)$ is hom. trivial if $cl(Z)=0$.

$Z_1 \sim_{hom} Z_2$ if $Z_1 - Z_2$ is hom. trivial.

Likewise, have the notions of algebraic equivalence and numerical eq.

Algebraic: $Z \sim_{alg} 0$ if $\exists Y \in Z^k(X \times T)$ s.t. $Z = Y_{(t_1)} - Y_{(t_0)}$ some $t_0, t_1 \in T$.

Numerical: $Z \sim_{num} 0$ if $\forall Z' \in Z^{n-k}(X)$ that intersect Z properly, $Z' \cdot Z = 0$.

Again, have (exercise):

$$Z^k(X)_{rat} \subseteq Z^k(X)_{alg} \subseteq Z^k(X)_0 \subseteq Z^k(X)_{num} \subseteq Z^k(X).$$

Def: $CH^k(X) = Z^k(X) / Z^k(X)_{rat}$

$CH^k(X)_0 = Z^k(X)_0 / Z^k(X)_{rat}$

Just as before, have an exact seq. $0 \to CH^k(X)_0 \to CH^k(X) \to Z^k(X)/Z^k(X)_0 \to 0$

$$\frac{Z^k(X)}{Z^k(X)_0} \xrightarrow{cl} H^{2k}(X,\mathbb{Z}) \cap H^{k,k}(X).$$

Questions

A) What is the image of cl?

B) What can be said about $CH^k(X)_0$? For example, is it the $\mathbb{C}$-points of an abelian variety?

Original Hodge conjecture: $Im(cl) = H^{2k}(X,\mathbb{Z}) \cap H^{k,k}(X)$.

This is false in general (there are counterexamples).

Hodge Conjecture: $Im(cl) \otimes \mathbb{Q} = H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$.

- What is known about the Hodge conj: 1) trivially true for $k=0,n$.

2) True for $k=1$ (even the original conjecture holds) by Lefschetz (1,1) thm.

3) If the conjecture is true for (k,k)-classes on X, $k \leq n$, then it is true for $(n-k,n-k)$-classes. (Hard Lefschetz Thm: $H^{2k}(X,\mathbb{C}) \to H^{2n-2k}(X,\mathbb{C})$
$\omega \longmapsto \underbrace{\omega\wedge\omega\wedge\cdots\wedge\omega}_{(n-2k)\text{ times}}$)

So since we know it for (1,1) classes, then we know it

Even in the case of $\dim X = 4$, we don't know the conjecture in general.

About question B:

Claim: $Z^k(X)_0 / Z^k(X)_{alg}$ is at most countable. In fact, $Z^k(X) / Z^k(X)_{alg}$ is at most countable.

Chow varieties: Fix an ample line bundle on X.

$$\mathrm{Chow}_{d,d'}(X) : S \mapsto \{ \text{families of effective algebraic cycles of dim } d' \text{ and degree } d, \text{ on } X \times S / S, \text{ flat over } S.$$

~~Theorem~~: The functor $\mathrm{Chow}_{d,d'}(X)$ is represented by a projective variety, the Chow variety, called $\mathrm{Chow}_{d,d'}(X)$ (so it has finitely many components).

This then implies that $Z^k(X)/Z^k(X)_{alg}$ is countable (run over all d, all components)

This also implies that any putative A-J map that detects the nontriviality of $Z^k(X)_0 / Z^k(X)_{alg}$ could not possibly be surjective.

Def: The Griffiths Intermediate Jacobian is defined as follows:

$$F^k H^{2k-1}(X, \mathbb{C}) := \bigoplus_{j \geq k} H^{j, 2k-1-j}(X, \mathbb{C}) \qquad (\text{so } H^{2k-1}(X, \mathbb{C}) = F^k \oplus \overline{F^k})$$

$$J^k(X) := \frac{F_k(H^{2k-1}(X,\mathbb{C}))}{\mathrm{Im}(H^{2k-1}(X,\mathbb{Z}))} = \frac{H^{2k-1}(X,\mathbb{C})}{F^k H^{2k-1}(X,\mathbb{C}) + \mathrm{im}\, H^{2k-1}(X,\mathbb{Z})}$$

By PD, can write it also as:

$$J^k(X) = \frac{\bigoplus_{j \geq n-k+1} H^{j, 2n-2k+1-j}(X,\mathbb{C})^\vee}{\mathrm{im}\, H_{2n-2k+1}(X, \mathbb{Z})} = \frac{(F^{n-k+1} H^{2n-2k+1}(X,\mathbb{C}))^\vee}{\mathrm{im}\, H_{2n-2k+1}(X,\mathbb{Z})}.$$

Rmk: The same argument as before shows that $J^k(X)$ is a cx torus. However, in general not an abelian variety.

Now we define A-J:

$$Z^k(X)_0 \to J^k(X).$$

Lemma: [Voisin, Prop 7.5] Let $F^r A^k(X)$ be the space of smooth cx-valued differential forms, which are sums of forms of type $(r', k-r')$, $r' \geq r$, at every point of X.

Then $F^p(H^k(X,\mathbb{C})) = \dfrac{\ker(d: F^p A^k(X) \to F^p A^{k+1}(X))}{\mathrm{Im}(d: F^p A^{k-1}(X) \to F^p A^k(X))}$

Let $[\omega] \in F^{n-k+1} H^{2n-2k+1}(X, \mathbb{C})$. Then $[\omega]$ is represented by

$$\omega \in \frac{\ker(d: F^{n-k+1} A^{2n-2k+1} \to \cdots)}{\mathrm{Im}(d: F^{n-k+1} A^{2n-2k} \to \cdots)}$$

Define:

$\Phi(Z)([\omega]) := \int_\Gamma \omega$, where $\partial\Gamma = Z$ and as before can check that this is well-defined.

<u>Thm (Griffiths)</u>: Let S be a complex variety, not necessarily complete. Let $Y \in Z^k(X \times S / S)$ be a flat family of cycles of codim k on X, parametrized by S.

Fix $s_0 \in S$. Then $Y_s - Y_{s_0} \in Z^k(X)_0$, and:

$$\begin{array}{l} s \mapsto \Phi^k_X(Y_s - Y_{s_0}) \in J^k(X) \\ \cap \\ S \end{array}$$

is a <u>holomorphic</u> map.

<u>Corollary</u>: If $Z \in Z^k(X)_{rat}$, then $\Phi^k_X(Z) = 0$.

(b/c there are no nontrivial maps $\mathbb{P}^1 \to T$ = a complex torus).

Therefore the map Φ^k_X factors through $CH^k(X)_0$.

<u>Obvious question</u>: what can one say about $\ker$, Im of this map.

<u>Auxiliary construction of cycles</u>:

Let X, X' be two smooth projective varieties of dims n and n', resp.

Let $Y \in Z^\ell(X \times X')$. Then have a map $\rho_Y : Z^k(X) \to Z^{k'}(X')$

by $\rho_Y(Z) = pr_{2,*}(pr_1^*(Z) \cdot Y)$ $\quad (k' = k + \ell - n)$.

$$\begin{tikzcd} & X \times X' \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & X' \end{tikzcd}$$

Can check: $\rho_Y|_{Z^k(X)_0} \subseteq Z^{k'}(X')_0$ and also it induces a map

$\rho_Y : CH^k(X)_0 \to CH^{k'}(X')_0$.

$$
\begin{array}{ccc}
CH^k(X)_0 & \xrightarrow{\rho_Y} & CH^{k'}(X')_0 \\
\Phi_X^k \downarrow & ? & \downarrow \Phi_{X'}^{k'} \\
J^k(X) & \dashrightarrow & J^{k'}(X')
\end{array} \qquad (*)
$$

Want to get a map

$$\tau_Y : H^{2n'-2k'+1}(X', \mathbb{C}) \longrightarrow H^{2n-2k+1}(X, \mathbb{C}).$$

$$\tau_Y(\omega) := pr_{1,*}\left(pr_2^*(\omega) \wedge cl(Y) \right)$$

Can check that τ_Y^t respects the corresponding lattices, thus giving a map $\tilde{\tau}_Y : J^k(X) \to J^{k'}(X')$.

(One can also check that τ_Y respects the filtrations:

$$\tau_Y : F^{n'-k'+1} H^{2n'-2k'+1}(X', \mathbb{C}) \longrightarrow F^{n-k+1} H^{2n-2k+1}(X, \mathbb{C}).\Big)$$

<u>Thm</u>: The diagram above $(*)$ commutes:

$$\Phi_{X'}^{k'} \circ \rho_Y = \tilde{\tau}_Y \circ \Phi_X^k.$$

<u>Prop</u>: $\Phi_X^k(CH^k(X)_{alg})$ is a subtorus of $J^k(X) = \dfrac{(H^{n-k+1,n-k})^\vee \oplus (H^{n-k+2,n-k-1})^\vee \oplus \cdots}{H_{2n-2k+1}(X, \mathbb{Z})}$

whose tangent space is contained inside $H^{n-k+1,n-k}(X)^\vee$.

<u>Rmk</u>: By what has been said so far, if one of the pieces $H^{n-k+i+1,n-k-i}$ is $\neq 0$, then AJ <u>cannot</u> be surjective, since the quotient alg/rat is countable.

<u>Pf</u> Let $Z \in Z^k(X)_{alg} \Rightarrow$ can find a smooth variety T, and two points $t_0, t_1 \in T$, and $Y \in CH^k(X \times T)$ s.t. $Z = Y_{t_1} - Y_{t_0}$.

May assume WLOG that Y is effective. We do first the case of T a <u>curve</u>, which is also <u>complete</u>.

Have the map

$$
\begin{array}{ccc}
CH^1(T)_0 & \longrightarrow & CH^k(X)_0 \\
\downarrow \; t_1 - t_0 & \longmapsto & Y_{t_1} - Y_{t_0} = Z \; \downarrow \\
J^1(T) & \longrightarrow & J^k(X)
\end{array}
\qquad \Rightarrow \Phi_X^k(Z) \in \operatorname{Im} \tilde{\tau}_Y : J^1(T) \to J^k(X).
$$

(cont'd)
For reasons of type, the image of $J'(T)$ has its tangent space contained in
$H^{n-k+1,n-k}(X)^\vee$.

In the general case, T, $t_0, t_1 \in T$, can join t_0 and t_1 by a smooth curve.
This reduces to the case of T being a smooth (possibly non-complete) curve.

This map: $T \longrightarrow$ Chow variety $\Rightarrow$ can complete the family and reduce to the previous case. ∎

Def: The "maximal abelian subvariety" of $J^k(X)$ is defined to be the largest subtorus of $J^k(X)$ whose tangent space is contained in $H^{n-k+1,n-k}(X)^\vee$. It is called $J^k_a(X)$.

Ex: 1) Show that $J^k_a(X)$ is "naturally" an abelian variety.
2) Show that if the Hodge conjecture is true then $J^k_a(X) = \Phi^k(CH^k(X)_{alg})$.

Now we have:

$$\begin{array}{ccc} CH^k(X)_{alg} & \subseteq & CH^k(X)_0 \\ \downarrow & & \downarrow \\ J^k_a(X) & \hookrightarrow & J^k(X) \end{array}$$

$$Z^k(X)_{alg} \subseteq Z^k(X)_0 \subseteq Z^k(X)_{num}.$$

Std conjectures $\Rightarrow$ $Z^k(X)_{num}/Z^k(X)_{hom}$ is torsion (we said it in the case of curves).

Griffiths: The k^{th} "Griffiths' group" is defined:

$$\mathrm{Griff}^k(X) := Z^k(X)_{hom}/Z^k(X)_{alg}.$$

He proved:
For the general hypersurface X of degree 5 in $\mathbb{P}^4$,
$\mathrm{Griff}^2(X)$ is not torsion.

Strategy: for general X, $J^2_a(X) = 0$. Then just need to show that AJ is nontorsion.
He shows: for any two lines $\ell_1, \ell_2 \subseteq X$, $\ell_1 - \ell_2$ is hom trivial,

Clemens: showed that, in fact, $\mathrm{Griff}^2(X)$ is not finitely generated (for general X).

Ceresa: if C is a generic curve of genus ≥ 3, $X = \mathrm{Jac}(C)$, &

$$C \in Z^{g-1}(X). \text{ Then } C - [-1]^* C \in \mathrm{Griff}^{g-1}(X) \text{ is}$$

an element of infinite order.

Bruno Harris: If C is the Fermat quartic, $x^4 + y^4 + z^4 = 0$, then $X = J(C)$ is 3-dim,
Showed that the Ceresa cycle $C - [-1]^* C$ is ~~nontorsion~~ nontrivial in $\mathrm{Griff}^2(X)$.

Spencer Bloch: Showed that B. Harris' example is nontorsion in $\mathrm{Griff}^2(X)$.
He does this by using the p-adic Abel-Jacobi map.

Thm (Nori): For $k \geq 3$, and $n > k$, $\exists$ varieties X of dim n, with cycle $Z \in Z^k(X)_{hom}$,
such that $[Z]$ in $\mathrm{Griff}^k(X)$ is nontorsion, but $\Phi_X^k(Z) = 0$.

Over # fields

Suppose that X is defined over a number field K.

Conj (Beilinson - Bloch).
1) $CH^*(X)_0$ is a finitely-generated abelian group.
2) Φ_X^k is injective on $CH^*(X)_0$, up to torsion.
3) $\mathrm{rk}\, CH^k(X)_0 = \mathrm{ord}_{s=k} L(H_{et}^{2k-1}(X), s)$.

Lecture 3: Hodge Conjecture.

Let V be a variety over $\mathbb{C}$. Consider the map:

$$CH^j(V)\otimes\mathbb{Q} \longrightarrow H^{j,j}_{dR}(V)\cap H^{2j}(V(\mathbb{C}),\mathbb{Q})$$

Conj: this is surjective.

The setup: K a quadratic imag field. For simplicity, $h(K)=1$, $-D=\mathrm{disc}(K)$ odd, $\mathcal{O}_K^\times=\pm1$.

(in particular, $D\in\{7,11,19,43,67,163\}$).

Let $\varepsilon_D:(\mathbb{Z}/D\mathbb{Z})^\times\to\pm1$ the quadratic character attached to K.

Let $A=A_D$ be an elliptic curve with CM by $\mathcal{O}_K$, defined over $\mathbb{Q}$, with conductor of $A=D^2$.

By Deuring, we know $L(A,s)=L(\psi_A,s)$,

where $\psi_A: I_K(\sqrt{-D})\to\mathbb{C}^\times$ (Hecke character), defined as:

$$\psi_A((a))=\varepsilon_D(a \bmod \sqrt{-D})\cdot a \quad \text{(by assumption, all ideals of } \mathcal{O}_K \text{ are principal!)}$$

Let $\Theta_{\psi_A}:=\frac12\sum_{a\in\mathcal{O}_K}\psi_A(a)\,q^{a\bar a}\in S_2(\Gamma_0(D^2))$, the weight-2 cusp form attached to A.

Fix an integer $r\geq 0$, and put $\psi=\psi_A^{r+1}$, consider then:

$$\Theta_\psi\in\begin{cases} S_{r+2}(\Gamma_0(D^2)) & \text{if } r \text{ is even (cond } \psi=\sqrt{-D})\\ S_{r+2}(\Gamma_0(D),\varepsilon_D) & \text{if } r \text{ is odd (cond } \psi=1). \leftarrow \text{assume this from now on.}\end{cases}$$

Geometric interpretation of Θ_ψ.

Let $\Gamma=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}) \mid D|c, \text{ and } \varepsilon_D(a)=1\}$. ← a is a square modulo D.

Let C_Γ be the associated modular curve, and Y_Γ the open modular curve.

So $Y_\Gamma(\mathbb{C})=\Gamma\backslash\mathcal{H}$.

Moduli interpretation:

$Y_\Gamma(F)=$ isomorphism classes of triples (E,Z,t) with:

- E ~~an~~ e.c $/F$.
- Z a sgp of order D $/F$.
- t an orbit in $Z\setminus\{0\}$ for the action of $\ker(\varepsilon_D)$ $/F$.

Note that $Y_\Gamma(\mathbb{C})=Y_0(D)(\mathbb{C})$, b/c $\langle\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\Gamma\rangle=\Gamma_0(D)$

Let $\mathcal{E} \to Y_\Gamma$ ~~be~~ the universal elliptic curve over Y_Γ, and let
open modular curve.

$W_r^\circ = \mathcal{E} \times_{Y_\Gamma} \mathcal{E} \times \cdots \times \mathcal{E}$ (r times), $\dim W_r^\circ = r+1$.

modular curve.

It has a smooth compactification, called the r^{th} Kuga-Sato variety of C_Γ, called W_r.

$$W_r^\circ(\mathbb{C}) = (\mathbb{Z}^{2r} \rtimes \Gamma) \backslash \mathbb{C}^r \times \mathcal{H} \quad \text{, where}$$

$$(m_1, n_1, \ldots, m_r, n_r)\ (w_1, \ldots, w_r, \tau) = (w_1 + m_1 + n_1\tau, \ldots, w_r + m_r + n_r\tau, \tau).$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (w_1, \ldots, w_r, \tau) = \left(\frac{w_1}{c\tau+d}, \ldots, \frac{w_r}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right).$$

If $f \in S_{r+2}(\Gamma_0(D), \varepsilon_D)$, then we can associate to it a differential:

$$\omega_f = (2\pi i)^{r+1} f(z)\, dw_1 \cdots dw_r\, dz \in \Omega^{r+1}(W_r^\circ).$$

The cuspidality of $f \Rightarrow \omega_f$ extends to $\Omega^{r+1}(W_r)$.

The q-expansion principle $\Rightarrow$ if f has Fourier coeffs in some field F, then ω_f is in $\Omega^{r+1}(W_r/F)$. (that's why we need the $(2\pi i)^{r+1}$).

Therefore $\omega_{\Theta_\psi} \in \Omega^{r+1}(W_r/\mathbb{Q})$.

<u>The de Rham cohomology of A.</u>

Recall the Hodge filtration:

$$0 \to \Omega^1(A/\mathbb{C}) \to H^1_{dR}(A/\mathbb{C}) \to H^1(A, \mathcal{O}_A) \to 0.$$

For any field F define:

$$H^1_{dR}(A/F) = \frac{\{\text{differentials on } A/F \text{ of the } 2^{nd} \text{ kind}\}}{(\text{exact differentials})\ dF(A)}$$

Then get

$$0 \to \Omega^1(A/F) \to H^1_{dR}(A/F) \to H^1(A, \mathcal{O}_A) \to 0.$$

Now assume $K \subseteq F$ (K is the CM-field of E). Then have an algebraic decomposition:

$$H^1_{dR}(A/F) = H^{10}_{dR}(A/F) \oplus H^{01}_{dR}(A/F).$$

This is done by exploiting the action of $\mathcal{O}_K$ on $H^1_{dR}(A/F)$.

For $\lambda \in \mathcal{O}_K$, let $[\lambda] \in \mathrm{End}_K(A)$. Fixing $K \subset F$, such that $[\lambda]^* \omega = \lambda \omega$, for all $\omega \in \Omega^1(A/F)$, $\forall \lambda \in \mathcal{O}_K$.

Then, for all $\eta \in H^1(A, \mathcal{O}_A)$, have $[\lambda]^* \eta = \bar{\lambda} \eta$.

Define $H^{01}_{dR}(A/F) = \{ \eta \in H^1_{dR}(A/F) \mid [\lambda]^* \eta = \bar{\lambda} \eta \} = \left([\sqrt{-D}] - \sqrt{-D} \right) H^1_{dR}(A/F)$.

<u>Conjecture</u> (Katz): Let E be any e-c def. over $\bar{\mathbb{Q}}$. Then

$$\dim_{\bar{\mathbb{Q}}} \left(H^1_{dR}(E/\bar{\mathbb{Q}}) \cap H^{01}_{dR}(E/\mathbb{C}) \right) = \begin{cases} 1 & \text{if } E \text{ has CM} \\ 0 & \text{otherwise.} \end{cases}$$

Given $\omega_A \in \Omega^1(A/K)$, let $\eta_A \in H^{01}_{dR}(A/K)$ be the unique class satisfying:

$$\langle \omega_A, \eta_A \rangle = 1. \qquad \left(\langle \omega, \eta \rangle := \frac{1}{2\pi i} \int_{A(\mathbb{C})} \omega \wedge \eta \right).$$

Consider then:

$$[\omega_{\theta_\psi} \wedge \eta_A^{r+1}] \in H^{2r+2}_{dR}(W_r \times A^{r+1}/K).$$

<u>Claim</u>: $\pi_{Hodge} := [\omega_{\theta_\psi} \wedge \eta_A^{r+1}] \in H^{r+1,r+1}_{dR}(W_r \times A^{r+1}) \cap H^{2r+2}_B(W_r \times A^{r+1}, K)$

i.e., $\omega_{\theta_\psi} \wedge \eta_A^{r+1}$ is basically a Hodge cycle.

<u>Q</u>: What can we say about the Hodge conjecture?

$\exists ?\ \Pi \in CH^{r+1}(W_r \times A^{r+1})$ s.t $cl(\Pi) = \pi_{Hodge}$?.

<u>Example</u>: in the case $D = 7$, $r = 1$, W_1 is a K3 surface, with Picard rank 20.

<u>Shioda-Inose</u>: $\exists$ an involution on W, say ι, s.t $W/\iota \simeq Km(A \times A) = A \times A / \{\pm I\}$.

$$\begin{tikzcd} W \arrow[dr] & & A \times A \arrow[dl] \\ & W/\iota & \end{tikzcd}$$

Let $\Pi := W \times_{W/\iota} (A \times A)$. Then $cl(\Pi) = \pi_{Hodge}$.

<u>Elkies-Kumar</u>: showed that Π is defined over $\mathbb{Q}$.

<u>Goal</u>: A numerical experiment to "test" for the presence of Π.

Let $\Omega_D \in \mathbb{C}^\times$ satisfy $\omega_A = \Omega_D(2\pi i\, dz)$.

Let Λ_ω be the period lattice of ω_A.

For any $\tau \in \mathcal{H} \cap K$, define:

$$J_{D,r}(\tau) := \Omega_D^{-r}\frac{(2\pi i)^{r+1}}{(\tau-\bar{\tau})^r}\int_{i\infty}^{\tau}(z-\bar{\tau})^r\,\theta_{\psi_A^{r+1}}(z)\,dz \in \mathbb{C}/\Lambda_\omega$$

It depends only on $\tau \in \Gamma\backslash\mathcal{H}$. Let then $P_{D,r}(\tau)$ be the image on $A_D(\mathbb{C}) = \mathbb{C}/\Lambda_\omega$.

Denote also $J_{D,r} := J_{D,r}\left(\frac{D+\sqrt{-D}}{2D}\right)$, $P_{D,r} := P_{D,r}\left(\frac{D+\sqrt{-D}}{2D}\right)$.

<u>Theorem A</u>: Assume the Hodge conjecture for $\omega_{\theta_{\psi_A^{r+1}}} \wedge \eta_A^{r+1}$. Then

$$P_{D,r}(\tau) \in A_D(K^{ab}) \otimes \mathbb{Q}.$$

<u>Theorem B</u>: Assume the Hodge conjecture for $\omega_{\theta_{\psi_A^{r+1}}} \wedge \eta_A^{r+1}$. Then:

$P_{D,r} \in A(K)\otimes\mathbb{Q}$, and $\exists\, P_D \in A_D(K)\otimes\mathbb{Q}$, depending on K but not r, s.t.:

$$P_{D,r} = m_{D,r}P_D, \quad \text{where} \quad m_{D,r}^2 = \frac{2r!\,(2\pi\sqrt{D})^r}{\Omega_D^{2r+1}} L(\psi_A^{2r+1}, r+1).$$

<u>Sketch of the proof of theorems A+B.</u>

<u>Step 1</u>: The Hodge conjecture and modular parametrizations.

Hodge Conj $\Rightarrow \Pi \in CH^{r+1}(W_r \times A^{r+1}) = CH^{r+1}(\underbrace{(W_r\times A^r)}_{X_r = X_{D,r}}\times A) = CH^{r+1}(X_r \times A)$.

Therefore Π induces a map:

$$\Phi : CH^{r+1}(X_r)_0 \longrightarrow CH^1(A)_0 \cong A$$

This is defined over any field F over which Π is defined.

Get also:

$$\Pi_{dR} : H^{r+1}_{dR}(A^{r+1}) \to H^{r+1}_{dR}(W_r), \quad \text{and} \quad \Pi_{dR}(\omega_A^{r+1}) = \omega_{\theta_{\psi_A^{r+1}}}.$$

For each $\sigma \in Gal(F/\mathbb{Q})$, get Π_{dR}^σ, and can average: replace Π by $\frac{1}{\#Gal(F/\mathbb{Q})}\sum \Pi^\sigma$

Φ respects fields of definition, and gives:

Step 2: Generalized Heegner cycles.

We need explicit elements in $CH^{r+1}(X_r)_0(K^{ab})$.

Let $\varphi : A \to A'$ be any isogeny defined over some ring class field H_φ.
s.t $D \nmid \deg\varphi$

$(\mathrm{Graph}\,\varphi)^r \subseteq (A\times A')^r = (A')^r \times A^r$

The triple $(A, A[\sqrt{-D}], t) \in Y_r(K)$ (fix it), induces via φ a level structure on A', and gives an embedding $\subset \mathcal{E}^r \times A^r \subseteq X_r$.

We need to project it to make it nullhomologous. Call the resulting $\Delta_\varphi = \epsilon_{Sym}(\mathrm{Graph}\,\varphi)^r$.

Then $\Delta_\varphi \in CH^{r+1}(X_r)_0(H_\varphi)$.

Fact: The collection $\{\Delta_\varphi\}_\varphi$ generates an infinite rank subgroup of $CH^{r+1}(X_r)_0(K)$

Def: The points $P_\varphi := \Phi(\Delta_\varphi) \in A(K^{ab})$ are called Chow-Heegner points in $A(K^{ab})$.

Step 3: Use the complex Abel-Jacobi map.

Recall: $AJ_A : CH^1(A)_0(\mathbb{C}) \longrightarrow \dfrac{\Omega^1(A/\mathbb{C})}{H_1(A(\mathbb{C}),\mathbb{Z})}$

$AJ_{X_r} : CH^{r+1}(X_r)_0(\mathbb{C}) \longrightarrow \mathrm{Fil}^{r+1} H^{2r+1}_{dR}(X_r/\mathbb{C})^\vee / \mathrm{im}\,H_{2r+1}(X_r(\mathbb{C}),\mathbb{Z})$.

Given $\varphi : A \to A'$, let $\omega' \in \Omega^1(A')$ be s.t $\varphi^*\omega' = \omega_A$. The period lattice is

$\Lambda_{\omega'} \approx \mathbb{Z} + \mathbb{Z}\tau'$

Claim: $J_{D,r}(\tau') = AJ_A(P_\varphi)(\omega_A)$

Proof (sketch):

$AJ_A(P_\varphi)(\omega_A) = AJ_A(\Phi\Delta_\varphi)(\omega_A) = AJ_{X_r}(\Delta_\varphi)(\Phi^*_{dR}(\omega_A))$. $(*)$

Recall: $\Phi^*_{dR}(\omega_A^{r+1}) = \omega_{\psi_A^{r+1}} \Rightarrow \Phi^*_{dR}(\omega_A) = \omega_{\psi_A^{r+1}} \wedge \eta_A^r$.

So $(*) = AJ_{X_r}(\Delta_\varphi)(\omega_{\psi_A^{r+1}} \wedge \eta_A^r)$

Just need to evaluate this, which will be done later.

At the end, get ... $J_{D,r}(\tau')$.

Prove of Theorem B.

Need p-adic methods.

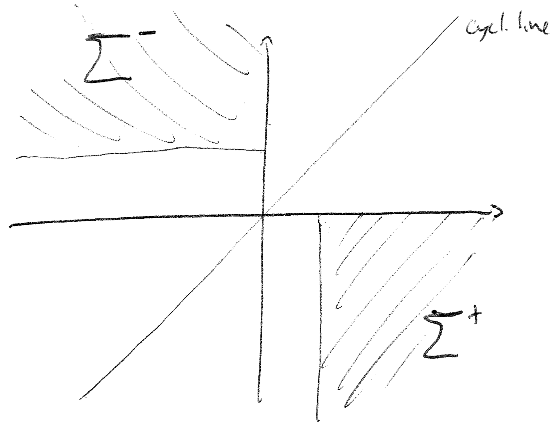
Consider the Katz's p-adic L-function

Recall ψ_A, the fixed Hecke character of conductor $\sqrt{-D}$. Can also consider ψ_A^* by $\psi_A^*(\alpha) = \psi_A(\bar{\alpha})$.

Consider $\Sigma := \{ \psi_A^{\ell_1} (\psi_A^*)^{\ell_2} \mid \ell_1, \ell_2 \in \mathbb{Z} \} \simeq \mathbb{Z} \times \mathbb{Z}$.

~~Then $\Sigma = \Sigma^+ \cup \dots$~~ $\Sigma^+ = \{ \psi_A^{\ell_1} (\psi_A^*)^{\ell_2} \mid \ell_1 \geq 1,\ \ell_2 \leq 0 \}$,

$\Sigma^- = (\Sigma^+)^*$



$\Sigma \subset \left(\mathbb{Z}/_{p-1} \times \mathbb{Z}_p \right)^2$ (p odd, p split in $\mathcal{O}_K$).

Let $\bar{\Sigma}$ be the completion of Σ wrt the p-adic metric.

Theorem: There exists a p-adic period Ω_p, and a continuous p-adic function $L_p(\eta)$, for $\eta \in \bar{\Sigma}$, satisfying: (a choice of $\mathfrak{p} \mid p$ is involved in this definition).

$$(\forall \eta \in \Sigma^+) \qquad \frac{L_p(\eta)}{\Omega_p^{\ell_1 - \ell_2}} = (\sqrt{-D})^{\ell_1} (\ell_1 - 1)! \left(1 - \frac{\eta(\mathfrak{p})}{p}\right)\left(1 - \frac{\eta(\bar{\mathfrak{p}})}{p^{\ell_1 + \ell_2}}\right) \frac{L(\eta^{-1}, 0)}{\Omega^{\ell_1 - \ell_2}}.$$

(since Σ^+ is dense in $\bar{\Sigma}$, this gives uniqueness of $L_p(\eta)$).

Basic Reference: de Shalit's book "Iwasawa Theory of elliptic curves with CM".

Main result: Consider $\omega_{\theta_{\psi_A}^{r+1}} \wedge \omega_A^j \eta_A^{r-j} \in \mathrm{Fil}^{r+1} H_{dR}^{2r+1}(X_r)$, $0 \leq j \leq r$.

$$AJ_{X_r}(\Delta_1)(\omega_{\psi_A^{r+1}} \wedge \omega_A^j \eta_A^{r-j}) \doteq L_p(\psi_A^{-j} (\psi_A^*)^{1+j}) / \Omega_p^{-1-2j} \cdot L_p(\psi_A^{1+(r-j)} (\psi_A^*)^{-(r-j)}) / \Omega_p^{1+2(r-j)}$$

$\doteq$: up to $K^\times$, independent of r.

outside the range of interpolation

inside the range of interpolation.

p-adic AJ: $CH^{r+1}(X_r)_0(F) \to \mathrm{Fil}^{r+1} H_{dR}^{2r+1}(X_r/F)^\vee$.

Ex: For $j=0$, find:

$$AJ_{X_r}(\Delta_1)(\omega_{\psi_A^{r+1}} \wedge \eta_A^r)^2 \sim \frac{L_p(\psi_A^*)}{\Omega_p^{-1}} \cdot \frac{L(\psi_A^{r+1}, r+1)}{\Omega^{1+2r}} \in K.$$

In fact, $AJ_{X_r}(\Delta_1)(\omega_{\psi_A^{r+1}}\wedge\eta_A^r)^2 = AJ_A(P_1)(\omega_A)^2 = \log_{\omega_A}(P_1)^2$.

The Hodge conjecture $\Rightarrow$ $L_p(\psi_A^*) = \Omega_p^{-1}\cdot\log_{\omega_A}(P_1)^2$.

This formula was already computed by K. Rubin, by other methods. This implies that this identity holds without assuming the Hodge conjecture.

Get a p-adic formula:

$$AJ(\Delta_1)(\omega_{\psi_A^{r+1}}\wedge\eta_A^r)^2 = \log^2_\omega(P)\cdot m_{D,r}^2 \qquad (P\in A(K)).$$

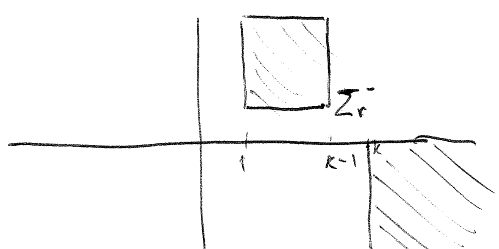
Lecture 4

Sketch (of proof of main result). Assume $\ell_1+\ell_2$ odd.

Uses a p-adic Rankin L-function. Let $f\in S_{r+2}(\Gamma_0(D),\varepsilon_D)$, write $k=r+2$.

Define $\Sigma_r^+ := \{\psi_A^{\ell_1}(\psi_A^*)^{\ell_2} \mid \ell_1\geq k,\ \ell_2\leq 0\}$

$\Sigma_r^- := \{\psi_A^{\ell_1}(\psi_A^*)^{\ell_2} \mid 1\leq \ell_1,\ell_2\leq k-1\}$.



$$\frac{L_p(f,\eta)}{\Omega_p^{2(\ell_1-\ell_2)}} = (\text{Euler factor})\times\frac{L(f/K,\eta^{-1},0)}{\Omega^{2(\ell_1-\ell_2)}} \quad \forall\eta\in\Sigma_r^+ \qquad (\text{assume } p \text{ splits in } K).$$

1st ingredient: A Gross-Zagier-type formula.

p-adic $\rightarrow$ $$AJ_{X_r}(\Delta_1)(\omega_f\wedge\omega_A^j\eta_A^{r-j})^2 = (*)\cdot\frac{L_p(f,\psi_A^{r+1-j}(\psi_A^*)^{1+j})}{\Omega_p^{2(r-2j)}} \qquad (\forall f).$$

(cf the paper in the handout).

2nd ingredient: Factorization, when $f=\Theta_{\psi_A^*}$. Then:

$$L_p(\Theta_\psi,\eta) = L_p(\eta\psi^{-1})\,L_p(\eta(\psi^*)^{-1}).$$

Tomorrow we'll see more details on this proof.

Recall $\Pi_{Hodge} = [\omega_{\Theta_{\psi_A^{r+1}}} \wedge \eta_A^{r+1}] \in H^{2r+1}_{dR}(W_r \times A^{r+1})$.

Choose $\omega_A \in \Omega^1(A/K)$, and recall the period, called Chowla-Selberg period Ω:

$$\omega_A = \Omega\,(2\pi i\, dz).$$

Take $\eta_A \in H^{0,1}_{dR}(A/K)$, $\langle \omega_A, \eta_A \rangle = 1$; then $\eta_A = \frac{\Omega^{-1}\, d\bar{z}}{\sqrt{D}}$.

Corollary: $\int_\gamma \eta_A^{r+1} \in \Omega^{-r-1} \cdot K$, as γ ranges over $H_{r+1}(A^{r+1}, \mathbb{Q})$.

Periods attached to ω_f, $f \in S_{r+2}(\Gamma_0(D), \varepsilon_D)$.

Shimura periods: Let $a \in \mathbb{P}_1(\mathbb{Q})$, $P \in \mathbb{Q}[x]^{\deg \le r}$. Define:

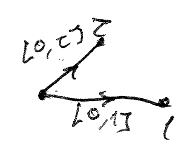
$$J_f(a;P) := (2\pi i)^{r+1} \int_a^{a+i\infty} P(z) f(z)\, dz \in \mathbb{C}.$$

Prop: The $\mathbb{Q}$-vs of periods of ω_f is generated by expressions $\{J_f(a,P)\}_{a \in \mathbb{Q},\, P \in \mathbb{Q}[x]^{\deg \le r}}$.

Pf

Consider the following regions:

$$\gamma_{a,j} := \left\{ (w_1, \dots, w_r, \tau) \;:\; \begin{array}{l} \tau = a + it \\ 0 \le t < \infty \end{array},\; \begin{array}{l} w_1, \dots, w_j \in [0,\tau] \\ w_{j+1}, \dots, w_r \in [0,1] \end{array} \right\}.$$



$\dim_{\mathbb{R}} \gamma_{a,j} = r+1$.

Thinking of $\gamma_{a,j} \subset W_r^\circ$, then $\partial \gamma_{a,j} = 0$.

In general, $\gamma_{a,j}$ need not extend to a closed $(r+1)$-chain on W_r. However, can take $\left(T_\ell - 1 - \varepsilon_D(\ell)\ell^{r+1}\right)\gamma_{a,j}$, which can be extended to W_r

(This is called the Manin trick used in this form by Chad Schoen).

$$\int_{(T_\ell - 1 - \varepsilon_D(\ell)\ell^{r+1})\gamma_{a,j}} \omega_f = \left(a_\ell(f) - 1 - \varepsilon_D(\ell)\ell^{r+1}\right) \int_a^{a+i\infty} (2\pi i)^{r+1} f(z)\, z^j\, dz \qquad 0 \le j \le r.$$

$$\text{"} \; J_f(a, z^j) \pmod{\mathbb{Q}^\times}$$

Now, we will focus in the case $J_{0,j} = J_{\Theta_{\psi_A^{r+1}}}(0, z^j)$.

__Theorem:__ For all $0 \le j \le \frac{r-3}{2}$,

$$J_{\theta, j} \in (2\pi i \Omega)^{r+1} K.$$

__Proof:__

Use $L(\theta_{\psi_A^{r+1}}, s)$ as a Mellin transform of $\theta_{\psi_A^{r+1}}$. This gives:

$$J_{\theta, j} = (-2\pi i)^j \, \Gamma(r+1-j) \, L(\theta_{\psi_A^{r+1}}, r+1-j)$$

On the other hand,

$$L(\theta_{\psi_A^{r+1}}, r+1-j) = \frac{1}{2} \sum_{a \in \mathcal{O}_K} \frac{a^{r+1}}{(a\bar{a})^{r+1-j}}, \text{ which converges if } 0 \le j \le \frac{r-1}{2}.$$

Up to $\Omega^\times$, can write:

$$J_{\theta_{\psi_A^{r+1}}, j} \sim (2\pi i)^0 \sum \frac{a^j}{\bar{a}^{r+1-j}}$$

Writing $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\tau$, for $j = 0$ we have:

$$J_{\theta, 0} \sim \sum \frac{1}{\bar{a}^{r+1}} = E_{r+1}(\tau),$$

and $$E_{r+1}(\tau) = \sum_{(m,n)}{}' \frac{1}{(m+n\tau)^{r+1}} \sim (2\pi i)^{r+1} G_{r+1}(\tau). \quad \left(G_{r+1}(\tau) = \zeta(-r) + 2\sum_{n \ge 1} \sigma_r(n) q^n \right)$$

For $1 \le j \le \frac{r-3}{2}$, can see also: $J_{\theta, j}$ is the value at τ of a __real-analytic__ Eisenstein series:

$$E_{k_1, k_2} := \sum_{(m,n)}{}' (m+n\tau)^{-k_1} (m+n\bar{\tau})^{-k_2} \cdot (\tau - \bar{\tau})^{k_2}.$$

__Exercise:__ 1) E_{k_1, k_2} has weight $k_1 - k_2$ (but, of course, it is not holomorphic).

2) $J_{\theta, j} \sim (2\pi i)^j \, E_{r+1-j, -j}(\tau)$

__Shimura-Maass operators__

$$\delta_k := \frac{1}{2\pi i} \left(\frac{d}{d\tau} + \frac{k}{\tau - \bar{\tau}} \right)$$

__Exercise:__ 1) If f is of wt k, then $\delta_k f$ is of weight $k+2$.

2) $\delta_{k_1 - k_2} E_{k_1, k_2} = \frac{k_1}{2\pi i} E_{k_1+1, k_2-1}$

3) $\delta_k^j := \delta_{k+2j-2} \cdots \delta_{k+2} \delta_k$. Then:

So we have, so far:

$$J_{\theta,j} \sim (2\pi i)^{2j} \delta^{j}_{r+1-2j} E_{r+1-2j}(\tau) = (2\pi i)^{r+1} \delta^{j}_{r+1-2j} G_{r+1-2j}(\tau)$$

Thm (Shimura's algebraicity):

If $f \in M_k(\Gamma)$ and f has Fourier coefficients in $F \supset K$, then:

$$\frac{(\delta_k^{j} f)(\tau)}{\Omega^{k+2j}} \in F.$$

Corollary: $J_{\theta,j} \in (2\pi i \Omega)^{r+1} K \quad \forall\ 0 \leq j \leq \frac{r-3}{2}$.

In particular, the periods of the cycle Π_{Hodge} belong to K.

Pf (of Shimura's alg)

Idea: interpret $\delta_k f$ geometrically. We can view f as a rule to which any point $(E,t) \in Y_\Gamma$, associates $f(E,t)_{/F} \in \mathrm{Sym}^k \Omega^1(E/F)$.

Can recover $f(\tau)$ by: $f(z) := f\left(\frac{\mathbb{C}}{\langle 1,\tau\rangle}, \frac{1}{N}, 2\pi i\, dz\right) \quad (\Gamma = \Gamma_1(N))$.

So f can be viewed as a global section, over Y_Γ, of the sheaf $\mathrm{Sym}^k \underline{\omega}$, where $\underline{\omega} = \pi_* \Omega^1(\mathcal{E}/Y_\Gamma)$.

The form f can be viewed as an algebraic family of elements in $\mathrm{Sym}^k \Omega^1(\mathcal{E}/Y_\Gamma)$.

Def: The Gauss-Manin connection is a map of sheaves:

$$\nabla : \mathcal{H}^1_{dR}(\mathcal{E}/Y_\Gamma) \longrightarrow \mathcal{H}^1_{dR}(\mathcal{E}/Y_\Gamma) \otimes \Omega^1_{Y_\Gamma},$$

satisfying:

If $\omega_\lambda \in H^1_{dR}(E_\lambda)$, $\nabla \omega_\lambda = (\nabla_\lambda \omega) \otimes d\lambda$, with

$$\int_{\gamma_j} \nabla_\lambda \omega_\lambda = \frac{d}{d\lambda} \int_{\gamma_j} \omega_\lambda ,$$ where γ_1, γ_2 is a horizontal basis for $H_1(E_\lambda)$.

↓

Important properties

- If $f \in H^0(Y_\Gamma, \mathrm{Sym}^r \mathcal{H}^1_{dR}(\mathcal{E}/Y_\Gamma))$ which is defined over F, then:

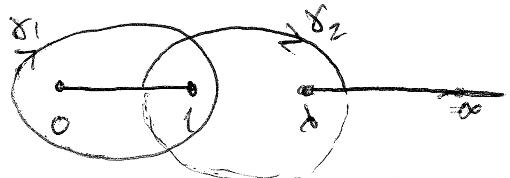
$$\nabla f(E,t)_{/F} \in \mathrm{Sym}^r \mathcal{H}^1_{dR}(E/F) \otimes \Omega^1(Y_\Gamma/F).$$

Proof by example:

Consider $\mathcal{E}_\lambda : y^2 = x(x-1)(x-\lambda)$ (Legendre family).

$$\omega_\lambda := \frac{dx}{y} \in \Omega^1(E_\lambda) \subseteq H^1_{dR}(E_\lambda).$$

Note that $\omega_\lambda = \dfrac{dx}{\sqrt{x(x-1)(x-\lambda)}}$.



So $\displaystyle \frac{d}{d\lambda}\int_{\gamma_j}\omega_\lambda = \int_{\gamma_j}\frac{d}{d\lambda}\frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \int_{\gamma_j}\frac{1}{2}\frac{1}{x-\lambda}\frac{dx}{y}$.

So $\nabla_\lambda \omega_\lambda = \left[\frac{1}{2}\frac{dx}{y}\cdot\frac{1}{x-\lambda}\right]$. ← has a pole of order 2 at $(\lambda,0)$. $\boxtimes$

- Leibniz rule: For $\lambda \in \mathcal{O}_{Y_\Gamma}$, $\omega \in \mathrm{Sym}^r\mathcal{H}^1_{dR}(\mathcal{E}/Y)$,

$$\nabla(\lambda\omega) = \cancel{d\lambda\otimes\omega + \nabla\omega}\ \omega\otimes d\lambda + \lambda\nabla\omega.$$

- Kodaira-Spencer map:

$$KS : \underset{\mathrm{Sym}^2\underline{\omega}}{\overset{\shortparallel}{\underline{\omega}^2}} \longrightarrow \Omega^1_Y \quad ; \quad KS(\omega_1\otimes\omega_2) := \langle \omega_1, \nabla\omega_2\rangle \in \Omega^1_Y.$$

It is an isomorphism. Have $KS^{-1} : \Omega^1_Y \to \underline{\omega}^2$.

Now, define:

$\delta_K f(E,t,\omega)_{/\mathbb{C}}$ using all this. $(\delta_K f)\left(\mathbb{C}/\langle 1,\tau\rangle, \frac{1}{N}, 2\pi i\, dz\right) := \delta_K f(\tau)$

and now $(\delta_K f)(E,t,\lambda\omega) := \lambda^{-k-2}(\delta_K f)(E,t,\omega)$

Theorem:

$\cancel{(\delta_K f)(E,t,\omega)}$ $(\delta_K f)(E,t,\omega) = \langle KS^{-1}\nabla f(E,t), \eta^{k+2}\rangle$, where

η is the unique class in $H^{01}_{dR}(E/\mathbb{C})$ s.t. $\langle\omega,\eta\rangle = 1$.

Suppose that $f \in H^0(Y_\Gamma, \underline{\omega}^k)$ is defined over F, and (E,t) also $/F$.

Then $f(E,t) \in \mathrm{Sym}^k\, \Omega^1(E/F)$, and $\nabla f(E,t) \in \mathrm{Sym}^k H^1_{dR}(E/F) \otimes \Omega^1(Y_\Gamma/F)$.

Next, $KS^{-1}\nabla f(E,t) \in \mathrm{Sym}^{k+2} H^1_{dR}(E/F)$.

In the case of E having CM, then $\eta \in H^1_{dR}(E/F)$ (if $F \supset K$).

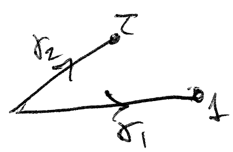
Therefore $(\delta_k f)(E,t,\omega) \in F$.

But $\omega = \Omega \cdot 2\pi i\, dz$, so $(\delta_k f)(E,t,\Omega\, 2\pi i\, dz) \in F$, and this proves Shimura's theorem. (for $j=1$, the rest is the same).

<u>Proof (of the Thm relating</u> δ_k to ∇):

We do the computation in the case $E = \mathbb{C}/\langle 1,\tau\rangle$, $\omega = 2\pi i\, dz$. Then:

$$\nabla f\left(\mathbb{C}/\langle 1,\tau\rangle, \tfrac{1}{N}\right) = \nabla\left(f(\tau)(2\pi i\, dz)^k\right) = f'(\tau)\,d\tau\,(2\pi i\, dz)^k + f(\tau)\nabla\left((2\pi i\, dz)^k\right) =$$
$$= f'(\tau)\,d\tau\,(2\pi i\, dz)^k + k\, f(\tau)(2\pi i\, dz)^{k-1}\nabla(2\pi i\, dz)$$

Taking , the periods of $2\pi i\, dz$ are $(2\pi i, 2\pi i\tau)$.

Differentiating gives $(0, 2\pi i)$. So get the class $2\pi i\dfrac{dz - d\bar z}{\tau - \bar\tau}$.

$$\therefore\ \nabla_\tau(2\pi i\, dz) = 2\pi i\,\frac{dz - d\bar z}{\tau - \bar\tau}.$$

And plugging it to the other formula get:

$$\nabla f\left(\mathbb{C}/\langle 1,\tau\rangle, \tfrac{1}{N}\right) = f'(\tau)\,d\tau\,(2\pi i\, dz)^k + k\, f(\tau)(2\pi i\, dz)^{k-1}(2\pi i)\,\frac{dz - d\bar z}{\tau - \bar\tau}\, d\tau.$$

<u>Exercise</u>: $KS(2\pi i\, dz \otimes 2\pi i\, dz) = 2\pi i\, d\tau$.

$$\therefore\ \nabla f(\quad) = \frac{1}{2\pi i}\left(f'(\tau)(2\pi i\, dz)^{k+2} + \frac{k f(\tau)}{\tau - \bar\tau}(2\pi i\, dz)^{k+2}\right) + \cancel{(\quad)\, d\bar z}^{\,0}$$
$$= (\delta_k f)(\tau)\cdot(2\pi i\, dz)^{k+2} + (-)\, d\bar z.$$

Pairing it with η^{k+2} yields $\delta_k(f)(\tau)$, as we wanted.

Lecture 5. p-adic L-functions.

This is an intro to p-adic L-functions.

Consider $\zeta(s)$, the Riemann zeta-function.

For negative integer, $\zeta(k)$ is a rational number: $\zeta(1-k) = -\frac{B_k}{k}$ $(\forall k \geq 1)$

(here we define B_k's by $\frac{t}{e^t - 1} = \sum_{k \geq 1} B_k \frac{t^k}{k!}$).

Question: Can be define a "p-adic analytic" function on $\mathbb{Z}_p$ that agrees with $\zeta(s)$ at negative integers?

(by p-adic analytic, we mean that it is of the form $\sum a_n X^n \in \mathbb{Z}_p[[X]]$, with $a_n \to 0$ p-adically).

Rk: if we had such a function "$f(s) = \sum a_n s^n$", then $\left[k \equiv k' \ (p^n) \Rightarrow f(k) \equiv f(k') \ (\text{mod } p^n) \right]$

so need to investigate congruences of those Bernoulli numbers.

Three flavours of p-adic L-functions.

(i) p-adic analytic functions.

(ii) (Mazur): measures on Galois groups. ← nicest one.

(iii) Power series $\in \mathbb{Z}_p[[X]]$

Preliminaries on p-adic distributions & measures.

Let I be a nonempty directed set (given i, i', $\exists j, j \geq i, j \geq i'$).

For each i, spz have X_i, $\pi_{ij} : X_i \to X_j$ (inverse system). $\# X_i < \infty$.

Let $X := \varprojlim_i X_i$ with the inverse limit topology (so compact).

Comes with $\pi_i : X \to X_i$.

Def: Let A be an abelian group. An A-valued distribution on X is a collection of maps $\varphi_i : X_i \to A$, such that for $i \geq j$,

$$\begin{tikzcd} X_i \arrow[r, "\varphi_i"] \arrow[d, "\pi_{ij}"'] & A \\ X_j \arrow[ur, "\varphi_j"'] & \end{tikzcd} \quad : \forall y \in X_j, \ \varphi_j(y) = \sum_{\substack{x \in X_i \\ \pi_{ij}(x) = y}} \varphi_i(x) .$$

Example (Bernoulli distribution): $I = \mathbb{Z}_{\geq 1}$.

$$\mathbb{Z}/N\mathbb{Z} \twoheadrightarrow \mathbb{Z}/M\mathbb{Z} \quad , \quad \text{and} \quad \varphi_N^{(N)} : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{Q}$$

defined by:

$$\varphi^{(N)}(x) = \left\langle \frac{x}{N} \right\rangle - \frac{1}{2} \text{, where } \langle \cdot \rangle \text{ is the fractional part.}$$
$$\in \{0, \tfrac{1}{N}, \dots, \tfrac{N-1}{N}\}.$$

If $M \mid N$, $\quad \mathbb{Z}/N\mathbb{Z} \quad x + Mi, \ 0 \leq i \leq N/M - 1$.

$$\begin{array}{cc} \mathbb{Z}/N\mathbb{Z} & x+Mi \\ \downarrow & \downarrow \\ \mathbb{Z}/M\mathbb{Z} & y \end{array}$$

Get: $$\sum_{\substack{y \in \mathbb{Z}/N\mathbb{Z} \\ y \mapsto x}} \varphi^{(N)}(y) = \sum_{i=0}^{\frac{N}{M}-1} \varphi^{(N)}(x+Mi) = \sum_{i=0}^{\frac{N}{M}-1} \left\langle \frac{x+Mi}{N} \right\rangle - \frac{1}{2} =$$

$$= \sum_{i=0}^{\frac{N}{M}-1} \left\langle \frac{x+Mi}{N} - \frac{1}{2} \right\rangle = \frac{x}{N}\left(\frac{N}{M}\right) + \frac{M}{N}\frac{\cancel{N}}{M}\left(\frac{N}{M}-1\right)\left(\frac{1}{2}\right) - \frac{1}{2}\frac{N}{M} = \frac{x}{M} - \frac{1}{2} \quad \checkmark.$$

Rmk: $B_1(T) = T - \frac{1}{2}$. So we've defined $\varphi^{(N)}(x) = B_1\left(\left\langle \frac{x}{N} \right\rangle\right)$.

Interested in distributions that take values in a field K which is complete wrt a p-adic absolute value (eg $\mathbb{Q}_p$, $K/\mathbb{Q}_p$ finite, $\mathbb{C}_p, \dots$). (p fixed from now on).

Let φ be a K-valued distribution.

Def: A <u>step function</u> on X (with values on K) is a function $f: X \to K$ that factors through one of the X_i.

Given a dist. φ and a step function, can define:

$$\int f \, d\varphi := \sum_{x \in X_i} f(x) \varphi_i(x).$$

Def: A K-valued distribution φ on X (with values on K) is said to be <u>bounded</u> if $\|\varphi\|$ is bounded. $\left(\|\varphi\| := \sup_{i, x \in X_i} \{ |\varphi_i(x)| \} \right)$.

Def: A <u>measure</u> is a bounded distribution.

Example: Let φ be a K-valued measure on X, and $f: X \to K$ a continuous function. Then show that $\int f\,d\varphi$ is well-defined.

(Since X is compact, can approximate f uniformly by step functions f_n, and then $\int f\,d\varphi := \varinjlim \int f_n\,d\varphi$.

Example (Bernoulli measure on $\mathbb{Z}_p$). $X = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$.

Let $c \in \mathbb{Z}_p^\times$, define $\varphi_c^{(N)}(x) := \varphi^{(N)}(x) - c\,\varphi^{(N)}(c^{-1}x) = (*)$

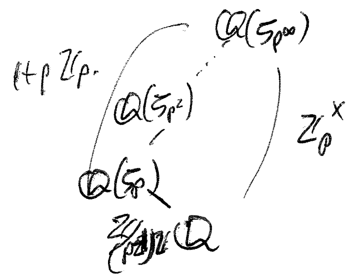
For $0 \le x < p^n$, $c^{-1}x \equiv y\ (p^n)$, $y \in \mathbb{Z}$, $0 \le y < p^n$. Then

$$(*) = \frac{x}{N} - \frac{1}{2} - c\left(\frac{y}{N} - \frac{1}{2}\right) = \underbrace{\frac{x - cy}{N}}_{\in\, \mathbb{Z}_p} + \frac{c-1}{2} \in \mathbb{Z}_p.$$

Def: The Kubota-Leopoldt p-adic L-function is the measure $d\mu_{KL,c}$ on $\mathbb{Z}_p^\times$, given by ~~the~~ restricting the Bernoulli measure to $\mathbb{Z}_p^\times$.

Fix embeddings $\overline{\mathbb{Q}} \xrightarrow{i_\infty} \mathbb{C}$, $\overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$. Let χ be a Dirichlet character of conductor lp^n (some n).

By CFT, can think of χ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$.



So χ can be thought of as a function on $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times$.

Thm: For all integers $k \ge 1$,

$$\int_{G = \mathbb{Z}_p^\times} \chi(a)\, a^{k-1}\, d\mu_{KL,c}(a) = \left(1 - \chi(c)c^k\right)\left(1 - \chi(p)p^{k-1}\right) L(1-k, \chi).$$

(This thm is due to Kubota-Leopoldt, Iwasawa, Mazur).

Now we look at the other two flavours.

<u>Teichmüller character</u>: $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mu_{p-1}$, s.t. $\omega(a) \equiv a$ (mod a prime above p).

Given $a \in \mathbb{Z}_p^\times$, define $\langle a \rangle$ by $a = \omega(a)\langle a \rangle$, $\langle a \rangle \equiv 1 \pmod p$.

Let χ be a Dirichlet character of conductor lp^n. For $s \in \mathbb{Z}_p$, define

$$L_p(s,\chi) \text{ by : } L_p(1-s,\chi) := \frac{1}{1-\chi(c)\langle c\rangle^s}\int \chi(a)\langle a\rangle^{s-1} a^{-1}\, d\mu_{\chi L, c}(a)$$

<u>Thm</u>: Suppose $\chi \neq 1$ (resp. $\chi = 1$). Then:

$L_p(s,\chi)$ (resp $(s-1)L_p(s,\chi)$) is a p-adic analytic function of s, satisfying:

$$L_p(1-k,\chi) = (1-\chi\omega^{-k}(p))\, L(1-k, \chi\omega^{-k}).$$

~~Pf~~ More generally:

<u>Lemma</u>: For any μ on $\mathbb{Z}_p^\times$, the function $s \longmapsto \int_{\mathbb{Z}_p^\times} \langle a \rangle^s d\mu$ is a p-adic analytic function on s.

Pf: First, reduce to assuming that μ is supported on $1+p\mathbb{Z}_p$ (by writing $\mathbb{Z}_p^\times$ as a finite union of subsets $\cong 1+p\mathbb{Z}_p$).

Then $s \longmapsto \int_{1+p\mathbb{Z}_p} \langle a\rangle^s d\mu = \int_{1+p\mathbb{Z}_p} a^s\, d\mu$ (*) But

$$\left[a^s = (1+a-1)^s = \sum_{n\geq 0} \binom{s}{n}(a-1)^n \right]$$

$\Rightarrow$ (*) $= \sum_{n \geq 0} \binom{s}{n} \cdot \int_{1+p\mathbb{Z}_p} (a-1)^n\, d\mu(a)$, which is p-adic analytic on s. ∎

Then $\chi(a)a^{-1}d\mu_{\chi L, c}(a)$ is a measure, and can apply the Lemma. ∎

Next we will see the 3rd flavour.

First, note that a measure on $\mathbb{Z}_p^\times$ is the same as giving $p-1$ measures on $\mathbb{Z}_p$.

A $\mathbb{Z}_p$-valued measure on $\mathbb{Z}_p$ is the giving of $\phi_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p$.

So it is an element of the group ring:

$$\varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}].$$

But $\mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}] \xrightarrow[1 \mapsto X]{\sim} \dfrac{\mathbb{Z}_p[T]}{(T^{p^n}-1)} \xrightarrow[T=1+X]{\sim} \dfrac{\mathbb{Z}_p[X]}{(1+X)^{p^n}-1}$

Therefore, get an element of $\varprojlim_n \dfrac{\mathbb{Z}_p[X]}{(1+X)^{p^n}-1} = \mathbb{Z}_p[[X]]$.

$$\begin{tikzcd}
\mathbb{Q}(\zeta_{p^\infty}) \arrow[d, dash, "\mathbb{Z}_p \cong \Gamma"'] \arrow[dd, dash, bend left, "\mathbb{Z}_p^\times"] \\
\mathbb{Q}(\zeta_p) \arrow[d, dash, "(\mathbb{Z}/p\mathbb{Z})^\times \cong \Delta"'] \\
\mathbb{Q}
\end{tikzcd}$$

A meas. on $\mathbb{Z}_p^\times$ is the same as $p-1$ measures on Γ.

called a character of the "first kind".

Fix a character θ of $(\mathbb{Z}/p\mathbb{Z})^\times$. If $\theta \neq 1$, there exists a power series $f^\theta(X) \in \mathbb{Z}_p[[X]]$ s.t for all characters χ "of the second kind" i.e. of Γ.

$$f^\theta(\chi^{-1}(\gamma)\gamma^s - 1) = L_p(s, \chi\theta), \quad \gamma = 1+p.$$

Suppose χ is an even character. Then one can show:

$$L(1,\chi) = \frac{-\tau(\chi)}{f} \sum_{a=1}^{f} \chi(a) \log(1-\zeta_f^a)$$

(with $f = f_\chi = \mathrm{cond}(\chi)$, and $\zeta_f = e^{2\pi i/f}$).

We also get:

with the branch $\log_p(p) = 0$.

$$L_p(1,\chi) = -\frac{\tau(\chi)}{f}\left(1 - \frac{\chi(p)}{p}\right) \sum_{a=1}^{f} \chi(a) \log_p(1-\zeta_p^a).$$

The fact that we get "log" is the first instance of the p-adic Beilinson conjectures.

Algebraic Hecke characters.

Let K, L be # fields, and $T \in \mathbb{Z}[\mathrm{Hom}(K, \bar{L})]$. So $T = \sum_{\sigma: K \hookrightarrow \bar{L}} n_\sigma \sigma$.

Def: An algebraic ~~character~~ Hecke character χ of K with values in L, of infinity type T, and of conductor dividing $\mathfrak{f}$ ($\mathfrak{f}$ a nonzero integral ideal in K).
is a homomorphism:

$$\chi : I(\mathfrak{f}) \to L^\times, \quad I(\mathfrak{f}) = \text{gp of fract. ideals in } K \text{ prime to } \mathfrak{f},$$

such that for all $\alpha \in K^\times$, totally positive and $\equiv 1 \bmod \mathfrak{f}$,

$$\chi((\alpha)) = \alpha^T := \prod_{\sigma: K \hookrightarrow \bar{L}} (\sigma\alpha)^{n_\sigma}$$

Rmk: Can think of χ as a character on the ideles of K, $\mathbb{A}_K^\times \to L^\times$,
as follows: given $x \in \mathbb{A}_K^\times$, ~~define~~ pick $\alpha \in K^\times$ s.t. $\alpha \gg 0$, $\alpha \equiv 1 \bmod \mathfrak{f}$, and let:

$$\tilde{\chi}(x) := \chi(x\alpha)\alpha^{-T}.$$

(Rmk that it is indep. of the choice of α, b/c $\chi\left(\left(\frac{\beta}{\alpha}\right)\right) = \left(\frac{\beta}{\alpha}\right)^T$.).

Then $\tilde{\chi}|_{K^\times} = T$.

Exercise: Show that $T: K^\times \to L^\times$ is an algebraic map. This gives:
$T_{\mathbb{A}} : \mathbb{A}_K^\times \to \mathbb{A}_L^\times$, write $\mathbb{A}_L^\times \to L_\lambda^\times$ the projection, and
T_λ for the composition. (λ a place of L).

~~If λ is any place of L, let T_λ~~

Define $\chi_\lambda : \mathbb{A}_K^\times / K^\times \to L_\lambda^\times$, by $\chi_\lambda(x) := \tilde{\chi}(x) / T_\lambda(x)$.

If λ is a finite place, the map χ_λ factors through the gp of connected components, which is $\mathrm{Gal}(K^{ab}/K)$.

Given χ an algebraic Hecke character of K (with values in L), get (using the fixed embeddings):

1) A complex character (~~usually~~ called a Grossencharacter)
2) A p-adic character

denote both

Example:

On $\mathbb{Q}$, define $\mathbb{N}(a) := a$, an algebraic Hecke char. on $\mathbb{Q}$.

The associated p-adic character is the p-adic cyclotomic character,

$\chi_{cyc} : Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \to \mathbb{Z}_p^{\times}$.

Recall now the formula $\int \chi(a) a^{k-1} d\mu(a) = \cdot L(1-k, \chi)$.

$$\int \chi(a) a^{k-1} d\mu(a) = \int (\chi \mathbb{N}^{k-1})(a)\, d\mu(a) \qquad L(1-k,\chi) = L(0, \chi\mathbb{N}^{k-1})$$

So can say: $\forall$ alg. Hecke characters,

$$\int \chi(a) d\mu(a) = \cdot L(0, \chi).$$

<u>Katz p-adic L-function:</u> Let K be an imag. quadratic field, $\sigma, \bar{\sigma} : K \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Write $T = k\sigma + j\bar{\sigma}$, for $k, j \in \mathbb{Z}$. Say that T has ∞-type (k, j).

Consider $\mathbb{N}_K := \mathbb{N} \circ N_{K/\mathbb{Q}}$, which has ∞-type $(1,1)$.

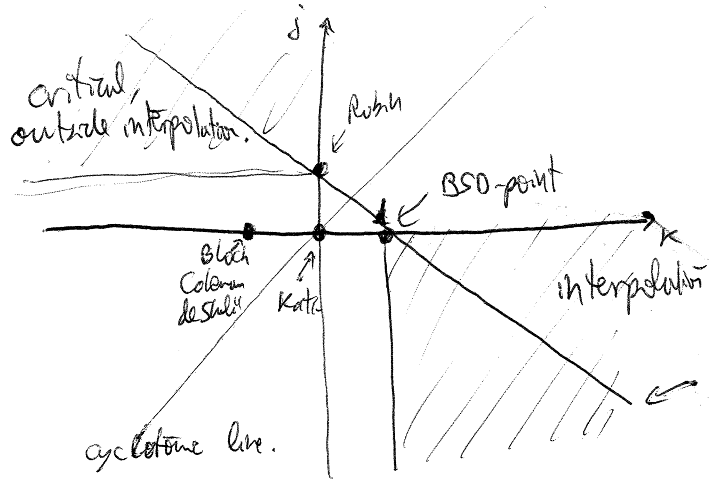
As a Grossencharacter, have:

$$L(\chi, s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \prod_{\mathfrak{q}} \frac{1}{(1 - \chi(\mathfrak{q}) N\mathfrak{q}^{-s})}$$

<u>Theorem</u> (de Shalit's book, originally proved by Katz). Let p be split in K. Let $\mathfrak{g}$ be an integral ideal in K. Then there exists a unique p-adic measure μ_{Katz} on $\mathcal{G} := Gal(K(\mathfrak{g}p^{\infty})/K)$, and a choice of complex and p-adic periods Ω, Ω_p, s.t.:

$\forall$ (alg) Hecke characters χ of conductor dividing $\mathfrak{g}p^{\infty}$ and of ∞-type (k,j), $k>0$, $j \leq 0$, the following interpolation property holds:

$$\frac{1}{\Omega_p^{j-k}} \int_{\mathcal{G}} \chi \, d\mu_{Katz} = \frac{1}{\Omega^{j-k}} (\cdots) \left(1 - \frac{\chi(\mathfrak{p})}{p}\right) L_{\mathfrak{g}\bar{\mathfrak{p}}}(\chi^{-1}, 0).$$



Katz already in his original paper showed that the point $(0,0)$ was related to elliptic units.

The point $(-1,0)$ is related to Beilinson's conjecture. Bloch did the complex case, and Coleman-de Shalit did the p-adic case.

Rubin showed, looking at $(0,1)$, that

$$L_p^{katz}(\psi^*) = \log_p^2(\text{rat point}), \text{ when } E \text{ has rk } 1/\mathbb{Q}.$$

Can look at critical points <u>outside</u> the range of interpolation (as Rubin did). These are the ones with $k \leq 0$, $j > 0$.

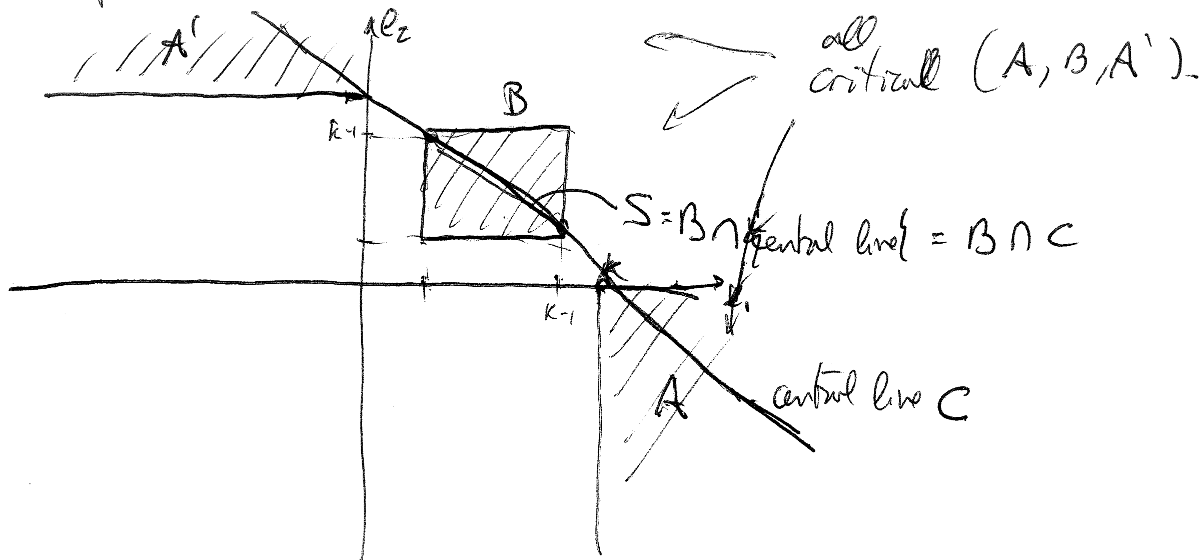
<u>Rankin-Selberg</u>

Fix a modular form f of weight k, and an imag. quadratic field K. Let χ be a Hecke character of K. Ass. to χ, f there are automorphic reps π_χ, π_f. Then there is:

$L(\pi_f \times \pi_\chi, s)$. Actually, there's a theta series $\Theta(\chi) = \sum \chi(a) e^{2\pi i N a z}$ and a deg 2 L-function, similarly for f, and can define $L(\pi_f \times \pi_\chi, s)$ as having all the roots in the Euler products in common, to get a deg 4 L-function.

We can also see it as $L(f/K, \chi, s)$ (base change f to K.) (twisted L-function).

The picture in this case is:



Look at $L(f, \chi^{-1}, 0)$.

The transcendental part is (a power of) a CM-period of χ, if in A

i.e. a power of $\langle f, f \rangle = \int_{\Gamma \backslash \mathcal{H}} f(z)\overline{f(z)}\, y^{k-2}\, dz$, if χ is in B

So there will be two p-adic L-functions, say $L_p^{(1)}$ and $L_p^{(2)}$.

Let's put ourselves in the Heegner situation:

f of conductor N squarefree, and K split or ramified at all $q \mid N$.

Assume k even, and consider the point $(k/2, k/2)$. Then $L(f, \chi^{-1}, 0)$ vanishes, so look at the derivative, say in the cyclotomic direction.

This should be related to p-adic L-functions ($k=2$ Perrin-Riou, greater k: Nekovar).

Concentrate on region A. So let $L_p = L_p^{(1)}$, and Hida showed: there exists a measure μ on $\mathcal{H}$ s.t. $\int \chi\, d\mu = (\cdot)\, L(f/K, \chi^{-1}, 0)$ for $\chi \in A$.

Question: What can we say about χ on the central segment S?

Idea (Waldspurger) For χ on C,

$$L(f/K, \chi^{-1}, 0) = *\,(P_\chi(F))^2, \text{ where } P_\chi \text{ is a period of } F \text{ twisted by } \chi.$$

for some form $F \in \pi_f$, $F: GL_2(\mathbb{A}) \to \mathbb{C}^\times$.

Can think of $F: GL_2(\mathbb{A}) \to \mathbb{C}^\times$, and Heegner hypothesis implies that there is an embedding $K \hookrightarrow M_2(\mathcal{O})$, $K^\times \hookrightarrow GL_2(\mathbb{A})$.

So $P_\chi(F) := \int F|_{/\mathbb{A}_K^\times} \cdot \chi$ = twisted sum of values of F at CM points.

If χ has weight $(k+j, -j)$, then

$$L(f/K, \chi^{-1}, 0) = \Big(\sum_{x\ \text{CM points}} (\delta^j f)\cdot\chi\Big)^2 \quad (\text{so } F = \delta^j f,\ \delta \text{ the Shimura-Maass diff op}).$$

(for "nice enough" χ).

In order to interpolate this, need to throw away the Euler factor at p, to get $\Big(\sum (\delta^j f^\flat)\chi\Big)^2$

(where if $f(q) = \sum a_n q^n$, then $f^\flat(q) = \sum_{p\nmid n} a_n q^n$.)

Can think of $f^\flat$ as a p-adic modular form. Set $\theta = q\frac{d}{dq}$ (Serre's θ).

Then $\delta^j f^\flat(\text{complex point}) = \theta^j f^\flat(\text{complex point})$

To study the region B, can take limits (can take j to be negative, in the previous formula). This turns out to be related to the p-adic A-J.

Lecture 6: Abel-Jacobi maps.

Complex setting:

Recall $\varphi : A \to A' \rightsquigarrow \Delta_\varphi \in CH^{r+1}(X_r)(H_\varphi)$.

$$\begin{array}{ccc} X_r & \supset & (A')^r \times A^r \\ \downarrow & & \downarrow \\ C_r & \ni & P_{A'} = (A', t') \end{array}$$

$\Delta_\varphi = \varepsilon(\mathrm{Graph}^t \varphi)$. What is ε?

Let $G = ((-1)^r \rtimes S_r) \times ((-1)^r \rtimes S_r)$, which acts on $(A')^r \times A^r$.

Let $j : G \to \pm 1$, j is $\begin{cases} \text{the sign function on } S_r \\ \text{the identity (sends } -1 \mapsto -1) \text{ on } (-1)^r. \end{cases}$

Define then $\varepsilon := \frac{1}{\# G} \sum_{\sigma \in G} j(\sigma)\sigma \quad (\varepsilon^2 = \varepsilon)$.

Want to compute AJ.

$$\begin{array}{ccc} \widetilde{X}_r & \longrightarrow & X_r^\circ \\ \pi \downarrow & \square & \downarrow \pi \\ \mathcal{H} & \xrightarrow{pr} & Y_r \end{array}$$

$\widetilde{X}_r = \widetilde{W}_r \times A^r(\mathbb{C})$, where $\widetilde{W}_r = \mathcal{E}^r \searrow \mathcal{H}$.

Prop: There exists a cycle (~~topological~~) $\widetilde{\Delta}_\varphi$ on $\widetilde{X}_r$ satisfying:

1) $pr_*(\widetilde{\Delta}_\varphi) = \Delta_\varphi + \partial \xi$, where ξ is supported on $\pi^{-1}(A', t') = \pi^{-1}(P_{A'})$.

2) $\widetilde{\Delta}_\varphi$ is homologically trivial.

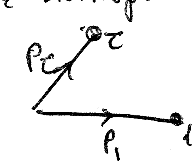
Proof:

Let $\mathcal{L}_r := \mathrm{Sym}^r \mathcal{H}^1_{dR}(\mathcal{E}/Y_r)$, $\mathcal{L}_{r,r} := \mathrm{Sym}^r \mathcal{H}^1_{dR}(\mathcal{E}/Y_r) \otimes \mathrm{Sym}^r H^1_{dR}(A)$.

Let $\widetilde{\mathcal{L}}_r$ and $\widetilde{\mathcal{L}}_{r,r}$ denote the pullbacks of $\mathcal{L}_r, \mathcal{L}_{r,r}$ to $\mathcal{H}$.

Since $\mathcal{H}$ is simply connected, these sheaves are trivial:

$\widetilde{\mathcal{L}}_r = \mathbb{H}_r \otimes \mathcal{O}_{\mathcal{H}}$ where $\mathbb{H}_r = H^0(\mathcal{H}, \mathcal{L}_r)^{\nabla = 0}$. Concretely, can write $\mathbb{H}_r = \bigoplus_{j=0}^{r} \mathbb{C}\, \eta_1^j \eta_\tau^{r-j}$, where η_1, η_τ correspond to p_1, p_τ.



Note that Γ acts naturally on $\mathbb{H}_r$ and $\mathbb{H}_{r,r}$.

Also, $\mathbb{L}_r / I_\Gamma \mathbb{L}_r = 0$ (and similar for $\mathbb{L}_{r,r}$). (I_Γ = augmentation ideal in $\mathbb{Q}[\Gamma]$).

Also,

$\mathcal{E}$ maps $H^{2r}((A')^r \times A^r) \longrightarrow \mathrm{Sym}^r H^1(A') \otimes \mathrm{Sym}^r H^1(A) \in H^{2r}((A')^r \times A^r)$.

Hence can write:

$$cl_{(A')^r \times A^r}(\Delta_\varphi) = \sum_{j=1}^{t} (\gamma_j^{-1} - 1) \cdot Z_j \quad , \text{where} \begin{cases} \gamma_j \in \Gamma \\ Z_j \in H^{2r}(\pi^{-1}(\tau_0), \mathbb{Q}) \end{cases}$$

$$\begin{array}{c} H^{2r}(\pi^{-1}(\tau_0), \mathbb{Q}) \\ \| \xleftarrow{\text{Poincaré duality.}} \\ H_{2r}(\tilde{X}_r, \mathbb{Q}) \\ \| \\ H_{2r}(\tilde{X}_r, \mathbb{Q}) \end{array}$$

~~Given now $\tau \in \mathcal{H}$, and a 2r-dimensional cycle on $\tilde{X}_r$ supported on $\pi^{-1}(\tau)$, let $Z(\tau, Z)$~~

Given $\tau \in \mathcal{H}$, and $Z \in \mathbb{L}_{r,r}$, let $\mathcal{Z}(\tau, Z)$ be any cycle on $\tilde{X}_r$, supported on $\pi^{-1}(\tau)$, whose image in $H_{2r}(\pi^{-1}(\tau))$ is equal to Z.

Let also $\tilde{\Delta}_\varphi := \sum_{j=1}^{t} \mathcal{Z}(\gamma_j \tau, Z_j) - \mathcal{Z}(\tau, Z_j)$

So $\tilde{\Delta}_\varphi = \partial\left(\sum_{j=1}^{t} \text{path}\,((\tau \to \gamma_j \tau) \times Z_j) \right)$.

<u>Def</u>: A divisor of degree 0 on $\mathcal{H}$ with coefficients in $\mathbb{L}_{r,r}$ is a formal sum:

$$\sum_{j=1}^{t} \tau_j \cdot Z_j \quad , \tau_j \in \mathcal{H}, \; Z_j \in \mathbb{L}_{r,r}, \text{ st } \sum_{j=1}^{t} Z_j = 0.$$

Write $\mathrm{Div}^0(\mathcal{H}, \mathbb{L}_{r,r})$ for the group of deg-0 divisors $\mathbb{L}_{r,r}$ valued.

Define the evaluation map:

$$[\cdot, \cdot] : H^0(\mathcal{H}, \mathcal{L}_{r,r}) \times \mathrm{Div}(\mathcal{H}, \mathbb{L}_{r,r}) \longrightarrow \mathbb{C}$$

$$(G, D = \sum \tau_j Z_j) \longmapsto \underset{[G,D]}{\underset{\|}{\sum}} \langle G(\tau_j), Z_j \rangle_{\mathbb{L}_{r,r}}$$

<u>Def</u> ~~Theorem~~ A <u>primitive</u> of $\omega_f \in H^0(\cancel{Y_r, \underline{\omega}^{r+2}})$, $\underline{\omega} = \Omega^1(\mathcal{E}/Y_r)$

$H^0(Y_r, \underline{\omega}^r \otimes \Omega^1_{Y_r}) \hookrightarrow H^0(Y_r, \mathcal{L}_r \otimes \Omega^1_{Y_r})$.

(here $\underline{\omega}^r = \mathrm{Sym}^r(\underline{\omega}) \subset \mathrm{Sym}^r(\mathcal{H}^1_{dR}(\mathcal{E}/Y_r)) = \mathcal{L}_r$).

Rmk: We can't integrate on Y_Γ, since ω_f is not cohomologically trivial in general. But we can do it on $\mathcal{H}$, because it is contractible.

Theorem: Let $\alpha \in H^r_{dR}(A^r)$.

$$AJ(\Delta_\varphi)(\omega_f \wedge \alpha) = [F_f \wedge \alpha, \tilde{\Delta}_\varphi] \qquad \left(\begin{array}{l} \tilde{\Delta}_\varphi \in \mathrm{Div}^0(\mathcal{H}, \mathbb{L}_{r,r}) \\ F_f \wedge \alpha \in H^0(\mathcal{H}, \mathcal{L}_{r,r}) \end{array} \right)$$

Rmk: F_f is well-defined up to $H^0(\mathcal{H}, \mathcal{L}_{r,r})^{\nabla = 0} = \mathbb{L}_{r,r}$. But, since $\tilde{\Delta}_\varphi$ has degree 0, the expression above doesn't depend on the choice of F_f.

Pf:

$$AJ(\Delta_\varphi)(\omega_f \wedge \alpha) = \int_{pr_*(\partial^{-1}\hat{\Delta}_\varphi)} \omega_f \wedge \alpha = \int_{\partial^{-1}\tilde{\Delta}_\varphi} pr^*(\omega_f \wedge \alpha) =$$

$$= \sum_{j=1}^{t} \int_{\tau}^{\gamma_j \tau} \underbrace{\langle pr^*(\omega_f \wedge \alpha), Z_j \rangle_{\mathbb{L}_{r,r}}}_{\hat{\Omega}^1(\mathcal{H})}$$

null duality on $\mathbb{L}_{r,r}$

Z_j is horizontal.

But $\langle pr^*(\omega_f \wedge \alpha), Z_j \rangle = \langle \nabla F_f \wedge \alpha, Z_j \rangle = d \langle F_f \wedge \alpha, Z_j \rangle$

$$\therefore AJ(\Delta_\varphi)(\omega_f \wedge \alpha) = \sum_{j=1}^{t} \Big(\langle F_f \wedge \alpha, Z_j \rangle (\gamma_j \tau) - \langle F_f \wedge \alpha, Z_j \rangle (\tau) \Big) =$$

$$= \langle (F_f \wedge \alpha)(\gamma_j \tau), Z_j \rangle - \langle (F_f \wedge \alpha)(\tau), Z_j \rangle =$$

$$= [F_f \wedge \alpha, \tilde{\Delta}_\varphi]$$

Calculation of F_f:

Let η_1, η_τ be the basis for $H^1_{dR}(\mathbb{C}/\langle 1, \tau \rangle)$ corresp. to P_1, P_τ, s.t. $\left(\begin{array}{l} \langle \omega, \eta_1 \rangle = \int_{P_1} \omega \\ \langle \omega, \eta_\tau \rangle = \int_{P_\tau} \omega \end{array} \right)$.

Also have $dw, d\bar{w}$, where w is the std coordinate in $\mathbb{C}/\langle 1, \tau \rangle$.

$\langle , \rangle$	dw	$d\bar{w}$	η_1	η_τ
dw	0	$\frac{-1}{2\pi i}(\tau - \bar{\tau})$	1	τ
$d\bar{w}$	$\frac{1}{2\pi i}(\tau - \bar{\tau})$	0	1	$\bar{\tau}$

so $2\pi i \, dw = \tau \eta_1 - \eta_\tau$ and let: $\omega = 2\pi i \, dw$

$2\pi i \, d\bar{w} = \bar{\tau} \eta_1 - \eta_\tau$ $\qquad \eta = \frac{d\bar{w}}{\tau - \bar{\tau}}$.

Prop: Choose a basepoint $\tau_0 \in \mathcal{H}$. Then the section of $\tilde{\mathcal{L}}_r$ over $\mathcal{H}$ given

by $\langle F_f(\tau), \omega^j \eta^{r-j} \rangle = (-1)^j (2\pi i)^{j+1} \int^{\tau} (z-\tau)^j (z-\bar{z})^{r-j} f(z) \, dz$ $(0 \le j \le r)$

Proof: $\eta_\tau^j \eta_1^{r-j}$ are horizontal.

Then:

$$d\langle F_f, \eta_\tau^j \eta_1^{r-j}\rangle = \langle \nabla F_f, \eta_\tau^j \eta_1^{r-j}\rangle = \langle (2\pi i)^{r+1} f(\tau)\, dw^r d\tau, \eta_\tau^j \eta_1^{r-j}\rangle =$$

$$= (2\pi i)^{r+1} \tau^j f(\tau)\, d\tau.$$

$$\therefore \langle F_f, \eta_\tau^j \eta_1^{r-j}\rangle = (2\pi i)^{r+1} \int_{\tau_0}^{\tau} f(z) z^j\, dz.$$

$$\therefore \langle F_f, P(\eta_\tau, \eta_1)\rangle = (2\pi i)^{r+1} \int_{\tau_0}^{\tau} f(z) P(z,1)\, dz \qquad \forall P(x,y) \text{ homog of degree } r.$$

Now, just express $\omega^j \eta^{r-j}$ in terms of $\eta_\tau^{j'} \eta_1^{r-j'}$ $(j'=0,\dots r)$.

Prop: If we set $\tau_0 = i\infty$, then ~~the resulting expression~~ one can replace $\tilde{\Delta}_\varphi$ by the class of Δ_φ in the fiber, in the above expression, although the result is defined up to some period lattice.

Theorem: Let $\varphi: (A, t, \omega_A) \longrightarrow (A', t', \omega')$ be an isogeny of triples $\begin{pmatrix} \varphi^*\omega' = \omega \\ \varphi(t) = t' \end{pmatrix}$.

$$(A', t', \omega') = \left(\frac{\mathbb{C}}{\langle 1,\tau\rangle}, \frac{1}{N}, 2\pi i\, dz\right)$$

Then:

$$AJ(\Delta_\varphi)(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \frac{(-\deg(\varphi))^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{r+1}} \int_{i\infty}^{\tau} (z-\tau)^j (z-\bar{\tau})^{r-j} f(z)\, dz$$

Pf/

$$AJ(\Delta_\varphi)(\omega_f \wedge \omega^j \eta^{r-j}) = \langle F_f(\tau) \wedge \alpha, cl(\Delta_\varphi)\rangle_{(A')^r \times A^r} = \int_{\Delta_\varphi} F_f(\tau) \wedge \alpha =$$

$$= \int_{\mathrm{Graph}(\varphi)^r} F_f(\tau) \wedge \alpha \left(\begin{array}{c}\text{Change of variables} \\ (\varphi^r, id^r): A^r \xrightarrow{\sim} \mathrm{Graph}(\varphi)^r\end{array}\right) = \int_{A^r} \varphi^* F_f(\tau) \wedge \alpha =$$

$$= \langle \varphi^* F_f(\tau), \alpha \rangle_A.$$

In particular, set $\alpha = \omega_A^j \eta_A^{r-j}$, then $\varphi^*\omega' = \omega$, and $\varphi^*\eta' = (\deg\varphi)\cdot\eta$.

Then just plug-in into the formula. $\blacksquare$

Remark on F_f:

Can define $G_j(\tau) := \langle F_f(\tau), \omega^j \eta^{r-j} \rangle$, which are the $r+1$ components of the primitive F_f. ($j = 0 \ldots r$).

$\text{height}(G_j) = r - 2j$. Also, $\delta_r G_0 = f(\tau)$, and $\delta_{r-2j} G_j = j G_{j-1}$ $\forall\, 1 \le j \le r$ (exercise).

So in some sense, "$G_j = \delta_{r+2}^{-j-1} f$".

P-adic setting: Fix an odd prime p, and let F be a p-adic field (eg a finite ext. of $\mathbb{Q}_p$).

$$AJ_{X_r} : CH^{r+1}(X_r)_0(F) \longrightarrow Fil^{r+1} H_{dR}^{2r+1}(X_r/F)^\vee$$

§1. Étale Abel-Jacobi map:

Let F be an arbitrary field (eg F a # field).

Define the étale ~~AJ~~ cycle class map:

$$cl : CH^{r+1}(X_r)(F) \to H^{2r+2}(\overline{X}_r, \mathbb{Q}_p)(r+1) \circlearrowleft G_F = Gal(\bar{F}/F).$$

and:

$$AJ_{et} : \ker cl = CH^{r+1}(X_r)_0(F) \longrightarrow H^1(G_F, H^{2r+1}(\overline{X}_r, \mathbb{Q}_p)(r+1))$$
$$\| $$
$$Ext^1_{Rep(G_F)}(\mathbb{Q}_p, H^{2r+1}(\overline{X}_r, \mathbb{Q}_p)(r+1)).$$

Remark: $H_{dR}^{2r+2}(X_r \times A/\mathbb{C})$ contains a Hodge cycle $\omega_{\Theta_{\psi_A}^{r+1}} \wedge \eta_A^{r+1}$.

On the étale side,

$$\left(H^{2r+2}(\overline{X_r \times A}, \mathbb{Q}_p)(r+1)\right)^{G_K}$$ contains a nontrivial element (Tate class)

Tate's conjecture predicts this element should come from $CH^{r+1}(X_r)(F) \otimes \mathbb{Q}_p$.

But $H^{2r+1}(\overline{X}_r, \mathbb{Q}_p)(r+1)$ contains a copy of $H^1(\bar{A}, \mathbb{Q}_p)(1) \cong (T_p A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

So composing AJ_{et} with this projection, get an element of $H^1(G_F, V_p(A))$. So even without producing a point they can produce a cohomology class. So Tate's conj or Hodge conj or finiteness of Ш implies that we get a point.

So $\{AJ_{et}(\Delta_\varphi)\}_{\varphi : A \to A'}$ give an Euler system of classes $\kappa_\varphi \in H^1(H_c, V_f \otimes \eta_A^r)$

Definition of the étale A-J:

- Consider the excision exact seq. in étale cohomology for the pair $(X_r, X_r \setminus \pi^{-1}(P_{A'}))$:

$$\begin{tikzcd}
\cancel{K}' \arrow[r] \arrow[d] & H^{2r+1}_{et}(\overline{X}_r, \mathbb{Q}_p)(r+1) \arrow[r] \arrow[d, equal] & V_\Delta \arrow[r, two heads] \arrow[d] & \mathbb{Q}_p \ni 1 \arrow[r] \arrow[d] & 0 \\
\cancel{K} \arrow[r] & H^{2r+1}_{et}(\overline{X}_r, \mathbb{Q}_p)(r+1) \arrow[r] & H^{2r+1}_{et}(X_r \setminus \pi^{-1}(P_{A'}), \mathbb{Q}_p)(r+1) \arrow[r] & H^{2r}_{et}(\pi^{-1}(P_{A'}), \mathbb{Q}_p) \arrow[r, "cl"] & H^{2r+2}(\overline{X}_r)
\end{tikzcd}$$

$1 \mapsto cl_{\pi^{-1}(P_{A'})}(\Delta)$

(but $\varepsilon K = 0$)

Get an extension in $\mathrm{Ext}^1_{\mathrm{Rep}_p(G_F)}(\mathbb{Q}_p, \varepsilon H^{2r+1}_{et}(\overline{X}_r, \mathbb{Q}_p)(r+1))$.

<u>Comparison theorem</u>: The p-adic A-J arises from the étale p-adic A-J in the case where F is a p-adic field. Assume now $p \nmid 2D$ (so X_r has a smooth model over $\mathcal{O}_F$) — say $\mathfrak{X}_r$.
Assume also $p \nmid (\deg \varphi)$. Then the cycle Δ_φ also extends to a smooth cycle on $\mathfrak{X}_r$.

<u>Fact</u>: the extension V_Δ is a crystalline representation of G_F.

Consider the cristalline Dieudonné-module functor:

$$D = D_{cris} : \mathrm{Rep}_{F,cris} \longrightarrow \underline{ffm}_F \leftarrow \text{category of filtered Frobenius modules (over } F)$$

assume $F/\mathbb{Q}_p$ unramified.

<u>Def</u>: A filtered Frobenius module is a finite dim'l F-vectorspace, equipped with the following structures.

1) An exhaustive decreasing filtration (the Hodge filtration). $V \supset \dots \supset F^q V \supset F^{q+1} V \supset \dots \supset 0$.
2) A σ-linear endomorphism $\phi : V \to V$, which is "invertible".

The Dieudonné module functor $\mathbb{D} : \mathrm{Rep}_{F,cris} \to \underline{ffm}_F$ is defined by:

- $\mathbb{D}(V) = (V \otimes B_{cris})^{G_F}$. V is cristalline if $\dim_F \mathbb{D}(V) = \dim_{\mathbb{Q}_p} V$.
- Gives an equivalence of categories between $\mathrm{Rep}_{F,cris} \xrightarrow{\sim} \underline{ffm}^{ad}_F$

(ad means admissible, and $\underline{ffm}^{ad}_F$ is a certain subcategory of $\underline{ffm}_F$.

- $\mathbb{D}_{cris}(H^{2r+1}_{et}(\overline{X}_r, \mathbb{Q}_p)(r+1)) = H^{2r+1}_{dR}(X_r/F)(r+1)$.

Therefore $D_\Delta := \mathbb{D}(V_\Delta)$

So $D_\Delta \in \mathrm{Ext}^1_{\mathrm{ffm}_F}\left(F, H^{2r+1}_{dR}(X_r/F)(r+1)\right)$

<u>General fact</u>: if H is any ffm of strictly negative weight ($\mathrm{Fil}^0 H = 0$), then s.t. $H^{\phi=1}=0$.

$$\mathrm{Ext}^1_{\mathrm{ffm}}(F, H) = H/\mathrm{Fil}^0 H .$$

<u>Sketch of Pf</u>: (of fact)

Given $0 \to H \to D \xrightarrow{\rho} F \to 0$, choose two elements η^{hol} and $\eta^{frob} \in D$,

s.t $\eta^{hol} \in \mathrm{Fil}^0 D$, $\eta^{frob} \in D^{\phi=1}$, s.t $\rho(\eta^{hol}) = \rho(\eta^{frob}) = 1$.

Note that η^{hol} is well-def up to elts of $\mathrm{Fil}^0 D = \mathrm{Fil}^0 H$.

η^{frob} is well-def up to elt of $H^{\phi=1} = 0$ (by assumption).

Then $\eta^{hol} - \eta^{frob} \in H/\mathrm{Fil}^0 H$ ☒

Let now

$$\mathrm{CH}^{r+1}(X_r)_0(F) \xrightarrow{\mathrm{AJ}_{et}} \mathrm{Ext}^1_{\mathrm{Rep}_{F,cris}}\left(\mathbb{Q}_p, H^{2r+1}(\bar{X}_r, \mathbb{Q}_p)(r+1)\right)$$

$\wr\!|$ comp. isom.

$$\mathrm{Ext}_{\mathrm{ffm}_F}\left(F, H^{2r+1}_{dR}(X_r/F)(r+1)\right)$$

$\wr\!|$

$$H^{2r+1}_{dR}(X_r/F) \Big/ \mathrm{Fil}^{r+1} H^{2r+1}_{dR}(X_r/F)$$

$\wr\!|$ duality

$$\left(\mathrm{Fil}^{r+1} H^{2r+1}_{dR}(X_r/F)\right)^\vee .$$

duality $\cong$

Lecture 7.

To compute AJ_p, need to understand the extension which arises from the cycle Δ:

$$H^{2r-1}_{dR}(\pi^{-1}(P)) \to H^{2r+1}_{dR}(X_r/F) \to H^{2r+1}_{dR}(X_r - \pi^{-1}(P_{A'})) \to H^{2r}_{dR}(\pi^{-1}(P)) \to H^{2r+2}_{dR}(X_r)$$

$\Delta_\varphi \in \varepsilon CH^{r+1}(X_r)$, so can apply ε to this exact sequence.

Facts:

1) ε kills the leftmost and rightmost entries.

2) $\varepsilon H^{2r+1}_{dR}(X_r) = H^1_{dR}(C_T, \mathcal{L}_{r,r}) := \text{"}\mathbb{H}^1\text{"}(0 \to \mathcal{L}_{r,r} \xrightarrow{\nabla} \mathcal{L}_{r,r} \otimes \Omega^1_C(\log \cdot \text{cusps}) \to 0)$.

3) $\varepsilon H^{2r+1}_{dR}(X_r \setminus \pi^{-1}(P_{A'})) \cong H^1_{dR}(C_T \setminus \pi^{-1}(P_{A'}), \mathcal{L}_{r,r})$.

4) $\varepsilon H^{2r}_{dR}(\cancel{X_r}\,\pi^{-1}(P_{A'})) = \mathcal{L}_{r,r}(P_{A'})$.

So get:

$$0 \to H^1_{dR}(C, \mathcal{L}_{r,r}) \to H^1_{dR}(C - \{P_{A'}\}, \mathcal{L}_{r,r}) \to \mathcal{L}_{r,r}(P_{A'}) \to 0$$

Can think of $H^1_{dR}(C, \mathcal{L}_{r,r})$ as:

$$\left\{ \begin{array}{c} \mathcal{L}_{r,r}\text{-valued differentials of} \\ \text{the second kind} \end{array} \right\} \Big/ \text{exact.}$$

Recall the complex formula:

$$AJ_{\mathbb{C}}(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle F_f \wedge \alpha\,(\tau_{A'}), cl(\Delta) \rangle_{(A')^r \times A^r}$$

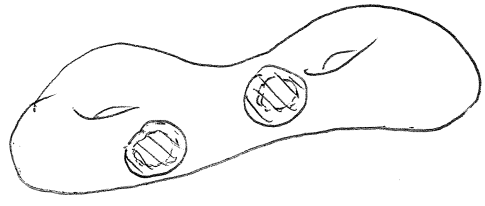
(for a "good" choice of primitive F_f of ω_f on $\mathcal{H}$).

<u>Theorem</u>: $AJ_p(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle F_f \wedge \alpha\,(P_{A'}), cl(\Delta) \rangle_{(A')^r \times A^r}$,
where F_f is the Coleman primitive of ω_f.

<u>Aside</u>: Coleman integration.

rigid-analytic spaces.

$P_{A'} \in \mathcal{A} \subset \mathcal{W} \subset C$, where $\mathcal{A}$ is an affinoid in C, $\mathcal{W}$ a wide open in C.



Let $\Phi : \mathcal{A} \to \mathcal{A}$ be a lift of the Frobenius morphism $\phi : C_{/k} \to C_{/k}$ (here, k is the residue field of $\mathcal{O}_F$).

This induces morphisms $\Phi : H^1_{dR}(W, \mathcal{L}^{rig}_{r,r}) \to H^1_{dR}(W, \mathcal{L}^{rig}_{r,r})$, and restricting get at the end an endo $\Phi : H^1_{dR}(C, \mathcal{L}^{rig}_{r,r}) \to H^1_{dR}(C, \mathcal{L}^{rig}_{r,r})$, which is the Frobenius...

<u>Lemma</u> (Coleman): There is a polynomial $P \in F[X]$ s.t.:

1) $P(\Phi)$ annihilates $H^1_{dR}(C, \mathcal{L}^{rig}_{r,r})$.

2) $P(\Phi)$ is injective on the space $H^0_{l.a}(C, \mathcal{L}^{rig}_r)^{\nabla = 0}$ (l.a : locally-analytic).

locally-analytic horizontal section of $\mathcal{L}_r$ on C.

3) $P(\Phi) \neq 0$.

(see Coleman's article in BU p-adic monodromy volume).

<u>Theorem</u> (Coleman): Let ω_f be in $H^0(C, \mathcal{L}_r \otimes \Omega^1_C(\text{log-cusps}))$. Then there is a <u>unique</u> locally-analytic primitive F_f over W satisfying:

1) $\nabla F_f = \omega_f$.

2) $P(\Phi)(F_f)$ is rigid-analytic section of $\mathcal{L}_r$ over $\mathcal{A}$.

<u>Remark</u>:

These properties characterize F_f uniquely: if F_f' is another one, then

$P(\Phi)(F_f - F_f')$ is rigid-analytic and <u>horizontal</u> section.

There are no such, so $P(\Phi)(F_f - F_f') = 0$.

But since $P(\Phi)$ is injective on $H^0_{l.a}(C, \mathcal{L}^{rig}_r)^{\nabla=0}$, $F_f = F_f'$.

<u>"Proof"</u> (of the result $AJ(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle F_f \wedge \alpha(P_{A'}), cl(\Delta) \rangle_{(A')^r \times A^r}$).

called "hol" before

$$AJ(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle \omega_f \wedge \alpha, \eta^{reg}_\Delta - \eta^{frob}_\Delta \rangle$$ where $\langle \cdot, \cdot \rangle$ is the pairing $H^1_{dR}(C, \mathcal{L}_{rr}) \times H^1_{dR}(C, \mathcal{L}_{rr}) \to H^2_{dR}(C, \mathcal{O}_C) = H^2_{dR}(C) = F$.

and where:

1) $\eta^{reg}_\Delta \in H^0(C \setminus \{P_{A'}\}, \Omega^1 \otimes \mathcal{L}_{r,r})$, and $res_{P_{A'}}(\eta^{reg}_\Delta) = cl(\Delta_\varphi)$.

2) η^{frob}_Δ is a meromorphic differential of the second kind, $res_{P_{A'}}(\eta^{frob}_\Delta) = cl(\Delta_\varphi)$,

and $res_P(\eta^{frob}_\Delta) = 0 \ (\forall P)$, Also, $\Phi[\eta^{frob}_\Delta] = [\eta^{frob}_\Delta]$

Definition of $\langle \cdot , \cdot \rangle$: can be written as a sum of residues:

$$AJ(\Delta\varphi)(\omega_f \wedge \alpha) = \sum_{P \in C} \mathrm{res}_P\left(\langle F_{f,P} \wedge \alpha, \eta_\Delta^{\mathrm{rig}} - \eta_\Delta^{\mathrm{frob}} \rangle\right)$$, where $F_{f,P}$ is any local primitive of ω_f.

$$= \sum_{P \in C} \mathrm{res}_P \langle F_{f,P} \wedge \alpha, \eta_\Delta^{\mathrm{rig}} \rangle - \sum_{P \in C} \mathrm{res}_P \langle F_{f,P} \wedge \alpha, \eta_\Delta^{\mathrm{frob}} \rangle = \mathrm{I} + \mathrm{II}.$$

Note that

$$\mathrm{I} = \mathrm{res}_{P_{A'}} \langle F_{f,P_{A'}} \wedge \alpha, \eta_\Delta^{\mathrm{rig}} \rangle = \langle F_{f,P_{A'}} \wedge \alpha (P_{A'}), cl(\Delta) \rangle_{\mathrm{Lrr}(P_{A'})}.$$

Now, the contribution of II is 0 if $F_{f,P}$ is the Coleman primitive at P:

$$\sum_{P \in C} \mathrm{res}_P \langle F_{f,P} \wedge \alpha, P(\Phi)\eta_\Delta^{\mathrm{frob}} \rangle = P(1)\left(\text{" } \mathrm{II}. \text{"} \right)$$

$$\shortparallel$$

$$\sum_{P \in C} \mathrm{res}_P \langle P(\Phi) F_{f,P} \wedge \alpha, \eta_\Delta^{\mathrm{frob}} \rangle \overset{\swarrow}{=} 0$$ residue thm for rigid differentials on curves.

Just use that $P(1) \neq 0$, to conclude the proof. $\blacksquare$

~~Doing~~ The same manipulations as over $\mathbb{C}$ show:

$$AJ(\Delta\varphi)(\omega_f \wedge \alpha) = \langle \varphi^* F_f(P_{A'}), \alpha \rangle_{A^r}.$$

In particular, $AJ(\Delta)(\omega_f \wedge \alpha) = \langle F_f(P_A), \alpha \rangle_{A^r}$.

<u>Now, need a formula for F_f.</u>

Idea: replace δ_K by θ.

<u>Key point</u>: if E is <u>ordinary</u> over F, then $H^1_{dR}(E)$ admits a canonical line, complementary to $\Omega^1(E/F)$.

Let $H^1_{dR}(E/F)^{\Phi} = \{ \eta \in H^1_{dR}(E) \mid \Phi\eta = (\text{p-adic unit}) \cdot \eta \}$ (slope-0 part).

<u>Over $\mathbb{C}$</u>, $(\delta_K f)(E,t,\omega) = \langle k s^{-1} \nabla f(E,t), \eta_\infty^{k+2} \rangle$, where $\eta_\infty \in H^{0,1}_{dR}(E/\mathbb{C})$, $\langle \omega, \eta_\infty \rangle = 1$.

<u>So, over F</u>: define $(\theta f)(E,t,\omega) = \langle k s^{-1} \nabla f(E,t), \eta_p^{k+2} \rangle$, where

$\eta_p \in H^1_{dR}(E/F)^{\Phi}$, normalized also so that $\langle \omega, \eta_p \rangle = 1$.

Since ~~η_p is defined only~~ η_p is defined only on the ordinary locus, θf is only a p-adic modular form (not classic anymore). (in the same sense that δf is a non-holomorphic m.f. b/c η_∞ is not holomorphic).

Let C^{ord} = complement of the supersingular residue discs on C.

Can check: $(\theta f)(\mathrm{Tate}_q, \xi_D, \frac{dt}{t}) = \sum n a_n q^n$ (if $f = \sum a_n q^n$).

(so θ corresponds to $q\frac{d}{dq}$ at the level of q-expansions).

Simple calculation: define

$$G_j(E,t,\omega) = \langle F_f(E,t), \omega^j \eta^{r-j} \rangle \quad \left(\begin{array}{c}\text{locally-analytic section of} \\ \underline{\omega}^{r-2j} \text{ on } C^{ord}\end{array}\right).$$

Then $\theta G_0 = \omega_f$, and $\theta G_j = j G_{j-1}$, $\forall 1 \leq j \leq r$.

So in some sense, G_j "=" $\theta^{-1-j} f$.

For $r < 0$, define $\theta^r f := \lim\limits_{\substack{j \to r \\ (j \geq 0) \\ (\text{in } \mathbb{Z}/p-1 \times \mathbb{Z}_p)}} \theta^j f$, ~~so~~ $\left(\theta^r f = \sum\limits_{\substack{n \\ p \nmid n}} n^r a_n q^n\right)$,

which is a p-adic modular form of wt $k+2r$.

Let $f^\flat :=$ "p-stabilization of f" $:= \sum\limits_{p \nmid n} a_n q^n$.

We can also construct a "p-stabilized" cycle $\Delta_f^\flat$.

Final formula:

$$AJ(\Delta_1^\flat)(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = \theta^{-1-j} f(A,t,\omega) \quad \left(\begin{array}{c}\text{on } \underline{\text{ordinary}} \text{ elliptic curves} \\ (\text{using } p \text{ split !})\end{array}\right).$$

A p-adic analogue of Ω_∞:

Defined by $\omega_A = \Omega_p \, \omega_{can}$ ($\omega_{can} = 2\pi i \, dz$).

Theorem: For all $0 \leq j \leq r$, ~~and for all $f \in$~~ $\left(f \in M_{r+2}(\Gamma_0(D), \varepsilon_D),\ \Psi_A^{k+j}(\Psi_A^*)^{-j}, j \in \mathbb{Z}\right)$.

$$\frac{L_p(f/K, \Psi_A^{k-1-j}(\Psi_A^*)^{1+j})}{\Omega_p^{2(r-2j)}} \overset{\text{up to rational factors}}{\sim} \left(AJ(\Delta_1^\flat)(\omega_f \wedge \omega_A^j \eta_A^{r-j})\right)^2.$$

Proof:

The formula of Waldspurger gives $(j \geq 0)$.

$$L\left(\ell/K, \left(\psi_A^{k+j}(\psi_A^*)^{-j}\right)^{-1}, 0\right) = \left(\left(\delta_k^j f\right)\left(\frac{\mathbb{C}}{\langle 1,\tau\rangle}, \frac{1}{D}, 2\pi i\, dz\right)\right)^2$$

Then get:

$$\frac{L\left(\ell/K, \left(\psi_A^{k+j}(\psi_A^*)^{-j}\right)^{-1}, 0\right)}{\Omega_\infty^{2(k-2j)}} \sim \delta_k^j f(A,t,\omega_A)^2 \quad (*) \quad \left(\omega \in \Omega^1(A/K)\right)$$

Since A has CM, both η_∞ and η_p belong to $H^1_{dR}(A/K)$, and they are equal.

So $(*) = \left(\theta^j f\right)(A,t,\omega_A)^2$

Putting-in the correct Euler factor, obtain:

$$\frac{L_p\left(\ell/K, \psi_A^{k+j}(\psi_A^*)^{-j}\right)}{\Omega_p^{2(k+2j)}} \sim \left(\theta^j f^\flat\right)(A,t,\omega_A)^2 .$$

~~Divide now by~~ $\cancel{\Omega_p^{2(k+2j)}}$ ~~to get~~:

$$\cancel{\frac{L_p\left(\ell/K, (\psi_A)^{k+j}(\psi_A^*)^{-j}\right)}{\Omega_p^{2(k+2j)}} \sim \theta^j f(A,t,\omega_{can})}$$

Multiplying back $\Omega_p^{2(k+2j)}$ get:

$$L_p\left(\ell/K, \psi_A^{k+j}(\psi_A^*)^{-j}\right) \sim \theta^j f^\flat(A,t,\omega_{can})^2 \qquad (j \geq 0.)$$

Replace now j by $-1-j$ $(j\geq 0)$, and get:

$$L_p\left(\ell/K, \psi_A^{k-1-j}(\psi_A^*)^{1+j}\right) = \theta^{-1-j} f^\flat(A,t,\omega_{can})^2 \qquad (\forall j \in \mathbb{Z}).$$

For $0 \leq j \leq r$, have a geometric interpretation of this values:

$$\frac{L_p\left(\ell/K, \psi_A^{k-1-j}(\psi_A^*)^{1+j}\right)}{\Omega_p^{2(r-2j)}} = \theta^{-1-j} f^\flat(A,t,\omega_A)^2$$

Now, if $0 \leq j \leq r$, $\dfrac{L_p\left(\ell/K, \psi_A^{k-1-j}(\psi_A^*)^{1+j}\right)}{\Omega_p^{2(r-2j)}} = AJ_p(\Delta_1^\flat)\left(\omega_f \wedge \omega_A^j \eta_A^{r-j}\right)$

A very special case

Spse $f = \Theta_{\psi_A^{r+1}}$. Then:

$$\frac{L_p(\Theta_{\psi_A^{r+1}}, \psi_A^{r+1-j}(\psi_A^*)^{(1+j)})}{\Omega_p^{2(r-2j)}} = AJ(\Delta_i^q)(\omega_\theta \wedge \omega_A^j \eta_A^{r-j})^2.$$

But the LHS factors into a product of two L-functions:

$$LHS \sim \frac{L_p^{katz}(\psi_A^{-j}(\psi_A^*)^{1+j})}{\Omega_p^{-1-j}} \quad \frac{L_p^{katz}(\psi_A^{1+(r-j)}(\psi_A^*)^{-(r-j)})}{\Omega_p^{1+2(r-j)}}$$

$\in K^\times$

So we find that

$$AJ(\Delta_i^q)(\omega_\theta \wedge \omega_A^j \eta_A^{r-j})^2 \sim \cancel{AJ(\Delta_i^*)} \frac{L_p(\psi_A^{-j}(\psi_A^*)^{1+j})}{\Omega_p^{-1-j}}$$

For $j=0$, have:

$$AJ(\Delta_i^q)(\omega_\theta \wedge \eta_A^r)^2 \sim \Omega_p \cdot L_p(\psi_A^*).$$

We recover a formula of Rubin: assuming the Hodge conjecture,

$$AJ(\Delta_i^q(\omega_\theta \wedge \eta_A^r)) = \log_\omega(P_1) \quad (P_1 \text{ some point on } A).$$

and Rubin proved $\Omega_p L_p(\psi_A^*) \sim (\log_\omega(P_1))^2$ unconditionally.

Ingredients in Rubin's proof:

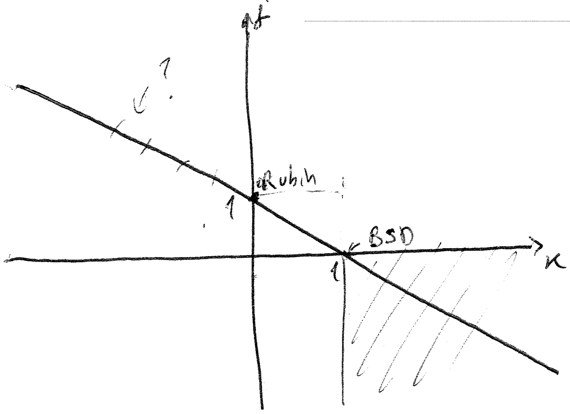
1) Elliptic units k, and show $\Omega_p^{-1} \dfrac{\log k^2}{\langle k,k\rangle} = \dfrac{L_p(\psi^*)}{L_p'(\psi_A)}$

2) If $L'(\psi_A^{-1},0) \neq 0$ (BSD expect $rk \geq 1$), $GZ, K \Rightarrow \exists P$ on $A(\mathbb{Q})$, s.t. P generates $Sel(A)$

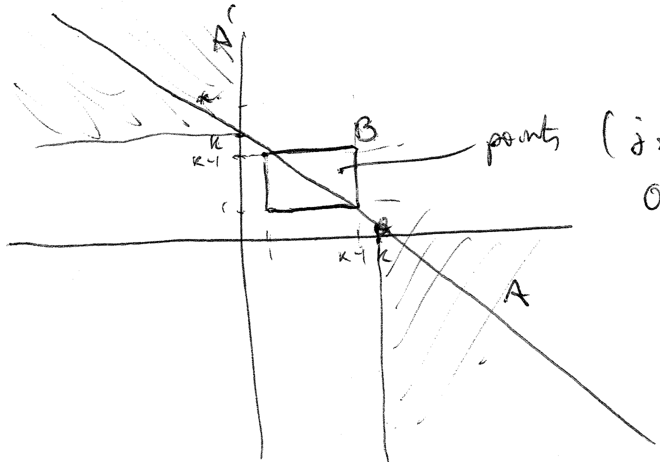
3) Can replace k by P; $\Omega_p^{-1} \dfrac{(\log p)^2}{\langle P,P\rangle} = \dfrac{L_p(\psi^*)}{L_p'(\psi_A)}$.

4) Perrin-Riou $\Rightarrow L_p'(\psi_A) \sim \langle P,P\rangle$.

Recall Katz's p-adic L-function. Consider ψ of infinity type (k,j).



We consider the Rankin-Selberg p-adic L-function of weight k.



points $(j+1, r+1-j)$

$0 \leq j \leq r \quad (r = k-2)$.

Then $$\underset{\text{type}(j+1,r+1-j)}{L_p^{RS}(1,\chi)} = \left(AJ(\Delta^4)(\omega^j\eta^{r-j}\wedge\omega_f)\right)^2$$

If f is of the form $\theta_{\psi^{r+1}}$, the LHS is:

$$L_p^{Katz}(\psi^{-r-1}, \chi)\, L_p^{Katz}(\psi^{k-r-1}\cdot\chi).$$

Write now $\chi = \psi^{j+1}(\psi^*)^{r+1-j}$. Then the RHS is:

$$L_p^{Katz}(\psi^{-r-1}\psi^{j+1}(\psi^*)^{r+1-j})\, L_p^{Katz}((\psi^*)^{-r-1}\psi^{j+1}(\psi^*)^{r+1-j})$$

$$= \underbrace{L_p^{Katz}(\psi^{j-r}(\psi^*)^{r-j+1})}_{\text{outside range of interpolation}}\ \underbrace{L_p^{Katz}(\psi^{j+1}(\psi^*)^{-j})}_{\text{in the range of interpolation}}$$

$$\sim L_p^{Katz}(\psi^{j-r}(\psi^*)^{r-j+1})\, L(\psi^{j+1}(\psi^*)^{-j}) \quad \text{(up to periods).}$$

This is a generalization of Rubin's result.

Next we will see application to the Griffiths' group.

Recall: if X is any variety,

$$\mathrm{Griff}^k(X) := CH^k(X)_0 \big/ CH^k(X)_{alg}$$

Work of Griffiths, Clemens gives examples of varieties $/\mathbb{C}$ for which $\mathrm{Griff}^k(X)$ is not torsion. Require transcendental elements in the base field, so doesn't work for number fields.

But B. Harris showed that, for $C : X^4 + Y^4 + Z^4 = 0$, $g(C) = 3$, $X = \mathrm{Jac}(C)$, then $C - [-1]^* C \in Z^2(X)_0$ is nontrivial in $\mathrm{Griff}^2(X)$.

In this case, $X = \mathrm{Jac}(C) \overset{\text{isog}}{\sim} A \times A \times A$, A an e.c. with CM by $\mathbb{Z}[i]$ $(A \simeq (Y^2 = X^3 - X))$.

Bloch showed: $C - [-1]^* C$ is non-torsion in $\mathrm{Griff}^2(X)$.

This is related to the fact that $L(\psi^3, s)$ has sign -1.

Over # fields there are very few examples of nontrivial elts. of $\mathrm{Griff}^k(X)$.

One of them is due to C. Schoen:

$$W \times W \downarrow X(N).$$

The usual Heegner cycles generate a subgroup of $\mathrm{Griff}^2(W \times W)$ which is not finitely-generated.

Beilinson-Bloch conjecture:

Let X be defined over some # field E.

Conj: $\operatorname{rk} CH^k(X)_0 = \operatorname{ord}_{s=k} L(H^{2k-1}(\bar{X}), s) = \operatorname{ord}_{s=0} L(H^{2k-1}(\bar{X})(k), s)$

There is an AJ map:

$$AJ : CH^k(X)_0 \longrightarrow H^1(E, H^{2k-1}(\bar{X})(k)).$$

And there is a refined version of Beilinson-Bloch, using filtrations on both sides:

$$F^i H^*(\bar{X}) = \varinjlim_{\substack{Y \subseteq \bar{X} \\ \text{closed subschemes} \\ \text{of codim } i}} \operatorname{Ker}\left(H^*(\bar{X}) \to H^*(\bar{X} \setminus Y)\right)$$

"= cohomology classes supported on a closed subscheme of codim i"

(Coniveau filtration)

Bloch-Ogus

$F^i (CH^k(X) \otimes \mathbb{Q}) = \{ x \in CH^k(X) \otimes \mathbb{Q} \mid x \text{ is nullhomologous in codim } i \}$

Rmk: nullhomologous in codim i means

$\exists$ a cycle c representing γ, and a closed subscheme $Y \subseteq \bar{X}$ of codim i, s.t

$\mathrm{Supp}(c) \subseteq Y$ and c is homologous to 0 in Y.

(See Bloch-Ogus '74 "Twisted Poincaré duality...")

One can see that $F^0 H^{2k-1}(\bar{X}) = H^{2k-1}(\bar{X})$, and $F^k H^{2k-1}(\bar{X}) = 0$.

On the Chow groups, $F^k CH^k(\bar{X})_{\mathbb{Q}} = 0$, and $F^0 CH^k(\bar{X})_{\mathbb{Q}} = CH^k(\bar{X})_0 \subsetneq CH^k(\bar{X})_{\mathbb{Q}}$

Moreover, $CH^k(\bar{X})_{alg} \otimes \mathbb{Q} \cong F^{k-1} CH^k(\bar{X})_{\mathbb{Q}}$.

Bloch shows that AJ is compatible with these filtrations, and get maps:

$$gr^i CH^k(X) \xrightarrow{AJ} H^1(E, gr^i H^{2k-1}(\bar{X})(k))$$

<u>Example</u>:

$X := A \times W$, and look at $CH^2(X)$.

$$\left.\begin{array}{l} F^0 = CH^2(X)_0 \otimes \mathbb{Q} \\ \quad\;\; \cup\!\mid \\ F^1 = CH^2(X)_{alg} \otimes \mathbb{Q} \end{array}\right) \mathrm{Griff}^2 \otimes \mathbb{Q}.$$

$$\begin{array}{l} \quad\;\; \cup\!\mid \\ F^2 = 0 \end{array}$$

So AJ gives a map $\mathrm{Griff}^2 \otimes \mathbb{Q} \to H^1(K, gr^0 H^3(\bar{X})(\,))$.

$H^3(A \times W)$ by Künneth contains $M_\psi \otimes M_{\psi^2} = M_{\psi^3} \oplus M_\psi(1)$. $\Rightarrow$ gets

$\omega \wedge \omega_f$

$$H^1(K, V_{\psi^3}(-)).$$

a generalized Heegner cycle

$$\downarrow$$

So can evaluate AJ on $V(\omega \wedge \omega_f)$, to

$$H^1(\mathbb{Q}_p, V_{\psi^3})$$

obtain $L_p(\psi^{-1}(\psi^*)^2) L_p(\psi)$. Deduce from this the following:

elliptic curve!

<u>Theorem</u>: If $L_p(\psi^{-1}(\psi^*)^2) \neq 0$ and $L_p(\psi) \neq 0$ (note $L_p(\psi) = L_p(A_{\mathbb{Q}}, 1)$)

Then: Δ_1 is nontorsion in $\mathrm{Griff}^2(X)$.

More generally:

Thm: If $L_p(\psi^{-r}(\psi^*)^{r+1}) \not\equiv 0$ and $L_p(\psi) \not\equiv 0$, then

Δ_1 (in X_r) is nontorsion in $\mathrm{Griff}^{r+1}(X_r)$.

This can be generalized to any of the pairs of points in the k-j plane, to get lots of nontorsion cycles.

Compare with G-Z:

They construct a point $P \in E(K)$, which is either in $\mathbb{Q}$ or not.

In our situation, if the L-function factors, then get two graded pieces in the Coniveau filtration..