Goal of the lectures: give a proof of Katok - Sarnak formula using spectral theory.

Kronecker limit formulas (KLF).

It is about the Epstein zeta function

$$G(\tau, s) = \frac{1}{2} \sum_{m,n}{}' \frac{y^s}{|m\tau + n|^{2s}} \qquad \operatorname{Re}(s) > 1.$$

$$= \zeta(2s)\, E(\tau, s) \qquad E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma \tau)^s \qquad \Gamma = PSL_2(\mathbb{Z}), \quad \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}$$

$$2G(\tau, s) = \frac{\pi}{s-1} + 2\pi\left(\gamma - \log 2 - \frac{1}{2} \log\left(y |\eta(\tau)|^4\right)\right) + O(s-1)$$

$$= -\left(1 + \log(2\pi y |\eta(\tau)|^4)\right) s + O(s^2)$$

We may write:

$$\bigstar \quad \lim_{s \to 1} \left( 2G(\tau, s) - \frac{\pi}{s-1} \right) = 2\pi(\gamma \ldots\ldots).$$

Let $D \neq 1$ be a fundamental discriminant

$$Q_D = \{ Q(x,y) = \underbrace{Ax^2 + Bxy + Cy^2}_{[A,B,C]} : B^2 - 4AC = D \} \qquad (\text{If } D < 0, \text{ require } A > 0).$$

$\Gamma$ acts on $Q_D$ by $g.Q(x,y) = Q(\delta x - \beta y, -\gamma x + \alpha y)$ if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

This is set up so that the roots of $g.Q$ are $g.$(roots of $Q$).

Let $\Lambda_D = Q_D / \Gamma$, which is a finite set, $|\Lambda_D| =: h_D$.

Let $\Gamma_Q = \{ g \in \Gamma : gQ = Q \}$. If $D < 0$, then $\Gamma_Q$ is finite (~~$\Gamma_Q = \{1\}$ except if $D = -3, -4$~~).

$|\Gamma_Q| = 1$ unless $\begin{cases} Q \sim [a, 0, a] & (|\Gamma_Q| = 2). \\ Q \sim [a, a, a] & (|\Gamma_Q| = 3). \end{cases}$ Call $w_Q := |\Gamma_Q|$.

Each class $[Q]$ ~~in~~ in $Q_D / \Gamma$ corresponds to a point $z_Q = \dfrac{-B + \sqrt{D}}{2A}$.

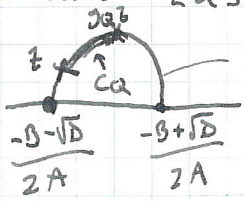
(note $Q(z_Q, 1) = 0$), called a CM point.

If $D > 0$, $\Gamma_Q$ is infinite cyclic with a generator $g_Q = \pm \begin{pmatrix} t + \frac{Bu}{2} & Cu \\ -Au & t - \frac{Bu}{2} \end{pmatrix}$

where $t^2 - Du^2 = 4$, $t, u \geq 1$, $u$ minimal.

$\varepsilon := \dfrac{t + u\sqrt{D}}{2} \in K = \mathbb{Q}(\sqrt{D})$ is the smallest unit with $\varepsilon > 1$, $\|\varepsilon\| = 1$.

Each class $[Q]\in\Lambda_D$ corresponds to a closed geodesic on $\mathbb{H}/\Gamma$




$S_Q$ : geodesic $A|z|^2+Bx+c$.

$C_Q$ : arc from $z$ to $g_Q z$, for a fixed $z$.

The first application of the KLF is to the class number formula.

$$\zeta_K(s) \doteq \zeta(s)\cdot L(s,\chi_D) \quad , \quad \chi_D(n)=\left(\frac{D}{n}\right). \qquad \alpha=\frac{1}{2}(1-\mathrm{sgn}(D)).$$

Let $\Lambda(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ ; $\Lambda(s,\chi_D)=\pi^{-s/2}\Gamma\left(\frac{s+\alpha}{2}\right)|D|^{s/2}L(s,\chi_D)$

<u>Thm</u> (Dirichlet, Hecke)

$$\Lambda(s)\Lambda(s,\chi_D)=\sum_{[Q]\in\Lambda_D}\begin{cases}2\sqrt{\pi}\, w_Q^{-1}E^*(\tau_Q,s) & \text{if } D<0\\ \int_{C_Q}E^*(\tau,s)\frac{d\tau}{Q(\tau,1)} & \text{if } D>0\end{cases}$$

and $E^*(\tau,s)=\Lambda(2s)E(\tau,s)$

Now, using KLF, residue is <u>independent</u> of $\tau$ !

$$2G(\tau,s)=\frac{\pi}{s-1}+\dots$$ + Thm of Dirichlet–Hecke, get:

<u>Thm</u> (Class Number Formula – CNF).

$$L(1,\chi_D)=\begin{cases}\frac{2\pi}{w_D}|D|^{-1/2}h_D & \text{if } D<0. \qquad \text{(Dirichlet)}\\ D^{-1/2}\cdot 2\cdot h_D\log(\varepsilon_D) & \text{if } D>0. \qquad \text{(Hecke)}\end{cases}$$

$\left(2\log(\varepsilon_D)=\int_{C_Q}\frac{d\tau}{Q(\tau,1)}\right)$ = length of the geodesic $C_Q$ ).

Using the functional equation, this can also be written as

- $L(0,\chi_D)=2h_D/w \qquad D<0$
- $L'(0,\chi_D)=\frac{1}{2}h_D\log\varepsilon \qquad D>0.$

For $D<0$ there is also a formula for $L'(0,\chi_D)$, due to Kronecker:

$$L'(0,\chi_D)=\frac{2h}{w}\left(-\frac{1}{2}\log|D|+\log 4\pi+\sum_{[Q]}\log\left(y_Q|\eta(\tau_Q)|^4\right)\right).$$

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One more formula (due to Lerch–Hurwitz) $(D<0)$.

$$L'(0,\chi_D) = -L(0,\chi_D)\log|D| + \sum_{n=1}^{|D|-1} \chi_D(n)\log\Gamma\left(\frac{n}{|D|}\right).$$

Putting all this together, we obtain the formula of Chowla–Selberg: (recall $\eta^{24}=\Delta$)

$$\prod_{j=1}^{h_D} \Delta(\tau_j) \doteq \left(\prod_{n=1}^{D} \Gamma\left(\frac{n}{|D|}\right)^{|D|/n}\right)^{3w}.$$

## § Genus characters

For any $D$, let $\chi$ be a real character of the class group. Let $L(s,\chi)$ be the associated $L$-function.

Any such character corresponds to ~~the~~ a factorization $D = d_1\cdot d_2$, $d_1, d_2$ fundamental discriminants.

$$\chi_{d_1}(Q) = \begin{cases} \left(\frac{d_1}{r}\right) & \text{if } (A,B,C,d_1)=1 \text{ \& } Q \text{ represents } r \text{ and } (r,d_1)=1 \\ 0 & \text{else.}\end{cases}$$

$Q=[A,B,C]$

Then we have $\chi_{d_1}(-Q) = \operatorname{sgn}(d_1)\,\chi_{d_1}(Q)$.

On $\mathcal{Q}_D/\Gamma$, $\chi_{d_1}(Q) = \chi_{d_2}(Q) = \chi(Q)$.

Another facts due to Kronecker:

$$L(s,\chi) = L(s,\chi_{d_1})\,L(s,\chi_{d_2}).$$

D<0 case:

$D = d_1\cdot d_2 < 0$. Assume $d_1 > 0$, $d_2 \leq -4$ ← to avoid ~~...~~ units...

① $L(s,\chi) = L(s,\chi_{d_1})\,L(s,\chi_{d_2})$

② $L(1,\chi_{d_2}) = 2\pi|d_2|^{-1/2} h_2$

③ $L(1,\chi_{d_1}) = (d_1)^{-1/2}\log(\varepsilon_{d_1})\, h_1$.

④ $\Lambda(s,\chi_{d_1})\,\Lambda(s,\chi_{d_2}) = \sum_{Q\in\mathcal{Q}_D/\Gamma} \chi(Q)\,E^*(z_Q,s)$

⑤ KLF

$\Longrightarrow$ ~~Solution~~ Kronecker's solution to Pell's equation:

$$\varepsilon = \prod_{[Q]} \left(y_Q\,|\eta(\tau_Q)|^4\right)^{-\chi(Q)/h_1h_2}.$$

Example: $D = -35 = 5\cdot(-7)$. $h_1 = h_2 = 1$, $h_D = 2$.

$\tau_Q \in \left\{ \frac{-5+\sqrt{-35}}{6}, \frac{-1+\sqrt{-35}}{18} \right\}$ $\qquad \varepsilon = \frac{3+\sqrt{5}}{2}$.

$$\frac{3+\sqrt{5}}{2} = 3\,\frac{\eta\left(\frac{-5+\sqrt{-35}}{6}\right)}{\eta\left(\frac{-1+\sqrt{-35}}{6}\right)}.$$

Case $D > 0$.

Case 1: $D = d_1 d_2$, with $d_1 > 0$, $d_2 > 0$

Case 2: $D = d_1 d_2$ with $d_1 < 0$, $d_2 < 0$.

In either case, one obtains a formula for $L(s,\chi)$ in terms of cycle integrals.

(The case of $J_K \rightsquigarrow \sum_{C_Q} \int_{C_Q} E^*(\tau,s) \frac{d\tau}{Q(\tau,1)}$ is a special case: $D = 1\cdot D$).

Thm (Hecke): $\Lambda(s,\chi) = \Lambda(s,\chi_{d_1})\Lambda(s,\chi_{d_2}) =$

$$= \sum_{[Q]} \chi(Q)\cdot \begin{cases} \int_{C_Q} E^*(\tau,s)\frac{d\tau}{Q(\tau,1)} & d_1,d_2>0 \\ \int_{C_Q} i\,\partial_\tau E^*(\tau,s)\,d\tau & d_1,d_2<0. \end{cases} \qquad (\star)$$

For the case $d_1<0$, $d_2<0$:

Using $\partial_\tau G(\tau,s) = \partial_\tau \sum_{m,n}' \frac{y^s}{|m\tau+n|^{2s}} = -\frac{is}{2} y^{s-1} \sum \frac{1}{(m\tau+n)^2|m\tau+n|^{2(s-1)}}$

Using Hecke's limit formula

$$\lim_{s\to 0} \partial_\tau(G(\tau,s)) = \frac{\pi}{3}\left(E_2(\tau) - \frac{3}{\pi y}\right) \quad \text{where } E_2(\tau) = 1 - 24\sum \sigma(n) q^n.$$

In this case, $L(0,\chi) = L(0,\chi_{d_1})L(0,\chi_{d_2}) = \frac{1}{2}\sum_Q \chi(Q) \underbrace{\int_{C_Q}\left(\left(E_2(\tau) - \frac{3}{\pi y}\right)d\tau\right)}_{\Psi(Q)}$

$\Psi(Q)$ can be written in terms of Dedekind sums.

RHS of $(\star)$ will be written $\mathrm{Tr}_{\chi,D}(E^*(\tau,s))$

LHS of $(\star)$ is just $\Lambda(s,\chi_{d_1})\Lambda(s,\chi_{d_2})$.

## Modern point of view:

We define a weight 1/2 Eisenstein series

$$E^*_{1/2}(z,s) := \Lambda(2s)\, 2^s\, y^{s/2-1/4} + \Lambda(2-2s)\, 2^{1-s}\, y^{3/4-s/2} + \sum_{\substack{n\equiv 0,1 \\ (\mathrm{mod}\ 4) \\ n\neq 0}} b(n,s) \underbrace{W(4\pi|n|y)e(nx)}_{\substack{\mathcal{E}(nz) \\ \uparrow \\ \text{some function} \\ \text{like exp}\ldots}}$$

where, for fundamental $d$, we define

$$b(d,s) \doteq \Lambda(s,\chi_d)$$

~~Th~~ For $b(dm^2,s)$ definition comes from Shimura-Hecke relation:

$$m^{s-\frac12}\sigma_{1-2s}(m)\, b(d,s) = m \sum_{\substack{n|m \\ n>0}} n^{-3/2}\left(\frac{d}{n}\right) b\left(\frac{m^2 d}{n^2}, s\right).$$

So $\Lambda(s,\chi_{d_1})$ is the $d_1$-coefficient of $E^*_{1/2}$ (and same for $d_2$).

Then $E^*_{1/2}(z,s)$ transforms with weight ~~½~~ 1/2 for $\Gamma_0(4)$.

(like the theta function $\Theta(z) = \sum e^{2\pi i n^2 z}$)

~~[illegible]~~

So the "trace" of a weight 0 object $E^*(\tau,s)$ is ~~the~~ related to the product of Fourier coefficients of a weight-1/2-object.

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Q: is there an analogue of the theorem of Hecke-Dirichlet for general "Maass forms" $\varphi(z)$ (non-holomorphic but satisfying an equation $\Delta\varphi = \lambda\varphi$, $\Delta$ = hyperbolic Laplacian)

Yes: D = 1.D by Maass & Katok-Sarnak.

Thm (Maass, Katok-Sarnak, ...) + Duke-Imamoglu-Toth.

$$\sum_{[Q]} \chi(Q) \begin{cases} \int_{C_Q} \varphi(\tau)\frac{d\tau}{Q(\tau,1)} & d_1, d_2, D>0, \\ \varphi(\tau_Q) & D<0 \\ \int_{C_Q} \partial_\tau \varphi(\tau)\, d\tau & d_1, d_2 < 0 \end{cases} = b(d_1)\, b(d_2)$$

Notation $\mathcal{H} \ni z$, $PSL_2(\mathbb{R})$ acts $z \mapsto \frac{az+b}{cz+d}$

$ds^2 = \frac{dx^2+dy^2}{y^2}$ invariant metric, $d\mu = \frac{dx\,dy}{y^2}$ invariant measure.

$\Delta_{\mathcal{H}} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, we'll just call it $\Delta$.

For $\Gamma = PSL_2(\mathbb{Z})$, on $\Gamma \backslash \mathcal{H}$ we have a measure $\mu$ as well.

Choose a fundamental domain $F = \{ z \in \mathcal{H} : |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2} \}$.

Can compute $\mathrm{Area}(F) = \pi/3$

Spectral theory: understand the $\Delta$-operator acting on functions on $\Gamma \backslash \mathcal{H}$.

Let $f, g : \Gamma \backslash \mathcal{H} \to \mathbb{C}$,

$\langle f, g \rangle = \int_F f(z)\overline{g(z)}\, d\mu$

Denote by $\mathcal{L}^2(\Gamma \backslash \mathcal{H}) = \{ f : \Gamma \backslash \mathcal{H} \to \mathbb{C} : \langle f, f \rangle < \infty \}$.

- Spectral theorem for $PSL_2(\mathbb{Z})$.

$$f(z) = c_0 + \frac{1}{4\pi}\int_{-\infty}^{\infty} c(t)\, E\left(z, \tfrac{1}{2}+it\right) dt + \sum_{\varphi} c_\varphi(f) \cdot \varphi.$$

The $\varphi$'s in the sum are the non-constant eigenfunctions of $\Delta$.

$$c(t) = \int_F f(z)\, \overline{E\left(z, \tfrac{1}{2}+it\right)}\, d\mu(z)$$ (Rmk: $E$ is not in $\mathcal{L}^2$).

$$c_\varphi = \frac{\langle f, \varphi \rangle}{\langle \varphi, \varphi \rangle}$$

Fourier coefficients

If $f$ is $\Gamma$-invariant, in particular it's invariant under translation.

So $f(x+iy) = \sum_{n \in \mathbb{Z}} a_n(y)\, e^{2\pi i n x}$

Def: $\mathcal{L}^2_{cusp} = \{ f \in \mathcal{L}^2 : a_0(y) \equiv 0 \} = \{ f : \int_0^1 f(z+t)\,dt = 0 \ \forall z \}$.

$\mathcal{E}$ = orthogonal complement of $\mathcal{L}^2_{cusp}$ in $\mathcal{L}^2$.

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If we are interested in equidistribution results, then functions on $\mathcal{E}$ can handle neighborhoods of cusps. For general regions, one needs the cuspidal spectrum.

$\mathcal{E}$ can be constructed directly, by averaging:

Let $k: \mathbb{H} \to \mathbb{C}$ be fast-decay (say, compactly supported). Then

$$\sum_{\gamma \in \Gamma} k(\gamma z) \text{ is invariant.}$$

A variation: $\psi(z+1) = \psi(z) \Rightarrow \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\gamma z)$ is $\Gamma$-invariant as well.

These are called "Poincaré series".

$\psi(y) e^{mx}$ is sufficient. (simplest case: $m=0$)

For example, $E(z, \psi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\operatorname{Im} \gamma z)$ for $\psi: (0,\infty) \to \mathbb{C}$ compactly-supported.

<u>Lemma</u>: $\langle f, E(z,\psi) \rangle =$

$$= \int_0^\infty a_0(y) \overline{\psi(y)} \, dy$$

If we use Mellin-transform, $\hat{\psi}(s) = \int_0^\infty \psi(y) y^{-s} \frac{dy}{y}$

Then $\psi(y) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{\psi}(s) y^{-s} \, ds$ $\qquad (\sigma > 1)$

$(\sigma)$ ← vertical line $\operatorname{Re}(s) = \sigma$

We get then $E(z, \psi) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{\psi}(s) E(z,s) \, ds$

(the $E(z,s)$) These are eigenfunctions of the Laplacian $\left( \Delta E(z,s) = s(s-1) E(z,s) \right)$

but they are <u>never</u> in $\mathcal{L}^2$.

$E(z,s)$ is originally defined for $\operatorname{Re}(s) > 1$, can be analytically continued, with a pole at $s=1$, $\operatorname{Res}_{s=1} E(z,s) = $ constant in $z$.

We can move $\sigma \to 1/2$, and then we have

$$E(z,\psi) = c_0 + \frac{1}{4\pi}\int_{(1/2)} E(z,s)\,\hat{\psi}(s)\,ds.$$

About cusp forms,

Claim: $L^2_{cusp}$ is spanned by eigenfunctions of $\Delta$.

$$(\Delta+\lambda)f = 0 \qquad (\Delta \text{ is symmetric \& negative-definite, so } \lambda \geq 0).$$

[ Detour: If $(\Delta+\lambda)f=0$, $\lambda = s(1-s) = \frac{1}{4}+r^2$, $s=\frac{1}{2}+ir$. ← Usual normalization, comes up often.

Rmk: if $f$ is as above,

$$f = \sum_{n\in\mathbb{Z}} a_n(y)e^{nx} \to \Delta\left(a_n(y)e^{nx}\right) = -y^2 a_n''(y)\cdot 4\pi^2 n^2 e(nx).$$

Therefore $a_n(y)$ has to satisfy $y^2 a_n''(y) + \left(s(1-s) - 4\pi^2n^2\right)a_n(y) = 0$.

Two l.i. solutions: $\sim \underset{\text{not in } L^2}{\cancel{e^y}},\ e^{-y}$. ]

We want to construct an integral kernel that represents the "resolvent".

$\lambda$ is in the resolvent set if $\exists$ bounded operator $R: L^2 \to L^2$ which is the inverse: $(\Delta+\lambda)Rf = f$, $R(\Delta+\lambda)f = f$ (whenever defined).

General construction of such operators.

$k: H\times H \to \mathbb{C}$ compactly supported (or with fast enough decay).

Define $L: L^2(\Gamma\backslash H) \to L^2(\Gamma\backslash H)$

$$f \mapsto \int_H k(z,w)f(w)\,d\mu(w)$$

Also, $Lf = \int_F K(z,w)f(w)\,d\mu(w)$, where $K(z,w) := \sum_{\gamma\in\Gamma} k(z,\gamma w)$.

We will use $k$ as $k(u(z,w))$, $u(z,w) = \dfrac{|z-w|^2}{\operatorname{Im} z\operatorname{Im} w}$ (hyp. distance between $z$ & $w$).

With such a choice, the Laplacian will satisfy

$$\Delta_z k(z,w) = \Delta_w k(z,w)$$, which shows that "$L$" operator commutes with $\Delta$.

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$K(z,w)$ is not a cusp form in $w$.

Let $H(z,w) = \int_0^1 K(z,w+t)\,dt$. This is again a Poincaré series

$$H(z,w) = \int_0^1 \sum_{\tau\in\Gamma_\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} k(z,\tau\gamma w+t)\,dt$$

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \underbrace{\sum_{n\in\mathbb{Z}}\int_0^1 k(z,\gamma w+n+t)\,dt}_{\int_{-\infty}^{\infty} k(z,\gamma w+t)\,dt}$$ ← It is an incomplete Eisenstein series.

Letting $\hat{K} = K - H$, then it acts on cusp forms in the same way as $K$

Fact: $\hat{K}$ is bounded on $\mathcal{F}\times\mathcal{F}$ — $\hat{K}$ is a Hilbert-Schmidt type operator on $L^2_{cusp}(\Gamma\backslash\mathbb{H})$.

$\Rightarrow$ $Lf = \int_{\mathcal{F}} \hat{K}(z,w) f(w)\,d\mu(w)$ is a compact operator

$\Rightarrow$ $L$ has a basis consisting of eigenfunctions.

A special ~~way of~~ choice of $k$ gives the ~~resolvent~~ kernel for the resolvent.

We want to choose $k$ to be $g_s(u)$ s.t.

$$(\Delta_w + s(1-s))g_s = \text{Dirac-}\delta \text{ at } z=w.$$

This is an ODE for $g_s$:

$$u(u+1)g_s'' + (2u+1)g_s' + s(1-s)g_s = 0$$

Want $g_s \sim C\log(u)$ as $u\to 0^+$.

If $\mathrm{Re}(s)>1$, then $G_s(z,w) := \sum_{\gamma\in\Gamma} g_s(u(z,\gamma w))$ is called the Green function,

and it's the kernel for the resolvent:

$$\boxed{(\Delta + s(1-s))\int_{\mathcal{F}} G_s(z,w)\,u(w)\,d\mu(w) = u(z)}$$

We may compute the spectral expansion of $G_s$. Fix $a$, with $\operatorname{Re}(a) > 1$.

$$G_a(z,w) = c_0(z) + \text{Eisenstein part} + \sum_j c_j(z)\overline{\varphi_j(w)} \qquad \varphi_j : \text{eigenfunctions of } \Delta.$$

Now, use the fact that

$$\left(\Delta + a(1-a)\right) \int_{\mathcal{F}} G_a(z,w)\varphi_j(w)\,d\mu(w) = \varphi_j(z).$$

$$\| $$

$$\left(\Delta + a(1-a)\right) c_j(z) \langle \varphi_j, \varphi_j \rangle$$

$$\Rightarrow c_j(z) = \frac{1}{s_j(1-s_j) - a(1-a)} \cdot \frac{1}{\langle \varphi_j, \varphi_j \rangle} \cdot \varphi_j(z).$$

The Eisenstein series part can be continued to $\operatorname{Re}(s) > 0$ with no poles.

The cuspidal part can be continued, but it has poles at $s_j = \frac{1}{2} + it_j$,

with $\displaystyle \operatorname*{Res}_{s=s_j} G_s(z,w) = \frac{1}{\langle \varphi, \varphi \rangle} \varphi(z)\bar{\varphi}(w)$.

We can take Fourier coefficients

$$G_s(z,w) = \sum_{m\in\mathbb{Z}} F_{-m}(z,s)\, W(my)\, e(mx), \qquad w = x+iy.$$

Here, $\displaystyle F_m(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f_m(\gamma z, s)$

$$\operatorname{Res}_{s=s_j} (2s-1) F_m(z,s) = \frac{2}{\langle \varphi_j, \varphi_j \rangle} a(m) \varphi_j(z)$$

$\operatorname{Re}(b) > 1$

So it's better to use $G_a(z,w) - G_b(z,w)$ ~~with~~ with fixed $b$ and varying $a$.

Then the spectral expansion coefficients become

$$\frac{1}{s(1-s) - a(1-a)} - \frac{1}{s(1-s) - b(1-b)} = O\left(\frac{1}{|s|^4}\right).$$

· Weight $1/2$: $\displaystyle \theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} = \sum_{n\in\mathbb{Z}} e(n^2 z)$

$\displaystyle J(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}$. Weight $1/2$ means $u(\gamma z) = J(\gamma, z)\, u(z)$.

Reference: Fay ~1974, Crelle's Journal.


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Again, one can write

$$G_{1/2,s}(z,w) = \sum_{n \in \mathbb{Z}} F_{1/2,n}(z,s)\, W^{\pm}_{1/2,s}(ny)\, e(nx) \qquad w = x+iy.$$

A different normalization: $s = \frac{1}{2} + i\frac{r}{2}$.

Then $\|\varphi_j\| = 1$, and

$$\operatorname{Res}_{s=\frac{1}{2}+i\frac{r_j}{2}} (2s-1)\, G_{1/2,s}(z,w) = \sum_{\substack{\psi \text{ basis (orthonormal)} \\ \text{of eigenspace} \\ s_j = \frac{1}{2} + i\frac{r_j}{2}}} \psi(z)\overline{\psi(w)}$$

and

$$\operatorname{Res}_{s=\frac{1}{2}+i\frac{r_j}{2}} F_{1/2,n}(z,s) = \sum_{\psi \text{ ON basis}} \overline{b(n)}\, \psi(z) \quad , \quad b(n) = \text{Fourier coefficients of } \psi.$$

Denote by $U_r$ the subspace of $L^2_{cusp}$ with eigenvalue $\frac{1}{4} + r^2$.

$T_m$ : Hecke operators, which commute with $\Delta$.

$\Rightarrow$ $U$ can be split up according to ~~t~~ eigenspaces for all the $T_m$.

If $T_m \varphi = \lambda(m)\varphi$ and $a_\varphi(1) = 0$, then $\varphi \equiv 0$.

So we can normalize the eigenfunctions so that $a_\varphi(1) = 1$.

Also, $a(-n) = a(-1)\,a(n)$ $\Rightarrow$ can split forms into "even" ($a(-1)=1$) and "odd" ($a(-1) = -1$).

For weight $1/2$ there are Hecke operators, but only parametrized by squares: ~~$T_{1/2}(m^2)$~~ $T_{1/2,m^2}$.

If $T_{1/2,m^2}\Psi = a_\psi(m)\cdot\Psi$, then

$$L(s+\tfrac{1}{2},\chi_d) \sum_{n=1}^{\infty} \frac{b(dn^2)}{n^{1-s}} = b(d)\Bigg(\underbrace{\prod \frac{1}{1 - a_\psi(p)p^{-s} + p^{-2s}}}_{(*)}\Bigg)$$

Fact: the Euler product $(*)$ is the $L$-function of a weight-0 modular form:

$$\prod\left(1 - a_\psi(p)p^{-s} + p^{-2s}\right)^{-1} =: \sum_{n=1}^{\infty} \frac{a_\psi(n)}{n^s}, \text{ and } \sum a_\psi(n) e^{2\pi i n z}$$

is a weight-0 m.f. which is "even".


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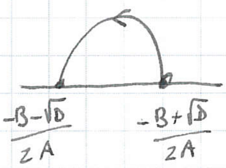
Technical lemma: cycle integral of the Poincaré series.

Let $f(z) = M(y)e(mx)$, $M: (0,\infty) \to \mathbb{C}$, $M(y) = O(y^\alpha)$ for some $\alpha > 1$ as $y \to 0$.

$P(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma z)$

If $Q$ is an indefinite quadratic form $Ax^2 + Bxy + Cy^2$ of disc $D = B^2 - 4AC$, orient the geodesic




$A > 0$ (opposite orientation if $A < 0$).

$\sigma_Q = \begin{pmatrix} \frac{t+Bu}{2} & Cu \\ -Au & \frac{t-Bu}{2} \end{pmatrix}$ $t^2 - Du^2 = 4$ generate the stabilizer of $Q$ in $\Gamma$, $\Gamma_Q$.

Want to compute:

$$\sum_{Q \in \Gamma \backslash Q_D} \chi(Q) \int_{C_Q} P(z)\, ds \qquad , ds = \frac{\sqrt{dx^2+dy^2}}{y}.$$

$C_Q$ : arc joining $z \in \mathbb{H}$ to $\sigma_Q z$.

Rmk: we can express it as $\frac{dz}{Q(z,1)}$ but only on $S_Q$ (the geodesic).

$\Gamma_Q$ acts freely on $\Gamma_\infty \backslash \Gamma$ on the right (b/c eigenvalues of $\sigma_Q$ are $\neq 1$).

So we can decompose

$$P(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_Q} \sum_{k=-\infty}^{\infty} f(\gamma \sigma^k z) \text{, and } \sum_{k=-\infty}^{\infty} \int_{C_Q} f(\gamma\sigma^k z)\, ds = \int_{S_Q} f(\gamma z)\, ds$$

To simplify, assume $\chi = 1$.

$$\sum_{Q \in \Gamma \backslash Q_D} \int_{C_Q} P(z)\, ds = \sum_{Q \in \Gamma \backslash Q_D} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \int_{S_Q} f(\gamma z)\, ds = \sum_{Q} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_Q} \int_{S_{\gamma Q}} f(z)\, ds$$

Recall $f(z) = M(y)e(mx)$, and parametrize $S_Q$ by circle of center $-B/2A$, radius $\frac{\sqrt{D}}{2A}$

So $f\left(\frac{-B}{2A}+\frac{\sqrt{D}}{2A}e^{i\theta}\right) = e\left(\frac{-mB}{2A}\right)e\left(\frac{-m\sqrt{D}}{2A}\cos\theta\right)M\left(\frac{\sqrt{D}}{2A}\sin\theta\right).$

This leads to the following

$$T_m M(t) := \int_0^\pi e(m\cos\theta)\, M(t\sin\theta)\,\frac{d\theta}{\sin\theta}$$

This can be evaluated for many choices of interesting $M$. (like Eisenstein series...)

There appears also an exponential sum

$$S_m(1,D;c) = \sum_{B^2\equiv D\,(\mathrm{mod}\ 4A)} e\left(\frac{-2mB}{4A}\right)$$

Collecting terms, get

$$\sum_{Q\in\Gamma\backslash Q_D}\int_{C_Q} P(z)\,ds = \sum_{c\equiv 0\,(4)} S_m(1,D;c)\,T_m M\left(\frac{\sqrt{D}}{2c}\right).$$

With genus characters, the same but use:

$$S_m(d_1,d_2,c)=\sum_{\substack{b\,(c)\\ b^2\equiv d_1d_2\,(4)}} \chi\left[\frac{c}{4}, b, \frac{b^2-d_1d_2}{c}\right] e\left(\frac{2mb}{c}\right).$$

Kloosterman sums $$K_{1/2}(m,n;c) = \sum_{a\,(c)} \left(\frac{c}{a}\right)\varepsilon_a\, e\left(\frac{ma+n\bar{a}}{c}\right)$$

where $\varepsilon_a = \begin{cases} 1 & a\equiv 1\ (4)\\ i & a\equiv 3\ (4)\end{cases}$

$$S_m(d_1,d_2;c) = \sum_{n\mid(m,\frac{c}{4})} \left(\frac{D}{n}\right)\sqrt{\frac{n}{c}}\, K^+\left(d_1, \frac{m^2}{n^2}d_2, \frac{c}{n}\right) \qquad (D=d_1d_2).$$

where $K^+ = (1-i)K_{1/2}\cdot\begin{cases}1 & \frac{c}{4} \text{ even}\\ 2 & \frac{c}{4}\text{ odd}\end{cases}.$