Arithmetic of Euler Systems

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Centro de Ciencias Pedro Pascual

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Foreword

The main aim of the workshop was to give an introduction to the recent developments in the area, in particular the work of Bertolini-Darmon-Rotger, Lei-Loeffler-Zerbes and Kings-Loeffler-Zerbes on Euler systems for Rankin convolutions, at a level accessible to graduate students and other younger researchers.

The instructional part of the workshop consisted of 12 lectures giving an account of the above works, together with a selection of more advanced talks on related areas of current research.

Students and postdocs volunteered to give talks themselves in the introductory lecture series; which were allocated and coordinated by David Loeffler and Sarah Zerbes, the scientific advisers.

Acknowledgements

The organizers wish to thank the participants for all the work they put in preparing the talks and typesetting the first version of these notes.

We wish to thank also the *Centro de Ciencias Pedro Pascual* for its hospitality and for providing us with exceptionally inspiring setting.

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Lectures

Arithmetic of Euler Systems

Introduction to Modular Curves

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Let $\mathcal{H} = \{\tau \in \mathbb{C} \colon \mathrm{Im}(\tau) > 0\}$ denote the complex upper half plane and $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. By the modular curve associated to Γ we refer to the quotient $\Gamma\backslash\mathcal{H}$ where the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ is given by linear fractional transformations.

For $\tau \in \mathcal{H}$, let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and E_τ be the elliptic curve $\mathbb{C}/\Lambda_\tau$. If $\Gamma = \Gamma_1(N)$, then $\Gamma_1(N)\backslash\mathcal{H}$ parametrises elliptic curves with a marked point of exact order N. Explicitly there is a bijection

$$\Gamma_1(N) \cdot \tau \leftrightarrow [E_\tau, P],$$

between cosets and equivalence classes of pairs $[E_\tau, P]$ where we may assume $P = \frac{1}{N} + \Lambda_\tau \in E_\tau[N]$.

We will rephrase the definition of a pair $[E_\tau, P]$ so that it makes sense with $\mathbb{C}$ replaced by any scheme S.

Definition 1.1. Let S be any scheme. Then an *elliptic curve over* S is a scheme E with a proper flat morphism $\pi \colon E \to S$ whose fibres are smooth genus 1 curves with a choice of section $O \colon S \to E$.

Definition 1.2. Let $Y_1(N)$ be the smooth $\mathbb{Z}[1/N]$-scheme representing the functor on $\mathbb{Z}[1/N]$-schemes

$$\mathcal{F} \colon S \mapsto \left\{ \begin{array}{c} \text{Isomorphism classes of pairs (E,P)} \\ \text{where } E \text{ is an elliptic curve } /S \text{ and} \\ P \in E(S) \text{ a point of exact order } N. \end{array} \right\}.$$

By "$Y_1(N)$ represents $\mathcal{F}$", we mean $\mathcal{F}(\cdot)$ is isormorphic to $\mathrm{Hom}(\cdot, Y_1(N))$ as functors. If $S = \mathbb{C}$ there is a natural bijection $\phi \colon \Gamma_1(N)\backslash\mathcal{H} \to Y_1(N)(\mathbb{C})$ which is an analytic isomorphism and $(Y_1(N), \phi)$ is a model for $\Gamma_1(N)\backslash\mathcal{H}$.

Remark 1.3. [LLZ14, 2.1.4] The cusp $\Gamma_1(N) \cdot \infty$ of $Y_1(N)(\mathbb{C})$ is usually not defined over $\mathbb{Q}[[q]]$ but rather over $\mathbb{Q}(\mu_N)[[q]]$ since it corresponds to the pair $(\mathbb{G}_m/q^{\mathbb{Z}}, \zeta_N)$ where $\zeta_N = e^{\frac{2\pi i}{N}}$.

This leads to the following alternative definition of $Y_1(N)$. A choice of $P \in E(S)$ of exact order N amounts to giving a closed immersion

$$\iota \colon (\mathbb{Z}/N\mathbb{Z})_S \to E$$

of group schemes, where $(\mathbb{Z}/N\mathbb{Z})_S$ denotes the constant group scheme of $\mathbb{Z}/N\mathbb{Z}$ over S. Similarly we can use a model for $Y_1(N)(\mathbb{C})$ which parameterises pairs (E, ι), where $\iota \colon (\mu_N)_S \hookrightarrow E$ is a closed immersion. The corresponding smooth scheme is denoted $Y_\mu(N)$, and thus we obtain a model $(Y_\mu(N), \phi_N)$ for $Y_1(N)(\mathbb{C})$. Here ϕ_μ is defined by $\tau \mapsto (E_\tau, \iota_\tau)$ for $\tau \in \mathcal{H}$ and ι_τ denotes the embedding defined by $\iota(\zeta_N) = \frac{1}{N} + \Lambda_\tau$. The cusp $\Gamma_1(N)$ is defined over $\mathbb{Q}[[q]]$ with respect to this model.

Definition 1.4. Let $L \geq 3$ and let $Y(L)$ be the smooth $\mathbb{Z}[1/L]$-scheme representing the functor sending,

$$S \mapsto \left\{ (E, e_1, e_2)/\sim \ : \ \begin{array}{c} E \text{ is an elliptic curve } /S \\ \text{and } e_1, e_2 \text{ generate } E[L]. \end{array} \right\}.$$

There is a left action of $\mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z})/\{\pm 1\}$ on $Y(L)$ given by the action on e_1, e_2 given by,

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Remark 1.5. If $S = \mathbb{C}$ we have an isomorphism of analytic spaces

$$(\mathbb{Z}/L\mathbb{Z})^\times \times \Gamma(L)\backslash\mathcal{H} \xrightarrow{\sim} Y(L)(\mathbb{C})$$
$$(a,\tau) \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \nu(\tau),$$

where ν is the canonical map $\nu\colon \mathcal{H} \to Y(L)(\mathbb{C}), \tau \mapsto (E_\tau, \tau/L, 1/L)$. This in particular tells us that $Y(L)$ is not geometrically connected.

Definition 1.6. Let $L \geq 3$, $M, N \geq 1$ and $M, N \mid L$. Set $Y(M,N) = G_{M,N}\backslash Y(L)$ where

$$G_{M,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) \;:\; \begin{matrix} (a,b) \equiv (1,0) \pmod{M}, \\ (c,d) \equiv (0,1) \pmod{N}. \end{matrix} \right\}.$$

If $M + N \geq 5$, $Y(M,N)$ represents the functor of triples (E, e_1, e_2) where e_1 has order M, e_2 order N and together they generate a subgroup of order MN. In order to define Hecke operators on $K_2(Y(M,N))$ and ther duals, we need also:

Definition 1.7. Let $A \geq 1, L \geq 3$, M and N s.t. $M \mid L$ and $AN \mid L$. Define $Y(M, N(A))$ to be the quotient of $Y(L)$ by the subgroup $G_{M,N(A)} \leq \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z})$ given by

$$G_{M,N(A)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) \;:\; \begin{matrix} a \equiv 1 \pmod{M},\ b \equiv 0 \pmod{M} \\ c \equiv 0 \pmod{NA},\ d \equiv 1 \pmod{N}. \end{matrix} \right\}.$$

Similarly define $Y(M(A), N)$ using $G_{M(A),N}$ given by

$$G_{M(A),N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) \;:\; \begin{matrix} a \equiv 1 \pmod{M},\ b \equiv 0 \pmod{MA} \\ c \equiv 0 \pmod{N},\ d \equiv 1 \pmod{N}. \end{matrix} \right\}.$$

The $\mathbb{Z}[1/L]$-scheme $Y(M, N(A))$ (resp. $Y(M(A), N)$ represents the functor which sends

$$s \mapsto \left\{ (E, e_1, e_2, C)/\sim \;:\; \begin{matrix} C \text{ is a cyclic subgroup of order } NA \text{ (resp. } MA), \\ e_2 \in C, \text{ (resp. } e_1 \in C) \text{ and the product } \langle e_1 \rangle C \\ \text{(resp. } \langle e_2 \rangle C) \text{ as subgroups of } E \text{ is a direct sum.} \end{matrix} \right\}.$$

There is an isomorphism

$$\phi_A\colon Y(M, N(A)) \to Y(M(A), N),$$
$$(E, e_1, e_2, C) \mapsto (E', e_1', e_2', C').$$

Given by letting E' be the quotient of E by NC, a cyclic subgroup of order A. Then e_1' is defined to be the image of e_1 which by the disjointness of $\langle e_1 \rangle$ and C is necessarily of order M. Define e_2' to be the image of $A^{-1}e_2$ in E'. There is necessarily such a point as C is cyclic of order NA and e_2' is independent of this choice. Lastly, set C' to be the image of $A^{-1}\langle e_1 \rangle$ in E'. This is a cyclic subgroup of E' of order MA.

1 Hecke Operators

The Hecke operators $T(n)$ on $K_2(Y(M,N))$ and their duals $T'(n)$ on $H^1(Y(M,N)(\mathbb{C}), \mathbb{Z})$ for $n \geq 1$, $(n, M) = 1$ are defined as follows:

- For $n = 1$, $T(1) = T'(1) = \mathrm{id}$,
- For $n = p$, $p \nmid M$ let $\pi_1\colon Y(M, N(p)) \to Y(M,N), \pi_2\colon Y(M(p), N) \to Y(M,N)$ be the projections defined by forgetting C. Then set $T(p) = (\pi_2)_* \circ (\phi_p^{-1})^* \circ (\pi_1)^*$ and $T'(p)$ to be $(\pi_1)_* \circ (\phi_p)^* \circ (\pi_2)^*$.
- For $n = p^e$, $p \nmid M$ (and $e \geq 2$) we set,

$$T(p^e) = \begin{cases} T(p)^e & \text{if } p \mid N \\ T(p)T(p^{e-1}) + T(p^{e-2}) \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & p \end{pmatrix}^* p & \text{if } p \nmid N \end{cases}.$$

For $T'(p^e)$ the formula is identical in T'.

- If $n = \prod_p p^{e(p)}$, where $e(p) \geq 0$ and p ranges over all prime numbers not dividing M, we define $T(n)$ and $T'(n)$ multiplicatively using the above definitions.

Theorem 1.8. *If p is a prime $p \nmid MN$, then Y(M,N) has a smooth model over $\mathbb{Z}_p$.*

Proof. Pick $L \in \mathbb{N}$ such that $M, N \mid$L. Since $Y(M,N)$ is a quotient of $Y(L) = Y(L,L)$ it is enough to check that $Y(L)$ has a smooth model over $\mathbb{Z}[1/L]$, so if we take $K = \mathrm{lcm}(M,N)$ then L is invertible in $\mathbb{Z}_p$. The functional criterion for smoothness shows that $Y_1(L)$ is smooth over $\mathbb{Z}[1/L]$ if and only if for all local $\mathbb{Z}[1/L]$-algebras A and nilpotent ideals I, the map $Y(L)(A) \to Y(L)(A')$ is surjective, where $A' = A/I$.

Now, let A be a local $\mathbb{Z}[1/L]$-algebra and $I \subseteq A$ a nilpotent ideal, and write A' for the quotient. Take $(E', e_1', e_2') \in Y(L)(A')$, the A' valued points of the modular curve. Let E/A be any lifting of E'/A' obtained by lifting the coefficients to A. Note $\Delta(E) \in A^\times$ as its image in A' is $\Delta(E')$ and thus a unit.

It remains to check that there exist lifts of e_1' and e_2' to points of $E[L]$. This is equivalent to checking that $E[L]$ is smooth. But $[L] : E \to E$ is smooth and $E[L]$ is obtained by composing $[L]$ with the structure map $E \to \operatorname{Spec} A$ which is also smooth. The composition of smooth morphisms is smooth, and so its kernel $E[L] \to \operatorname{Spec} A$ is also. Hence we have a lift $(E, e_1, e_2) \in Y(L)(A)$ of (E', e_1', e_2') as required. □

Hida Theory

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Introduction

A classical problem of modern number theory is that of p-adic interpolation. In particular, given a set of 'classical' objects that are in some sense 'algebraic', one asks if there is a p-adic object that can be 'specialised' at certain values to give these classical objects. As an example of this, consider the Riemann ζ-function. We know that the value of $\zeta(s)$ at a negative integer $-m$ is rational ([Was97], Theorem 4.2). Kubota and Leopoldt proved a version of the following:

Theorem 2.1. *There is a space of p-adic weights $\mathcal{W}$ containing $\mathbb{N}$ and a p-adic function ζ_p on $\mathcal{W}$ such that for each integer $k \geq 1$, we have*

$$\zeta_p(k) = (1 - p^{k-1})\zeta(1-k).$$

This talk is concerned with the p-adic interpolation of spaces of modular forms, and in particular the pioneering work of Haruzo Hida in this field. First, I'll give some examples of the kind of question we're looking to answer.

1) If f is a modular form of some weight and level, can we find a set of modular forms of varying weights, containing f, that varies 'p-adic analytically' in the weight? (This question will be made more precise in the sequel).

2) On a larger scale, is there a space $\mathbb{S}$ of 'p-adic modular forms' that somehow interpolates the spaces of cusp forms of fixed level and weight k for each integer k? To be more precise, can we find such a space with surjective specialisation maps

$$\rho_k : \mathbb{S}(\Gamma) \to S_k(\Gamma)$$

for each k?

Hida considered these problems in the *ordinary* case. This, colloquially, is the subspace of the space of modular forms on which the Hecke operator T_p acts invertibly. We will see in this talk that there are positive answers to both of the above questions in this case, and that moreover these questions are in a sense interlinked. Indeed, if we have such a space $\mathbb{S}$ as in 2), then taking an element $F \in \mathbb{S}$, the set $\{\rho_k(F) : k \in \mathbb{N}\}$ is a set of modular forms that is 'interpolated' by F. We'll later construct such an $\mathbb{S}$, which will be the space of Λ-*adic modular forms.*

Remark 2.2. Throughout, we'll make a couple of simplifying assumptions that don't really affect the overall results. First, we talk only about cusp forms. The theory goes through basically identically for general modular forms, and is covered in [Hid93], Chapter 7.

Secondly, we'll assume p is an odd prime. The case $p = 2$ is not really any different, but it introduces a slightly annoying piece of notation; namely, the isomorphism $1 + p\mathbb{Z}_p \cong p\mathbb{Z}_p$ via the p-adic logarithm only holds for odd p, whilst we need to consider $1 + 4\mathbb{Z}_2 \cong 4\mathbb{Z}_2$ for $p = 2$. In [Hid93], this is dealt with by considering a new variable $\mathbf{p}$ equal to p for odd p and 4 when $p = 2$. As this talk is only an overview, for simplicity I prefer not to introduce this additional variable here!

Notation 2.3. We routinely talk about the space $S_k(\Gamma_0(p),\chi;\mathbb{Z}_p)$ throughout this article. This has a simple definition. For nice enough Γ, the space $S_k(\Gamma)$ has a basis of eigenforms with rational integer coefficients ([Shi94], Theorem 3.53 or [DS05], Corollary 6.5.6 for weight 2 and level $\Gamma_1(N)$), and we define $S_k(\Gamma;\mathbb{Z})$ to be the $\mathbb{Z}$-span of these eigenforms. Then for any $\mathbb{Z}$-algebra A, we define $S_k(\Gamma;A) := S_k(\Gamma;\mathbb{Z})\otimes_{\mathbb{Z}} A$. Note that $S_k(\Gamma;\mathbb{C}) = S_k(\Gamma)$. This notation will be used for any space where we can define a reasonable integral subspace, which we then tensor with $\mathbb{Z}_p$ - or, more generally, the ring $\mathcal{O}_K$ of integers in a finite extension $K/\mathbb{Q}_p$.

1 Λ-adic Forms and p-adic Families of Modular forms

We begin with a general overview of the theory of p-adic families of modular forms and the associated concept of Λ-adic forms, where $\Lambda := \mathbb{Z}_p[[X]]$ is the Iwasawa algebra.

1.1 Example: Eisenstein Series

We start with a motivating example. Consider the family of classical Eisenstein series of even weight k, with q-expansions

$$E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1}(n)q^n,$$

where

$$\sigma_{k-1}(n) := \sum_{\substack{d|n \\ d>0}} d^{k-1}.$$

We want to put these modular forms into a p-adic family. One way of doing this is to p-adically interpolate the Fourier coefficients; and in this case, that means p-adically interpolating the function

$$k \longmapsto d^k,$$

for an integer d.

There is a neat trick for doing this, which then provides motivation for an object called *weight space*, which I will discuss in greater detail in the following section. Rather than thinking of the *integer* k as the weight of this Eisenstein series, we can shift to think of the weight as being a 'thing we do to d', or rather as the *continuous homomorphism*

$$\begin{aligned} k : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ z &\longmapsto z^k. \end{aligned}$$

So, to write down a set of p-adic weights, it's natural to consider continuous endomorphisms of $\mathbb{Z}_p$. In fact, the function $k : z \mapsto z^k$ is deeply horrible at elements $z \in \mathbb{Z}_p$ that are not units and there is absolutely no hope of p-adically interpolating the function $k \mapsto p^k$. Thus, to ensure our weight space contains the integers, we really want to restrict to continuous endomorphisms of ${\mathbb{Z}_p}^\times$. Thus, we define:

Definition 2.4. Define (the $\mathbb{Z}_p$ points of) *p-adic weight space* to be

$$\mathcal{W}(\mathbb{Z}_p) := \mathrm{Hom}_{\mathrm{cts}}({\mathbb{Z}_p}^\times, {\mathbb{Z}_p}^\times).$$

Define the *even points of weight space* to be the subset $\mathcal{W}^+(\mathbb{Z}_p)$ of those $\kappa \in \mathcal{W}(\mathbb{Z}_p)$ that satisfy

$$\kappa(-1) = 1.$$

It's easy to see that $\mathcal{W}(\mathbb{Z}_p)$ does indeed contain a copy of the integers via the map $\kappa_k : z \mapsto z^k$, and that the even integers lie in $\mathcal{W}^+(\mathbb{Z}_p)$.

We now want to write down an 'Eisenstein series of weight κ', where $\kappa \in \mathcal{W}^+(\mathbb{Z}_p)$, such that

E_{κ_k} is a familiar Eisenstein series. We have problems at coefficients of q^n where $p|n$; to get around this, we need to take a p-stabilisation of E_k, namely

$$\begin{aligned} E_k^{\{p\}} &:= E_k(z) - p^{k-1}E_k(pz) \\ &= \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1}^p(n)q^n \in M_k(\Gamma_0(p)), \end{aligned}$$

where

$$\sigma_{k-1}^p(n) = \sum_{\substack{0<d|n \\ p\nmid d}} d^{k-1}.$$

It is now straightforward to see how to interpolate the non-constant coefficients. We define, for $\kappa \in \mathcal{W}^+(\mathbb{Z}_p)$, the function

$$\sigma_{\kappa-1}^p(n) = \sum_{\substack{0<d|n \\ p\nmid d}} \frac{\kappa(d)}{d}.$$

To interpolate the constant coefficients, we fall back on the theory of p-adic L-functions. Recall Theorem 2.1; this said that there exists a p-adic zeta function ζ_p on $\mathcal{W}(\mathbb{Z}_p)$ such that

$$\zeta_p(\kappa_k) = (1-p^{k-1})\zeta(1-k).$$

Hence, if we let $\kappa \in \mathcal{W}^+(\mathbb{Z}_p)$ and define

$$E_\kappa(z) = \frac{\zeta_p(\kappa)}{2} + \sum_{n\geq 0}\sigma_{\kappa-1}^p(n)q^n,$$

then

$$E_{\kappa_k}(z) = E_k^{\{p\}}$$

as formal q-expansions.

To sum up what we've done:

Theorem 2.5. *Let $\mathcal{O}(\mathcal{W}^+(\mathbb{Z}_p))$ be the ring of rigid analytic functions on $\mathcal{W}^+(\mathbb{Z}_p)$. Then there is a formal q-expansion*

$$E(X,z) = \sum_{n\geq 0} A_n(X)q^n \in \mathcal{O}(\mathcal{W}^+(\mathbb{Z}_p))[[q]],$$

where $A_n(X) \in \mathcal{O}(\mathcal{W}^+(\mathbb{Z}_p))$ and X is a parameter on $\mathcal{W}^+(\mathbb{Z}_p)$, such that if we put $X = \kappa_k$, we have

$$E(\kappa_k, z) = E_{\kappa_k}(z) = E_k^{\{p\}}.$$

Thus we have obtained a 'p-adic family of Eisenstein series.'

This example will later be generalised to the example of Λ-*adic forms*. For more details about this p-adic family, and the arguments above, see [Hid93], Chapter 7.1, [Maz12], Section 2.3. The construction of the p-adic ζ-function is covered in detail in [Hid93] and [Was97].

1.2 Weight Space

Here, we examine weight space through a slightly more explicit lens. In the previous section, we defined weight space to be $\mathcal{W}(\mathbb{Z}_p) := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p{}^\times, \mathbb{Z}_p{}^\times)$; this is simplifying matters significantly. What we've actually written down is the set of $\mathbb{Z}_p$-points of a rigid p-analytic space $\mathcal{W}$. For the construction of this space, see [Buz04]. It is more common to speak about the $\mathbb{C}_p$-points, or rather, $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p{}^\times, \mathbb{C}_p^\times)$, where $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$. This then contains all p-power Dirichlet characters, viewed as characters of $\mathbb{Z}_p{}^\times$ in the natural way.

As mentioned previously, we know that $\mathcal{W}(\mathbb{Z}_p)$ contains a copy of the integers. Via the decomposition ${\mathbb{Z}_p}^\times \cong (\mathbb{Z}/p)^\times \times (1+p\mathbb{Z}_p)$, we see that

$$\mathcal{W}(\mathbb{Z}_p) = \mathrm{End}((\mathbb{Z}/p)^\times) \times \mathrm{Hom}_{\mathrm{cts}}(1+p\mathbb{Z}_p, 1+p\mathbb{Z}_p).$$

For the first factor, we know that any endomorphism of $(\mathbb{Z}/p)^\times$ is simply a Dirichlet character χ of conductor p, which - for a fixed primitive $(p-1)$th root of unity ζ_{p-1} - is determined entirely by $\chi(\zeta_{p-1}) \in (\mathbb{Z}/p)^\times$. For the second factor, consider the topological generator $u := 1+p$ of $1+p\mathbb{Z}_p$. Any continuous endomorphism ψ of $1+p\mathbb{Z}_p$ is determined entirely by the image of u. Thus we can identify

$$\mathcal{W}(\mathbb{Z}_p) = \bigsqcup_{\chi} \mathcal{W}_\chi(\mathbb{Z}_p) \cong (\mathbb{Z}/p)^\times \times (1+p\mathbb{Z}_p),$$

where the sum is over all characters $\chi \pmod p$, via

$$(\chi,\psi) \longmapsto (\chi(\zeta_{p-1}), \psi(u)).$$

That is, $\mathcal{W}(\mathbb{Z}_p)$ is topologically the disjoint union of $p-1$ open unit discs in $\mathbb{Z}_p$ - each centred at a $(p-1)th$ root of unity ζ_{p-1}^a - and each corresponding to different Dirichlet character χ of conductor p determined by $\chi(\zeta_{p-1}) = \zeta_{p-1}^a$. We have denoted the disc corresponding to χ by $\mathcal{W}_\chi(\mathbb{Z}_p)$. The identification of $\mathcal{W}_\chi(\mathbb{Z}_p)$ with $1+p\mathbb{Z}_p$ is non-canonical, depending on the choice of topological generator u.

Note that if $k \in \mathbb{Z}$ is an integer, then the component k lands in corresponds to the value of $k \pmod{p-1}$. We can consider a different copy of the integers inside $\mathcal{W}(\mathbb{Z}_p)$ for each Dirichlet character χ of conductor p as follows:

Definition 2.6. let ω denote the Teichmuller character of ${\mathbb{Z}_p}^\times$, that is, the projection of $x \in {\mathbb{Z}_p}^\times \cong (\mathbb{Z}/p)^\times \times (1+p\mathbb{Z}_p)$ onto the first component under this isomorphism. For a fixed Dirichlet character χ of conductor p, corresponding to a fixed component $\mathcal{W}_\chi(\mathbb{Z}_p) \subset \mathcal{W}(\mathbb{Z}_p)$, we can consider a copy of the integers living inside $\mathcal{W}_\chi$ by considering the homomorphism

$$\kappa_{\chi,k} : z \longmapsto \chi(z)w(z)^{-k}z^k, \quad z \in {\mathbb{Z}_p}^\times,$$

or rather

$$(x,y) \longmapsto (\chi(x), y^k) \in {\mathbb{Z}_p}^\times, \quad x \in (\mathbb{Z}/p)^\times, y \in 1+p\mathbb{Z}_p.$$

We end this section by making question 1) from above more precise.

Definition 2.7. Let $\Gamma := \Gamma_0(p)$ be the standard congruence subgroup of level p, and let χ be a character of conductor p. Then given a set

$$A := \{f_k \in S_k(\Gamma, \chi\omega^{-k}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p\},$$

a *p-adic interpolation of A* is a formal q-expansion

$$F(\mathbf{X}) = \sum_{n\geq 0} A_n(\mathcal{X})q^n \in \mathcal{O}(\mathcal{W}_\chi(\mathbb{Z}_p))[[q]],$$

where $\mathcal{O}(\mathcal{W}_\chi(\mathbb{Z}_p))$ is the ring of p-adic analytic functions on $\mathcal{W}_\chi(\mathbb{Z}_p)$, and $\mathbf{X}$ is a parameter over $\mathcal{W}_\chi(\mathbb{Z}_p)$, with the property that

$$F(\kappa_{\chi,k}) = \sum_{n\geq 0} a_n(\kappa_{\chi,k})q^n = f_k \in S_k(\Gamma_0(p), \chi\omega^{-k}, \mathbb{Z}_p)$$

as formal q-expansions.

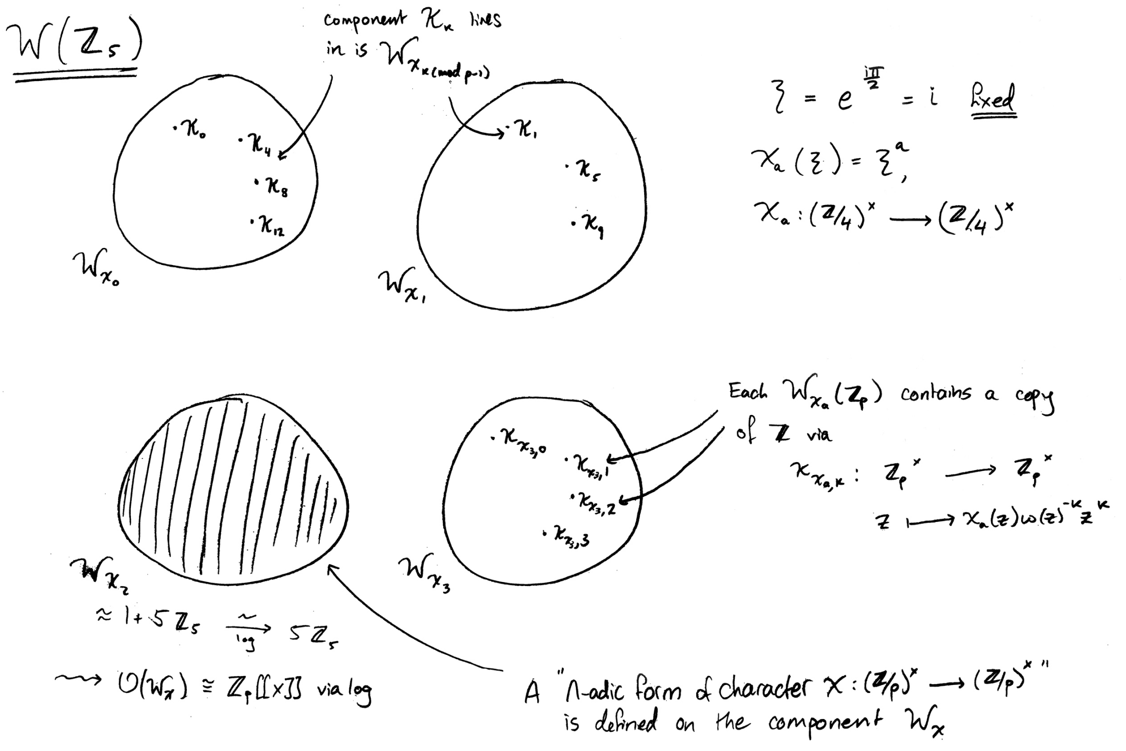
Proposition 2.8. *We can identify $\mathbb{Z}_p[[X]]$ with a subspace of $\mathcal{O}(\mathcal{W}_\chi(\mathbb{Z}_p))$ by attaching, to such a power series $A(X) \in \mathbb{Z}_p[[X]]$, the analytic function $\mathcal{A}(\mathbf{X})$ defined by*

$$\mathcal{A}(\kappa) := A(\psi(u)-1)$$

for $u = 1+p$ and

$$\kappa = (\chi,\psi) \in \mathcal{W}_\chi(\mathbb{Z}_p) \subset \mathcal{W}(\mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cts}}({\mathbb{Z}_p}^\times, {\mathbb{Z}_p}^\times) \cong \mathrm{End}((\mathbb{Z}/p)^\times) \times \mathrm{Hom}_{\mathrm{cts}}(1+p\mathbb{Z}_p).$$

Figure 1: Weight space for $p = 5$



Note that since $\psi(u) \in 1 + p\mathbb{Z}_p$, any such power series converges at $\psi(u) - 1$. This Proposition gives us the basis for the study of Λ-adic forms in the next section.

Remark 2.9. Indeed, via the p-adic logarithm, we can fix a topological generator u and identify $\mathcal{W}_\chi(\mathbb{Z}_p) \cong 1 + p\mathbb{Z}_p \cong p\mathbb{Z}_p$. Under this identification, we get $\mathcal{O}(\mathcal{W}_\chi(\mathbb{Z}_p)) \cong \mathbb{Z}_p[[X]]$. This is *not* the copy of $\mathbb{Z}_p[[X]]$ we embedded above, however. We stick to this latter subspace in the definitions that follow to avoid going into the calculations coming from the p-adic logarithm (and for the simpler reason that we are following [Hid93], which does this!).

1.3 Λ-adic forms

Let $\Lambda := \mathbb{Z}_p[[X]]$ be the Iwasawa algebra. Then from Definition 2.7 and Proposition 2.8, we want to look at functions $F(X) = F(X, q)$ such that:

Definition 2.10. Let χ be a character of $(\mathbb{Z}/p)^\times$. A formal q-expansion

$$F(X, q) = F(X) = \sum_{n \geq 0} A_n(X) q^n \in \Lambda[[q]] = \mathbb{Z}_p[[X, q]]$$

is called Λ-*adic cusp form of character* χ if

$$F(u^k - 1) \in S_k(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p)$$

for all sufficiently large k, where $u := 1 + p$ is a *fixed* topological generator. We call an integer k that this condition holds an *admissible* weight.

Remark 2.11. Note that from our above arguments, evaluating at $u^k - 1$ is the same as taking a fuction $F \in \mathcal{O}(\mathcal{W}_\chi(\mathbb{Z}_p))[[q]]$ and evaluating it at the weight $\kappa_{\chi,k}$.

Each Λ-adic cusp form then gives a p-adic family of classical modular forms $\{F(u^k - 1)\}_k$, where we index over all admissible k. For later purposes, we'll want to look at the space of *all* Λ-adic forms, as defined in:

Definition 2.12. (i) Write $\mathbb{S}(\chi, \Lambda)$ for the Λ-module of all Λ-adic cusp forms of character χ (noting that this is indeed a Λ-module).

(ii) We write $\mathbb{S}(1, \Lambda) = \mathbb{S}(\Gamma_1(p), \Lambda)$ for the Λ-module

$$\mathbb{S}(1, \Lambda) := \bigoplus_{\chi} \mathbb{S}(\chi, \Lambda),$$

where the sum is over all Dirichlet characters mod p (including the identity).

Note that $F \in \mathbb{S}(1, \Lambda)$ implies that $F(u^k - 1) \in S_k(\Gamma_1(p), \mathbb{Z}_p)$ for sufficiently large k.

In the third section of this talk, we'll also want to introduce more general levels. Let N be an integer coprime to p. Then define $\mathbb{S}(N, \Lambda)$ to be the space of Λ-adic cusp forms of level $\Gamma_1(N)$, that is, the space of formal q-expansion $F(X) \in \Lambda[[q]]$ such that

$$F(u^k - 1) \in S_k(\Gamma_1(Np), \mathbb{Z}_p)$$

for all sufficiently large k. (We shall later see that in the cases we care about this is also sufficient to interpolate all forms of level Np^r as well).

2 Hida's Work on Ordinary Modular Forms

We've now given a sort of theoretical answer to Question 1) of the introduction, by introducing Λ-adic forms. At the moment, we've only exhibited one such example of a p-adic family of modular forms, and even this appears in a slightly different guise to that introduced in Section 1.3. Rather than trying to generate more, we'll now focus on the structure of $\mathbb{S}(\chi, \Lambda)$ and use it in relation to Question 2), the hunt for some space that p-adically interpolates all of the spaces of modular forms of all weights. Suppose we take any finitely generated submodule $\mathbb{S}'$ of $\mathbb{S}(\chi, \Lambda)$, and note that $\mathbb{S}'$ has a *specialisation property*; namely that we can choose some k_0 such that for all $k \geq k_0$, evaluation at $u^k - 1$ gives rise to a map

$$\mathbb{S}' \longrightarrow S_k(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p).$$

This is exactly the kind of specialisation map we're looking for.

As for any congruence subgroup the dimension of $S_k(\Gamma)$ grows linearly with k, we cannot hope to find a finitely generated module that interpolates all of these spaces. Our aim, then, is to identify suitable subspaces of $S_k(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p)$ and $\mathbb{S}(\chi, \Lambda)$ that have growth bounded independently of k, and then we *can* hope that we can provide a simultaneous interpolation via these specialisation maps. One of Hida's great contributions to modern number theory was showing that if we restrict to the *ordinary* subspaces, then this is indeed the case.

2.1 Ordinary Forms and the Ordinary Projector

Colloquially, the *p-ordinary subspace* of a space of modular forms with level divisible by p is the subspace on which the U_p Hecke operator acts invertibly. A Hecke eigenform $f \in S_k(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p)$ with U_p eigenvalue a_p is ordinary precisely when a_p is a p-adic unit. We make this more precise.

Definition 2.13. Let M be a classical space of modular forms with an action of the Hecke operator U_p. Define *Hida's ordinary projector* to be the limit

$$e_p := \lim_{n \longrightarrow \infty} U_p^{n!}.$$

We skip the necessary work involved with convergence issues (see [Hid93], Chapter 7.2). If f is a Hecke eigenform, then its U_p eigenvalue a_p is algebraic, and hence can be viewed as living in a finite extension of $\mathbb{Z}_p$. It thus makes sense to take its p-adic valuation (which is well-defined). If $|a_p| < 1$, then it is easy to see that $e_p f = 0$. Suppose instead that $|a_p| = 1$. Then a_p has finite multiplicative order d, and then we see that $(U_p)^{n!}f = f$ for all $n \geq d$; hence, in this case, $e_p f = f$. To summarise:

$$e_p f = \begin{cases} f & : |a_p| = 1 \\ 0 & : |a_p| < 1 \end{cases}$$

Definition 2.14. Define the *space of ordinary cuspidal modular forms of weight k, level Γ and character χ* to be

$$S_k^{\mathrm{ord}}(\Gamma, \chi; \mathbb{Z}_p) := e_p S_k(\Gamma, \chi; \mathbb{Z}_p).$$

A cusp form is *ordinary* if it lies in the ordinary subspace. We say a Λ-adic cusp form $F \in \mathbb{S}(\chi, \Lambda)$ is *ordinary* if, for all sufficiently large k, the specialisation $F(u^k - 1)$ is an ordinary modular form. Define $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)$ to be the Λ-module of ordinary Λ-adic cusp forms.

Proposition 2.15. *There is a unique idempotent e_p^Λ on $\mathbb{S}(\chi, \Lambda)$ such that*

$$\mathbb{S}^{ord}(\chi, \Lambda) = e_p^\Lambda \mathbb{S}(\chi, \Lambda),$$

and such that, for $F \in \mathbb{S}(\chi, \Lambda)$, we have

$$(e_p^\Lambda \cdot F)(u^k - 1) = e_p(F(u^k - 1)).$$

Proof. See [Hid93], Proposition 7.3.1. □

2.2 The Structure of the Ordinary Subspaces

Hida's remarkable theorem for the spaces of ordinary modular forms is as follows.

Theorem 2.16. *For any $k \geq 2$, and any character χ modulo p^n, we have*

$$\mathrm{rank}_{\mathbb{Z}_p} S_k^{ord}(\Gamma_0(p^n), \chi\omega^{-k}; \mathbb{Z}_p) = \mathrm{rank}_{\mathbb{Z}_p} S_2^{ord}(\Gamma_0(p^n), \chi\omega^{-2}; \mathbb{Z}_p),$$

that is, the rank is constant as the weight varies.

As such, we *can* hope to p-adically interpolate the spaces $S_k^{\mathrm{ord}}(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p)$ as the weight varies, and we already have a perfect candidate space to do so: namely $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)$. It turns out that:

Theorem 2.17. *Let χ be a character of conductor p. Then:*

(i) The space $\mathbb{S}^{ord}(\chi, \Lambda)$ is free of finite rank over Λ, and in fact we have

$$\mathrm{rank}_\Lambda \mathbb{S}^{ord}(\chi, \Lambda) = \mathrm{rank}_{\mathbb{Z}_p} S_2^{ord}(\Gamma_0(p), \chi\omega^{-2}; \mathbb{Z}_p),$$

the constant rank from Theorem 2.16.

*(ii) After a suitable extension of coefficients to a finite extension **K** of Frac(Λ), the space $\mathbb{S}^{ord}(\chi, \Lambda) \otimes_\Lambda$ **K** has a basis consisting of Hecke eigenforms, and specialisation of this basis at weight k, for $k \geq 2$, gives a basis of eigenforms for the space $S_k^{ord}(\Gamma_0(p), \chi\omega^{-k}; \mathcal{O})$, where $\mathcal{O}$ is the ring of integers in some finite extension of $\mathbb{Q}_p$.*

The following is, perhaps, the most important result of this talk.

Theorem 2.18. *[Control Theorem, fixed level] We have the following* control theorem*: for each $k \geq 2$, let $\mathfrak{p}_k$ be the prime ideal of Λ generated by the polynomial $X - (u^k - 1)$. Then evaluation at $u^k - 1$ induces an isomorphism*

$$\mathbb{S}^{ord}(\chi, \Lambda)/\mathfrak{p}_k \xrightarrow{\sim} S_k^{ord}(\Gamma_0(p), \chi\omega^{-k}; \mathbb{Z}_p).$$

Therefore $\mathbb{S}^{\mathrm{ord}}(\chi, \Lambda)$ is the interpolating space we wanted in the case of ordinary forms.

2.3 Varying Levels

So far we've defined familes of modular forms that vary with regard to the weight. Such families also interpolate modular forms of varying (p-power) levels. Indeed, let ε be a finite order character of $1+p\mathbb{Z}_p$ factoring through the quotient $(1+p\mathbb{Z}_p)/(1+p\mathbb{Z}_p)^{p^\alpha}$, with α minimal. Then it transpires that if F is an ordinary Λ-adic cusp form of character χ, then

$$F(\varepsilon(u)u^k - 1) \in S_k^{\mathrm{ord}}(\Gamma_0(p^{\alpha+1}), \chi\varepsilon\omega^{-k}; \mathbb{Z}_p[\varepsilon]).$$

As such, the control theorem above can be extended further:

Theorem 2.19. *[Control Theorem, varying levels] We have the following* control theorem*: for each $k \geq 2$ and ε as above, let $\mathfrak{p}_{k,\varepsilon}$ be the prime ideal of Λ generated by the polynomial $X - (\varepsilon(u)u^k - 1)$. Then evaluation at $\varepsilon(u)u^k - 1$ induces an isomorphism*

$$\mathbb{S}^{ord}(\chi, \Lambda)/\mathfrak{p}_{k,\varepsilon} \xrightarrow{\sim} S_k^{ord}(\Gamma_0(p^{\alpha+1}), \chi\varepsilon\omega^{-k}; \mathbb{Z}_p[\varepsilon]).$$

Thus a Λ-adic cusp form interpolates modular forms of varying weights *and* levels.

2.4 Hida Families

We come at last to a definition of a Hida family. There are a number of definitions in the literature, but we'll start with the one that has been heavily implied throughout the above:

Definition 2.20. A *Hida family* is the set $\{F(u^k - 1) : k \text{ admissible for } F\}$ attached to an element $F \in \mathbb{S}^{\mathrm{ord}}(\chi, \Lambda) \otimes_\Lambda \Lambda_{\mathbf{K}}$, where $\Lambda_{\mathbf{K}}$ is the integral closure of Λ in a finite extension $\mathbf{K}$ of $\mathrm{Frac}(\Lambda)$. If f is an ordinary cusp form of some weight and level, then a *Hida family passing through f* is a Hida family containing f.

Proposition 2.21. *If f is an ordinary cusp form of p-power level, then there is a Hida family passing through f.*

3 Passing to cohomology

In the last section of this talk, we relate the results above to those summarised in [KLZ15a], section 7, namely relating the control theorems 2.18 and 2.19 to the work of Ohta, in the process putting the results above into the form to be used later on in this workshop. A source of motivation for this work is to write down a Λ-adic version of the Eichler-Shimura map relating spaces of modular forms to cohomology groups of modular curves. In [Oht95] and [Oht99], the correct cohomology groups are given; namely, inverse limits of étale cohomology groups for the modular curves $Y_1(Np^r)_{\overline{\mathbb{Q}}}$. Firstly, to see why this might be an appropriate thing to consider, we look at the Hecke algebras.

3.1 Hecke algebras

Definition 2.22. (i) Define $\mathbb{S}^{\mathrm{ord}}(N, \Lambda) := e_p\mathbb{S}^{\mathrm{ord}}(N, \Lambda)$ to be the space of *ordinary Λ-adic forms of level N on Γ_1.*

(ii) Define *Hida's ordinary/universal Hecke algebra of level N* to be the subring $\mathbb{T}^{\mathrm{ord}}(N, \Lambda)$ of $\mathrm{End}(\mathbb{S}^{\mathrm{ord}}(N, \Lambda))$ generated over Λ by the Hecke operators T_n.

We note that the pairing

$$(,) : \mathbb{T}^{\mathrm{ord}}(N, \Lambda) \times \mathbb{S}^{\mathrm{ord}}(N, \Lambda) \longrightarrow \Lambda$$

defined by

$$(T, F) = a_1(F|T)$$

gives us a natural duality ([Hid93], Theorem 7.3.5), that is,

$$\mathbb{S}^{\mathrm{ord}}(N, \Lambda) \cong \mathrm{Hom}_\Lambda(\mathbb{T}^{\mathrm{ord}}(N, \Lambda), \Lambda)$$

and

$$\mathbb{T}^{\mathrm{ord}}(N,\Lambda) \cong \mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(N,\Lambda),\Lambda).$$

Now we quote a theorem of Hida:

Theorem 2.23. *For each fixed $k \geq 2$, we have*

$$\mathbb{T}^{ord}(N,\Lambda) \cong \varprojlim_{r\geq 1} \mathbb{T}_k^{ord}(\Gamma_1(Np^r);\mathbb{Z}_p),$$

where $\mathbb{T}^{ord}(\Gamma_1(Np^r);\mathbb{Z}_p)$ is the subring of $\mathrm{End}(S_k^{ord}(\Gamma_1(Np^r);\mathbb{Z}_p))$ generated over $\mathbb{Z}_p$ by the Hecke operators.

Proof. [Hid86], Theorem 1.1. □

A little thought show that this is not that surprising. After all, we've shown first that the rank of the space of ordinary forms of level $\Gamma_1(N)$ and weight k is independent of the weight, and is equal to the rank of $\mathbb{S}^{ord}(N,\Lambda)$ over Λ, and secondly that the space of ordinary Λ-adic forms also sees forms of p-power level. So we should be able to build the universal Hecke algebra from the Hecke algebras of any *fixed* weight and varying p-power level, as indeed this theorem tells us we can.

3.2 Ohta's 'anti-ordinary' reformulation

If we're looking for an Eichler-Shimura style isomorphism, then, the natural candidate for a cohomology group would be to fix weight 2 (for simplicity, since we can work with any single weight) and take the inverse limit

$$\varprojlim_{r\geq 1} \mathrm{H}^1(\Gamma_1(Np^r),\mathbb{Z}_p),$$

where the limit is taken with respect to the trace maps. It turns out this doesn't work. Unfortunately, the usual Hecke operators on the individual components do *not* commute with the trace maps ([LLZ14], Section 6.9), so the Hecke operator T_n is not well-defined on the limit. Instead, we must use the adjoint operator T_n^*. This is defined in much the same way, but using the matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)$ rather than the more usual $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\right)$. This then commutes with the trace maps ([Oht99]).

In much the same way as before, we can define an idempotent $e_p^* = \lim_{n\to\infty}(T_p^*)^{n!}$, the *anti-ordinary projector.*

3.3 Galois representations and the control theorem

We conclude this article with the final definition of the cohomology group we will use in later talks. Note that there is a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{et}}(Y_1(Np^r)_{\overline{\mathbb{Q}}},\mathbb{Z}_p) \cong \mathrm{H}^1(\Gamma_1(Np^r),\mathbb{Z}_p)$$

([Oht95], equation (1.2.3)). We use the étale groups, since this gives us a natural Galois action. We also introduce a twist by the cyclotomic character in the coefficients, as in [KLZ15a], for later use.

Definition 2.24. Define

$$\mathrm{H}^1_{\mathrm{ord}}(Np^\infty) := \varprojlim_{r\geq 1} e_p^* \mathrm{H}^1_{\mathrm{et}}(Y_1(Np^r)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)),$$

which is a finitely generated projective Λ-module.

This can be made to play the role of the cohomology group in an Eichler-Shimura style isomorphism for Λ-adic forms that is equivariant with respect to the adjoint Hecke action on cohomology and the usual Hecke action on Λ-adic forms; for this result, see [Oht99]. Our final result, quoted in the form given in [KLZ15a] (Proposition 7.2.1), shows that this module satisfies the control properties that we desire it to.

Theorem 2.25 (Control theorem, cohomological version)**.** *(i) The space* $\mathrm{H}^1_{ord}(Np^\infty)$ *has* Λ*-linear actions of* $G_{\mathbb{Q},S}$ *and the adjoint Hecke operators* T_n^* *that commute (where* S *is the set of primes dividing* Np*).*

(ii) We have the following control theorem: Let $\mathfrak{p}_{r,k}$ *denote the ideal of* Λ *generated by* $(1+X)^{p^{r-1}} - u^{kp^{r-1}}$ *(generalising the ideals* $\mathfrak{p}_k$ *from before). Then there is a canonical isomorphism*

$$\begin{aligned}\mathrm{H}^1_{ord}(Np^\infty)/\mathfrak{p}_{r,k} &\cong e_p^*\mathrm{H}^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k(\mathbb{Z}_p)(1)) \\ &\cong e_p^*\mathrm{H}^1(\Gamma_1(Np^r), \mathrm{Sym}^k(\mathbb{Z}_p)(1))\end{aligned}$$

of $\mathbb{Z}_p$*-modules that is compatible with the actions of* $G_{\mathbb{Q},S}$ *and the Hecke operators.*

Proof. See [Oht99] and [KLZ15a], Section 7. The description of the isomorphism is contained in [Oht95], Section 1.3, using results on cohomology from earlier in Section 1. □

Siegel units and Eisenstein classes

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After giving a brief introduction to Kato's Siegel units and the norm compatibility relations, we discuss the existence of the Eisenstein classes in motivic cohomology (as a generalisation of Kato's Siegel units) and state results concerning the image of the Eisenstein classes under the de Rham regulator.

1 Units on elliptic curves

Let $N \geq 3$ be an integer, we denote by $Y(N)$ the smooth irreducible affine curve over $\mathbb{Q}$ (modular curve of level N) which represents the functor

$$F : Sch^{op} \longrightarrow Sets$$
$$S \mapsto \left\{ \begin{matrix} \text{isomorphism classes of } (E, e_1, e_2), \text{ where } E \text{ is an elliptic curve over } S \\ \text{and } e_1, e_2 \in E(S) \text{ form a } \mathbb{Z}/N\mathbb{Z}\text{-basis of } E[N] \end{matrix} \right\}.$$

Let N' be a multiple of N, then any degeneracy map $Y(N') \to Y(N)$ induces an inclusion $\mathcal{O}(Y(N)) \hookrightarrow \mathcal{O}(Y(N'))$.

The goal of this talk is to define elements

$${}_cg_{\alpha,\beta} \in \bigcup_{N|N'} \mathcal{O}(Y(N))^*, \quad g_{\alpha,\beta} \in \bigcup_{N|N'} \mathcal{O}(Y(N))^* \otimes_{\mathbb{Z}} \mathbb{Q},$$

for $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$ and c a positive integer prime to 6 and to the orders of α and β, such that,

$${}_cg_{\alpha,\beta} \in \mathcal{O}(Y(N))^*, \quad g_{\alpha,\beta} \in \mathcal{O}(Y(N))^* \otimes_{\mathbb{Z}} \mathbb{Q} \text{ if } N\alpha = N\beta = 0.$$

In order to construct the **Siegel units**, we invoke a result of Siegel on elliptic curves over a general scheme S, which have nice compatibility conditions, and translate them in the modular curve setting, thanks to the existence of the universal elliptic curve of $Y(N)$.

Theorem 3.1 (Kato)**.** *Let E be an elliptic curve over a scheme S and fix a positive integer c prime to 6. Then, there exists a unique element ${}_c\theta_E \in \mathcal{O}(E \smallsetminus E[c])^*$ such that:*

(i) ${}_c\theta_E$ has divisor $c^2(0) - E[c]$ on E, where (0), the zero section of E, and $E[c]$, the kernel of the multiplication by c morphism, are regarded as Cartier divisors;

(ii) for any positive integer a coprime to c, let $N_a : \mathcal{O}(E \smallsetminus E[ac])^* \to \mathcal{O}(E \smallsetminus E[c])^*$ denote the norm map associated to the pullback of the multiplication by a map $E \smallsetminus E[ac] \to E \smallsetminus E[c]$. Then ${}_c\theta_E$ is compatible with N_a, i.e.

$$N_a({}_c\theta_E) = {}_c\theta_E.$$

Remark 3.2. Before sketching the proof, note that if $\pi : X \to Y$ is a finite, locally free morphism of smooth, projective, geometrically connected schemes of rank d, then there exists a norm map of degree d whose formation commutes with base-change. In the case of elliptic curve E/K, for K algebraically closed field of characteristic 0, the norm map N_a is the one associated to the field extension $K(E)/[a]^*K(E)$.

Sketch of the proof. **Uniqueness**. Suppose that f and g satisfy (i) and (ii), then

$$g = uf, \text{ for } u \in \mathcal{O}(S)^{\times}.$$

Hence, by (ii),

$$uf = g = N_a(g) = N_a(uf) = N_a(u)f = u^{a^2} f.$$

This is necessary to force u to be 1. Indeed, for $a = 2, 3$ we have that $u^3 - 1 = 0$ and $u^8 - 1 = 0$, conditions that imply $u = 1$.

Existence. We verify the existence locally and then glue the local pieces to obtain the required unit (which is possible since we have the norm relation). Recall we have the isomorphism on the S-rational points

$$\frac{\text{invertible sheaves of degree 0 divisors on } E}{\text{pullback of the ones on } S} \xrightarrow{\sim} E(S).$$

Fix now an integer a coprime to c; then, the image of $c^2(0) - E[c]$ under multiplication by a is $c^2(0) - E[c]$ itself. By choosing $a = 2$, we get that the image of $c^2(0) - E[c]$ in $E(S)$ is 0 (we use that c is coprime with 2). In other words, this implies that $c^2(0) - E[c]$ is locally principal on S, so locally there exists $f \in \mathcal{O}(E \smallsetminus E[c])^*$, with divisor $c^2(0) - E[c]$. Similarly as above, the divisor of $N_a(f)$ is $c^2(0) - E[c]$, hence

$$N_a(f) = u_a f, \text{ for } u_a \in \mathcal{O}(S)^*.$$

In order to get units invariant under the norm maps N_a, we simply take $g = u_2^{-3} u_3 f$. This function has the required property, since $u_a^{b^2-1} = u_b^{a^2-1}$ for a, b coprime with c (the equality comes from the fact that $N_b \circ N_a = N_a \circ N_b$):

$$\begin{aligned} N_a(g) &= u_2^{-3a^2} u_3^{a^2} u_a f \\ &= (u_2^{a^2-1})^{-3} u_3^{a^2-1} u_a (u_2^{-3} u_3 f) \\ &= u_a^{-9} u_a^{8} u_a g = g. \end{aligned}$$

We can then glue these pieces together to obtain the desired unit. □

These units have fundamental properties:

Proposition 3.3. *Let d be an integer prime to 6 and $E/S, c$ as in Theorem 3.1, then we have the following properties:*

(i) In $\mathcal{O}(E \smallsetminus E[cd])^*$,

$$({}_d\theta_E)^{c^2}([c]^*({}_d\theta_E))^{-1} = ({}_c\theta_E)^{d^2}([d]^*({}_c\theta_E))^{-1}. \tag{3.1}$$

(ii) The functions ${}_c\theta_E$ are invariant under base change, i.e. for any morphism $S' \to S$ and $g : E' = E \times_S S' \to E$,

$$g^*{}_c\theta_E = {}_c\theta_{E'}.$$

(iii) If $h : E \to E'$ is an isogeny between elliptic curves over S with degree prime to c, then the norm map N_h maps ${}_c\theta_E$ to ${}_c\theta_{E'}$.

Remark 3.4. Let $\phi : E \to E'$ be a separable isogeny between elliptic curves over an algebraically closed field K and denote by τ_P the translation by P map for $P \in E$, then $K(E)/\phi^*K(E')$ is Galois and there is an isomorphism

$$\begin{aligned} \operatorname{Ker}(\phi) &\xrightarrow{\sim} \operatorname{Gal}(K(E)/\phi^*K(E')) \\ P &\mapsto \tau_P^*. \end{aligned}$$

Hence, for any $Q \in E'(K)$, the norm relation gives

$${}_c\theta_{E'}(Q) = \prod_{\substack{T \in E(K) \\ \phi(T) = Q}} {}_c\theta_E(T).$$

2 Definition of Siegel units

Let $N \geq 3$ be an integer. In order to define the **Siegel units** for $Y(N)$, let $\mathcal{E}$ be the universal elliptic curve over $Y(N)$, i.e.

$$\begin{array}{c} \mathcal{E} \\ \downarrow \, \uparrow e_1 \; \uparrow e_2 \\ Y(N), \end{array}$$

where e_1 and e_2 denote sections corresponding to the choice of a basis for $\mathcal{E}[N]$.

Definition 3.5. Let $N \geq 3$ be an integer such that $N\alpha = N\beta = 0$ in $\mathbb{Q}/\mathbb{Z}$, where $(\alpha, \beta) = (\frac{a}{N}, \frac{b}{N}) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z}) \smallsetminus \{(0,0)\}$ with $a, b \in \mathbb{Z}$. Then, for $c > 1$ integer prime to $6N$, define

$${}_c g_{\alpha,\beta} = i^*_{\alpha,\beta}({}_c\theta_{\mathcal{E}}) \in \mathcal{O}(Y(N))^*,$$

where $i^*_{\alpha,\beta}$ denotes the pullback of the map

$$i_{\alpha,\beta} : Y(N) \to \mathcal{E} \smallsetminus \mathcal{E}[c], \quad i_{\alpha,\beta} = ae_1 + be_2.$$

Moreover, let $r > 1$ be an integer such that

- $(r, 6) = 1$,
- $r \neq \pm 1$,
- $r \equiv 1 \pmod{N}$.

Then

$${}_r g_{\alpha,\beta} \otimes \frac{1}{r^2 - 1} \in \bigcup_N \mathcal{O}(Y(N))^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The elements ${}_c g_{\alpha,\beta}$ and $g_{\alpha,\beta}$ are called **Siegel units**.

Remark 3.6. (i) $g_{\alpha,\beta}$ is independent of the choice of r;

(ii) For any integer c such that $(c, 6N) = 1$, then

$${}_c g_{\alpha,\beta} = (g_{\alpha,\beta})^{c^2} / g_{c\alpha,c\beta} \text{ in } \mathcal{O}(Y(N))^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

3 The action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ ans some quotients of $Y(N)$

The action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ on $Y(N)$ is given in the following way: let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/N\mathbb{Z})$, then σ acts on $Y(N)$ sending

$$(E, e_1, e_2) \mapsto (E, e_1', e_2'), \text{ where } \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

We will denote by σ^* the induced action on $\mathcal{O}(Y(N))^*$.

The following result will play an important role on the various proofs of the norm compatibility relations.

Lemma 3.7. *Let $\sigma \in GL_2(\mathbb{Z}/N\mathbb{Z})$ and let $(\alpha, \beta) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$. Then,*

$$\sigma^*({}_c g_{\alpha,\beta}) = {}_c g_{\alpha',\beta'},$$
$$\sigma^*(g_{\alpha,\beta}) = g_{\alpha',\beta'},$$

where c is an integer prime to $6N$ and $(\alpha', \beta') = (\alpha, \beta) \cdot \sigma$.

Remark 3.8. Note that we have a natural map

$$f : \mathfrak{h} \longrightarrow Y(N)(\mathbb{C}), \quad \tau \mapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \tau/N, 1/N), \tag{3.2}$$

such that the action of $SL_2(\mathbb{Z})$ on $\mathfrak{h}$ and the one of $GL_2(\mathbb{Z}/N\mathbb{Z})$ on $Y(N)(\mathbb{C})$ are compatible under f.

Moreover, we have

$$\begin{aligned}(\mathbb{Z}/N\mathbb{Z})^* \times \Gamma(N)\backslash\mathfrak{h} &\xrightarrow{\sim} Y(N)(\mathbb{C}),\\ (a,\tau) &\mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f(\tau).\end{aligned}$$

The space $\Gamma(N)\backslash\mathfrak{h}$ parametrizes isomorphism classes of triples consisting of elliptic curves with a choice of a basis for the N-torsion subgroup: $(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}), \mathbb{Z}/N\mathbb{Z}$-basis of $E[N])$, such that the two elements of the basis are mapped to a chosen primitive N^{th}-root of unity via the Weil pairing. The fibre of $Y(N)(\mathbb{C})$ over $e^{\frac{2\pi i}{N}}$, under the morphism defined by the Weil pairing, is identified with $\Gamma(N)\backslash\mathfrak{h}$.

Remark 3.9. The **Siegel units** can be seen as holomorphic functions on the upper half-plane $\mathfrak{h}$. In order to do it, we need to fix a N^{th}-root of unity ζ_N. As a consequence, their q-expansion has coefficients in $\mathbb{Q}(\zeta_N)$ and not $\mathbb{Q}$. Indeed, let $q = e^{2\pi i\tau}$ and $\zeta_N = e^{\frac{2\pi i}{N}}$, then

$$g_{\alpha,\beta} = q^{\frac{1}{12} - \frac{a}{2N} + \frac{a^2}{2N^2}} \prod_{n\geq 0}(1 - q^n q^{\frac{a}{N}} \zeta_N^b) \prod_{n\geq 1}(1 - q^n q^{-\frac{a}{N}} \zeta_N^{-b}).$$

Definition 3.10. Let M, N be integers greater than 1 and let $L \geq 3$ be a multiple of N and M. Consider the subgroup $G_{M,N}$ of $GL_2(\mathbb{Z}/L\mathbb{Z})$ given by

$$G_{M,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/L\mathbb{Z}) \colon \begin{matrix} a \equiv 1 \bmod M, & c \equiv 0 \bmod N, \\ b \equiv 0 \bmod M, & d \equiv 1 \bmod N \end{matrix} \right\},$$

and define

$$Y(M,N) = G_{M,N} \setminus Y(L).$$

Remark 3.11. If $N \geq 3$, then $Y(N,N) = Y(N)$.

Remark 3.12. Consider M, N such that $M + N \geq 5$; the scheme $Y(M,N)$ over $\mathbb{Q}$ represents the functor

$$\begin{aligned} F : Sch^{op} &\longrightarrow Sets \\ S &\mapsto \left\{ \begin{matrix} \text{isomorphism classes of } (E, e_1, e_2), \text{ where } E \text{ elliptic curve over } S \\ \text{and } e_1, e_2 \in E(S) \text{ such that } Me_1 = Ne_2 = 0 \text{ and the morphism} \\ \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E \text{ mapping } (a,b) \mapsto ae_1 + be_2 \text{ is injective} \end{matrix} \right\}. \end{aligned}$$

This follows since the $G_{M,N}$-equivalence classes of the level L structure $(\mathbb{Z}/L\mathbb{Z})^2 \to E[L]$ correspond to injections $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E$. Moreover, the canonical morphism $Y(L) \to Y(M,N)$ on S-rational points is given by

$$(E, e_1, e_2) \to (E, \frac{L}{M}e_1, \frac{L}{N}e_2).$$

Proposition 3.13. *Consider* ${}_d g_{0,\frac{1}{N}} \in \mathcal{O}(Y(N))^*$, *for* d *such that* $(d, 6N) = 1$; *then,*

$${}_d g_{0,\frac{1}{N}} \in \mathcal{O}(Y(1,N))^*.$$

4 The behaviour under norm morphisms

Firstly, we describe the *distribution relations* that arise from the structure of the norm map and the norm compatibility of ${}_c\theta_{\mathcal{E}}$.

Proposition 3.14. *Let* $(\alpha,\beta)\in(\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2\smallsetminus\{(0,0)\}$ *and let* a *be a non-zero integer. Then*

$$
\begin{aligned}
{}_cg_{\alpha,\beta} &= \prod_{\alpha',\beta'} {}_cg_{\alpha',\beta'},\\
g_{\alpha,\beta} &= \prod_{\alpha',\beta'} g_{\alpha',\beta'},
\end{aligned}
$$

where c *is an integer prime to* $6Na$ *and* α',β' *run through all the elements of* $\mathbb{Q}/\mathbb{Z}$ *such that* $a\alpha'=\alpha$ *and* $a\beta'=\beta$.

We now describe the norm relations of ${}_dg_{0,\frac{1}{N}}\in\mathcal{O}(Y(M,N))^*$.

Let $N'\geq 1$ be a multiple of N. Then the natural morphism $\alpha:Y(M,N')\to Y(M,N)$ induces a norm map

$$N_\alpha:\mathcal{O}(Y(M,N'))^*\to\mathcal{O}(Y(M,N))^*.$$

Remark 3.15. For any $f\in\mathcal{O}(Y(M,N'))^*$,

$$N_\alpha(f)=\prod_{\sigma\in S}\sigma^*(f),$$

where the finite set S consists of a system of right coset representatives for $G_{M,N'}\setminus G_{M,N}$.

Theorem 3.16. *Let* $M,N,N'\geq 1$ *be integers such that* $N|N'$ *and suppose that*

$$\mathrm{Rad}(N)=\mathrm{Rad}(N'),$$

where Rad *denotes the radical. Then,*

$$N_\alpha({}_dg_{0,\frac{1}{N'}})={}_dg_{0,\frac{1}{N}}.$$

Proof. The strong assumption on the prime divisors of N' allow us to find a set of right coset representatives for $G_{M,N'}\setminus G_{M,N}$. Let $a=\frac{N'}{N}$ and let $L\geq 3$ be an integer such that $M|L$ and $N'|L$. For every $(x,y)\in(\mathbb{Z}/a\mathbb{Z})^2$, we choose an element

$$\sigma_{x,y}=\begin{pmatrix}1 & 0\\ Nu & 1+Nv\end{pmatrix}\in GL_2(\mathbb{Z}/L\mathbb{Z}),$$

where

$$
\begin{aligned}
u&\equiv x \pmod a,\\
v&\equiv y \pmod a.
\end{aligned}
$$

The set $\{\sigma_{x,y}\}_{(x,y)\in(\mathbb{Z}/a\mathbb{Z})^2}$ forms a set of right coset representatives for $G_{M,N'}\setminus G_{M,N}$, where both are seen as subgroups of $GL_2(\mathbb{Z}/L\mathbb{Z})$). Hence, we have

$$N_\alpha({}_dg_{0,\frac{1}{N'}})=\prod_{(x,y)\in(\mathbb{Z}/a\mathbb{Z})^2}\sigma_{x,y}^*({}_dg_{0,\frac{1}{N'}})=\prod_{(x,y)\in(\mathbb{Z}/a\mathbb{Z})^2}{}_dg_{\frac{u}{a},\frac{1}{N'}+\frac{v}{a}}={}_dg_{0,\frac{1}{N}},$$

where we used Lemma 3.7 for the second equality and Proposition 3.14 for the last one. □

Remark 3.17. Removing the assumption on the set of prime divisors of N' makes the calculation harder. We compute the norm relation in a few steps. Note the morphism $\alpha:Y(M,Nl)\to Y(M,N)$ factors through some intermediate quotients

$$Y(M,Nl)\xrightarrow{\alpha_1}Y(M,N(l))\xrightarrow{\alpha_2}Y(M,N),$$

thus

$$N_\alpha({}_dg_{0,\frac{1}{Nl}})=N_{\alpha_2}\left(N_{\alpha_1}({}_dg_{0,\frac{1}{Nl}})\right).$$

Recall

$$Y(M, N(A)) = G_{M,N(A)} \setminus Y(L),$$

where

- $L \geq 3$ is divisible by M and AN;
- $G_{M,N(A)}$ is the subgroup of $GL_2(\mathbb{Z}/L\mathbb{Z})$ defined by

$$G_{M,N(A)} \colon = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/L\mathbb{Z}) \colon \begin{matrix} a \equiv 1 \bmod M, & c \equiv 0 \bmod AN, \\ b \equiv 0 \bmod M, & d \equiv 1 \bmod N \end{matrix} \right\}.$$

Remark 3.18. Consider the canonical morphisms induced by f in (3.2)

$$\mathfrak{h} \longrightarrow Y(M, N(A))(\mathbb{C}), \quad \mathfrak{h} \longrightarrow Y(M(A), N)(\mathbb{C}).$$

Then, there is a unique (iso)morphism

$$\phi_A : Y(M, N(A)) \longrightarrow Y(M(A), N),$$

which makes the following diagram commute.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{[A]} & \mathfrak{h} \\ \downarrow & & \downarrow \\ Y(M, N(A))(\mathbb{C}) & \stackrel{\phi_A}{\dashrightarrow} & Y(M(A), N)(\mathbb{C}), \end{array}$$

where $[A]$ denotes multiplication by A.

Lemma 3.19. *Let $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$ and let A be an integer greater than 1 and let c be a positive integer prime to $6A$ and to the orders of α and β. Then, as functions on the upper half-plane $\mathfrak{h}$,*

$$_c g_{\alpha,\beta}(A\tau) = \prod_{\beta'} {_c g_{\alpha,\beta'}}(\tau), \tag{3.3}$$

where β' ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $A\beta' = \beta$.

$$_c g_{\alpha,\beta}(\tau/A) = \prod_{\alpha'} {_c g_{\alpha',\beta}}(\tau), \tag{3.4}$$

where α' ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $A\alpha' = \alpha$.

Theorem 3.20. *Let $M, N, N' \geq 1$ be integers such that $N' = Nl$, where l is a prime such that $l \nmid MN$. Then, we have*

$$N_\alpha({_d g_{0,\frac{1}{Nl}}}) = {_d g_{0,\frac{1}{N}}} \left({_d g_{0,\frac{r}{N}}}\right)^{-1},$$

where r denotes the inverse of l modulo N.

Proof. In what follows, r denotes the inverse of l modulo N.

Step 1. We have that

$$N_{\alpha_1}({_d g_{0,\frac{1}{Nl}}}) = \phi_l^*({_d g_{0,\frac{1}{N}}}) \left({_d g_{0,\frac{r}{N}}}\right)^{-1},$$

where $\phi_l : Y(M, N(l)) \to Y(M(l), N)$ is defined in Remark 2.1.7.

Step 2. We have that

$$N_{\alpha_2}(\phi_l^*({_d g_{0,\frac{1}{N}}})) = {_d g_{0,\frac{1}{N}}} \left({_d g_{0,\frac{r}{N}}}\right)^{l}.$$

Step 3.

$$N_{\alpha_2}({}_dg_{0,\frac{r}{N}}) = \left({}_dg_{0,\frac{r}{N}}\right)^{l+1}.$$

Putting all together, we have

$$\begin{aligned} N_\alpha({}_dg_{0,\frac{1}{Nl}}) &= N_{\alpha_2}\left(N_{\alpha_1}({}_dg_{0,\frac{1}{Nl}})\right) = N_{\alpha_2}\left(\phi_l^*({}_dg_{0,\frac{1}{N}})\left({}_dg_{0,\frac{r}{N}}\right)^{-1}\right) \\ &= {}_dg_{0,\frac{1}{N}}\left({}_dg_{0,\frac{r}{N}}\right)^{l}\left({}_dg_{0,\frac{r}{N}}\right)^{-l-1} = {}_dg_{0,\frac{1}{N}}\left({}_dg_{0,\frac{r}{N}}\right)^{-1}. \end{aligned}$$

□

5 Eisenstein classes in Motivic cohomology

Consider the universal elliptic curve

$$\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ Y_1(N), \end{array}$$

Definition 3.21. For the cohomology theory $\mathcal{T} \in \{\acute{e}t, dR\}$, we define the appropriate category of coefficient sheaves on $Y_1(N)$, $\mathscr{H}_{\mathcal{T}}$, as follows:

$$\mathscr{H}_{\acute{e}t} = R^1\pi_*\mathbb{Q}_p(1),$$

$$\mathscr{H}_{dR} = R^1\pi_*[\mathcal{O}_{\mathcal{E}} \xrightarrow{d} \Omega_{\mathcal{E}/Y_1(N)}](1).$$

Remark 3.22.

- Let $f : X \to Y$ be a morphism of schemes. Given a positive integer i and a sheaf $\mathcal{F}$ on X, the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}),$$

  on Y and it can be seen as cohomology along the fibres of X over Y. In our case, $R^1\pi_*\mathbb{Q}_{\mathcal{T}}$ is the sheaf whose fiber at a point x is the cohomology

$$H^1_{\mathcal{T}}(\mathcal{E}_x, \mathbb{Q}_{\mathcal{T}}).$$

- Note again that one may form de Rham cohomology $\mathcal{H}^\bullet_{dR}(\mathcal{E}/Y_1(N))$ as a sheaf on the base $Y_1(N)$ exactly as the higher derived functor $R^\bullet\pi_*[\Omega^*_{\mathcal{E}/Y_1(N)}]$.
- The twist in the de Rham sheaf $\mathscr{H}_{dR}$ is a shift of the Hodge filtration attached to it, which becomes

$$\begin{aligned} F^{-1}\mathscr{H}_{dR} &= \mathscr{H}_{dR}, \\ F^0\mathscr{H}_{dR} &= \underline{\omega}, \\ F^1\mathscr{H}_{dR} &= 0, \end{aligned}$$

  where $\underline{\omega}$ is the subsheaf of $\mathscr{H}_{dR}$

$$\underline{\omega} = \pi_*\Omega^1_{\mathcal{E}/Y_1(N)}.$$

We want to define non-zero cohomology classes which generalise Siegel units:

$$Eis^k_{mot,b,N} \in H^1_{mot}(Y_1(N), TSym^k\mathscr{H}_{\mathbb{Q}}(1)).$$

Remark 3.23. We are looking for elements in motivic cohomology with coefficients in $TSym^k\mathscr{H}_{\mathbb{Q}}(1)$ because we are dealing with modular forms of weight $k+2$ and, looking at the etale cohomology with coefficients in $TSym^k\mathscr{H}_{et}(1)$ the connection is visible as we will see in Francesc's talk.

Remark 3.24. Why do we consider Eisenstein classes? The *dlog* map

$$\mathcal{O}(Y_1(N))^* \xrightarrow{dlog} H^0(Y_1(N)(\mathbb{C}), \Omega),$$

will map $g_{0,\frac{1}{N}}$ to an explicit weight 2 eisenstein series: norm compatibility tells us that the object obtained this way will transform like an Eisenstein series under Hecke operators. The Kummer map

$$\mathcal{O}(Y_1(N))^* \longrightarrow H^1_{\acute{e}t}(Y_1(N), \mathbb{Q}_p(1)),$$

induces a morphism in cohomology, similarly, also the previous map does. Therefore, we are looking for an object which is the common source of these maps and this is the first motivic cohomology group:

$$\begin{array}{ccc} H^1_{\acute{e}t}(Y_1(N), TSym^k\mathscr{H}_{\acute{e}t}(1)) & \longleftarrow & H^1_{mot}(Y_1(N), TSym^k\mathscr{H}_{\mathbb{Q}}(1)) \\ & & \downarrow \\ & & H^1(Y_1(N)(\mathbb{C}), TSym^k\mathscr{H}_{dR} \otimes \Omega). \end{array}$$

Suppose to take $k = 0$, then we have

$$Eis^0_{mot,b,N} \in H^1_{mot}(Y_1(N), \mathbb{Q}(1)) = \mathcal{O}(Y_1(N))^* \otimes \mathbb{Q}.$$

The natural candidate is the Siegel unit $g_{0,b/N}$.
In general for $k \geq 1$, we need to first define the motivic cohomology groups $H^1_{mot}(Y_1(N), TSym^k\mathscr{H}_{\mathbb{Q}}(1))$.

Definition 3.25. For an integer $k \geq 0$, let $\mathfrak{I}_k = \mu_2^k \rtimes \Sigma_k$, where Σ_k denotes the symmetric group on k letters. Define the character

$$\varepsilon_k : \mathfrak{I}_k \longrightarrow \mu_2, \quad (a_1, \dots, a_k, \sigma) \mapsto a_1 \cdots a_k \cdot \mathrm{sign}(\sigma).$$

Then, denote by $\mathcal{E}^k$ the k-fold fibre product of $\mathcal{E}$ over $Y_1(N)$. We have an action of $\mathfrak{I}_k$ on $\mathcal{E}^k$, since

- μ_2 acts on $\mathcal{E}$ as multiplication by -1,
- Σ_k acts on $\mathcal{E}^k$ permuting the factors.

Using an argument involving the Leray spectral sequence, we have the following.

Theorem 3.26. *For any cohomology theory $\mathcal{T} \in \{\acute{e}t, dR\}$, we have*

$$H^{i+k}_{\mathcal{T}}(\mathcal{E}^k, \mathbb{Q}_{\mathcal{T}}(j+k))(\varepsilon_k) \simeq H^i_{\mathcal{T}}(Y_1(N), TSym^k\mathscr{H}_{\mathcal{T}}(j)).$$

Thus, it is reasonable to define

$$H^i_{mot}(Y_1(N), TSym^k\mathscr{H}_{\mathbb{Q}}(j)) = H^{i+k}_{mot}(\mathcal{E}^k, \mathbb{Q}(j+k))(\varepsilon_k).$$

The choice is also motivated by the next fundamental result.

Proposition 3.27. *The regulator map $r_{\mathcal{T}}$ commutes with the action of $\mathfrak{I}_k$ and so we get a map*

$$r_{\mathcal{T}} : H^i_{mot}(Y_1(N), TSym^k\mathscr{H}_{\mathbb{Q}}(j)) \longrightarrow H^i_{\mathcal{T}}(Y_1(N), TSym^k\mathscr{H}_{\mathcal{T}}(j))$$

Let $k \geq 0$ and $b \in \mathbb{Z}/N\mathbb{Z}$. The cohomology classes

$$Eis^k_{mot,b,N} \in H^1_{mot}(Y_1(N), TSym^k\mathscr{H}_{\mathbb{Q}}(1))$$

can be constructed in two different ways:

(i) They can be obtained as specialization of the elliptic polylogarithm at the order N section $b\varepsilon_N$ (here ε_N is the canonical N torsion section);

(ii) They can be introduced as cup-products of several ${}_c\theta_{\mathcal{E}}$'s that are pulled back to

$$H^{k+1}_{mot}(\mathcal{E}^k, \mathbb{Q}(k+1))(\varepsilon_k)$$

by certain maps depending on the choice of an torsion N section. Here, we give a brief description of the second (more intuitive) approach for the classes in $Y_1(N)$, constructing them in $K_{k+1}(\mathcal{E}^k)(\varepsilon_k) \otimes \mathbb{Q}$, hence defining classes in

$$H^{k+1}_{mot}(\mathcal{E}^k, \mathbb{Q}(k+1))(\varepsilon_k).$$

- For any section $x \in \mathcal{E}(Y_1(N))$ and $k \geq 1$, we define the morphism

$$i_x : \mathcal{E}^k \to \mathcal{E}^{k+1}, \quad (p_1, \ldots, p_k) \mapsto (x - p_1, p_1 - p_2, \ldots, p_{k-1} - p_k, p_k).$$

If $k = 0$, we set i_x to be x.

Recall that the definition of the **Siegel units** consists of the pullback under the sections $ae_1 + be_2$ of canonical units in $\mathcal{O}(\mathcal{E} \setminus \mathcal{E}[c])^*$. Hence, the first try in defining the **Eisenstein symbols** would be to replace the section $ae_1 + be_2$ by $i_{ae_1+be_2}$ and the role of ${}_c\theta_{\mathcal{E}}$ by cup-product of it.

- Denote by pr_i the projection of $\mathcal{E}^{k+1}$ into the i^{th}-component. For any positive integer c coprime to 6, define

$${}_c\theta^{[k+1]}_{\mathcal{E}} = pr_1^* {}_c\theta_{\mathcal{E}} \cup \cdots \cup pr_{k+1}^* {}_c\theta_{\mathcal{E}} \in K_{k+1}((\mathcal{E} \smallsetminus \mathcal{E}[c])^{k+1}).$$

- For any positive integer c such that $x \in \mathcal{E}(Y_1(N))$ is disjoint from $\mathcal{E}[c]$, define the open subschemes
  (i) ${}_cV^k_x = i_x^{-1}((\mathcal{E} \smallsetminus \mathcal{E}[c])^{k+1}) \subset \mathcal{E}^k$;
  (ii) ${}_cU^k_x = \bigcap_{\gamma \in \mathfrak{I}_k} \gamma({}_cV^k_x) \subset \mathcal{E}^k$.

First, we pullback ${}_c\theta^{[k+1]}_{\mathcal{E}} \in K_{k+1}((\mathcal{E} \smallsetminus \mathcal{E}[c])^{k+1})$ by i_x, for x as in the previous definition, in order to get

$${}_cg^{[k]}_x = i_x^*({}_c\theta^{[k+1]}_{\mathcal{E}}) \in K_{k+1}({}_cV^k_x).$$

- If $z \in \mathcal{E}(Y_1(N))$ is a section disjoint from the zero section, the inclusion ${}_1U^k_z \hookrightarrow (\mathcal{E} \smallsetminus \{\pm z\})^k$ induces an isomorphism

$$K_\bullet((\mathcal{E} \smallsetminus \{\pm z\})^k)(\varepsilon_k) \xrightarrow{\sim} K_\bullet({}_1U^k_z)(\varepsilon_k).$$

In order to map the elements ${}_cg^{[k]}_x$ into $K_{k+1}({}_1U^k_z)(\varepsilon_k)$:

1. define the elements

$${}_c\bar{g}^{[k]}_x = \frac{1}{|\mathfrak{I}_k|} \sum_{\gamma \in \mathfrak{I}_k} \varepsilon(\gamma)\gamma^*({}_cg^{[k]}_x) \in K_{k+1}({}_cU^k_x)(\varepsilon_k);$$

2. note that $[c] : {}_cU^k_x \to {}_1U^k_{cx}$ is an étale covering and we can talk about the induced pushforward in K-theory.

- Take the elements

$${}_c\,\mathrm{Eis}^{[k]}_x = [c]_*({}_c\bar{g}^{[k]}_x) \in K_{k+1}((\mathcal{E} \smallsetminus \{\pm cx\})^k)(\varepsilon_k).$$

- Define

$$Eis^k_{mot,b,N} = (c^2 - c^{-k})^{-1} {}_cEis^k_{0,b/N},$$

for $c > 1$ integer coprime to 6 and congruent 1 modulo N, $N \geq 5$, $b \in \mathbb{Z}/N\mathbb{Z}$ non-zero.

Remark 3.28. (i) If $h_1, \dots, h_{k+1} : E \to E'$ are isogenies between elliptic curves over $Y_1(N)$ of degree prime to c, we have

$$(h_1 \times \cdots \times h_{k+1})_{*\,c}\theta_E^{[k+1]} = {}_c\theta_{E'}^{[k+1]}.$$

(ii) These classes satisfy all the norm compatibility relations as the Siegel units do.

Definition 3.29. For any cohomology theory $\mathcal{T} \in \{ét, dR\}$, define

$$Eis^k_{\mathcal{T},b,N} = r_{\mathcal{T}}(Eis^k_{mot,b,N}),$$

so, we have

$$Eis^k_{et,b,N} \in H^1_{ét}(Y_1(N)_{\mathbb{Z}[1/Np]}, TSym^k \mathscr{H}_{ét}(1)),$$
$$Eis^k_{dR,b,N} \in H^1_{dR}(Y_1(N)_{\mathbb{Q}}, TSym^k \mathscr{H}_{dR}(1)).$$

6 The de Rham Eisenstein Class

We want to give a formula for the de Rham Eisenstein class, comparing its q-expansion with one of an "Eisenstein series" of weight $k+2$. Kato defines a norm-compatible family of "Eisenstein series" algebraically. The idea under the construction is to give an additive equivalent to Siegel units. There are two natural ways to define modular forms from the units we described above.

1. Take the logarithmic derivative of ${}_c\theta_{\mathcal{E}}$,

$$dlog({}_c\theta_E) \in \Gamma(\mathcal{E} \smallsetminus \mathcal{E}[c], \Omega^1_{\mathcal{E}/Y_1(N)}),$$

and pull it back by $i_{\alpha,\beta}$. It's a weight 1 modular form

$${}_cE^{(1)}_{\alpha,\beta} = i^*_{\alpha,\beta}(dlog({}_c\theta_{\mathcal{E}})) \in \Gamma(Y_1(N), i^*_{\alpha,\beta}\Omega^1_{\mathcal{E}/Y_1(N)}) = \Gamma(Y_1(N), \underline{\omega}),$$

where the last equality follows from the fact that, since $\Omega^1_{\mathcal{E}/Y_1(N)}$ is free on the fibres of $\pi : \mathcal{E} \to Y_1(N)$,

$$\underline{\omega} = 0^*\Omega^1_{\mathcal{E}/Y_1(N)}$$

is isomorphic to $x^*\Omega^1_{\mathcal{E}/Y_1(N)}$, for any section $x \in \mathcal{E}(Y_1(N))$.

2. Take the logarithmic derivative

$$dlog({}_cg_{\alpha,\beta}) \in \Gamma(Y_1(N), \Omega^1_{Y_1(N)/\mathbb{Q}}).$$

In this particular setting, the Kodaira-Spencer map, i.e. the $\mathcal{O}_{Y_1(N)}$-linear morphism

$$KS : \underline{\omega}^{\otimes 2} \longrightarrow \Omega^1_{Y_1(N)/\mathbb{Q}}$$

is an isomorphism, hence

$$dlog({}_cg_{\alpha,\beta}) \in \Gamma(Y_1(N), \underline{\omega}^{\otimes 2})$$

gives a weight 2 modular form.

Starting from ${}_cE^{(1)}_{\alpha,\beta}$, we can construct modular forms of higher weight $k \neq 2$. In particular, for an integer $k \geq 1$ and $(\alpha, \beta) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$, we wish to define elements

1. $E^{(k)}_{\alpha,\beta} \in M_k(\Gamma(N))$, where $k \geq 1$, $k \neq 2$;
2. $\tilde{E}^{(2)}_{\alpha,\beta} \in M_2(\Gamma(N))$;
3. $F^{(k)}_{\alpha,\beta} \in M_k(\Gamma(N))$, where $k \geq 1$ and $(\alpha, \beta) \neq (0, 0)$ if $k = 2$.

For lack of time, we only define $F^{(k+2)}_{0,b/N}$ by their q-expansions.

Definition 3.30. Let $\zeta_N = e^{2\pi i/N}$ and $q = e^{2\pi i\tau}$. For $k \geq -1$ and $b \in \mathbb{Z}/N\mathbb{Z}$ not zero, define

$$F^{(k+2)}_{0,b/N} = \zeta(-1-k) + \sum_{n>0} q^n \sum_{\substack{dd'=n \\ d,d'>0}} d^{k+1}(\zeta_N^{bd'} + (-1)^k \zeta_N^{-bd'}).$$

In the case $k = 0$, we have the following.

Proposition 3.31. *There is a map* $dlog : \mathcal{O}(Y_1(N))^* \otimes \mathbb{Q} \to M_2(\Gamma_1(N))$ *which, as functions on* $\mathfrak{h}$, *maps* $f(\tau) \mapsto \frac{f'(\tau)}{f(\tau)}$. *Then, for any* $b \in \mathbb{Z}/N\mathbb{Z}$ *not zero, we have,*

$$dlog(g_{0,b/N}) = -F^{(2)}_{0,b/N}.$$

Proof. Compare the q-expansions. □

For $k > 0$, first note that $Eis^k_{dR,b,N}$ belongs to the degree zero piece of the Hodge filtration of

$$H^1_{dR}(Y_1(N)_{\mathbb{Q}}, TSym^k \mathscr{H}_{dR}(1)),$$

which is given by

$$\Gamma(Y_1(N)_{\mathbb{Q}}, \underline{\omega}^{\otimes k} \otimes \Omega_{Y_1(N)}).$$

Let $(Tate(q), \zeta_N)$ denote the $Spec(\mathbb{Z}[\zeta_N, 1/N])((q))$-point of $Y_1(N)$, given by the Tate curve $Tate(q)$ with canonical differential ω_{can}. Then, ω_{can} gives a section of

$$Fil^0 \mathscr{H}_{dR}|_{Tate(q)} = \underline{\omega}|_{Tate(q)}.$$

Denote by $v^{[0,k]}$ the k^{th} tensor power of this section. Then, we have the following.

Theorem 3.32. *The pullback of the de Rham Eisenstein class to* $(Tate(q), \zeta_N)$ *is given by*

$$Eis^k_{dR,b,N} = -N^k F^{(k+2)}_{0,b/N} \cdot v^{[0,k]} \otimes \frac{dq}{q}.$$

The essential ingredient of the proof is the residue formula at the cusp ∞ of the motivic class.

Lemma 3.33. *The Eisenstein classes satisfy the following residue formula*

$$res_\infty(Eis^k_{mot,b,N}) = -N^k \zeta(-1-k),$$

for all $k \geq 0$.

Definition of the global classes

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1 Beilinson-Kato elements and Global galois cohomology

1.1 Modular curves

Definition 4.1. For $N \geq 3$, let $Y(N)/\mathbb{Q}$ be the modular curve such that

$$S \longmapsto \{^{\text{isomorphic classses of triples (E,e}_1\text{,e}_2\text{)}}_{E/S\text{:elliptic curve,e}_1\text{,e}_2\text{ sections of E/S generating E[N]}}\}$$

where $Y(N)$ is equipped with a left action of $GL_2(\mathbb{Z}/N\mathbb{Z})$: the element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps (E, e_1, e_2) to (E, e_1', e_2') such that

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

μ_N^0 is the scheme of primitive N-th roots of unity. we have a canonical surgection

$$Y(N) \twoheadrightarrow \mu_N^0$$

which maps (E, e_1, e_2) to $< e_1, e_2 >_{E[N]}$ under the Weil pairing. This induced action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ on μ_N^0 is given by $\sigma(\xi_N) = \xi_N^{\det \sigma}$. $Y(N)(\mathbb{C})$ over the point ξ_N is canonically identified with $\Gamma(N)\backslash\mathcal{H}$ via the map

$$\tau \mapsto (\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau, \tau/N, 1/N).$$

Definition 4.2. For $M, N \geq 1$, define $Y(M, N)$ to be the quotient of $Y(L)(L \geq 3$, M and N divide $L)$ by the group:

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) : (a, b) \equiv (1, 0) \bmod M, (c, d) \equiv (0, 1) \bmod N\}.$$

If $M+N \geq 5$, $Y(M, N)$ represents (E, e_1, e_2) where e_1 has order M, e_2 has order N and (e_1, e_2) generates a subgroup of E of order MN.

Denote $Y_1(N) = Y(1, N)$ such that

$$S \longmapsto \{^{\text{isomorphic classses of (E,}\Gamma\text{),}}_{E/S\text{:elliptic curve, Gamma is a section of E/S of exact order N}}\}$$

Proposition 4.3. *$N \geq 3, m \geq 1$ and $L \geq 3$ is divisible by both N and m, then the map*

$$Y(L) \to Y_1(N) \otimes \mu_M^0$$

sends (E, e_1, e_2) to $[(E, \frac{L}{N}e_2), < \frac{L}{m}e_1, \frac{L}{m}e_2 >_{E(M)}]$.

Consider the Steinberg symbol $\{,\} : O(Y)^\times \times O(Y)^\times \to K_2(Y)_\mathbb{Q}$. Define ${}_{c,d}Z_{M,N} = K_2(Y(M, N)) \otimes \mathbb{Q}$.

Lemma 4.4. *Let $N \mid N', M \mid M'$ and prime(M')=prime(M), prime(N')=prime(N). Then the trace map*

$$O(Y(M,N'))^{\times} \to O(Y(M,N))^{\times}$$

takes $_{d}g_{0,1/N'}$ *to* $_{d}g_{0,1/N}$

Let $\pi : \mathcal{E} \to Y(M,N)$. Define $\mathcal{H}^1 = R^1\pi_*\mathbb{Z}$, $\mathcal{H}_p = \mathcal{H}^1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathcal{H}_{\mathbb{Q}_p} = \mathcal{H}_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Denote $T_p\mathcal{E}$ the relative $p-adic$ Tate module of $\mathcal{E}$ over Y viewed as a sheaf $T_p\mathcal{E} \cong \mathcal{H}_p(1)$ by Poincare duality. Which deduces $\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}\mathcal{H}_p(1) \cong \mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(T_p\mathcal{E})(2-k)$.

Definition 4.5. We define the morphism $ch_{M,N}(k,r,r')$:

$$\varprojlim_n K_2(Y(Mp^n, Np^n)) \to H^1(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p(Y(M,N))(k-r)})$$

by several steps.

Firstly, define $ch_{2,2} : K_2(Y(Mp^n, Np^n) \to H^2(Y(Mp^n, Np^n), \mathbb{Z}/p^n(2))$, where $ch_{2,2}$ is the Chern chlass map. Consider the Kummer sequence

$$0 \to (\mathbb{Z}/p^n)(1) \to O_X^{\times} \to O_X^{\times} \to 0$$

Let $h : O_X^{\times} \to H^1(X, \mathbb{Z}/p^n(1))$ be the connecting homomorphism. $ch_{2,2} : \{f,g\} \to h(f) \cup h(g) \in H^2(X, \mathbb{Z}/p^n\mathbb{Z}(1))$ where U is the cup product.

Let $(e_{1,n}, e_{2,n})$ be the canonical basis of Ω. $T_p\mathcal{E}/p^n$ is for $Y(Mp^n, Np^n)$. We have this map $e_{1,n}^{\otimes(r'-1)} \otimes e_{2,n}^{\otimes(k-r'-1)} \otimes \zeta_{p^n}^{\otimes(-r)}$:

$$H^2(Y(Mp^n, Np^n), \mathbb{Z}/p^n(2)) \to H^2(Y(Mp^n, Np^n), \mathrm{Sym}^{k-2}(T_p\mathcal{E}/p^n))(2-r).$$

And the second item above is isomorphic to

$$H^2(Y(Mp^n, Np^n), \mathrm{Sym}^{k-2}(\mathcal{H}_p/p^n))(k-r).$$

Now, consider the trace map τ_r :

$$\varprojlim_n H^2(Y(Mp^n, Np^n), \mathrm{Sym}^{k-2}(\mathcal{H}_p/p^n))(k-r) \to \varprojlim_n H^2(Y(M,N), \mathrm{Sym}^{k-2}(\mathcal{H}_p/p^n))(k-r)$$

and finally we have the Hochschild-Serre spectral sequence $^{et}E_2^{a,b}$:

$$H^a_et(\mathbb{Q}, H^b(Y(M,N) \otimes \bar{\mathbb{Q}}, -)) \to H^{a+b}_et(Y(M,N), -)$$

and $H^q_{et}(Y(M,N)_{\bar{\mathbb{Q}}}, -) = 0$ for any $q \geq 2$ by Deligne's theorem. So, we get an edge map

$$H^2(Y(M,N), \mathrm{Sym}^{k-2}(\mathcal{H}_p/p^n))(k-r) \to H^1(\mathbb{Q}, H^1_{et}(Y(M,N)_{\bar{\mathbb{Q}}}, \mathrm{Sym}^{k-2}(\mathcal{H}_p/p^n))(k-r)) \to H^1(\mathbb{Q}, V_f(k-r)).$$

1.2 Symmetric Tensors

Let H be an abelian group, $TSym^kH$ is the submodule of S_k-invariant elements in the k-fold thnsor product $H^{\otimes k}$. Let $\sigma : H^k \to H^k$ be a permutation and $\varphi : H^k \to H^{\otimes k}$ be the natural embedding. The map associated to the permutation σ is the unique isomorphism $\tau_\sigma : H^{\otimes k} \to H^{\otimes k}$ such that $\varphi \circ \sigma = \tau_\sigma \circ \varphi$. Define

$$TSym^kH = \{T \in H^{\otimes k} | \tau_\sigma(T) = T \text{ for any } \sigma \in \mathrm{S_k}\}$$

where S_k is the symmetric group

$\bigcup_{k\geq 0} TSym^kH$ has a structure

$$h^{\otimes m} \cdot h^{\otimes n} = \frac{(m+n)!}{m!n!} h^{\otimes(m+n)}.$$

1.3 Eisenstein classes on $Y_1(N)$

Theorem 4.6 (Beilinson)**.** *Assume $N \geq 5$ and $b \in \mathbb{Z}/N\mathbb{Z}$ are non-generated. Then there exists non-zero cohomology classes*

$$Eis_{mot,b,N} \in H^1_{mot}(Y_1(N), TSym^k \mathcal{H}_{\mathbb{Q}}(1)).$$

Remark 4.7. For $k = 0$, $H^1_{mot}(Y_1(N), \mathbb{Q}(1))^\times \equiv O(Y_1(N))^\times \otimes \mathbb{Z}_p$. So $Eis^0_{mot,b,N} = g_{0,b/N}$ and met for $k = 0$ coincides with $h : O(Y_1(N))^\times \to H^1(Y_1(N), \mathbb{Z}/p^n(1))$.

Let $Eis^0_{et,b,N} = \eta_{et}(Eis^0_{mot,b,N}) \in H^1_{et}(Y_1(N)\mathbb{Z}[1/Np], TSym^k \mathcal{H}_{\mathbb{Q}_p}(1))$.

1.4 Rankin-Eisenstein classes on products of modules curves

The Clebsch-Gordoon map:

$$k \geq 0, k' \geq 0 \textit{ and } 0 \leq j \leq \min(k, k').$$

Consider the map$\bigwedge^2 H \to H \otimes H$ which sends $x \wedge y \mapsto x \otimes y - y \otimes x$.

Hence we get a map

$$TSym^j(\bigwedge^2 H) \to TSym^j(H) \otimes TSym^j(H)$$

by taking $TSym^j$ on both sides.

Conclusion above, we have

$$TSym^{k+k'-2j}(H) \to TSym^k(H) \otimes TSym^{k'}(H) \otimes \det(H)^{-j}.$$

Let (e_1, e_2) be a basis for $\mathbb{Z}^2$. Then $e_1 \wedge e_2$ is a basis for $\bigwedge^2 \mathbb{Z}^2$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$, let $v_1 = ae_1 + be_2$ and $v_2 = ce_1 + de_2$. Then $v_1 \wedge v_2 = \det A e_1 \wedge e_2$.

Definition 4.8. Define $CG^{[k,k'j]}$: $TSym^{k+k'-2j}(H) \to TSym^k(H) \otimes TSym^{k'}(H) \otimes \det(H)^{-j}$ to be a geometric *Etale* realization.

Let $\mathcal{E} \to S$ be an elliptic curve over a base S. let S be a T-scheme. Let p be invertible on T.

Define an etale $\mathbb{Q}_p$-sheaf on $S \times_T S$ by

$$TSym^{[k,k']}\mathcal{M}_{\mathbb{Q}_p} = \pi_1^*(TSym^k \mathcal{M}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \pi_2^*(TSym^{k'} \mathcal{M}_{\mathbb{Q}_p}).$$

Diagram?????

where $\pi_1^*(TSym^k \mathcal{M}_{\mathbb{Q}_p})$ is a sheaf on $S \times_T S$ and $\pi_2^*(TSym^{k'} \mathcal{M}_{\mathbb{Q}_p})$ is a sheaf on $S \times_T S$. $\Delta : S \hookrightarrow S \times_T S$, then

$$\Delta_*(TSym^{[k,k']}\mathcal{H}_{\mathbb{Q}_p}) = TSym^k(\mathcal{H}_{\mathbb{Q}_p}) \otimes TSym^{k'}(\mathcal{H}_{\mathbb{Q}_p}).$$

If S is smooth with relative dimension d over T, we have along exact sequence called the Gysin sequence. We get maps Δ_* :

$$H^i_{et}(S, TSym^k(\mathcal{H}_{\mathbb{Q}_p}) \otimes TSym^{k'}(\mathcal{H}_{\mathbb{Q}_p})(j)) \to H^{i+2d}_{et}(S \times_T S, TSym^{[k,k']}\mathcal{H}_{\mathbb{Q}_p}(j + d)).$$

which follows from cohomological duality (Milne's).

Let $S = Y_1(N)$. Recall the Cletsch-Gordon map $CG^{[k,k',j]}$:

$$H^1(Y_1(N), TSym^{k+k'-2j}(\mathcal{H}_{\mathbb{Q}_p})(1)) \to H^1(Y_1(N), TSym^k(\mathcal{H}_{\mathbb{Q}_p}) \otimes TSym^{k'}(\mathcal{H}_{\mathbb{Q}_p}))(1 - j).$$

Now composing with the pushforward under the Gysin sequence, we know $\Delta_* \circ CG^{[k,k',j]}$:

$$H^1(Y_1(N), TSym^{k+k'-2j}(\mathcal{H}_{\mathbb{Q}_p})(1)) \to H^3_{et}(Y_1(N)^2, TSym^{[k,k']}(\mathcal{H}_{\mathbb{Q}_p}))(2 - j).$$

Note that $Eis^{k+k'-2j}_{et,b,N} \in H^1_{et}(Y_1(N)/\mathbb{Z}[1/Np], TSym^{k+k'-2j}\mathcal{H}_{\mathbb{Q}_p})(1))$.

Let $k, k' \geq 0$ and $0 \leq j \leq \min(k, k')$

Definition 4.9. Define the etale Rakin-Eisenstein class by

$$Eis^{[k+k'-2j]}_{et,b,N} = \Delta_* \circ CG^{[k,k',j]}(Eis^{k+k'-2j}_{et,b,N})$$

which belongs to $H^3_{et}(Y_1(N)^2, TSym^{[k,k']}(\mathcal{H}_{\mathbb{Q}_p}))(2 - j)$. Using the Hochschild-Serre spectral sequence, we get an edge map like before. Then we get a class in $H^1(\mathbb{Q}, V_f \times V_g(-j))$.

Compatibility in p-adic families

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1 Introduction

The goal of this talk is to construct Beilinson-Flach elements in Hida families which p-adically interpolate étale Rankin-Eisenstein classes at level $\Gamma(M,N)$ for a pair of modular forms f,g of weights $k+2, k'+2 \geq 2$ twisted by a cyclotomic variable. These Beilinson-Flach elements project to those constructed by Bertolini, Darmon and Rotger at level $\Gamma_1(N)$ and for weights $k+2 = k'+2 = 2$ (see [BDR15a]). The interpolation property in the case $k+2 = k'+2 = 2$ is shown in [LLZ14], and generalizes the main result of [BDR15b] (in which f is fixed, g varies in a Hida family, and no cyclotomic variable is considered). The proof of the interpolation property in the general case is considered in [KLZ15a]. This proof being too long to be reproduced here, we will content ourselves with giving some ideas on the case of a single modular curve (which is treated in [Kin13] by means of a detailed study of the elliptic polylogarithm), that is, we will sketch how Eisenstein-Iwasawa classes interpolate Eisenstein classes.

2 Preliminaries on linear algebra

Let H denote the profinite group ${\mathbb{Z}_p}^d$ for $d \geq 1$. We will be interested in the spaces

$$\mathrm{TSym}^k H \quad \text{and} \quad \mathrm{Sym}^k H\,.$$

The first denotes the $\mathbb{Z}_p$-algebra of symmetric k-tensors, that is, the space of $\mathfrak{S}_k$-invariants of $H \otimes \overset{k}{\cdots} \otimes H$. In contrast, by the second we denote the kth symmetric power of H, that is, the space of $\mathfrak{S}_k$-coinvariants of $H \otimes \overset{k}{\cdots} \otimes H$. For $m \leq k$ and $h \in H$, write $h^{[m]} = h^{\otimes m} \in \mathrm{TSym}^m H$. If $(e_1, \dots, e_d)$ is a basis for H, then $(e_1^{[n_1]} \cdots e_d^{[n_d]} | n_1 + \dots n_d = k)$ is a basis for $\mathrm{TSym}^k H$. We have a $\mathbb{Z}_p$-homomorphism

$$\mathrm{Sym}^k H \to \mathrm{TSym}^k H\,, \qquad e_1^{n_1} \cdots e_d^{n_d} \mapsto k! \cdot e_1^{[n_1]} \cdots e_d^{[n_d]}\,,$$

which becomes an isomorphism after tensoring with $\mathbb{Q}_p$. However, we will keep the distinction between these two spaces, because often we will have to work integrally.

2.1 The Clebsch-Gordan map

We wish to define the Clebsch-Gordan map for $k, k' \geq 0$ and $0 \leq j \leq \min\{k, k'\}$

$$\mathrm{CG}^{[\mathrm{k},\mathrm{k}',\mathrm{j}]}\colon\ \mathrm{TSym}^{k+k'-2j} H \otimes \mathrm{TSym}^j(\wedge^2 H) \to \mathrm{TSym}^k H \otimes \mathrm{TSym}^{k'} H\,.$$

We have an obvious inclusion

$$\mathrm{TSym}^{k+k'-2j} H \subseteq \mathrm{TSym}^{k-j} H \otimes \mathrm{TSym}^{k'-j} H\,.$$

By taking jth powers, the map $\wedge^2 H \to H \otimes H$ that sends $x \wedge y$ to $x \otimes y - y \otimes x$, yields a map

$$\mathrm{TSym}^j(\wedge^2 H) \to \mathrm{TSym}^j H \otimes \mathrm{TSym}^j H\,.$$

The map $\mathrm{CG}^{[\mathrm{k},\mathrm{k}',\mathrm{j}]}$ is obtained as the tensor product of the two previous maps.

2.2 The kth moment map

Let $(x_1,\dots,x_d)$ be the dual basis of $(e_1,\dots,e_d)$, where $x_i : H \to \mathbb{Z}_p$ is seen as a $\mathbb{Z}_p$-valued function on H.

Consider the space of $\mathbb{Z}_p$-valued measures on H

$$\Lambda(H) = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}(C(H,\mathbb{Z}_p),\mathbb{Z}_p)\,,$$

where $C(H,\mathbb{Z}_p)$ denotes the space of continuous $\mathbb{Z}_p$-valued functions on H.

Definition 5.1. The kth moment map is the $\mathbb{Z}_p$-algebra homomorphism

$$\mathrm{mom}^k\colon \Lambda(H)\to \mathrm{TSym}^k H\,,\qquad \mathrm{mom}^k(\mu)=\sum_{n_1+\dots n_d=k}\mu(x_1^{n_1}\cdots x_d^{n_d})e_1^{[n_1]}\cdots e_d^{[n_d]}\,.$$

3 Étale Eisenstein and Rankin-Eisenstein classes

Let Y denote a modular curve corresponding to a representable moduli problem. It comes equipped with a universal elliptic curve $\pi\colon \mathcal{E}\to Y$. Fix a prime p throughout the talk. We define lisse étale sheaves on $Y[1/p]$:

- $\mathscr{H}_{\mathbb{Z}_p} = R^1\pi_*\mathbb{Z}_p(1)\simeq R^1\pi_*{\mathbb{Z}_p}^\vee$,
- $\mathscr{H}_{\mathbb{Q}_p} = R^1\pi_*\mathbb{Q}_p(1)$,
- $\mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Z}_p}$,
- $\mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Q}_p}\simeq \mathrm{Sym}^k\,\mathscr{H}_{\mathbb{Q}_p}$.

Remark 5.2. If P is a geometric point on Y corresponding to an elliptic curve E, we should think of the stalk of $\mathscr{H}_{\mathbb{Z}_p}$ at P as the p-adic Tate module $M_p(E)$ of E. Similarly, we should think of $\mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Z}_p,P}$ as $\mathrm{TSym}^k\,M_p(E)$.

For $f=\sum_{n\geq 1}a_n(f)q^n\in S_{k+2}(\Gamma_1(N_f))$ a normalized cuspidal eigenform, L a number field containing $\mathbb{Q}(\{a_n(f)\}_{n\geq1})$, N divisible by N_f, and $\mathfrak{P}$ a prime of L lying over p, let:

- $M_{L_\mathfrak{P}}(f)$ be the maximal subspace of $H^1_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}},\mathrm{Sym}^k\,\mathscr{H}^\vee_{\mathbb{Q}_p})\otimes_{\mathbb{Q}_p}L_\mathfrak{P}$ on which the Hecke operator T_ℓ acts as multiplication by a_ℓ for every prime ℓ.
- $M_{L_\mathfrak{P}}(f)^*$ be the maximal quotient of $H^1_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}},\mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Q}_p}(1))\otimes_{\mathbb{Q}_p}L_\mathfrak{P}$ on which the Hecke operator T'_ℓ acts as multiplication by a_ℓ for every prime ℓ.

If $\mathcal{O}_\mathfrak{P}$ denotes the ring of integers of $L_\mathfrak{P}$, then one defines integral versions $M_{\mathcal{O}_\mathfrak{P}}(f)$ and $M_{\mathcal{O}_\mathfrak{P}}(f)^*$ of the previous obtects in the obvious way.

Definition 5.3. Let $N\geq 5$, $b\in\mathbb{Z}/N\mathbb{Z}\setminus\{0\}$, and $k\geq 0$. The *étale Eisenstein class* $\mathrm{Eis}^k_{et,b,N}$ is defined as the image of the motivic Eisenstein class $\mathrm{Eis}^k_{mot,b,N}$ by the étale regulator map

$$H^1_{mot}(Y_1(N),\mathrm{TSym}^k\,\mathscr{H}_\mathbb{Q}(1))\to H^1_{et}(Y_1(N)_{\mathbb{Z}[1/Np]},\mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Q}_p}(1))$$

Example 5.4. As we saw in Antonio's talk, for $k=0$, $H^1_{mot}(Y_1(N),\mathbb{Q}(1))=\mathcal{O}(Y_1(N))^*\otimes\mathbb{Q}$ and the motivic Eisenstein class $\mathrm{Eis}^0_{mot,b,N}$ is the Siegel unit $g_{0,b/N}$.

Let $f\in S_{k+2}(\Gamma_1(N_f))$ and $g\in S_{k'+2}(\Gamma_1(N_g))$ for $k,k'\geq 0$. To shorten notation, until the end of this §, let us write $Y=Y_1(N)[1/Np]$, where N is an integer divisible by N_f and N_g. For $0\leq j\leq \min\{k,k'\}$, we will be interested in the following maps:

- The Clebsch-Gordan map:

$$H^1_{et}(Y,\mathrm{TSym}^{k+k'-2j}\,\mathscr{H}_{\mathbb{Q}_p}(1))\xrightarrow{\mathrm{CG}_*^{[k,k',j]}} H^1_{et}(Y,\mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Q}_p}\otimes\mathrm{TSym}^{k'}\,\mathscr{H}_{\mathbb{Q}_p}(1-j))\,.$$

  At the level of stalks, this is the map defined in 2.1. Indeed, note that in our situation $\wedge^2 H\simeq\det(H)\simeq\mathbb{Q}_p(1)$.

- The push-forward of the diagonal embedding:

$$H^1_{et}(Y, \mathrm{TSym}^k \mathscr{H}_{\mathbb{Q}_p} \otimes \mathrm{TSym}^{k'} \mathscr{H}_{\mathbb{Q}_p}(-j)) \xrightarrow{\Delta_*} H^3_{et}(Y^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j))\,.$$

Here, for $\mathcal{A}, \mathcal{B}$ sheaves on Y and $\pi_1, \pi_2 \colon Y^2 \to Y$ the two distinct projections, we write $\mathcal{A} \boxtimes \mathcal{B}$ for the sheaf $\pi_1^*\mathcal{A} \otimes \pi_2^*\mathcal{B}$ on Y^2. We then write $\mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p} = \mathrm{TSym}^k \mathscr{H}_{\mathbb{Q}_p} \boxtimes \mathrm{TSym}^{k'} \mathscr{H}_{\mathbb{Q}_p}$.

- There is an edge map coming from the Hochschild-Serre spectral sequence

$$H^3_{et}(Y^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)) \overset{\simeq}{\to} H^1(\mathbb{Z}[\frac{1}{Np}], H^2_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}}^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)))\,.$$

- Projection to the (f,g)-isotypic component

$$H^1(\mathbb{Z}[\frac{1}{Np}], H^2_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}}^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j))) \overset{\mathrm{pr}_{f,g}}{\to} H^1(\mathbb{Z}[\frac{1}{Np}], M_{L_{\mathfrak{P}}}(f)^* \otimes M_{L_{\mathfrak{P}}}(g)^*(-j))$$

Definition 5.5.
- The *Rankin-Eisenstein class* $\mathrm{Eis}^{f,g,j}_{et,b,N}$ is defined as the image of the étale Eisenstein class $\mathrm{Eis}^{k+k'-2j}_{et,b,N}$ by the concatenation of all the previous maps.
- The *Rankin-Eisenstein class* $\mathrm{Eis}^{[k,k',j]}_{et,b,N}$ at stage

$$H^3_{et}(Y^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)) \simeq H^1(\mathbb{Z}[\frac{1}{Np}], H^2_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)))$$

is defined as the image of the étale Eisenstein class $\mathrm{Eis}^{k+k'-2j}_{et,b,N}$ by the map $\Delta_* \circ \mathrm{CG}^{[k,k',j]}_*$.

4 Eisenstein-Iwasawa and Rankin-Iwasawa classes

Recall that as at the beginning of §3, if Y is a modular curve corresponding to a representable moduli problem, we have a universal elliptic curve $\pi \colon \mathcal{E} \to Y$. Let us see $\mathcal{E}$ as a covering of itself by means of

$$[p^r] \colon \mathcal{E}_r = \mathcal{E} \to \mathcal{E}$$

the p^r-multiplication map, with $r \geq 1$. Define the pro-system of étale lisse sheaves

$$\mathcal{L} = ([p^r]_*(\mathbb{Z}/p^r\mathbb{Z}))_{r\geq 1}\,,$$

which we call the *elliptic polylogarithm*. The transition maps are constructed in the following manner. First consider the composition

$$[p]_*\mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z} \tag{5.1}$$

of maps of sheaves on $\mathcal{E}_r$, where the first map is the trace map induced by $[p] \colon \mathcal{E}_{r+1} \to \mathcal{E}_r$ and the second is the reduction map. The transition map is now obtained by projecting (5.1) on $\mathcal{E}$ by $[p^r]_*$

$$[p^{r+1}]_*\mathbb{Z}/p^{r+1}\mathbb{Z} \to [p^r]_*\mathbb{Z}/p^r\mathbb{Z}\,.$$

Write $\mathcal{L}_{\mathbb{Q}_p} = \mathcal{L} \otimes \mathbb{Q}_p$. For a section $t \colon Y \to \mathcal{E}$, define the sheaf of Iwasawa modules

$$\Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle t\rangle) = t^*\mathcal{L}\,,$$

$$\Lambda(\mathscr{H}_{\mathbb{Z}_p}) = \Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle e\rangle)\,,$$

where $e \colon Y \to \mathcal{E}$ denotes the trivial section.

Remark 5.6. If P is a geometric point on Y corresponding to an elliptic curve E, we should think of the stalk of $\Lambda(\mathscr{H}_{\mathbb{Z}_p})$ at P as the Iwasawa algebra of the p-adic Tate module $M_p(E)$ of E, that is, the space of $\mathbb{Z}_p$-valued measures on $M_p(E)$. This justifies the notation and terminology used.

Remark 5.7. There exist sheafified moment maps

$$\mathrm{mom}^k\colon \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \to \mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Z}_p}$$

such that if P is a geometric point on Y corresponding to an elliptic curve E, then

$$\mathrm{mom}^k_P\colon \Lambda(\mathscr{H}_{\mathbb{Z}_p})_P = \Lambda(M_p(E)) \to \mathrm{TSym}^k\,\mathscr{H}_{\mathbb{Z}_p,P} = \mathrm{TSym}^k\,M_p(E)$$

coincides with the kth moment map of Definition 5.1.

In Antonio's talk, we have defined the Kato units ${}_c\theta_{\mathcal{E}} \in \mathcal{O}(\mathcal{E}\setminus\mathcal{E}[c])^*$ for $c>1$ and $(c,6)=1$. Observe that

$$H^1_{et}(\mathcal{E}\setminus\mathcal{E}[c],\mathcal{L}(1)) \simeq \lim_{\leftarrow r} H^1_{et}(\mathcal{E}\setminus\mathcal{E}[c],[p^r]_*(\mathbb{Z}/p^r\mathbb{Z})(1)) \simeq \lim_{\leftarrow r} H^1_{et}(\mathcal{E}_r\setminus\mathcal{E}_r[cp^r],\mathbb{Z}/p^r\mathbb{Z}(1))\,.$$

Thanks to the norm relations that we saw that Kato units satisfy, if $p\nmid c$ the following limit is well defined

$${}_c\Theta_{\mathcal{E}} = \lim_{\leftarrow r} \partial_r({}_c\theta_{\mathcal{E}_r}) \in \lim_{\leftarrow r} H^1_{et}(\mathcal{E}_r\setminus\mathcal{E}_r[cp^r],\mathbb{Z}/p^r\mathbb{Z}(1))\,,$$

where $\partial_r\colon \mathcal{O}(\mathcal{E}_r\setminus\mathcal{E}_r[cp^r])^* \to H^1(\mathcal{E}_r\setminus\mathcal{E}_r[cp^r],\mathbb{Z}/p^r\mathbb{Z}(1))$ is the connecting morphism for the exact sequence

$$1 \to \mu_{p^r} \to \mathbb{G}_m \xrightarrow{\cdot p^r} \mathbb{G}_m \to 1\,.$$

Until the definition of Rankin-Iwasawa class, for $M,N\geq 1$, $M|N$, and $M+N\geq 5$, let Y be the curve $Y(M,N)[1/MNp]$ defined in Kezuka's talk.

Definition 5.8. Let $c>1$ with $(c,6Np)=1$ and $b\in\mathbb{Z}/N\mathbb{Z}\setminus\{0\}$. Let $t_N\colon Y(M,N)\to\mathcal{E}\setminus\mathcal{E}[c]$ be the canonical section of order N (note that it takes values in $\mathcal{E}\setminus\mathcal{E}[c]$ by our choice of c). The *Eisenstein-Iwasawa class* ${}_c\mathcal{EI}_t$ is defined as the image of ${}_c\Theta_{\mathcal{E}}$ by the map

$$H^1_{et}(\mathcal{E}\setminus\mathcal{E}[c],\mathcal{L}(1)) \xrightarrow{(bt_N)^*} H^1_{et}(Y,\Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle bt_N\rangle)(1)) \xrightarrow{[N]_*} H^1_{et}(Y,\Lambda(\mathscr{H}_{\mathbb{Z}_p})(1))\,.$$

We will be interested in the following maps:

- The map induced by $\Lambda(\mathscr{H}_{\mathbb{Z}_p})\to\Lambda(\mathscr{H}_{\mathbb{Z}_p})\otimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})$

$$H^1(Y,\Lambda(\mathscr{H}_{\mathbb{Z}_p})(1)) \to H^1_{et}(Y,\Lambda(\mathscr{H}_{\mathbb{Z}_p})\otimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(1))\,.$$

- The push-forward of the diagonal embedding $\Delta\colon Y\to Y^2$

$$H^1_{et}(Y,\Lambda(\mathscr{H}_{\mathbb{Z}_p})\otimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(1)) \xrightarrow{\Delta_*} H^3_{et}(Y^2,\Lambda(\mathscr{H}_{\mathbb{Z}_p})\boxtimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(2))\,.$$

- For $a\in\mathbb{Z}/M\mathbb{Z}$, the map

$$H^3_{et}(Y^2,\Lambda(\mathscr{H}_{\mathbb{Z}_p})\boxtimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)) \xrightarrow{u_{a*}} H^3_{et}(Y^2,\Lambda(\mathscr{H}_{\mathbb{Z}_p})\boxtimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(2))\,,$$

  where $u_a\colon Y^2\to Y^2$ is the automorphism that is the identity on the first factor and the map that sends a triple (E,e_1,e_2) to the triple $(E,e_1+a\frac{N}{M}e_2,e_2)$ on the second factor.

- The edge map

$$H^3_{et}(Y^2,\Lambda(\mathscr{H}_{\mathbb{Z}_p})\boxtimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)) \xrightarrow{\simeq} H^1(\mathbb{Z}[\frac{1}{MNp}],H^2_{et,c}(Y(M,N)_{\overline{\mathbb{Q}}},\Lambda(\mathscr{H}_{\mathbb{Z}_p})\boxtimes\Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)))\,.$$

Definition 5.9. The *Rankin-Iwasawa class* ${}_c\mathcal{RI}_{M,N,a}$ is defined as the image of the Eisenstein-Iwasawa class ${}_c\mathcal{EI}_{1,N}$ by the concatenation of all the previous maps.

In §5, we will see that Rankin-Iwasawa classes (or even more generally, Beilinson-Flach elements) interpolate Rankin-Eisenstein classes (see Theorem 5.12). To conclude the section, we will see an intermediate result, which shows that Eisenstein-Iwasawa classes (for $M=1$) interpolate Eisenstein classes.

Theorem 5.10 (Thm. 4.7.1 of [Kin13])**.** *For $N \geq 5$, $b \in \mathbb{Z}/N\mathbb{Z}\backslash\{0\}$, and $c > 1$ with $(c, 6Np) = 1$, one has*

$$\mathrm{mom}^k({}_c\mathcal{EI}_{b,N}) = c^2 \,\mathrm{Eis}^k_{et,b,N} - c^{-k}\,\mathrm{Eis}^k_{et,cb,N}$$

as elements of $H^1_{et}(Y_1(N)[1/Np], \mathrm{TSym}^k \mathcal{H}_{\mathbb{Q}_p}(1))$.

Sketch of proof. In the course of the proof, let us write $t\colon Y_1(N) \to \mathcal{E}$ for a section of order N, $e\colon Y_1(N) \to \mathcal{E}$ for the trivial section, and $Y = Y_1(N)[1/Np]$. The proof uses the following crucial properties of the elliptic polylogarithm

- For an isogeny $\varphi\colon \mathcal{E} \to \mathcal{E}$, one has $\varphi^*\mathcal{L}_{\mathbb{Q}_p} \simeq \mathcal{L}_{\mathbb{Q}_p}$.
- $e^*\mathcal{L}_{\mathbb{Q}_p} \simeq t^*\mathcal{L}_{\mathbb{Q}_p} \simeq (\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t\rangle) \otimes \mathbb{Q}_p) \simeq \prod_{k\geq 0} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}$.
- There is a multiplication map $\mathrm{mult}\colon \pi^*\mathcal{H}_{\mathbb{Q}_p} \otimes \mathcal{L}_{\mathbb{Q}_p} \to \mathcal{L}_{\mathbb{Q}_p}$.

Consider the following diagram (the first vertical arrow of which we take as a black box[1])

$$\begin{array}{ccc}
\mathrm{Hom}_Y(\mathcal{H}_{\mathbb{Q}_p}, \prod_{k\geq 1}\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}) & & \\
\Big\downarrow \simeq & & \\
\mathrm{Ext}^1_{\mathcal{E}\backslash\{e\}}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, \mathcal{L}_{\mathbb{Q}_p}(1)) & \xrightarrow{[c]^*} & \mathrm{Ext}^1_{\mathcal{E}\backslash\mathcal{E}[c]}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, [c]^*\mathcal{L}_{\mathbb{Q}_p}(1)) \\
\Big\downarrow t^* & & \Big\downarrow \simeq \\
\mathrm{Ext}^1_Y(\mathcal{H}_{\mathbb{Q}_p}, \prod_{k\geq 0}\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)) & & \mathrm{Ext}^1_{\mathcal{E}\backslash\mathcal{E}[c]}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, \mathcal{L}_{\mathbb{Q}_p}(1)) \\
\Big\downarrow \simeq & & \Big\uparrow \mathrm{mult}_{\mathcal{H}_{\mathbb{Q}_p}} \\
\mathrm{Ext}^1_Y(\mathbb{Q}_p, \mathcal{H}^\vee_{\mathbb{Q}_p} \otimes \prod_{k\geq 0}\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)) & & \mathrm{Ext}^1_{\mathcal{E}\backslash\mathcal{E}[c]}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, \mathcal{L}_{\mathbb{Q}_p} \otimes \pi^*\mathcal{H}_{\mathbb{Q}_p}(1)) \\
\Big\downarrow \simeq & & \Big\uparrow \otimes\pi^*\mathcal{H}_{\mathbb{Q}_p} \\
H^1_{et}(Y, \mathcal{H}^\vee_{\mathbb{Q}_p} \otimes \prod_{k\geq 0}\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)) & & \mathrm{Ext}^1_{\mathcal{E}\backslash\mathcal{E}[c]}(\mathbb{Q}_p, \mathcal{L}_{\mathbb{Q}_p}(1)) \\
\Big\downarrow \mathrm{contr} & & \Big\uparrow \simeq \\
H^1_{et}(Y, \prod_{k\geq 1}\mathrm{Sym}^{k-1} \mathcal{H}_{\mathbb{Q}_p}(1)) & & H^1_{et}(\mathcal{E}\setminus\mathcal{E}[c], \mathcal{L}_{\mathbb{Q}_p}(1)) \\
\Big\downarrow \mathrm{pr}_{k-1} & & \\
H^1_{et}(Y, \mathrm{Sym}^{k-1} \mathcal{H}_{\mathbb{Q}_p}(1)) & &
\end{array}$$

At the level of stalks the contraction map is defined in the following way

$$\mathrm{contr}\colon H^\vee \otimes \mathrm{Sym}^k H \to \mathrm{Sym}^{k-1} H\,, \qquad h^\vee \otimes h_1 \otimes \cdots \otimes h_k \mapsto \frac{1}{k+1}\sum_{j=1}^{k} h^\vee(h_j) h_1 \otimes \cdots \otimes \hat{h}_j \otimes \cdots \otimes h_k\,.$$

Let pol denote the image of the canonical immersion

$$\mathcal{H}_{\mathbb{Q}_p} \hookrightarrow \prod_{k\geq 0} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}$$

by the very first isomorphism in the above diagram and write $t^*\mathrm{pol} = (t^*\mathrm{pol}^k)_{k\geq 1}$. The first step of the proof is to show that if $t = bt_N$, where t_N is the canonical section of order N, then the Eisenstein class $\mathrm{Eis}^k_{et,b,N}$ is the image of $t^*\mathrm{pol}^{k+1}$ by the concatenation of the maps in the first column of the the previous diagram.

[1] It follows from the Leray spectral sequence for $\mathcal{L}_{\mathbb{Q}_p}$ and π, the localization sequence, and the vanishing of $R^i\pi_*\mathcal{L}_{\mathbb{Q}_p}$ except for $i = 2$.

Note that we had defined Kato elements ${}_c\Theta_{\mathcal{E}} \in H^1_{et}(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\mathbb{Q}_p}(1))$. If we detnote by $\mathrm{mult}_{\mathcal{H}_{\mathbb{Q}_p}}$ the concatenation of the maps on the second column, then the second step of the proof consists of establishing the following fundamental relation

$$c^2\mathrm{pol}|_{\mathcal{E}\setminus\mathcal{E}[c]} - c[c]^*\mathrm{pol} = \mathrm{mult}_{\mathcal{H}_{\mathbb{Q}_p}}({}_c\Theta_{\mathcal{E}})$$

in $\mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\pi^*\mathcal{H}\mathbb{Q}_p, \mathcal{L}_{\mathbb{Q}_p}(1))$. Now the theorem follows from the following two facts:

- The concatenation of the maps on the second and first column coincide with the sheafified kth moment map mom^k (once tensored with $\mathbb{Q}_p$); and
- The isomorphism $t^*\mathcal{L}_{\mathbb{Q}_p} \simeq t^*[c]^*\mathcal{L}_{\mathbb{Q}_p}$ is multiplication by c^k on the graded piece $\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}$.

□

5 Beilinson-Flach elements: Projection to $Y_1(N)$

Let $m \geq 1$ and $N \geq 5$. Let μ_m° be the scheme of primitive mth roots of unity, that is, $\mu_m^\circ = \mathrm{Spec}(\mathbb{Z}[\zeta_m])$, where ζ_m is a primitive mth root of unity. In Vivek's talk we have seen that there exists a map[2]

$$\alpha_m \colon Y(m, mN) \to Y_1(N) \times \mu_m^\circ .$$

Definition 5.11. We will write ${}_c\mathcal{BF}^{[0]}_{m,N}$ for the image of the Rankin-Iwasawa class ${}_c\mathcal{RI}_{m,mN,1}$ by the map

$$(\alpha_m \times \alpha_m)_* \colon H^3_{et}(Y(m,mN)^2, \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathcal{H}_{\mathbb{Z}_p})(2)) \to H^3_{et}(Y_1(N)^2 \times \mu_m^\circ, \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathcal{H}_{\mathbb{Z}_p})(2))$$

We still need to introduce one more sheaf of Iwasawa modules. Let

$$\mathrm{pr}_r \colon \mathrm{Spec}(\mathbb{Z}[1/p]) \times \mu_{p^r}^\circ \to \mathrm{Spec}(\mathbb{Z}[1/p])$$

the natural projection for $r \geq 1$. Define the pro-étale sheaf

$$\Lambda_\Gamma(-\mathbf{j}) = (\mathrm{pr}_{r*}(\mathbb{Z}/p^r\mathbb{Z}))_{r\geq 1} .$$

The notation is justified by the fact that the stalk of $\Lambda_\Gamma(-\mathbf{j})$ at a geometric point is the Iwasawa algebra Λ_Γ of the Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$, equipped with an action of Γ by the inverse of the canonical character $\mathbf{j} \colon \Gamma \to \Lambda_\Gamma^*$. There are moment maps

$$\mathrm{mom}^j_\Gamma \colon \Lambda_\Gamma(-\mathbf{j}) \to \mathbb{Z}_p(-j) .$$

The key property of $\Lambda_\Gamma(-\mathbf{j})$ is that it permits to transfer variations on the level to the sheaf side. More precisely, there is an isomorphism

$$\varprojlim_r H^3(Y_1(N)^2 \times \mu_{mp^r}^\circ, \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathcal{H}_{\mathbb{Z}_p})(2)) = H^3(Y_1(N)^2 \times \mu_m^\circ, \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \otimes \Lambda_\Gamma(2-\mathbf{j})) ,$$

Let us write

$$e' = e'_{\mathrm{ord}} = \lim_{n\to\infty} (U'_p)^{n!}$$

for Ohta's anti-ordinary operator. The operator (U'_p, U'_p) is invertible on the image of (e', e'), and the so-called "Second norm relation", seen in Vivek's talk, shows that the inverse limit

$${}_c\mathcal{BF}_{m,N} = \varprojlim_r (U'_p, U'_p)^{-r}(e', e')\,{}_c\mathcal{BF}^{[0]}_{mp^r,N} ,$$

which is an element of

$$(e', e')H^3(Y_1(N)^2 \times \mu_m^\circ, \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathcal{H}_{\mathbb{Z}_p}) \otimes \Lambda_\Gamma(2-\mathbf{j})) ,$$

is well defined. The classes ${}_c\mathcal{BF}_{m,N}$ are called *Beilinson-Flach elements.* The following theorem establishes the interpolation property of the Beilinson-Flach elements. It should be seen as a generalization of Theorem 5.10.

[2] There, this map was denoted by t_m, but in the present talk we reserve this notation for the canonical section of order m.

Theorem 5.12 (Thm. 6.3.3 of [KLZ15a])**.** *Let $k, k' \geq 0$ and $0 \leq j \leq \min\{k, k'\}$. For a prime $p \geq 3$, $N \geq 1$, $m \geq 1$ and $c > 1$ with $p|N$, $(p, m) = 1$, and $(c, 6mNp) = 1$, we have that*

$$\mathrm{mom}^k \otimes \mathrm{mom}^{k'} \otimes \mathrm{mom}^j_\Gamma({}_c\mathcal{BF}_{m,N}) =$$

$$= (1 - p^j(U'_p, U'_p)^{-1}\sigma_p)(c^2 - c^{-k-k'+2j}\sigma_c^2(\langle c\rangle, \langle c\rangle))\frac{(e', e')\,\mathrm{Eis}^{[k,k',j]}_{et,m,N}}{(-1)^j j!\binom{k}{j}\binom{k'}{j}}\,,$$

where σ_c is the arithmetic Frobenius at c in $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$.

6 Beilinson-Flach elements in Hida families

Set

$$H^1_{\mathrm{ord}}(Np^\infty) = \varprojlim_{r} e' H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))\,.$$

It is finitely generated and projective over $\Lambda_D = \mathbb{Z}_p[[\mathbb{Z}_p{}^*]]$. For $r \geq 1$, recall the existence of Ohta's twisting map

$$H^1_{\mathrm{ord}}(Np^\infty) \overset{\mathrm{Ohta}}{\to} e' H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle t_N\rangle)(1))\,.$$

Remark 5.13. For $r = 1$, it is easy to see that Ohta's twisting map is an isomorphism. Indeed, if one defines $\mathcal{E}[p^r]\langle t_N\rangle$ by the cartesian diagram

$$\begin{array}{ccc} \mathcal{E}[p^r]\langle t_N\rangle & \longrightarrow & \mathcal{E}_r = \mathcal{E} \\ \Big\downarrow \mathrm{pr}_{r,t} & & \Big\downarrow [p^r] \\ Y_1(N) & \xrightarrow{\ t_N\ } & \mathcal{E} \end{array}\ ,$$

it is not difficult to see that $\mathcal{E}[p^r]\langle t_N\rangle \simeq Y_1(Np^r)$. Set

$$\Lambda(\mathscr{H}_r\langle t_N\rangle) = t_N^*([p^r]_*\mathbb{Z}/p^r\mathbb{Z}) = \mathrm{pr}_{r,t*}(\mathbb{Z}/p^r\mathbb{Z})\,.$$

It follows that

$$H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \Lambda(\mathscr{H}_r\langle t_N\rangle)(1)) \simeq H^1_{et}(\mathcal{E}[p^r]\langle t_N\rangle_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^r\mathbb{Z}(1)) \simeq H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^r\mathbb{Z}(1))\,.$$

By taking limits we get

$$H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle t_N\rangle)(1)) \simeq H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))\,.$$

As we saw in Chris' talk, Ohta's control Theorem states that the compostion of the moment map $\mathrm{mom}^k \circ [N]_!$ with Ohta's twisting map induces an isomorphism

$$H^1_{\mathrm{ord}}(Np^\infty)/I_{k,r} \to e' H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k(\mathscr{H}_{\mathbb{Z}_p})(1))\,, \tag{5.2}$$

where $I_{k,r}$ is the ideal of Λ_D generated by $[1 + p^r] - (1 + p^r)^k$. Write $\mathbb{T}_{Np^\infty}$ for the Hecke algebra generated by the Hecke operators T'_ℓ acting on $H^1_{\mathrm{ord}}(Np^\infty)$. There are Λ_D-linear commuting actions of $\mathbb{T}_N p^\infty$ and $G_{\mathbb{Q},S}$, the Galois group of the maximal unramified extension outside the set S of primes dividing Np. $\mathbb{T}_N p^\infty$ is a finite projective Λ_D-algebra.

Definition 5.14.

- A *Hida family* $\mathbf{f}$ is any of the finitely many maximal ideals of $\mathbb{T}_N p^\infty$.
- If $\mathbf{f}$ is a Hida family, set

$$M(\mathbf{f})^* = H^1_{\mathrm{ord}}(Np^\infty)_{\mathbf{f}}\,, \qquad \Lambda_{\mathbf{f}} = (\mathbb{T}_N p^\infty)_{\mathbf{f}}\,.$$

- An *arithmetic prime* is a prime ideal $\mathfrak{p}$ of Λ_D of height 1 lying over an ideal of the form $I_{k,r}$ for some k, r.

Associated to an arithmetic prime $\mathfrak{p}$, there is an eigenform $f_\mathfrak{p}$ of level Np^r and weight $k+2$ such that

$$M_{\mathcal{O}_\mathfrak{P}}(f_\mathfrak{p})^* = M(\mathbf{f})^* \otimes_{\Lambda_\mathbf{f}} \mathcal{O}_\mathfrak{P}\,,$$

where $\mathfrak{P}$ is a prime of $\mathbb{T}_{Np^\infty}$ above $\mathfrak{p} \subseteq \Lambda_D$ and the tensor product is taken with respect to the projection map

$$\Lambda_\mathbf{f} \to \mathcal{O}_\mathfrak{P} = \Lambda_\mathbf{f}/\mathfrak{P}\,.$$

Definition 5.15. For Hida families $\mathbf{f}$ and $\mathbf{g}$ of tame levels N_f and N_g, $m \geq 1$ coprime to p, and $c > 1$ coprime to $6mN_fN_gp$, we define

$${}_c\mathcal{BF}_m^{\mathbf{f},\mathbf{g}} \in H^1(\mathbb{Z}[\frac{1}{mpN_fN_g}, \mu_m], M(\mathbf{f})^* \otimes M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$$

to be the image of the class ${}_c\mathcal{BF}_{m,N}$ for $N = \mathrm{Lcm}(N_f, N_g)$ under the edge map coming from the Hochschild-Serre spectral sequence, the projection map $Y_1(N)^2 \to Y_1(N_f) \times Y_1(N_g)$, the Künneth formula, and localization at $\mathbf{f}$ and $\mathbf{g}$.

The main and final theorem of this talk is the following.

Theorem 5.16 (Thm. 8.1.4 of [KLZ15a])**.** *If f and g are ordinary newforms of levels N_f and N_g which are specializations of the Hida families $\mathbf{f}$ and $\mathbf{g}$ of weights $k+2$ and $k'+2$, then for every $0 \leq j \leq \min\{k, k'\}$ the specialization*

$${}_c\mathcal{BF}_1^{\mathbf{f},\mathbf{g}}(f, g, j) \in H^1(\mathbb{Z}[1/pN_fN_g], M_{L_\mathfrak{P}}(f)^* \otimes M_{L_\mathfrak{P}}(g)^*(-j))$$

is equal to

$$\frac{\left(1 - \frac{p^j}{\alpha_f\alpha_g}\right)\left(1 - \frac{\alpha_f\beta_g}{p^{1+j}}\right)\left(1 - \frac{\beta_f\alpha_g}{p^{1+j}}\right)\left(1 - \frac{\beta_f\beta_g}{p^{1+j}}\right)}{(-1)^j j! \binom{k}{j}\binom{k'}{j}} \left(c^2 - \frac{c^{-k-k'+2j}}{\varepsilon_c(f)\varepsilon_c(g)}\right)(\mathrm{Eis}^{f,g,j}_{et,1,N})\,,$$

where α_f, β_f are the roots of the Hecke polynomial $X^2 - a_p(f)X + p^{k-1}\varepsilon_p(f)$, and analogously for α_g, β_g.

Proof. Except of three Euler factors, the other factors come from Theorem 5.12 applied to the p-stabilizations of f and g. Let N be divisible by N_f, N_g, and p. The remaining three Euler factors are obtained by relating the Beilinson-Flach elements ${}_c\mathcal{BF}_{1,N/p}$ and ${}_c\mathcal{BF}_{1,N}$ relative to f and g; or equivalently, the Rankin-Iwasawa classes ${}_c\mathcal{RI}_{1,N/p,1}$ and ${}_c\mathcal{RI}_{1,N,1}$ relative to f and g. □

Norm compatibility relations

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1 Aim

[1]

These notes reproduce the talk given by Vivek Pal on the norm-compatibility relations satisfied by the Beilinson-Flach elements studied during the workshop.

2 Modular curves

We start by recalling and fixing some notations concerning modular curves that will be used throughout this chapter.

For $N \geq 3$, let $Y(N)$ be the smooth affine modular curve over $\mathbb{Q}$ representing the moduli functor on the category of $\mathbb{Q}$-schemes which associates to every scheme S the set of isomorphism classes of triples (E, e_1, e_2), where E is an elliptic curve over S and e_1, e_2 are sections of E/S generating $E[N]$.

Recall that $Y(N)$ is equipped with a natural left action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ (factoring through $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle \pm 1 \rangle$), by which an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{G}L_2(\mathbb{Z}/N\mathbb{Z})$$

sends a point corresponding to the moduli of a triple (E, e_1, e_2) to the point corresponding to the moduli of (E, e_1', e_2'), where

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Besides, we also have a surjective morphism

$$Y(N) \longrightarrow \mu_N^\circ, \quad (E, e_1, e_2) \longmapsto \langle e_1, e_2 \rangle_{E[N]},$$

where μ_N° is the group scheme of primitive N-th roots of unity and $\langle\,,\,\rangle_{E[N]}$ denotes the Weil pairing on $E[N]$. Using that the Weil pairing is non-degenerate and alternating, one can check that the action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ induced on μ_N° is given by the rule $\sigma \cdot \zeta = \zeta^{\det \sigma}$. The fibre of $Y(N)(\mathbb{C})$ over $e^{2\pi i/N} \in \mu_N^\circ(\mathbb{C})$ is then canonically (and $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariantly) identified with $\Gamma(N)\backslash\mathcal{H}$ via

$$\tau \longmapsto (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \tau/N, 1/N).$$

Here $\mathcal{H}$ denotes the complex upper half-plane, and $\Gamma(N)$ stands for the principal congruence subgroup of level N in $\mathrm{SL}_2(\mathbb{Z})$.

We will be interested in certain quotients of the curves $Y(N)$, which have already appeared but whose definition is recalled here.

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Definition 6.1. Given integers $M, N \geq 1$, the curve $Y(M,N)$ is defined to be the quotient of the modular curve $Y(L)$, for any integer $L \geq 3$ divisible by both M and N, by the action of the group

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) : \begin{array}{c} a \equiv 1, b \equiv 0 \,(\mathrm{mod}\, M), \\ c \equiv 0, d \equiv 1 \,(\mathrm{mod}\, N) \end{array} \right\}.$$

The curve $Y(M,N)$ admits a moduli interpretation induced from $Y(L)$: it classifies triples (E, e_1, e_2), where now e_1 has order M, e_2 has order N, and e_1, e_2 generate a subgroup of E of order MN.

Definition 6.2. If $N \geq 3$, we denote by $Y_1(N)$ the smooth affine modular curve over $\mathbb{Q}$ representing the moduli functor on the category of $\mathbb{Q}$-schemes which associates to every scheme S the set of isomorphism classes of pairs (E, P), where E is an elliptic curve over S and P is a section of E/S of exact order N.

From the very definitions, $Y_1(N)$ coincides with $Y(1,N)$. More generally, if $N \geq 3$, $m \geq 1$, and $L \geq 3$ is divisible by both N and m, then the morphism

$$\begin{array}{rcl} Y(L) & \longrightarrow & Y_1(N) \times \mu_m^{\circ} \\ (E, e_1, e_2) & \longmapsto & \left[(E, \frac{L}{N} e_2), \langle \frac{L}{m} e_1, \frac{L}{m} e_2 \rangle_{E[m]} \right] \end{array}$$

identifies $Y_1(N) \times \mu_m^{\circ}$ with the quotient of $Y(L)$ by the action of the group

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) : \begin{array}{c} c \equiv 0, d \equiv 1 \,(\mathrm{mod}\, N), \\ ad - bc \equiv 1 \,(\mathrm{mod}\, m) \end{array} \right\}.$$

In the following, we will focus mainly on the curves $Y(m, mN)$ for $m, N \geq 1$. Notice that $Y(m, mN)$ maps naturally to μ_m°, with geometrically connected fibres, and it is endowed with a left action of

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \,(\mathrm{mod}\, N) \right\},$$

compatible with the determinant action on μ_m°. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } N \mid c,$$

then $g(E, e_1, e_2) = (E, ae_1 + bNe_2, c/Ne_1 + de_2)$.

Definition 6.3. Let $m, N \geq 1$ be as before.

i) We define a morphism $t_m : Y(m, mN) \to Y_1(N) \times \mu_m^{\circ}$ by the rule

$$(E, e_1, e_2) \longmapsto \left[(E/\langle e_1 \rangle, [me_2]), \langle e_1, Ne_2 \rangle_{E[m]} \right],$$

where $[me_2]$ denotes the image of me_2 in the quotient $E/\langle e_1 \rangle$.

ii) For each $a \geq 1$, we define a morphism $\tau_a : Y(am, amN) \to Y(m, mN)$ by the recipe

$$(E, e_1, e_2) \longmapsto (E/C, [e_1], [ae_2]),$$

where C is the cyclic subgroup (of order a) generated by me_1, and similarly as before here $[x]$ denotes the image of x in E/C.

Remark 6.4. We will be concerned with products of two modular curves. By a slight abuse of notation, $Y(N)^2$ will denote the fibre product $Y(N) \times_{\mu_N^{\circ}} Y(N)$, which is a subvariety of $Y(N) \times_{\mathrm{Spec}(\mathbb{Q})} Y(N)$ preserved by the action of the group

$$\{(\sigma, \tau) \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})^2 : \det(\sigma) = \det(\tau)\}.$$

In the same fashion, we will write $Y(m, mN)^2$ for the variety $Y(m, mN) \times_{\mu_m^{\circ}} Y(m, mN)$, which is endowed with a natural action of the group

$$\left\{ (\sigma, \tau) \in \mathrm{GL}_2(\mathbb{Z}/mN\mathbb{Z})^2 : \det(\sigma) \equiv \det(\tau) \,(\mathrm{mod}\, m), \quad \sigma, \tau \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right) (\mathrm{mod}\, N) \right\}.$$

3 Siegel units

Definition 6.5. Given $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 - \{(0,0)\}$ of order dividing N, and an integer $c > 1$ coprime to $6N$, we denote by ${}_cg_{\alpha,\beta} \in \mathcal{O}(Y(N))^\times$ Kato's Siegel unit as in previous lectures.

Remark 6.6. The element ${}_cg_{\alpha,\beta}$ is defined over $\mathbb{Q}$ as a function on $Y_1(N)$, but only after a choice of N-th root of unity one can view it as a holomorphic function on $\mathcal{H}$ whose q-expansion coefficients lie in $\mathbb{Q}(\mu_N)$. Indeed, this identification is made via the identification of the fibre of $Y(N)(\mathbb{C})$ over $e^{2\pi i/N} \in \mu_N^\circ(\mathbb{C})$ with $\Gamma(N)\backslash\mathcal{H}$.

In this lecture we are mostly interested in the units ${}_cg_{0,1/N}$, which descend to units on $Y_1(N)$. The next theorem of Kato establishes a compatibility property satisfied by them.

Theorem 6.7 (Kato)**.** *Let $M, N, N' \geq 1$ be integers, with $N \mid N'$, and let*

$$\alpha : Y(M, N') \longrightarrow Y(M, N), \quad (E, e_1, e_2) \longmapsto (E, e_1, \frac{N'}{N} e_2)$$

be the natural projection inducing the norm map

$$\alpha_* : \mathcal{O}(Y(M, N'))^\times \longrightarrow \mathcal{O}(Y(M, N))^\times.$$

i) If $\mathrm{prime}(N') = \mathrm{prime}(N)$*, then*

$$\alpha_*({}_cg_{0,1/N'}) = {}_cg_{0,1/N}.$$

ii) If $N' = N\ell$ for a prime ℓ not dividing MN, then

$$\alpha_*({}_cg_{0,1/N'}) = {}_cg_{0,1/N} \cdot ({}_cg_{0,\ell^{-1}/N})^{-1},$$

where by a slight abuse of notation ℓ^{-1} means the inverse of ℓ modulo N.

4 Chow groups, motivic cohomology and The Gersten complex

Let X be a smooth variety over a field k of characteristic 0, by which we mean a separated scheme of finite type over k, and let $\mathrm{CH}^2(X, 1)$ be the higher Chow group of X as defined by Bloch. We shall briefly recall how this group is related to the motivic cohomology and the Gersten complex of X, since it will be via this complex that we will construct elements in $\mathrm{CH}^2(X, 1)$.

Recall the Gersten complex of X,

$$\mathrm{Gerst}_2(X) : \coprod_{x \in X^0} K_2(k(x)) \xrightarrow{d^0} \coprod_{x \in X^1} k(x)^\times \xrightarrow{d^1} \coprod_{x \in X^2} \mathbb{Z},$$

where X^i denotes the codimension i cycles in X, $k(x)$ is the residue field at x, d^0 denotes the tame symbol map and d^1 maps a function to its divisor. Then we have the following comparison isomorphisms (cf. [LLZ14, Prop. 2.5.8]):

Theorem 6.8. *For a smooth variety X over k, there are isomorphisms*

$$\mathrm{H}^1(\mathrm{Gerst}_2(X)) \simeq \mathrm{H}^1(X, \mathscr{K}_2) \simeq \mathrm{CH}^2(X, 1) \simeq \mathrm{H}^3_{mot}(X, \mathbb{Z}(2)),$$

where $\mathscr{K}_2$ denotes the sheafification of $U \mapsto K_2(U)$ on X.

Definition 6.9. Write $Z^2(X, 1)$ for the kernel of the boundary morphism d^1 in the Gersten complex $\mathrm{Gerst}_2(X)$, so that

$$Z^2(X, 1) = \left\{ \sum_i (C_i, \phi_i) : C_i \in X^1, \phi_i \in k(C_i)^\times, \sum_i \mathrm{div}(\phi_i) = 0 \right\}.$$

5 Zeta elements

Definition 6.10. For integers $m, N \geq 1$, we define the curve $\mathcal{C}_{m,N,j} \subseteq Y(m, mN)^2$ as the subvariety

$$\left(u, v : v = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u \right).$$

If $c > 1$ is an integer coprime to $6mN$, then we define

$${}_c\mathcal{Z}_{m,N,j} = (\mathcal{C}_{m,N,j}, \phi) \in Z^2(Y(m, mN)^2, 1),$$

where $\phi \in \mathcal{O}(\mathcal{C}_{m,N,j})^\times$ is the pullback of ${}_cg_{0,1/mN}$ along either of the projections $\mathcal{C}_{m,N,j} \to Y(m, mN)$.

Proposition 6.11. *The zeta elements ${}_c\mathcal{Z}_{m,N,j}$ satisfy the following properties.*

i) If ρ denotes the involution of $Y(m, mN)^2$ which interchanges the two factors, then

$$\rho^*({}_c\mathcal{Z}_{m,N,j}) = {}_c\mathcal{Z}_{m,N,-j}.$$

ii) If $c, d > 1$ are coprime to $6mN$, then

$$\left[d^2 - \left(\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \right)^* \right] \cdot {}_c\mathcal{Z}_{m,N,j}$$

is symmetric in c and d. Hence there exists a unique

$$\mathcal{Z}_{m,N,j} \in Z^2(Y(m, mN)^2, 1) \otimes \mathbb{Q}$$

such that for every c

$${}_c\mathcal{Z}_{m,N,j} = \left[c^2 - \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}^* \right] \mathcal{Z}_{m,N,j}.$$

iii) For every $b \in (\mathbb{Z}/mN\mathbb{Z})^\times$, we have

$$\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \right)^* {}_c\mathcal{Z}_{m,N,j} = {}_c\mathcal{Z}_{m,N,b^{-1}j}.$$

Proof. Statement i) is clear by using that the relation

$$v = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u$$

is translated under ρ into the relation

$$u = \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} v.$$

Part ii) is consequence of the fact that for any $\alpha, \beta \in \frac{1}{N}\mathbb{Z}/\mathbb{Z} - \{(0,0)\}$ and any $c, d > 1$ coprime to $6mN$, one has the following identity involving Siegel units:

$$(d^2 {}_cg_{\alpha,\beta} - {}_cg_{d\alpha,d\beta}) = (c^2 {}_dg_{\alpha,\beta} - {}_dg_{c\alpha,c\beta}).$$

It thus makes sense to define the zeta element $\mathcal{Z}_{m,N,j}$ as the element $(c^2-1)^{-1} {}_c\mathcal{Z}_{m,N,j}$ for any $c \equiv 1 \,(\mathrm{mod}\, mN)$. This is the element claimed in the statement.

Finally, the assertion in iii) follows from the identity

$$\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b^{-1}j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Indeed, this identity is translated into the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}_{M,N,j} & \xhookrightarrow{\left(1,\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)\right)} & Y(m,mN)^2 \\ \big\downarrow & \nearrow_{\left(1,\left(\begin{smallmatrix}1 & b^{-1}j\\ 0 & 1\end{smallmatrix}\right)\right)} & \\ \mathcal{C}_{M,N,b^{-1}j} & & \end{array}$$

from which the statement follows. □

Lemma 6.12. *Let $m, N \geq 1$ be integers with $m^2N \geq 5$, and let $j \in \mathbb{Z}$. Then there is a unique morphism*

$$\kappa_j : Y_1(m^2N) \otimes \mathbb{C} \longrightarrow Y_1(N) \otimes \mathbb{C}$$

of algebraic varieties over $\mathbb{C}$ such that the diagram of morphisms of complex-analytic manifolds

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{z \mapsto z+j/m} & \mathcal{H} \\ \big\downarrow & & \big\downarrow \\ Y_1(m^2N)(\mathbb{C}) & \xrightarrow{\kappa_j} & Y_1(N)(\mathbb{C}) \end{array}$$

commutes. Furthermore, the morphism κ_j is defined over $\mathbb{Q}(\mu_m)$, and depends only on the residue class of j modulo m.

Proof. The existence of the map κ_j at the level of quotients of the upper half-plane $\mathcal{H}$ follows from the inclusion of subgroups of $\mathrm{SL}_2(\mathbb{Z})$

$$\begin{pmatrix} 1 & m^{-1}j \\ 0 & 1 \end{pmatrix} \Gamma_1(m^2N) \begin{pmatrix} 1 & -m^{-1}j \\ 0 & 1 \end{pmatrix} \subseteq \Gamma_1(N).$$

Hence it remains to descend to an algebraic morphism over $\mathbb{Q}(\mu_m)$, for which we use the canonical models of the involved modular curves. First of all, consider the map

$$Y(m^2N) \longrightarrow Y(m, mN), \quad (E, e_1, e_2) \longmapsto (E/\langle me_2\rangle, [mNe_1], [e_2]),$$

where as usual here $[mNe_1]$ and $[e_2]$ denote the images of mNe_1 and e_2, respectively, in the quotient $E/\langle me_2\rangle$. Observe that this map is compatible with $z \mapsto mz$ on $\mathcal{H}$. Besides, it is invariant by the action on $Y(m^2N)$ of the subgroup

$$\left\{ \begin{pmatrix} u & * \\ 0 & 1 \end{pmatrix} : u \equiv 1 \,(\mathrm{mod}\, m) \right\} \subseteq \mathrm{GL}_2(\mathbb{Z}/mN\mathbb{Z}),$$

hence it factors through the quotient, giving rise to a map

$$Y_1(m^2N) \otimes \mathbb{Q}(\mu_m) \longrightarrow Y(m, mN). \tag{6.1}$$

Now consider the composition

$$Y_1(m^2N) \otimes \mathbb{Q}(\mu_m) \longrightarrow Y(m, mN) \xrightarrow{\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)} Y(m, mN) \xrightarrow{t_m} Y_1(N) \otimes \mathbb{Q}(\mu_m),$$

where the first map is (6.1). The three of these maps are morphisms of varieties over $\mathbb{Q}(\mu_m)$, and on the fibre over $\zeta_m \in \mu_m(\mathbb{C})$ they correspond to

$$z \mapsto mz, \quad z \mapsto z + j \quad \text{and} \quad z \mapsto z/m,$$

respectively, so the composition is compatible with $z \mapsto z+j/m$ on $\mathcal{H}$, and is the desired morphism κ_j. □

Definition 6.13. Let m, N and j be as above. We denote by $\iota_{m,N,j}$ the map

$$(\kappa_0, \kappa_j) : Y_1(m^2N) \times \mu_m \longrightarrow Y_1(N)^2 \times \mu_m,$$

and write $C_{m,N,j}$ for the irreducible curve in $Y_1(N)^2$ which is the image of this map.

The curves $C_{m,N,j}$ can be used to define classes in $\mathrm{CH}^2(Y_1(N)^2 \times \mu_m, 1)$ using the representation of this Chow group in terms of the Gersten complex:

Definition 6.14. Let $N \geq 5$, $m \geq 1$ and $j \in \mathbb{Z}/m\mathbb{Z}$. Let also $c \geq 1$ be coprime to $6mN$ and let $\alpha \in \mathbb{Z}/m^2N\mathbb{Z}$. The *generalized Beilinson-Flach element*

$${}_c\Xi_{m,N,j,\alpha} \in \mathrm{CH}^2(Y_1(N)^2 \otimes \mathbb{Q}(\mu_m), 1)$$

is defined as the class of the pair

$$\big(C_{m,N,j}, (\iota_{m,N,j})_*({}_c g_{0,\alpha/m^2N})\big) \in Z^2(Y_1(N)^2 \times \mu_m, 1).$$

For $\alpha = 1$, we simply write ${}_c\Xi_{m,N,j}$.

Proposition 6.15. *The generalized Beilinson-Flach element ${}_c\Xi_{m,N,j,\alpha}$ is the pushforward of*

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right)^* {}_c\mathcal{Z}_{m,N,j} \in Z^2(Y(m, mN)^2, 1)$$

along $t_m \times t_m : Y(m, mN)^2 \longrightarrow Y_1(N)^2 \times \mu_m$.

Proof. By construction, $C_{m,N,j}$ is the image of the curve $\mathcal{C}_{m,N,j}$ under the morphism $t_m \times t_m$. Then the proof is reduced to show that the pushforward of the Siegel unit ${}_c g_{0,1/m^2N}$ from $Y_1(m^2N) \otimes \mathbb{Q}(\mu_m)$ to $Y(m, mN)$ along the map in (6.1) is the Siegel unit ${}_c g_{0,1/mN}$. We refer the reader to [LLZ14, Prop. 2.7.4], for the details. □

6 First norm relations

We first consider the relation between zeta elements at different levels N (for fixed values of m and j).

Theorem 6.16 (First norm relation)**.** *Let N and N' be positive integers such that $N \mid N'$, and let α denote the natural projection $Y(m, mN') \longrightarrow Y(m, mN)$.*

i) If $\mathrm{prime}(N') \subseteq \mathrm{prime}(mN)$, then the pushforward

$$(\alpha \times \alpha)_* : \mathrm{CH}^2(Y(m, mN')^2, 1) \longrightarrow \mathrm{CH}^2(Y(m, mN)^2, 1)$$

sends ${}_c\mathcal{Z}_{m,N',j}$ to ${}_c\mathcal{Z}_{m,N,j}$.

ii) If $N' = N\ell$, where ℓ is a prime not dividing mN, then we have instead

$$(\alpha \times \alpha)_*({}_c\mathcal{Z}_{m,N\ell,j}) = \left[1 - \left(\begin{pmatrix} \ell^{-1} & 0 \\ 0 & \ell^{-1} \end{pmatrix}, \begin{pmatrix} \ell^{-1} & 0 \\ 0 & \ell^{-1} \end{pmatrix}\right)^*\right] {}_c\mathcal{Z}_{m,N,j},$$

where we view the matrices in $\mathrm{GL}_2(\mathbb{Z}/mN\mathbb{Z})$, hence ℓ^{-1} denotes the inverse of ℓ modulo mN.

Proof. First of all, we check that α commutes with the action of $\left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$. Indeed, on the one hand we have

$$(E, e_1, e_2) \overset{\left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)}{\longmapsto} (E, e_1 + N'je_2, e_2) \overset{\alpha}{\longmapsto} (E, e_1 + N'je_2, \frac{N'}{N}e_2),$$

and on the other hand

$$(E, e_1, e_2) \overset{\alpha}{\longmapsto} (E, e_1, \frac{N'}{N}e_2) \overset{\left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)}{\longmapsto} (E, e_1 + Nj\frac{N'}{N}e_2, \frac{N'}{N}e_2) = (E, e_1 + N'je_2, \frac{N'}{N}e_2).$$

As a consequence, $(\alpha \times \alpha)(\mathcal{C}_{m,N',j}) = \mathcal{C}_{m,N,j}$, and the following diagram commutes:

$$\begin{array}{ccc} Y(m,mN') & \xrightarrow{\left(1,\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)\right)} & \mathcal{C}_{m,N',j} \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha\times\alpha} \\ Y(m,mN) & \xrightarrow{\left(1,\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)\right)} & \mathcal{C}_{m,N,j} \end{array}$$

Now part i) follows because if $\mathrm{prime}(mN) = \mathrm{prime}(mN')$ then $\alpha_*({}_cg_{0,1/mN'}) = {}_cg_{0,1/mN}$ by part i) of Theorem 6.7. As for ii), we have

$$(\alpha \times \alpha)_*({}_c\mathcal{Z}_{m,N',j}) = {}_c\mathcal{Z}_{m,N,j} - {}_c\mathcal{Z}_{m,N,j,\ell^{-1}}$$

by Theorem 6.7, part ii), where ${}_c\mathcal{Z}_{m,N,j,\ell^{-1}}$ denotes the zeta element obtained from the Siegel unit ${}_cg_{0,\ell^{-1}/mN}$ instead of ${}_cg_{0,1/mN}$. But notice that

$${}_cg_{0,\ell^{-1}(mN} = \begin{pmatrix} \ell^{-1} & 0 \\ 0 & \ell^{-1} \end{pmatrix}^* {}_cg_{0,1/mN} \quad \text{in } \mathcal{O}(Y(m,mN))^\times,$$

and the action of $\left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell^{-1} \end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$, from which the claim follows. □

Secondly, one can deduce the following compatibility relation for zeta elements on $Y_1(N)^2 \otimes \mathbb{Q}(\mu_m)$:

Theorem 6.17 (First norm relation on $Y_1(N)$)**.** *Let N and N' be positive integers such that $N \mid N'$ as above, and let $\alpha : Y_1(N') \to Y_1(N)$ be the natural projection.*

i) If $\mathrm{prime}(mN') = \mathrm{prime}(mN)$*, then*

$$(\alpha \times \alpha)_*({}_c\Xi_{m,N',j}) = {}_c\Xi_{m,N,j}.$$

ii) If $N' = N\ell$, where ℓ is a prime not dividing mN, then

$$(\alpha \times \alpha)_*({}_c\Xi_{m,N\ell,j}) = [1 - (\langle \ell^{-1} \rangle, \langle \ell^{-1} \rangle)^* \sigma_\ell^{-2}]{}_c\Xi_{m,N,j},$$

where σ_ℓ denotes the arithmetic Frobenius automorphism at ℓ, and $\langle\,\rangle$ is the diamond operator.

Proof. This is a direct consequence of the previous theorem, because

$$\pi_{m,N} : Y(m,mN) \longrightarrow Y_1(N) \times \mu_m$$

intertwines $\left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right)$ with the diamond operator $\langle \ell \rangle$, and $\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with σ_ℓ. □

7 Hecke operators

Let ℓ be a prime, and $M, N \geq 1$ be integers (we allow $\ell \mid M$ or $\ell \mid N$).

Definition 6.18. We define the curve $Y(M(\ell), N)$ (resp. $Y(M, N(\ell))$) to be the quotient of the modular curve $Y(L)$, for some integer L with $M\ell \mid L$ and $N \mid L$ (resp. $M \mid L$ and $N\ell \mid L$), by the action of the group

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} a \equiv 1 \,(\mathrm{mod}\, M), b \equiv 0 \,(\mathrm{mod}\, M\ell) \\ c \equiv 0, d \equiv 1 \,(\mathrm{mod}\, N) \end{array} \right\},$$

$$\text{(resp. } \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} a \equiv 1, b \equiv 0 \,(\mathrm{mod}\, M) \\ c \equiv 0 \,(\mathrm{mod}\, N\ell), d \equiv 1 \,(\mathrm{mod}\, N) \end{array} \right\} \text{).}$$

The curve $Y(M(\ell), N)$ represents the moduli functor on $\mathbb{Q}$-schemes which associates to a $\mathbb{Q}$-scheme S the set of isomorphism classes of tuples (E, e_1, e_2, C), where $(E, e_1, e_2) \in Y(M, N)$ and C is a cyclic subgroup of E/S of order $M\ell$ with $e_1 \in C$ and such that the morphism

$$C \times \mathbb{Z}/N\mathbb{Z} \hookrightarrow E, \qquad (x, y) \mapsto x + ye_2$$

is injective. A similar moduli interpretation holds for $Y(M, N(\ell))$, requiring C to be a cyclic subgroup of E/S of order $N\ell$ with $e_2 \in C$ and such that the analogous morphism

$$C \times \mathbb{Z}/M\mathbb{Z} \hookrightarrow E, \qquad (x, y) \mapsto ye_1 + x$$

is injective.

Fixed M, N, ℓ, there is a natural isomorphism

$$\varphi_\ell : Y(M, N(\ell)) \xrightarrow{\simeq} Y(M(\ell), N)$$

defined by sending the moduli of the tuple (E, e_1, e_2, C) to that of (E', e_1', e_2', C'), where $E' = E/NC$, e_1' is the image of e_1 in E', e_2' is the image of $\ell e_2' \cap C$ in E' and C' is the image of $\ell^{-1}\mathbb{Z}e_1$ in E'.

In order to define a correspondence on $Y(M, N)$ associated to the prime ℓ, we consider the diagram of modular curves

$$\begin{array}{ccccc} & & Y(M(\ell), N) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ Y(M, N) & & & & Y(M, N) \end{array}$$

where π_1 is the natural degeneracy map induced by the identity on $\mathcal{H}$, and π_2 is the composite of φ_ℓ^{-1} with the natural degeneracy map $Y(M, N(\ell)) \to Y(M, N)$, corresponding to the map $z \mapsto z/\ell$ on $\mathcal{H}$.

By using the above diagram:

Definition 6.19. We write T_ℓ' (resp. U_ℓ') for the correspondence $(\pi_2)_*(\pi_1)^*$ on $Y(M, N)$ if ℓ does not divide MN (resp. if $\ell \mid MN$). Besides, the correspondence $(\pi_1)_*(\pi_2)^*$ is denoted by T_ℓ (resp. U_ℓ)) if ℓ does not divide MN (resp. if $\ell \mid MN$).

8 Second norm relations

Theorem 6.20 (Second norm relation). *Let $m \geq 1$, $N \geq 5$ and let ℓ be a prime dividing N. Write*

$$\tau_\ell : Y(m\ell, m\ell N) \longrightarrow Y(m, mN)$$

for the degeneracy map induced by $z \mapsto z/\ell$ on $\mathcal{H}$. Then for any $j \in (\mathbb{Z}/\ell m\mathbb{Z})^\times$ and any $c > 1$ coprime to $6\ell mN$, we have

$$(\tau_\ell \times \tau_\ell)_*({}_c\mathcal{Z}_{\ell m,N,j}) = \begin{cases} (U_\ell' \times U_\ell')({}_c\mathcal{Z}_{m,N,j}) & \text{if } \ell \mid m, \\ (U_\ell' \times U_\ell' - \Delta_\ell^*)({}_c\mathcal{Z}_{m,N,j}) & \text{if } \ell \nmid m. \end{cases}$$

Here, Δ_ℓ denotes the action of any element of $\mathrm{GL}_2(\mathbb{Z}/mN\mathbb{Z})^2$ of the form

$$\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right)$$

with $x \equiv \ell \,(\mathrm{mod}\, m)$.

Proof. Assume m, N and ℓ are as in the statement. Set $N' = N/\ell$ and consider the diagram

$$\begin{array}{ccccc} Y(\ell m, \ell mN) & \xrightarrow{\alpha} & Y(\ell m, mN) & \xrightarrow{\mathrm{pr}} & Y(m(\ell), mN) \\ & \searrow^{\tau_\ell} & & \swarrow^{\pi_2} & \downarrow{\scriptstyle \pi_1} \\ & & Y(m, mN) & & Y(m, mN) \end{array}$$

where $\alpha : Y(\ell m, \ell mN) \to Y(\ell m, \ell mN') = Y(\ell m, mN)$ and pr are the natural projection maps. Let us check that this diagram commutes. It is enough to check that $\pi_2 \circ \mathrm{pr} \circ \alpha = \tau_\ell$. Indeed, by using the moduli interpretation note first that

$$(E, e_1, e_2) \overset{\alpha}{\longmapsto} (E, e_1, \ell e_2) \overset{\mathrm{pr}}{\longmapsto} (E, \ell e_1, \ell e_2, \mathbb{Z}e_1).$$

Now we use that $\pi_2 = \pi_1 \circ \varphi_\ell^{-1}$, so that

$$(E, \ell e_1, \ell e_2, \mathbb{Z}e_1) \overset{\varphi_\ell^{-1}}{\longmapsto} (E/\langle me_1\rangle, [e_1], [\ell e_2], [e_2]) \overset{\pi_1}{\longmapsto} (E/\langle me_1\rangle, [e_1], [\ell e_2]).$$

Summing up, $\pi_2 \circ \mathrm{pr} \circ \alpha$ sends the point corresponding to the moduli of (E, e_1, e_2) to that of $(E/\langle me_1\rangle, [e_1], [\ell e_2])$, and from the definitions τ_ℓ is described by the same rule. Thus the above diagram commutes as we claimed. Further, it induces a commutative diagram of surfaces

$$\begin{array}{ccccc} Y(\ell m, \ell mN)^2 & \xrightarrow{\alpha\times\alpha} & Y(\ell m, mN)^2 & \xrightarrow{\mathrm{pr}\times\mathrm{pr}} & Y(m(\ell), mN)^2 \\ & {\scriptstyle \tau_\ell\times\tau_\ell}\searrow & & \swarrow{\scriptstyle \pi_2\times\pi_2} & \downarrow{\scriptstyle \pi_1\times\pi_1} \\ & & Y(m, mN)^2 & & Y(m, mN)^2 \end{array}$$

By the first norm relation, $(\alpha\times\alpha)_* {}_c\mathcal{Z}_{\ell m,N,j} = {}_c\mathcal{Z}_{\ell m,N',j}$. Using that $Y(m(\ell), mN)^2$ is the quotient of $Y(\ell m, mN)^2$ by the action of the subgroup

$$\left\{ \left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) : x \in \mathbb{Z}/\ell m\mathbb{Z}, x \equiv 1 \,(\mathrm{mod}\, m) \right\},$$

we find out that

$$(\mathrm{pr}\times\mathrm{pr})^*(\mathrm{pr}\times\mathrm{pr})_*({}_c\mathcal{Z}_{\ell m,N',j}) = \sum_{\substack{x\in(\mathbb{Z}/\ell m\mathbb{Z})^\times \\ x\equiv 1 \bmod m}} {}_c\mathcal{Z}_{\ell m,N',xj}.$$

Besides, $Y(m, mN)^2$ is the quotient of $Y(\ell m, mN)^2$ by the action of the subgroup

$$\left\{ \left(\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x & z \\ 0 & 1 \end{smallmatrix}\right) \right) : \begin{array}{l} x, y, z \in \mathbb{Z}/\ell m\mathbb{Z}, x \equiv 1 \,(\mathrm{mod}\, m), \\ y, z \equiv 0 \,(\mathrm{mod}\, m) \end{array} \right\},$$

and from this it follows that

$$(\mathrm{pr}\times\mathrm{pr})^*(\pi_1\times\pi_1)^* {}_c\mathcal{Z}_{m,N,j} = \sum_{\substack{k\in\mathbb{Z}/\ell m\mathbb{Z} \\ k\equiv j \bmod m}} {}_c\mathcal{Z}_{\ell m,N',k}.$$

By assumption, j is invertible modulo ℓm. If $\ell \mid m$, we thus have

$$\{xj : x \equiv 1 \,(\mathrm{mod}\, m)\} = \{k : k \equiv j \,(\mathrm{mod}\, m)\},$$

and since $(\mathrm{pr}\times\mathrm{pr})^*$ is injective, we conclude that

$$(\mathrm{pr}\times\mathrm{pr})_*({}_c\mathcal{Z}_{\ell m,N',j}) = (\pi_1\times\pi_1)^* {}_c\mathcal{Z}_{m,N,j}.$$

By applying $(\pi_2\times\pi_2)_*$, we obtain the result when $\ell \mid m$. The case $\ell \nmid m$ is a bit more involved, although the main ideas are similar. We refer the reader to [LLZ14, Theorem 3.3.1].

There is an alternative (more concise) proof, which in the case $\ell \mid m$ consists of showing that the diagram

$$\begin{array}{ccc} \mathcal{C}_{\ell m,N',j} & \xrightarrow{\mathrm{pr}\circ i_1} & Y(m(\ell), mN)^2 \\ {\scriptstyle \pi_1\circ\mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \pi_1\times\pi_1} \\ \mathcal{C}_{m,N,j} & \xrightarrow{i_2} & Y(m, mN)^2. \end{array}$$

is cartesian. From this one deduces that

$$((\mathrm{pr} \times \mathrm{pr}) \circ i_1)_*(\pi_1 \circ \mathrm{pr})^*(\mathcal{C}_{m,N,j}) = (\pi_1 \times \pi_1)^* i_{2,*}(\mathcal{C}_{m,N,j}),$$

that is to say, $(\mathrm{pr} \times \mathrm{pr})_*(\mathcal{C}_{m,N',j}) = (\pi_1 \times \pi_1)^*(\mathcal{C}_{m,N,j})$. Applying $(\pi_2 \times \pi_2)_*$, one obtains the desired results (cf. [KLZ15a, Section 5.4]). The proof also works when $\ell \nmid m$, but the above diagram must be slightly modified (see *ibid.* for the precise diagram). □

Finally, the previous theorem is translated into a norm relation for $Y_1(N)$, which we state below. The proof is a direct consequence of Theorem 6.20 (cf. [LLZ14, Theorem 3.3.2]).

Theorem 6.21 (Second norm relation for $Y_1(N)$)**.** *Let $m \geq 1$, $N \geq 5$, and ℓ be a prime dividing N as before. For every $j \in (\mathbb{Z}/\ell m\mathbb{Z})^\times$ and $c \in (\mathbb{Z}/\ell mN\mathbb{Z})^\times$,*

$$\mathrm{norm}_m^{\ell m}({}_c\Xi_{\ell m,N,j}) = \begin{cases} (U'_\ell \times U'_\ell)({}_c\Xi_{m,N,j}) & \text{if } \ell \mid m, \\ (U'_\ell \times U'_\ell - \sigma_\ell)({}_c\Xi_{m,N,j}) & \text{if } \ell \mid m, \end{cases}$$

where $\mathrm{norm}_m^{\ell m}$ denotes the Galois norm map, and σ_ℓ (for $\ell \nmid m$) denotes the arithmetic Frobenius at ℓ in $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$.

p-adic Hodge theory and Bloch–Kato theory

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1 Notation

We begin by setting up some notation that will be used later on.

- Let K be a finite extension of $\mathbb{Q}_p$, and let $K_n = K(\mu_{p^n})$, where μ_{p^n} is a p^n-th root of unity. Moreover we let F denote an unramified extension of $\mathbb{Q}_p$ and we have the convention that when we talk about both K and F, then we assume F is the maximal unramified extension of $\mathbb{Q}_p$.
- Let $K_\infty = \cup_n K_n$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.
- Let $\chi : \Gamma_K \to \mathbb{Z}_p^\times$ be the cyclotomic character. Note that it identifies Γ_K with an open subgroup of $\mathbb{Z}_p^\times$.
- Lastly we let $H_F = \ker \chi = \mathrm{Gal}(\overline{K}/K_\infty)$.

2 Review of some p-adic Hodge theory

The aim of p-adic Hodge theory is to understand p-adic representations V of the absolute Galois group G_K. If we take a prime $\ell \neq p$ then the ℓ-adic representations of G_K are well understood, but in the case $\ell = p$ we get many more representations as the topologies of the $\mathbb{Q}_p$-vector spaces and G_K are compatible, so we need more theory in order to study these representations.

The main strategy of p-adic Hodge theory is to construct rings of periods B, which come equipped with an action of G_K, such that

$$D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$$

is an interesting invariant of the representation V.

These rings of periods should satisfy the following requirements:

- B should be a domain.
- $\mathrm{Frac}(B)^{G_K} = B^{G_K}$ (in particular B^{G_K} is a field).
- If $y \in B$ is such that $y.\mathbb{Q}_p \subseteq B$ is stable under G_K, then $y \in B^\times$.

Definition 7.1. If V is a p-adic representation of G_K, we define

$$D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Note that, since B is a domain, we have a natural injective map

$$B \otimes_{B^{G_K}} D_B(V) \longrightarrow B \otimes \mathbb{Q}_p V.$$

From which is follows that $\dim_{B^{G_K}}(D_B(V)) \leq \dim_{\mathbb{Q}_p}(V)$. We say that V is B-**admissible** if we have equality of dimensions (i.e the above map is an isomorphism).

One can show that there exists a category of B-admissible representations of G_K, which is stable under subquotients, sums, tensor products an duals. But note that in general this category is not stable under extensions. But, note that if B has extra structure (a grading, filtration, an action of an operator), and this extra structure is compatible with the Galois action, then $D_B(V)$ will inherit this structure.

2.1 The ring B_{dR} and de Rham representations.

One of the main examples of a field of periods was constructed by Fontaine and is denoted by B_{dR}. This is the field of fractions of a complete valuation ring B_{dR}^+ which has residue field $\mathbb{C}_p$. For the sake of brevity we will not define B_{dR} explicitly, but only list some of its properties that we will use later.

- The maximal ideal of B_{dR}^+ is generated by an element t, which is a p-adic analogue of $2\pi i$, which depends on a choice $\varepsilon = (\varepsilon^{(n)})$ of compatible p-power roots of unity.
- The action of $G_{\mathbb{Q}_p}$ on t is via the cyclotomic character χ, i.e. $g(t) = \chi(g)t$ for $g \in G_{\mathbb{Q}_p}$.
- We have a descending filtration $\mathrm{Fil}^i = t^i B_{\mathrm{dR}}^+$, which is stable by the action of $G_{\mathbb{Q}_p}$.
- Since B_{dR}^+ is a complete discrete valuation ring, we can use Hensel's lemma to see that $\overline{\mathbb{Q}_p} \subseteq B_{\mathrm{dR}}^+$, and this is compatible with the action of $G_{\mathbb{Q}_p}$. However, note that B_{dR}^+ is not naturally a $\mathbb{C}_p$-algebra and in fact a theorem of Colmez says that $\overline{\mathbb{Q}_p}$ is dense in B_{dR} in a suitable topology.
- We have $(B_{\mathrm{dR}})^{G_K} = (B_{\mathrm{dR}}^+)^{G_K} = K$. Thus, if V is a p-adic representation of G_K, $D_{\mathrm{dR}}(V) = D_{B_{\mathrm{dR}}}(V)$ is a filtered K-vector space.

Definition 7.2. We say that a representation V of G_K is de Rham if it is B_{dR}-admissible. In other words, if we have an isomorphism

$$B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \cong B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V.$$

We now have the de Rham comparison theorem, which was conjectured by Fontaine and proven by Faltings.

Theorem 7.3. (de Rham comparison theorem) *Let X/K be a smooth proper variety, and let $V = H^i_{et}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$, so that V is a p-adic representation of G_K. Then V is de Rham, and there is a natural isomorphism of filtered K-vector spaces*

$$D_{dR}(V) \cong H^i_{dR}(X/K),$$

where $H^i_{dR}(X/K)$ is de Rham cohomology equipped with the Hodge filtration.

From B_{dR} we can also define a period ring B_{HT}, which is the graded ring associated to the filtered ring B_{dR}. We have

$$B_{\mathrm{HT}} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_p(j),$$

since $(t)^i/(t)^{i+1} = \mathbb{C}_p(j)$, where by (t) we denote the ideal generated by t.

Definition 7.4. We say a p-adic representation V of G_K is **Hodge–Tate** if it is B_{HT}-admissible.

Note that, if V is Hodge–Tate, then $D_{\mathrm{HT}}(V)$ is a graded K-vector space. Moreover if V is de Rham, then $D_{\mathrm{HT}}(V)$ is the graded vector space associated to the filtered vector space $D_{\mathrm{dR}}(V)$. In particular, a de Rham representation is Hodge–Tate.

Definition 7.5. Let V be a Hodge–Tate p-adic representation of G_K. A Hodge–Tate weight of V is an integer j such that $D_{\mathrm{HT}}(V)_{-j} \neq 0$. Equivalently, if V is de Rham, $\mathrm{Fil}^{-j} D_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-j+1} D_{\mathrm{dR}}(V)$.

Note that with this convention, $\mathbb{Q}_p(j)$ has Hodge–Tate weight j.

2.2 The ring B_{cris} and crystalline representations.

Next we study the ring of periods B_{cris} which a subring of B_{dR} equipped with an induced filtration and Galois action.

As before, we will not define B_{cris}, but only note that it satisfies the following properties:

- B_{cris} contains the element t from before, and we have $B_{\text{cris}} = B_{\text{cris}}^{+}[1/t]$, where $B_{\text{cris}}^{+} = B_{\text{dR}}^{+} \cap B_{\text{cris}}$.
- Let $F = \mathbb{Q}_p^{unr} \cap K$. We have $B_{\text{cris}}^{G_K} = F$, and the action of φ on $B_{\text{cris}}^{G_K}$ is the Frobenius on F.
- The ring B_{cris} is equipped with a Frobenius $\varphi : B_{\text{cris}} \to B_{\text{cris}}$ commuting with the action of $G_{\mathbb{Q}_p}$, and inducing the usual Frobenius on $\mathbb{Q}_p^{\text{unr}} \subseteq B_{\text{cris}}$.
- Frobenius acts on t by $\varphi(t) = pt$.

Definition 7.6. Let V be a p-adic representation V of G_K. We call V is **crystalline** if it is B_{cris}-admissible.

One can show that, if V is a crystalline representation, then

$$D_{\text{cris}}(V) = D_{B_{\text{cris}}}(V) = (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

is an F-vector space equipped with a filtration and a semi-linear Frobenius. Furthermore

$$K \otimes_F D_{\text{cris}}(V) = D_{\text{dR}}(V),$$

as filtered K-vector spaces. Thus we see that a crystalline representation is also de Rham.

It follows from the above that we have:

$$\text{crystalline} \Rightarrow \text{de Rham} \Rightarrow \text{Hodge-Tate}.$$

The property of being crystalline, for a p-adic representation V, is analogous to the property of being unramified for an ℓ-adic representation (for $\ell \neq p$). For example, if V is the p-adic Tate module of an abelian variety A/K, then V is crystalline if and only if V has good reduction (Iovita).

An important theorem one has is the *crystalline comparison isomorphism*:

Theorem 7.7. *Let k be a perfect field of characteristic p, let $\mathcal{O}_F = W(k)$ and $F = \mathrm{Frac}(\mathcal{O}_F)$. Let $X/\mathcal{O}_F$ be a smooth, proper scheme, geometrically irreducible. Then for every $i \geq 0$, there is a functorial isomorphism*

$$H^i_{et}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^i_{cris}(X_k/\mathcal{O}_F) \otimes_{\mathcal{O}_F} B_{cris},$$

compatibly with filtrations, G_F-action and action of Frobenius (which acts diagonally on the right hand side).

A classical result of crystalline cohomology states that $H^i_{\text{cris}}(X_k/\mathcal{O}_F) \cong H^i_{\text{dR}}(X/\mathcal{O}_F)$. Thus $H^i_{\text{cris}}(X_k/\mathcal{O}_F)$ has a natural filtration and $H^i_{\text{dR}}(X/\mathcal{O}_F)$ has a natural action of a Frobenius.

Also note that it follows from the crystalline comparison isomorphism that

$$V = H^i_{\text{et}}(X_{\overline{F}}, \mathbb{Q}_p)$$

is crystalline, and moreover that $D_{\text{cris}}(V) \cong H^i_{\text{cris}}(X_k/\mathcal{O}_F) \otimes_{\mathcal{O}_F} F \cong H^i_{\text{dR}}(X/F)$.

2.3 The subgroups H^1_e, H^1_f, H^1_g

We begin by recalling the basic fact from homological algebra, that there is a natural identification of $H^1(K, V)$ with the set of isomorphism classes of extensions of $\mathbb{Q}_p$ by V.

Explicitly, if we are given an extension

$$0 \longrightarrow V \longrightarrow W \xrightarrow{p} \mathbb{Q}_p \to 0,$$

we choose $w \in W$ with $p(w) = 1$. Then for all $\sigma \in G_K$,

$$p(\sigma w) = \sigma p(w) = \sigma \cdot 1 = 1,$$

so $p(w - \sigma w) = 0$. In other words $w - \sigma w \in V$. Then the map $\sigma \mapsto w - \sigma w$ is a cocycle $G_K \to V$, whose cohomology class depends only on the isomorphism class of the extension. Therefore, it is natural, given a property of p-adic representations, to look at the subset of $H^1(K, V)$ consisting of those extensions of $\mathbb{Q}_p$ by V having that property.

With this in mind we have the following definition:

Definition 7.8.

- Let V be crystalline. Then we define the subgroup

$$H^1_e(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\mathrm{cris}}^{\varphi=1} \otimes V)).$$

- Let V be crystalline. Then we define the subgroup

$$H^1_f(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\mathrm{cris}} \otimes V))$$

  consisting of the extensions of $\mathbb{Q}_p$ by V which are crystalline.

- Let V be de Rham. Then we define subgroup

$$H^1_g(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\mathrm{dR}} \otimes V))$$

  consisting of the extensions of $\mathbb{Q}_p$ by V which are de Rham.

3 The Bloch-Kato exponential, logarithm, and dual exponential maps

3.1 The exponential map

Let V be a de Rham representation. Then Bloch and Kato have defined an "exponential map"

$$\exp_{K,V} : D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V) \longrightarrow H^1(K, V).$$

Note that, if $G/\mathcal{O}_K$ is a formal Lie group of finite height (eg. $G = \widehat{A}$ for $A/\mathcal{O}_K$), and $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p G$, then

$$D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V) \cong \tan(G(K)),$$

and we have a commutative diagram

$$\begin{array}{ccc} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V) & \xrightarrow{\exp_{K,V}} & H^1(K, V) \\ {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\delta_G} \\ \tan(G(K)) & \xrightarrow{\quad\exp\quad} & G(K) \end{array}$$

Here δ_G is the Kummer map, exp is the usual exponential map and $\tan(G(K))$ is the tangent group of $G(K)$.

In order to construct the Bloch-Kato exponential for an arbitrary de Rham representation V, one uses the *fundamental exact sequence*

$$0 \to \mathbb{Q}_p \to B_{\mathrm{cris}}^{\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$

where the second map is simply the composite.

$$B_{\mathrm{cris}}^{\varphi=1} \hookrightarrow B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+.$$

If we now tensor this exact sequence with V over $\mathbb{Q}_p$ and take invariants under G_K, then we get an exact sequence

$$0 \to V^{G_K} \to D_{\mathrm{cris}}^{\varphi=1}(V) \to ((B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes V)^{G_K} \to H^1_e(K,V) \to 0.$$

Now, since V is de Rham, we have

$$((B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes V)^{G_K} = D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V),$$

and therefore we deduce an isomorphism

$$\exp_{K,V} : \frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V) + D_{\mathrm{cris}}^{\varphi=1}(V)} \xrightarrow{\sim} H^1_e(K,V).$$

We denote by

$$\log_{K,V} : H^1_e(K,V) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V) + D_{\mathrm{cris}}^{\varphi=1}(V)}$$

the inverse of this isomorphism.

3.2 The dual exponential map

We begin by describing in more detail the filtered φ-module $D_{\mathrm{cris}}(\mathbb{Q}_p(j))$. Note that the choice of $\epsilon = (\epsilon^{(n)})$ determines the element $t \in B_{\mathrm{dR}}$ and a basis e_j of $\mathbb{Q}_p(j)$ for each j, such that $e_j \otimes e_{j'} = e_{j+j'}$. Furthermore, note that the element

$$t^{-j} e_j \in B_{\mathrm{cris}} \otimes \mathbb{Q}_p(j)$$

is Galois invariant and determines a canonical basis of $D_{\mathrm{cris}}(\mathbb{Q}_p(j))$, which does not depend on the choice of ϵ.

We have

$$\mathrm{Fil}^k D_{\mathrm{cris}}(\mathbb{Q}_p(j)) = \begin{cases} D_{\mathrm{cris}}(\mathbb{Q}_p(j)) = F & \text{if } k \leq -j \\ 0 & \text{if } k > -j \end{cases}$$

and the Frobenius φ on $D_{\mathrm{cris}}(\mathbb{Q}_p(j))$ acts as multiplication by p^{-j}, because $\varphi(t^{-j}e_j) = \varphi(t^{-j})e_j = p^{-j}t^{-j}e_j$.

In particular, the element $t^{-1}e_1 \in D_{\mathrm{dR}}(\mathbb{Q}_p(1))$ gives an isomorphism

$$D_{\mathrm{dR}}(\mathbb{Q}_p(1)) \xrightarrow{\sim} F.$$

From this, we obtain a perfect pairing

$$D_{\mathrm{dR}}(V) \otimes D_{\mathrm{dR}}(V^*(1)) \cong D_{\mathrm{dR}}(V \otimes V^*(1)) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p(1)) \cong F \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p,$$

and using this pairing, we identify $D_{\mathrm{dR}}(V)^*$ with $D_{\mathrm{dR}}(V^*(1))$. Moreover, the cup-product

$$H^1(K,V) \otimes H^1(K,V^*(1)) \longrightarrow H^2(K,\mathbb{Q}_p(1)) = \mathbb{Q}_p$$

identifies $H^1(K,V)^*$ with $H^1(K,V^*(1))$. Therefore, we can view

$$(\exp_{K,V})^* : H^1(K,V)^* \to (D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V))^*$$

as a map

$$\exp^*_{K,V} : H^1(K,V^*(1)) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(V^*(1))$$

This is the **dual exponential map** of V.

4 Perrin-Riou's big logarithm

4.1 Iwasawa cohomology

Definition 7.9. Let V be a p-adic representation of G_K of dimension d, and $T \subseteq V$ a $\mathbb{Z}_p$-lattice stable by G_K. The **Iwasawa cohomology** of V is defined as

$$H^i_{\mathrm{Iw}}(K, V) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim H^i(K_n, T)$$

where the inverse limit is taken with respect to the corestriction maps

$$\mathrm{cor}_{K_{n+1},K_n} : H^i(K_{n+1}, T) \to H^i(K_n, T).$$

Note that each $H^i(K_n, T)$ is naturally a $\mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$-module, so $H^i_{\mathrm{Iw}}(K, V)$ has a natural structure of a $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_K$-module, where $\Lambda_K = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$. Moreover, the module $H^i_{\mathrm{Iw}}(K, V)$ is idependent of the choice of lattice T.

Now, the ring $\mathbb{Q}_p \otimes \Lambda_K$ identifies with the space of p-adic measures on Γ_K, i.e.

$$\mathbb{Q}_p \otimes \Lambda_K = \mathrm{Hom}(C(\Gamma_K, \mathbb{Q}_p), \mathbb{Q}_p)$$

where $C(\Gamma_K, \mathbb{Q}_p)$ is the Banach space of continuous $\mathbb{Q}_p$-valued functions on Γ_K. As such it is equipped with a structure of $C(\Gamma_K, \mathbb{Q}_p)$-module, where if $f \in C(\Gamma_K, \mathbb{Q}_p)$ and $\mu \in \mathbb{Q}_p \otimes \Lambda_K$, then $f\mu$ is the measure $h \mapsto \int hf\, \mu$. In particular, if $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ is a continuous character and μ is a measure, the product $\eta\mu$ is a measure.

We also note that Λ_K is equipped with an action of G_K, given by $g(\mu) = [\overline{g}]\mu$, where $\overline{g} = g|_{K_\infty}$ is the image of g in Γ_K and $[\overline{g}]$ is the corresponding group-like element (the Dirac measure at $\overline{g}$).

Now, the structure of H^i_{Iw} has been determined by Perrin-Riou:

Theorem 7.10 (Perrin-Riou)**.** *We have $H^i_{Iw}(K, V) = 0$ for $i \neq 1, 2$; moreover:*

1. *The torsion submodule of $H^1_{Iw}(K, V)$ is a $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_K$-module isomorphic to V^{H_K}, and $H^1_{Iw}(K, V)/V^{H_K}$ is a free $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_K$-module of rank $[K : \mathbb{Q}_p]d$*
2. *$H^2_{Iw}(K, V) = (V^*(1)^{H_K})^*$*

In particular the $H^i_{\mathrm{Iw}}(K, V)$ are $\mathbb{Q}_p \otimes \Lambda_K$-modules of finite type. We also have natural projection maps $H^i_{\mathrm{Iw}}(K, V) \to H^i(K_n, V)$ which are compatible with corestriction maps.

4.2 Twisting Euler systems

In this section we describe $H^1_{\mathrm{Iw}}(K, V)$ as a single cohomology group, which can help us understand the "twisting" operation on Euler systems.

Let $\mathrm{Meas}(\Gamma_K, V) = \Lambda_K \otimes_{\mathbb{Q}_p} V$ be the space of V-valued measures on Γ_K, where we equip it with the diagonal action of G_K. Then we have a natural isomorphism

$$H^1(K, \Lambda_K \otimes_{\mathbb{Q}_p} V) \xrightarrow{\sim} H^1_{\mathrm{Iw}}(K, V),$$

which is defined as follows:

If $\sigma \mapsto \mu_\sigma$ is a cocycle $G_K \to \mathrm{Meas}(\Gamma_K, V)$ representing a cohomology class

$$\zeta \in H^1(K, \mathrm{Meas}(\Gamma_K, V)),$$

then, for any n, the map

$$\sigma \mapsto \int_{\Gamma_{K_n}} \mu_\sigma$$

is a cocycle

$$G_{K_n} \longrightarrow V$$

representing a cohomology class $\zeta_n \in H^1(K_n, V)$. This collection $(\zeta_n) \in H^1(K_n, V)$ is compatible under corestriction maps and determines an element of $H^1_{\mathrm{Iw}}(K, V)$.

If we let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be a continuous character, then there is a natural map

$$H^1_{\mathrm{Iw}}(K, V) \to H^1_{\mathrm{Iw}}(K, V(\eta^{-1})).$$

To define this map, first note that if $\mu \in \mathrm{Meas}(\Gamma_K, V)$ and $\sigma \in G_K$, then we have

$$\sigma(\eta\mu) = \eta(\sigma^{-1}) \cdot \eta\mu,$$

so multiplication by η is a Galois-equivariant map when viewed as a map

$$\mathrm{Meas}(\Gamma_K, V) \to \mathrm{Meas}(\Gamma_K, V(\eta^{-1})).$$

This then induces a map

$$H^1_{\mathrm{Iw}}(K, V) \to H^1_{\mathrm{Iw}}(K, V(\eta^{-1}))$$

on cohomology.

Now, for $x \in H^1_{\mathrm{Iw}}(K, V)$, we write x_η for the image of x in $H^1_{\mathrm{Iw}}(K, V(\eta^{-1}))$, and $x_{\eta,n}$ for its image in $H^1(K_n, V(\eta^{-1}))$. If $\eta = \chi^j$ we simply write $x_{n,j} \in H^1_{\mathrm{Iw}}(K_n, V(-j))$, where we identify $V(\chi^{-j})$ and $V(-j)$ using the basis ϵ.

4.3 The Perrin-Riou big logarithm (or regulator) map

Let p be an odd prime, and $F/\mathbb{Q}_p$ a finite unramified extension. Let V be a crystalline representation of G_F, with non-negative Hodge-Tate weights and no quotient isomorphic to the trivial representation. Furthermore, let $\mathcal{H}_{\mathbb{Q}_p}(\Gamma_F)$ be the algebra of $\mathbb{Q}_p$-valued distributions on Γ_F (dual to the space of locally analytic $\mathbb{Q}_p$-valued functions on Γ_F). Now under the above conditions, Perrin-Riou has constructed a "big logarithm" or "regulator" map

$$\mathcal{L}_{V,F} : H^1_{\mathrm{Iw}}(F, V) \to \mathcal{H}(\Gamma_F) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V).$$

This map interpolates the Bloch-Kato dual exponential and logarithm maps for twists of V in the cyclotomic tower[1] The importance of this map is that it provides a bridge between arithmetic and analysis. In particular, it allows us to construct p-adic L-functions from Euler systems.

Next we will explain how the regulator map interpolates the Bloch-Kato logarithm and dual exponential maps. We will assume for simplicity that $F = \mathbb{Q}_p$. Here we are following Appendix B of [LZ14].

If $\nu \in \mathcal{H}(\Gamma_K) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$, and $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ is a character, we shall write $\int_\Gamma \eta\nu$ for $\nu(\eta)$. Note that this is an element of $D_{\mathrm{cris}}(V)$.

Theorem 7.11. *Let V be a cristalline representation of $G_{\mathbb{Q}_p}$ with non-negative Hodge-Tate weights and no quotient isomorphic to $\mathbb{Q}_p$. Let η be a continuous character of Γ of the form $\chi^j\omega$, where ω is a finite-order character of conductor n. Let $x \in H^1_{Iw}(\mathbb{Q}_p, V)$. Then:*

- *If $j \geq 0$, we have*

$$\int_\Gamma \eta\mathcal{L}_V(x) =$$

$$j! \times \begin{cases} (1 - p^j\varphi)(1 - p^{-1-j}\varphi^{-1})^{-1} \left(\exp^*_{\mathbb{Q}_p, V(\eta^{-1})^*(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n = 0 \\ \tau(\omega)^{-1}p^{n(j+1)}\varphi^n \left(\exp^*_{\mathbb{Q}_p, V(\eta^{-1})^*(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n \geq 1 \end{cases}$$

- *If $j < 0$, we have*

$$\int_\Gamma \eta\mathcal{L}_V(x) =$$

$$\frac{(-1)^{-j-1}}{(-j-1)!} \times \begin{cases} (1 - p^j\varphi)(1 - p^{-1-j}\varphi^{-1})^{-1} \left(\log_{\mathbb{Q}_p, V(\eta^{-1})(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n = 0 \\ \tau(\omega)^{-1}p^{n(j+1)}\varphi^n \left(\log_{\mathbb{Q}_p, V(\eta^{-1})(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n \geq 1. \end{cases}$$

[1] This is a deep theorem/ conjecture

The second part of the theorem is due to Perrin-Riou; whereas the first part is a consequence of Perrin-Riou's explicit reciprocity conjecture $\mathrm{Rec}(V)$, proved independently by Benois and Colmez. The explicit formulae for the regulator can be viewed as a generalization of Coleman's formulae for the special values at integers of the Kubota-Leopoldt p-adic L-function in terms of polylogarithms of cyclotomic units.

It is from this formulae that one can deduce the nonvanishing of an Euler system from the nonvanishing of a p-adic L-function (when we know that this p-adic L-function arises from an Euler system).

5 A fundamental example

5.1 Coleman series

In this section, we describe a classical result of Coleman which was Perrin-Riou's inspiration for the construction of the regulator map. Let $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\epsilon^{(n)})$. Then we have:

Theorem 7.12 (Coleman)**.** *Let $u = (u_n)_{n\geq 0}$ be an element of $\varprojlim \mathcal{O}_{\mathbb{Q}_{p,n}}^\times$, where the inverse limit is taken with respect to the norm maps. Then there exists a unique series $\mathrm{Col}_u(T) \in \mathbb{Z}_p[[T]]^\times$ such that $\mathrm{Col}_u(\epsilon^{(n)} - 1) = u_n$ for every n.*

Moreover, if we set $G_u(T) = \log(\mathrm{Col}_u(T))$, then there exists a unique measure $\lambda_u \in \Lambda$ on Γ such that

$$\int_\Gamma (1+T)^{\chi(x)} \lambda_u(x) = G_u(T) - \frac{1}{p}\sum_{\zeta^p=1} G_u(\zeta(1+T)-1).$$

Note that $\varprojlim \mathcal{O}_{\mathbb{Q}_{p,n}}^\times$ is equipped with a structure of Λ-module and by Kummer theory, we have a natural map

$$\varprojlim \mathcal{O}_{\mathbb{Q}_{p,n}}^\times \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathbb{Q}_p(1)).$$

The map $u \mapsto \lambda_u$ is almost an isomorphism of Λ-modules (its kernel and cokernel are $\mathbb{Z}_p$-modules of rank 1). Now, Coleman's idea was that for a suitable choice of u, one could construct the Kubota-Leopoldt p-adic L-function.

The idea is to choose $\gamma \in \Gamma$, and define $u = (u_n)$ by

$$u_n = \frac{\gamma\epsilon^{(n)} - 1}{\epsilon^{(n)} - 1}.$$

Then one can check that the u_n's are norm-compabible, and moreover that if $k \in \mathbb{N} - \{0\}$, we have

$$\int_\Gamma \chi(x)^k \lambda_u(x) = (\chi(\gamma)^k - 1)(1 - p^{k-1})\zeta(1-k).$$

Now, if γ is chosen to have infinite order, then the Kubota-Leopoldt p-adic L-function is given by the (pseudo-) measure $(1-\gamma)^{-1}\lambda_u$.

6 Constructing the big logarithm: a sketch

We briefly describe the construction of Perrin-Riou's big logarithm. For this we will need a number of ingredients:

1. The theory of (φ, Γ)-modules.

2. Wach modules.

3. Fontaine's isomorphism.

6.1 (φ, Γ)-modules

Let $F/\mathbb{Q}_p$ be an unramified extension. We shall need the period rings A_F, A_F^+, B_F, B_F^+ and $B^+_{\mathrm{rig},F}$. Our choice of ϵ determines an element $\pi \in A_F^+$, and we have

$$A_F^+ = \mathcal{O}_F[[\pi]].$$

Set $A_F = \widehat{A_F[\pi^{-1}]}$, $B_F^+ = A_F^+[1/p]$ and $B_F = A_F[1/p]$. We define $B^+_{\mathrm{rig},F}$ as the ring of power series $f \in F[[\pi]]$ which converge on the open unit disc.

These rings are equipped with an $\mathcal{O}_F$-linear action of $\Gamma(= \Gamma_F)$, defined by

$$\gamma(\pi) = (\pi + 1)^{\chi(\gamma)} - 1,$$

and extended by linearity and continuity. Moreover, they are also equipped with a Frobenius φ, acting as the usual Frobenius on $\mathcal{O}_F$ and on π by $\varphi(\pi) = (\pi + 1)^p - 1$. We define a left inverse ψ for φ, characterized by the property that

$$(\varphi \circ \psi)(f(\pi)) = \frac{1}{p} \sum_{\zeta^p = 1} f(\zeta(1 + \pi) - 1).$$

Note that one can show that there is a natural identification of $(B^+_{\mathrm{rig},F})^{\psi=0}$ with $\mathcal{H}(\Gamma_F)$.

Definition 7.13. If V is a p-adic representation of G_F, we define

$$D_F(V) = (B \otimes_{\mathbb{Q}_p} V)^{H_F},$$

where B a certain period ring with $B^{H_F} = B_F$.

Note that $D_F(V)$ is a B_F-module equipped with commuting actions of Γ and φ, i.e. it is a (φ, Γ)**-module**. Furthermore, one can recover V from $D_F(V)$ by observing that

$$V = (B \otimes_{B_F} D_F(V))^{\varphi=1}.$$

6.2 The Wach module

Let V be a crystalline representation of G_F, and $T \subseteq V$ be a stable lattice. Then Wach and Berger have shown that there exists a unique A_F^+-submodule

$$N_F(T) \subseteq D_F(T)$$

such that:

1. $N_F(T)$ is A_F^+-free of rank $\dim V$;
2. Γ_F preserves $N_F(T)$ and acts trivially on $N_F(T)/\pi N_F(T)$;
3. There exists $b \in \mathbb{Z}$ such that $\varphi(\pi^b N_F(T)) \subseteq \pi^b N_F(T)$, and $\pi^b N_F(T)/\varphi^*(\pi^b N_F(T))$ is killed by a power of $\varphi(\pi)/\pi$.

We set $N_F(V) = B_F^+ \otimes_{A_F^+} N_F(T)$. This is independent of T and moreover, by a theorem of Berger, one can also recover $D_{\mathrm{cris}}(V)$ from $N_F(V)$ as

$$D_{\mathrm{cris}}(V) = (B^+_{\mathrm{rig},F} \otimes_{B_F^+} N_F(V))^{G_F}.$$

6.3 The Fontaine isomorphism

Fontaine has showed that we can recover the Iwasawa cohomology of V from its (φ, Γ)-module. More precisely, he has proved that there exists a canonical isomorphism of $\Lambda_{\mathbb{Z}_p}(\Gamma)$-modules

$$D_F(T)^{\psi=1} \xrightarrow{\sim} H^1_{\mathrm{Iw}}(F, T).$$

Moreover, if V has non-negative Hodge-Tate weights and no trivial quotient, then a result of Berger implies that

$$D_F(T)^{\psi=1} = N_F(T)^{\psi=1},$$

and therefore we have

$$N_F(T)^{\psi=1} \xrightarrow{\sim} H^1_{\mathrm{Iw}}(F,T).$$

If we now specialize to $F = \mathbb{Q}_p$, and let $x \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$, then by Fontaine's isomorphism, we can view

$$1 \otimes x \in \left(B^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{B^+_{\mathbb{Q}_p}} N(V)\right)^{\psi=1} \subseteq \left(B^+_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{B^+_{\mathbb{Q}_p}} D_{\mathrm{cris}}(V)\right)^{\psi=1}$$

Moreover, $\mathcal{L}_V(x)$ is the unique element of $\mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$ such that

$$\mathcal{L}_V(x)(1+\pi) = (1-\varphi)x.$$

For more details see [LZ14] for a beautiful account of the construction, and for the proof of the explicit formulae.

p-adic Eichler–Shimura Isomorphisms

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There are several works in the literature concerning p-adic Eichler–Shimura. In this talk we will follow Masami Ohta's original approach, which relies on the study of "good quotients" of p-divisible groups of modular Jacobians as introduced by Mazur and Wiles. Similar results are also obtained by Bryden Cais and, idependently, by Preston Wake.

Notations

- p is a prime ≥ 5, $(N,p)=1$, $N_r = Np^r$.
- $\Gamma_r = \Gamma_1(N_r)$, $X_r = X_1(N_r)$, $Y_r = Y_1(N_r)$; here $X_1(N_r))$ (resp. $Y_1(N_r))$ is the canonical model over $\mathbb{Q}$ of the complete (resp. open) modular curve associated with Γ_r.
- $\overline{X}_r = X_r \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ and $H^1(\overline{X}_r, \mathbb{Z}_p)$ always means the p-adic etale cohomology group (similarly for Y_r).
- $\mathrm{ES}_p(N)_{\mathbb{Z}_p} = \varprojlim_{r\geq 1} H^1(\overline{X}_r, \mathbb{Z}_p)$ (the p-adic Eichler–Shimura cohomology group of level N).
- $\mathrm{GES}_p(N)_{\mathbb{Z}_p} = \varprojlim_{r\geq 1} H^1(\overline{Y}_r, \mathbb{Z}_p)$ (the generalized p-adic Eichler–Shimura cohomology group of level N).
- J_r (resp. GJ_r) denotes the Jacobian variety of X_r (resp. the generalized Jacobian variety of X_r whose modulus is $C_r = X_r \setminus Y_r$).
- We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$; I_p denotes the inertia group of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let K be a complete subfield of $\mathbb{C}_p$, and let $\mathcal{O}$ be its ring of integers. Let $\Lambda_{\mathcal{O}}$ be the Iwasawa algebra. (We fix a topological generator u of the multiplicative group $1+p\mathbb{Z}_p$ and identify $\Lambda_{\mathcal{O}}$ with the formal power series ring $\mathcal{O}[[T]]$ via $u \mapsto 1+T$.) We assume also that $\mathcal{O}$ contains all roots of unity. Recall that the natural projection induces an isomorphism of $\Lambda_{\mathbb{Z}_p}$-modules (via $\Lambda_{\mathbb{Z}_p} \to e^*\mathcal{H}^*(N;\mathbb{Z}_p)$)

$$e^*\mathrm{GES}_p(N)_{\mathbb{Z}_p}/\omega_r \xrightarrow{\sim} e^*H^1(\overline{Y}_r, \mathbb{Z}_p), \qquad r \geq 1,$$

where $\omega_r = (1+T)^{p^{r-1}} - 1$ and e^* be Hida's idempotent attached to $T^*(p)$. This gives us the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U}^*_\infty/\omega_r & \longrightarrow & e^*\mathrm{GES}_p(N)_{\mathbb{Z}_p}/\omega_r & \longrightarrow & \tilde{B}^*_\infty/\omega_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\simeq & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U}^*_r & \longrightarrow & e^*H^1(\overline{Y}_r, \mathbb{Z}_p) & \longrightarrow & \tilde{B}^*_r & \longrightarrow & 0
\end{array}$$

where:

- $\mathcal{U}^*_r = e^*H^1(\overline{X}_r, \mathbb{Z}_p)^{I_p} = e^*H^1(\overline{Y}_r, \mathbb{Z}_p)^{I_p}$;
- $\tilde{B}^*_r = e^*H^1(\overline{Y}_r, \mathbb{Z}_p)/\mathcal{U}^*_r$, $B^*_r = e^*H^1(\overline{X}_r, \mathbb{Z}_p)/\mathcal{U}^*_r$;

- $\mathcal{U}_\infty^* = e^*\mathrm{GES}_p(N)_{\mathbb{Z}_p}^{I_p} = e^*\mathrm{ES}_p(N)_{\mathbb{Z}_p}^{I_p}$;
- $\tilde{B}_\infty^* = e^*\mathrm{GES}_p(N)_{\mathbb{Z}_p}/\mathcal{U}_\infty^*$, $B_\infty^* = e^x\mathrm{ES}_p(N)_{\mathbb{Z}_p}/\mathcal{U}_\infty^*$.

So the projection mapping induces a surjective homomorphism:

$$\tilde{B}_\infty^* \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} \longrightarrow \tilde{B}_r^* \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} = \tilde{B}_r^* \otimes_{\mathbb{Z}_p} \mathcal{O}, \quad r \geq 1.$$

Similarly for $e^*\mathrm{ES}_p(N)_{\mathbb{Z}_p}$, we get a surjective homomorphism

$$B_\infty^* \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} \longrightarrow B_r^* \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} = B_r^* \otimes_{\mathbb{Z}_p} \mathcal{O}, \quad r \geq 1.$$

Taking projective limits we get

$$B_\infty^* \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} \hookrightarrow \varprojlim_{r\geq 1}(B_r^* \otimes_{\mathbb{Z}_p} \mathcal{O}) \tag{8.1}$$

$$\tilde{B}_\infty^* \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} \hookrightarrow \varprojlim_{r\geq 1}(\tilde{B}_r^* \otimes_{\mathbb{Z}_p} \mathcal{O}) \tag{8.2}$$

(The injection follows from some commutative algebra.)

Construction of quotients of J_r

Let

$$X_r \xrightarrow{\pi_r} X_r^{r-1} \xrightarrow{p_r} X_{r-1}$$

$$Y_r \xrightarrow{\pi_r} Y_r^{r-1} \xrightarrow{p_r} Y_{r-1}$$

be the natural maps of modular curves, where X_r^{r-1} and Y_r^{r-1} are defined similarly as X_r and Y_r with respect to $\Gamma_{r-1} \cap \Gamma_0(p^r)$.

Define the quotient varieties as follows:

$$\alpha_r \colon J_r \longrightarrow \mathcal{A}_r$$

$$\tilde{\alpha}_r \colon GJ_r \longrightarrow \mathcal{Q}_r,$$

via:

- $\mathcal{A}_1 = J_1$, $\mathcal{Q}_1 = GJ_1$, $\alpha_1, \tilde{\alpha}_1$ the identities;
- $K_r = \ker(\tilde{\alpha}_{r-1} \circ p_{r*} \colon GJ_{Y_r^{r-1}} \longrightarrow \mathcal{Q}_{r-1})$;
- $K_r' = \ker(\alpha_{r-1} \circ p_{r*} \colon J_{Y_r^{r-1}} \longrightarrow \mathcal{A}_{r-1})$;
- $\mathcal{Q}_r = GJ_r/\pi_r^*(K_r)^0$ (the script 0 denotes connected component);
- $\mathcal{A}_r = J_r/\pi_r^*(K_r')^0$;
- α_r, $\tilde{\alpha}_r'$ the natural projections.

Denote ω_r the involution of J_r (and GJ_r, by abuse of notation) induced by the action of $\tau_r = \begin{pmatrix} 0 & -1 \\ N_r & 0 \end{pmatrix}$ on X_r.

Set

$$\mathcal{A}_r^* = J_r/\omega_r(\ker \alpha_r)$$

$$\mathcal{Q}_r^* = GJ_r/\omega_r(\ker \tilde{\alpha}_r)$$

$$\mathcal{N}_r^* = \ker(\mathcal{Q}_r \longrightarrow \mathcal{A}_r^) * .$$

The associated p-divisible group

$\mathcal{A}_r$ has semistable reduction. Denote by $\mathcal{A}_{r/\mathbb{Z}_p[\zeta_{p^r}]}$ the Neron model of $\mathcal{A}_r$ over $\mathbb{Z}_p[\zeta_{p^r}]$, and we use similar notation for other abelian varieties.

Denote by "(p)" the associated p-divisible group. In general, if G is a group scheme over a scheme S which is an extension of an abelian scheme by a torus, then the kernels ${}_{p^n}G$ of multiplication by p^n form a p-divisible group over S. We denote it by $G(p)$. Note that here we include factors of Jacobians which have bad(but multiplicative) reduction at p in our quotient varieties. Thus the system of kernels of multiplication by p^n on the Neron model of our quotient does not form a p-divisible group. However, its fixed part (in Groethendieck's sense) does.

Denote $\mathcal{A}^0_{r/\mathbb{Z}_p[\zeta_{p^r}]}(p)^f = \left(\left({}_{p^n}\mathcal{A}^0_{r/\mathbb{Z}_p[\zeta_{p^r}]}\right)^f\right)_{n\geq 0}$, where f means the fixed part in the sense of Groethendieck. Define the following p-divisible groups over $\mathbb{Z}_p[\zeta_{p^r}]$ by:

$$\mathcal{G}_r = e^*\mathcal{A}^{*0}_{r/\mathbb{Z}_p[\zeta_{p^r}]}(p)^f$$

$$\tilde{\mathcal{G}}_r = e^*\mathcal{Q}^{*0}_{r/\mathbb{Z}_p[\zeta_{p^r}]}(p)^f$$

We have the following canonical mappings:

$$B_r^* \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq \mathrm{Cot}(\mathcal{G}^0_{r/\mathcal{O}})(-1) \simeq e^*\,\mathrm{Cot}(\mathcal{A}^{*0}_{r/\mathcal{O}})(-1) \hookrightarrow e^*S_2^*(\Gamma_r;\mathcal{O})(-1), \tag{8.3}$$

$$\tilde{B}_r^* \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq \mathrm{Cot}(\tilde{\mathcal{G}}^0_{r/\mathcal{O}})(-1) \simeq e^*\,\mathrm{Cot}(\mathcal{Q}^{*0}_{r/\mathcal{O}})(-1) \hookrightarrow e^*M_2^*(\Gamma_r;\mathcal{O})(-1). \tag{8.4}$$

Here Cot denotes the cotangent space along the unit section, and $/\mathcal{O}$ denotes the base extension from $\mathbb{Z}_p[\zeta_{p^r}]$ to $\mathcal{O}$ for the p-divisible groups and the Neron models.

In (8.3) and (8.4) the first isomorphism comes from Tate's p-divisible group theory, the second by the definition of translation, and the third is due to Ohta's work.

Define the following notation:

$$M_k^*(\Gamma_r;\mathcal{O}) = \{f \in M_k(\Gamma_r;\mathbb{C}_p)\colon f|\tau_r \in M_k(\Gamma_r;\mathcal{O})\};$$

$$\mathfrak{M}_k^*(\Gamma_r;\mathcal{O}) = \varprojlim_{r\geq 1} M_k^*(\Gamma_r;\mathcal{O}).$$

Combining (8.1), (8.2), (8.3), and (8.4) and taking projective limit we get:

$$B^*_\infty \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} \hookrightarrow \varprojlim_{r\geq 1} B_r^* \otimes_{\mathbb{Z}_p} \mathcal{O} \hookrightarrow \varprojlim_{r\geq 1} e^*S_2^*(\Gamma_r;\mathcal{O})(-1);$$

$$\tilde{B}^*_\infty \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} \hookrightarrow \varprojlim_{r\geq 1} \tilde{B}_r^* \otimes_{\mathbb{Z}_p} \mathcal{O} \hookrightarrow \varprojlim_{r\geq 1} e^*M_2^*(\Gamma_r;\mathcal{O})(-1).$$

Theorem 8.1 (Ohta [Oht00])**.** *The above maps are isomorphisms.*

A remark on the structure of the proof: Note that by Nakayama's lemma it suffices to prove the case after modulo T, i.e in

$$\begin{array}{ccccc}
\tilde{B}^*_\infty \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda_{\mathcal{O}} & \longrightarrow & \varprojlim_{r\geq 1} \tilde{B}_r^* \otimes_{\mathbb{Z}_p} \mathcal{O} & \longrightarrow & \varprojlim_{r\geq 1} e^*M_2^*(\Gamma_r;\mathcal{O})(-1) \\
 & \searrow^{\text{By (8.1)}} & \downarrow{\scriptstyle \text{proj}} & & \simeq\downarrow{\scriptstyle \text{proj}} \\
 & & B_1^* \otimes_{\mathbb{Z}_p} \mathcal{O} & \longrightarrow & e^*M_2^*(\Gamma_1;\mathcal{O})(-1)
\end{array}$$

it suffices to show the surjectivity of the bottom arrow; this descent to the $r=1$ case which is the tricky part.

Comparison of Eichler–Shimura isomorphisms

To get the explicit reciprocity law we need a compatibility between Ohta's Λ-adic Eichler–Shimura isomorphism and the de Rham comparison isomorphism. The weight 2 case comes directly from Ohta's theorem.

In the following theorem we change a bit of the notations used so far. We denote by $\mathcal{F}^- H^1_{\mathrm{ord}}(Np^\infty)$ what Ohta denotes $\tilde{B}^*_\infty$.

Theorem 8.2 (Theorem 9.5.1 of [KLZ15a]). *For any $N \geq 1$ there is an isomorphism*

$$\mathrm{Oh}\colon e^*_{\mathrm{ord}}\mathfrak{M}^*_2(N,\mathbb{Z}_p) \longrightarrow \mathrm{D}(\mathcal{F}^- H^1_{\mathrm{ord}}(Np^\infty))$$

such that the following diagram commutes:

$$\begin{array}{ccc} e^*_{\mathrm{ord}}\mathfrak{M}^*_2(N,\mathbb{Z}_p) & \xrightarrow{\mathrm{Oh}} & \mathrm{D}(\mathcal{F}^- H^1_{\mathrm{ord}}(Np^\infty)) \\ {\scriptstyle \mathrm{pr}_r}\downarrow & & \downarrow \\ e^*_{\mathrm{ord}}M^*_2(Np^r,\mathbb{Z}_p) & \longrightarrow & \mathrm{D}(\mathcal{F}^- e'_{\mathrm{ord}}H^1(Y_1(Np^r)_{\overline{\mathbb{Q}}_p}),\mathbb{Z}_p(1)), \end{array}$$

where the bottom row is $\mathrm{pr}_{\mathcal{F}^-}\circ\mathrm{comp}_{\mathrm{dR}}$, $\mathrm{D}(M) = (M\hat{\otimes}_{\mathbb{Z}_p}\hat{\mathbb{Z}}_p^{ur})^{G_{\mathbb{Q}_p}}$, $\mathrm{pr}_{\mathcal{F}^-}$ *is the projection to* $\mathcal{F}^-$ *and* $\mathrm{com}_{\mathrm{dR}}$ *is the Faltings-Tsuji comparison isomorphism* $H^i(Y_{\mathbb{Q}_p}, \mathrm{TSym}^k\mathcal{H}_{\mathrm{dR}}) \simeq \mathrm{D}_{\mathrm{dR}}H^i(Y_{\bar{\mathbb{Q}}_p}, \mathrm{TSym}^k\mathcal{H}_{\mathbb{Q}_p})$

Now the goal is to extend this interpolating property of the map Oh to higher weights. Note that for any $k \geq 0$ we have an isomorphism

$$t_k\colon e^*_{\mathrm{ord}}\mathfrak{M}^*_2(N,\mathbb{Z}_p) \simeq e^*_{\mathrm{ord}}\mathfrak{M}_{k+2}(N,\mathbb{Z}_p)$$

(see Theorems 2.2.3 and 2.4.5 of Ohta's paper [Oht99]).

Theorem 8.3 (Theorem 9.5.2 of [KLZ15a]). *For every $k \geq 1$ and $r \geq 0$ the diagram*

$$\begin{array}{ccc} e^*_{\mathrm{ord}}\mathfrak{M}^*_2(N,\mathbb{Z}_p) & \xrightarrow[\simeq]{\mathrm{Oh}} & \mathrm{D}(\mathcal{F}^- H^1_{\mathrm{ord}}(Np^\infty)) \\ \downarrow{\scriptstyle \mathrm{pr}_r\circ t_k} & & \downarrow{\scriptstyle \mathrm{pr}_{k,r}} \\ e^*_{\mathrm{ord}}M^*_{k+2}(Np^r,\mathbb{Z}_p) & \longrightarrow & \mathrm{D}(V_{k,r}), \end{array}$$

commutes modulo the Eisenstein subspace of $M^*_{k+2}(Np^r,\mathbb{Z}_p)$, where

$$V_{k,r} = \mathcal{F}^- e'_{\mathrm{ord}}H^1(Y_1(Np^r)_{\overline{\mathbb{Q}}_p}, \mathrm{TSym}^k(\mathcal{H}_{\mathbb{Z}_p}(1)))$$

and $\mathrm{pr}_{k,r}$ is the natural map

$$H^1_{\mathrm{ord}}(Np^\infty) \longrightarrow V_{k,r}$$

given by the control theorem.

(Modified) syntomic and FP cohomology

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1 Introduction

Let K be a field and let X/K be a proper smooth variety. Then there exist *étale chern maps*

$$\mathrm{ch}_{\mathrm{et}} = {\mathrm{ch}_{\mathrm{et}}}^{i,j} : K_j(X) \to H^1\big(K, H^{2i-j-1}_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p(i))\big)$$

from the algebraic K-theory (resp. motivic cohomology) of the variety to Galois cohomology. We have seen how to construct Euler systems $(ES_n)_n$ for $K = \mathbb{Q}$ using this "étale chern method", or rather a version of it for affine schemes. But so far, we don't know that these Euler systems are not trivial.

For proving that $ES_n \neq 0$ it suffices to prove that $\mathrm{res}_p(ES_1) \neq 0$, where

$$\mathrm{res}_p : H^1(\mathbb{Q}, -) \to H^1(\mathbb{Q}_p, -)$$

is the restriction map in Galois cohomology. Since etale chern maps are compatible with base change it is enough to study étale chern maps for $K = \mathbb{Q}_p$.

From now on K will always denote a local field of characteristic 0 with valuation ring $\mathcal{O}_K$ and residue field $k \simeq \mathbb{F}_q$, $q = p^r$. Suppose that $\mathcal{X}$ is a smooth, proper scheme over $\mathcal{O}_K$, whose generic fiber $\mathcal{X}_K$ is isomorphic to X. One of the main aims of this note is to introduce *syntomic chern maps* $\mathrm{ch}_{\mathrm{syn}} = {\mathrm{ch}_{\mathrm{syn}}}^{i,j}$ such that the following diagram commutes:

$$\begin{array}{ccc}
H^{2i-j-1}_{\mathrm{dR}}(X)/F^i H^{2i-j-1}_{\mathrm{dR}}(X) & \xleftarrow{\quad \mathrm{ch}_{\mathrm{syn}} \quad} & K_j(\mathcal{X}) \\
{\scriptstyle \mathrm{Faltings}}\big\downarrow \simeq & & \big\downarrow {\scriptstyle \mathrm{ch}_{\mathrm{et}}} \\
D_{\mathrm{dR}}(V)/F^i D_{\mathrm{dR}}(V) & \xrightarrow{\quad \exp_{BK} \quad} & H^1\big(K, H^{2i-j-1}_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p(i))\big)
\end{array}$$

where $V = H^{2i-j-1}_{\mathrm{et}}\big(X_{\bar{K}}, \mathbb{Q}_p(i))\big)$ and $\exp_{BK}$ is the Bloch-Kato exponential map.

Remark 9.1. Let us assume that such maps $\mathrm{ch}_{\mathrm{syn}}$ existed. Then we remark that:

- we can basically forget about the étale regulators;
- for $c \in K_j(\mathcal{X})$, if $j \neq 1$ then $\mathrm{ch}_{\mathrm{syn}}(c) \neq 0$ implies that $\mathrm{ch}_{\mathrm{et}}(c) \neq 0$;
- we actually have that $\mathrm{Im}(\mathrm{ch}_{\mathrm{et}}) \subset H^1_f(K, -)$. This is true simply because we already know that $\mathrm{Im}(\exp_{\mathrm{BK}}) \subset H^1_f(K, -)$.

After Berthelot's construction of rigid cohomology (for varieties over characteristic p fields) given in [Ber96] and [Ber97], Gros introduced in [Gro94] the *rigid syntomic cohomology* for a scheme X smooth over an unramified base. He also constructed the syntomic regulators in the affine case. The goal here is to introduce Besser's modified version of syntomic cohomology as in [Bes00b]. We then see relations with de Rham and étale cohomology, compare it with the syntomic cohomology and underline its usefulness, expecially when it comes to perform computations.

2 Construction of (modified) syntomic cohomology

In order to define (modified) syntomic cohomology we need two ingredients:

(i) algebraic de Rham cohomology;

(ii) rigid cohomology (after Berthelot).

2.1 de Rham cohomology

Let X/K be a smooth variety. The de Rham complex of X/K is given by:

$$\Omega^{\cdot}_{X/K} : \mathcal{O}_X \to \Omega^1_{X/K} \to \Omega^2_{X/K} \to \dots$$

Definition 9.2. The de Rham cohomology of X over K is the hypercohomology of the de Rham complex, i.e. :

$$H^i_{\mathrm{dR}}(X) = \mathbb{H}^i(X, \Omega^{\cdot}_{X/K}).$$

Let us choose a smooth compactificaton Y of X, such that the complement $D = Y - X$ is a normal crossing divisor. We can then define the *logarithmic de Rham complex* $\Omega^{\cdot}_{Y/K}\langle \log D\rangle$. Locally, $\Omega^1_{Y/K}\langle \log D\rangle$ is generated by $\Omega^1_{Y/K}$ and $\mathrm{dlog}\, f_i = df_i/f_i$, where f_i are local equations for the divisor D.

Theorem 9.3 (Déligne)**.** *Let X, Y, D as above, then we have the following:*

$$H^i_{dR}(X) = \mathbb{H}^i(Y, \Omega^{\cdot}_{Y/K}\langle \log D\rangle)$$

In particular $\mathbb{H}^i(Y, \Omega^{\cdot}_{Y/K}\langle \log D\rangle)$ is a finite dimensional vector space.

Definition 9.4. We define $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y = \mathbb{R}\Gamma(Y, \Omega^{\geq n}_{Y}\langle \log D\rangle) \in D^b(\mathrm{Vect}_K)$

Thanks to a stronger version of Déligne's theorem above, this definition is well-defined up to quasi-isomorphisms, i.e. it is independent of the choice of D. We define $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ as the limit over all compactifications as above.

2.2 The rigid cohomology (after Berthelot)

Let X be a smooth variety over k. We take an open embedding $X \hookrightarrow \overline{X}$ of X in a proper compactification and a closed embedding $\overline{X} \to \mathcal{P}$ into the special fiber of a p-adic formal schemes, which is smooth around X. We have a specialisation (reduction) map

$$\mathrm{sp} : \mathcal{P}_K \to \overline{X}$$

and can define the inverse image

$$]X[= \mathrm{sp}^{-1}(X) \subset \mathcal{P}_K$$

of X under it.

Definition 9.5 (Strict neighborhood)**.** An open subscheme $V \subset \mathcal{P}_K$ is a *strict neighborhood* for $]X[$ if $]X[\subset V$ and the covering of $\mathcal{P}$ given by V and $P_K - X$ is admissible.

Take now $\mathcal{F}$ a sheaf on $\mathcal{P}_K$ and define:

$$j^+(\mathcal{F}) = \varinjlim_{U} j_{U*}(\mathcal{F})$$

where U runs through the *strict neighborhood* of $]X[$. Sections of this sheaf can be seen as sections of $\mathcal{F}$ on U, which slightly extend to bigger neighborhood. Hence, they are commonly named *overconvergent* sections.

Definition 9.6. We define $\mathbb{R}\Gamma_{\mathrm{rig}}(X, K)_{\overline{X},\mathcal{P}} = \mathbb{R}\Gamma(\mathcal{P}_K, j^+\Omega^{\cdot}_{\mathcal{P}_K})$.

By a theorem of Besser the definition does not depend on the choices made, up to quasi-isomorphism. So as before, we define $\mathbb{R}\Gamma_{\text{rig}}(X, K)$ as the limit over all possible choices.

Let us now start with a smooth scheme $\mathcal{X}/\mathcal{O}_K$ of finity type. We write $X = \mathcal{X}_K$ for its generic fiber and $\mathcal{X}_s = \mathcal{X}_k$ for its special fiber. Let us choose a compactification $\overline{X}$ of X as before. We write $\widehat{\overline{X}}$ for its completion along the special fiber. For every polynomial $P \in \mathbb{Q}[t]$ we have the following chain of morphims in the derived category:

$$F^n\mathbb{R}\Gamma_{\text{dR}}(X, K)_Y \to \mathbb{R}\Gamma_{\text{dR}}(Y, \Omega^{\cdot}_{Y/K}\langle \log D\rangle) \xrightarrow{(1)\text{ pull-back}} \mathbb{R}\Gamma(X, \Omega^{\cdot}_X) \to \tag{9.1}$$

$$\to \mathbb{R}\Gamma(X^{\text{an}}, \Omega^{\text{an},\cdot}_X) \xrightarrow{(2)} \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}, K) \to \tag{9.2}$$

$$\to \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s, K) \xrightarrow{(3) P(\psi^*)} \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s, K) \tag{9.3}$$

where:

(1) the pull-back of differential forms with logarithmic singularity gives holomorphic differentials;

(2) everywhere convergent forms are overconvergent;

(3) $\psi = \text{Fr}^r : \mathcal{X}_s \to \mathcal{X}_s$ is the r-th power Frobenius, especially ψ is k-linear.

Following the chain of morphisms we get a map:

$$P : F^n\mathbb{R}\Gamma_{\text{dR}}(X, K)_Y \to \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s, K)$$

which we still denote by P by abuse of notation.

Definition 9.7. Taking the limit over all compactifications we can define

$$\mathbb{R}_{f,P} = \text{the mapping cone}\big(F^n\mathbb{R}\Gamma_{\text{dR}}(X, K)_Y \to \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s, K)\big).$$

and its cohomology groups

$$H^i_{f,P}(\mathcal{X}, n) = H^i(\mathbb{R}_{f,P}(\mathcal{X}, n)).$$

This might look scary, but since it is defined as a mapping cone we can find a lot of useful short exact sequences. In particular we have, for each n:

$$0 \to \frac{H^{i-1}_{\text{rig}}(\mathcal{X}_s)}{P(\psi^*)F^nH^{i-1}_{\text{dR}}(X)} \to H^i_{f,P}(\mathcal{X}, n) \to F^nH^i_{\text{dR}}(X)^{P(\phi^*)=0} \to 0. \tag{9.4}$$

Moreover, if we are given $P \mid Q \in \mathbb{Q}[t]$, then we have the following commutative diagram:

$$\begin{array}{ccc} F^n\mathbb{R}\Gamma_{\text{dR}}(X/K)_Y & \xrightarrow{P(\psi^*)} & \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s, K) \\ \downarrow{=} & & \downarrow{Q/P} \\ F^n\mathbb{R}\Gamma_{\text{dR}}(X/K)_Y & \xrightarrow{Q(\psi^*)} & \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s, K) \end{array}$$

Thus, putting together the two informations we get maps of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{H^{i-1}_{\text{rig}}(\mathcal{X}_s)}{P(\psi^*)F^nH^{i-1}_{\text{dR}}(X)} & \longrightarrow & H^i_{f,P}(\mathcal{X}, n) & \longrightarrow & F^nH^i_{\text{dR}}(X)^{P(\phi^*)=0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{H^{i-1}_{\text{rig}}(\mathcal{X}_s)}{Q(\psi^*)F^nH^{i-1}_{\text{dR}}(X)} & \longrightarrow & H^i_{f,Q}(\mathcal{X}, n) & \longrightarrow & F^nH^i_{\text{dR}}(X)^{Q(\phi^*)=0} & \longrightarrow & 0 \end{array} \tag{9.5}$$

Definition 9.8 (Modified syntomic cohomology)**.** Let us define $P_{n,i}(t) = 1 - t^i/p^{ni}$, so in particular $P_{n,i} \mid P_{n,ji}$ for all j. The cohomology groups $H^i_{\text{ms}}(\mathcal{X}, n)$ of the complex

$$\mathbb{R}_{\text{ms}}(\mathcal{X}, n) = \varinjlim_i \mathbb{R}_{f,P_{n,i}}(\mathcal{X}, n).$$

are called the *modified syntomic cohomology groups* of $\mathcal{X}$.

Definition 9.9 (FP cohomology)**.** Let's define:

$$\mathbb{R}_{\text{fp}}(\mathcal{X}, n, m) = \varinjlim_{P} \mathbb{R}_{f,P}(\mathcal{X}, n).$$

where P runs over all monic polynomials in $\mathbb{Q}[t]$ such that all roots have complex absolute value equal to $q^{m/2}$. We define the *FP cohomology* of $\mathcal{X}$ to be the cohomology of the complex $\mathbb{R}_{\text{fp}}(\mathcal{X}, n, m)$, i.e. :

$$H^i_{\text{fp}}(\mathcal{X}, n) = H^i(\mathbb{R}_{\text{fp}}(\mathcal{X}, n)).$$

Remark 9.10. It exists also $H^i_{\text{syn}}(\mathcal{X}, n)$ defined using the Frobenius instead of the q-Frobenius and there exist functorial maps:

$$H^i_{\text{syn}}(\mathcal{X}, n) \to H^i_{\text{ms}}(\mathcal{X}, n)$$

One of the reason for which it is more convenient to use the modified syntomic cohomology instead of the syntomic cohomology is that computations are easier, in particular, one can use (9.4) and (9.5) in order to get results.

Proposition 9.11. *(i) Let $\mathcal{X}$ be proper over $\mathcal{O}_K$, $2n \neq i, i-1, i-2$, then:*

$$\begin{array}{ccc} H^i_{ms}(\mathcal{X}, n) & \xrightarrow{\simeq} & H^{i-1}_{dR}(X)/F^n H^{i-1}_{dR}(X) \\ \uparrow & \nearrow_{\simeq} & \\ H^i_{syn}(\mathcal{X}, n) & & \end{array}$$

(ii) If $\mathcal{X}/\mathcal{O}_K$ is proper, then we have the following short exact sequence:

$$0 \to \frac{H^{i-1}_{dR}(X)}{F^n H^{i-1}_{dR}(X)} \to H^i_{fp}(\mathcal{X}, n, i) \to F^n H^i_{dR}(X) \to 0.$$

Idea. We need two steps:

(1) $H^i_{\text{cris}}(\mathcal{X}_s, K) \cong H^i_{\text{dR}}(X, K)_Y \cong H^i_{\text{rig}}(\mathcal{X}_s, K)$

(2) Katz-Messing: the eigenvalues of ψ on cristalline cohomology have complex absolute values $q^{i/2}$

□

Theorem 9.12. *There exist chern maps $c^i_j : K_i(\mathcal{X}) \to H^{2j-i}(\mathcal{X}, j)$ such that:*

$$\begin{array}{ccc} K_i(\mathcal{X}) & \xrightarrow{c^i_j} & H^{2j-i}(\mathcal{X}, j) \\ \downarrow{\scriptstyle c_{dR}} & & \downarrow \\ F^j H^{2j-i}_{dR}(X) & \longleftarrow & H^{2j-i}_{ms}(\mathcal{X}, j) \end{array}$$

is commutative.

Idea. Compute the rigid and deRham cohomology of the classifying simplicial scheme $\mathbf{B.GL_n}$. □

Theorem 9.13. *If $\mathcal{X}$ is quasi-projective, then there exist functorial maps*

$$H^i_{syn}(\mathcal{X}, n) \to H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p(n))$$

compatible with the chern classes.

Theorem 9.14. *If X is projective, then:*

$$\begin{array}{ccc} H^{i-1}_{dR}(X)/F^n H^{i-1}_{dR}(X) & \longrightarrow & H^i_{syn}(\mathcal{X}, n) \\ & \overset{=0}{\searrow} & \downarrow \\ & & H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p(n)) \end{array}$$

is commutative.

Hence, by the Hochschild-Serre spectral sequence we get a map

$$H^{i-1}_{\mathrm{dR}}(X)/F^n H^{i-1}_{\mathrm{dR}}(X) \to H^1(K, H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p(n))).$$

This turns out to be exactly the Bloch-Kato exponential map by work of Niziol.

Example 9.15. If $\mathcal{X} = \operatorname{Spec} A$ is affine, we have the following explicit computation of modified syntomic cohomology:

$$H^i_{\mathrm{ms}}(\mathcal{X}, i) = \varinjlim \{(\omega, h) \mid \omega \in \Omega^i_{\mathrm{rig}}(\mathcal{X}), h \in \Omega^{i-1}_{A^+}/d\Omega^{i-2}_{A^+}, \text{s.t.} dh = (1 - \psi^i/q^{ni})\omega\},$$

where A^+ is some *ring of overconvergent functions.* So we can see these cohomology groups as the solutions to differential equations.

Let us now specialize to the case $\mathcal{X} = \mathbb{G}_m$ and the syntomic regulators of invertible functions:

$$\mathcal{O}_K[T, T^{-1}]^\times \hookrightarrow K_1(\mathbb{G}_m) \xrightarrow{c_1^1} H^1_{\mathrm{ms}}(\mathbb{G}_m, 1)$$

We know that $c_1^1(T) = \{(\omega, h)\}_n$, where $\omega = c^1_{1,\mathrm{dR}}(T)$ by Theorem 9.12. But the de Rham regulator of a function is simply the logarithmic derivative

$$c_1^1(T) = c^1_{1,\mathrm{dR}}(T) = \mathrm{dlog}(T) = dT/T.$$

Now we have to solve the following simple differential equation:

$$dh = \operatorname{dlog} T = \frac{1}{q} \operatorname{dlog} T^q = 0$$

So h clerly has to be constant. If we consider the inversion map $\tau : \mathbb{G}_m \to \mathbb{G}_m$, we get by functoriality:

$$c_1^1(T^{-1}) = \tau^*(\operatorname{dlog} T, h) = (-\operatorname{dlog} T, h).$$

Moreover, since $c_1^1(1) = 0$, we find:

$$0 = c_1^1(T \cdot T^{-1}) = (\operatorname{dlog} T, h) + (\operatorname{dlog} T, h) = (0, 2h)$$

which implies that $h = 0$. Indeed we found that $c_1^1(T) = (\operatorname{dlog} T, 0)$. In this case the syntomic regulator only describes the de Rham regulator.

Remark 9.16. We can use this techniques to get general formulas for the first chern class for invertible functions on affine schemes $X = \operatorname{Spec} A$ by studying the graph morphism $\Gamma_f : X \to X \times \mathbb{G}_m$

Evaluation of the regulators

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The goal of this talk is to show that the image of $\mathrm{Eis}^{[k,k',j]}_{\mathrm{et},1,N}$ under the étale Abel–Jacobi map $\mathrm{AJ}_{\mathrm{et},f,g}$ is a non-zero class in ${\mathrm{H}_{\mathrm{et}}}^1(\mathbb{Z}[\frac{1}{Np}], M(f\otimes g)^*(-j))$. We have seen that there is a diagram

$$\begin{array}{c} {\mathrm{H}_{\mathrm{et}}}^1(Y, (\mathrm{TSym}^{k+k'-2j}\,\mathcal{H})(1)) \\ \downarrow \Delta_* \circ \mathrm{CG}^{[k,k',j]} \\ {\mathrm{H}_{\mathrm{et}}}^3(Y\times Y, (\mathrm{TSym}^{[k,k']}\,\mathcal{H})(2-j)) \\ \downarrow \mathrm{AJ} \\ \mathrm{Ext}^1(\mathbb{Q}_p, M(f\otimes g)^*(-j)), \end{array}$$

where Δ is the diagonal embedding $Y \to Y\times Y$ and CG is the Clebsh–Gordon map introduced before.

Recall from Bruno's talk that for any filtered ϕ-module M satifying $M^{\phi=1} = 0$, there is an isomorphism

$$\mathrm{Ext}^1(\mathbb{Q}_p, M) \simeq M/F^0M.$$

Hence the cup product in de Rham cohomology gives a well defined pairing

$$\Big\langle , \Big\rangle : \mathrm{Ext}^1_{\mathrm{fil},\phi}(\mathbb{Q}_p, M(f\otimes g)^*(-j)) \times F^{1+j}M(f\otimes g) \to \mathbb{Q}_p.$$

The aim of the next two talks is to show that there exists $\lambda = \eta\otimes\omega$ in $F^{1+j}M(f\otimes g)$ such that the pairing

$$\Big\langle\, \mathrm{AJ}_{\mathrm{syn}}\,\mathrm{Eis}^{[k,k',j]}_{\mathrm{syn},b,N}, \eta\otimes\omega \Big\rangle$$

is non-zero. In more concrete terms, in this talk we will

1. Say what the class λ is;
2. Say what $\mathrm{Eis}^{[k,k',j]}_{\mathrm{syn},b,N}$ is, and
3. Say what the pairing of $\mathrm{AJ}_{\mathrm{syn}}\,\mathrm{Eis}^{[k,k',j]}_{\mathrm{syn},b,N}$ with λ is.

The proof that the pairing is nonzero is then roughly:

- Relate $\eta\otimes\omega$ coefficient of the p-adic realisation of the Rankin-Eisenstein class to a cup product in finite polynomial cohomology.
- Relate the cup product in finite polynomial cohomology to a cup product in de Rham cohomology.
- Relate the cup product in de Rham cohomology to a Petersson inner product.

The final stage is then to relate Petersson inner products to special values of p-adic L-functions, which we leave to the next talk. The method of proof is a p-adic analogue of [Bei86], which relates the extensions of mixed Hodge structure associated to $\mathrm{Eis}^{[0,0,0]}$ to special values of L-functions via cup products in absolute Hodge cohomology. This was generalised in unpublished work of Scholl to the case of higher weights. The natural analogue of absolute Hodge cohomology in a p-adic context is syntomic cohomology. However, the insight of Besser [Bes00a] is that to view periods of extensions of filtered ϕ-modules as cup products in a p-adic absolute cohomology, it is necessary to relax the definition of syntomic cohomology to allow for a more general "finite polynomial" cohomology. This was first used by Bertolini, Darmon and Rotger in [BDR15a]. The generalisation of this argument to the context above is in [KLZ15b].

1 Modular forms and de Rham cohomology

In this section we recall some facts about the algebraic theory of modular forms and the de Rham cohomology of modular curves [Kat73]. We then discuss the relation between cup products on modular curves and Petersson inner products. More details for the material in this section can be found in [DR14],[BDR15a],[KLZ15b]. Consider the universal elliptic curve $\pi\colon \mathcal{E} \to Y$, with the differential sheaf $\underline{\omega} = \pi_* \Omega^1_{\mathcal{E}/Y}$. We obtain a vector bundle $\mathcal{H}_{\mathrm{dR}} = (R^1\pi_*\Omega^\bullet_{\mathcal{E}/Y})^\vee$ endowed with a flat filtered connection on Y (the Gauss–Manin connection), which extends canonically to $\mathcal{H}$ and $\mathcal{H}^\vee$ on X. Moreover, we have the Hodge filtration exact sequence

$$0 \to \underline{\omega} \to \mathcal{H}^\vee \to \underline{\omega}^{-1} \to 0,$$

and we can pick a generator ω_{can} of $\underline{\omega}$. In this setting, we recall that a classical modular form of f of weight $k+2$ corresponds to a section

$$\omega_f = f(\tau)\frac{dq}{q}\omega_{\mathrm{can}}^k \in H^0(X, \underline{\omega}^k \otimes \Omega^1).$$

via the Kodaira-Spencer morphism

$$\Omega^1_X \xrightarrow{\simeq} \omega^2$$
$$\frac{dq}{q} \mapsto \omega_{\mathrm{can}}^2$$

On the other hand, recall that associated to an eigenform f defined over a number field L we also have a motive over L, which may be viewed as a direct summand of $H^1_{\mathrm{par}}(X, \mathrm{Sym}^k \mathcal{H})$. With enough formalism, this group may be viewed as $H^1_{\mathrm{dR}}(X, \iota_* \mathrm{Sym}^k \mathcal{H})$, where ι is the inclusion $Y \hookrightarrow X$. Somewhat more intuitively, it may be viewed as the subgroup of extensions which are trivial in a formal neighbourhood of each cusp. We will sporadically omit the par subscript, which by the above is equivalent to saying that we'll define $\mathcal{H}$ on X to be $\iota_*\mathcal{H}$. Anyway, consider the short exact sequence

$$0 \to H^0(X_L, \underline{\omega}^k \otimes \Omega^1) \to H^1_{\mathrm{par}}(X_L, \mathrm{Sym}^k \mathcal{H}^\vee) \to H^1(X_L, \underline{\omega}^{-k} \to 0.$$

Then f defines a direct summand $M_{\mathrm{dR}}(f)$ of $\mathrm{H_{dR}}^1_{\mathrm{par}}(X_L, \mathrm{Sym}^k \mathcal{H}^\vee)$ and projecting onto the f-isotypical component the sequence above becomes

$$0 \to F^1 \mathrm{M_{dR}}(f) \to \mathrm{M_{dR}}(f) \to \mathrm{M_{dR}}(f)/F^1 \to 0.$$

We can decompose

$$\mathrm{M_{dR}}(f \otimes g) = \mathrm{M_{dR}}(f) \otimes \mathrm{M_{dR}}(g) \ni \omega_f' \otimes \omega_g'.$$

Over $\mathbb{C}$, we can split the exact (using the Hodge splitting) to get a basis for the extension of scalars $\mathrm{M_{dR}}(f)_{\mathbb{C}}$:

$$\{\omega_f, \bar{\omega}_f\}.$$

In weight 2, we have $\bar{\omega}_f = f(-\bar{\tau})d\bar{\tau}$. For general weights,

$$\mathrm{H_{dR}}^1_{\mathrm{an,par}}(X^{\mathrm{an}}, \mathrm{Sym}^k \mathcal{H}^\vee) \ni \bar{\omega}_f = (-1)^k f(-\tau)\bar{\omega}_{\mathrm{can}} d\bar{\tau}.$$

Serre duality gives a pairing

$$\langle \cdot, \cdot \rangle \colon H^0(X_{\mathbb{C}}, \underline{\omega}^k \otimes \Omega^1) \times H^1(X_{\mathbb{C}}, \underline{\omega}^{-k}) \to \mathbb{C}$$

and therefore given cusp forms f and g we may consider $\langle \omega_f, \bar{\omega}_g \rangle$. On the other hand, the Petersson inner product gives another pairing

$$(f, g) = \int_X f(\tau)\overline{g(\tau)} y^k dx dy,$$

and one might wonder if these are related. The next proposition clarifies this (see [Shi94]).

Proposition 10.1. *1. Write η_g for the image of $\bar{\omega}_g$ in $H^1(X_{\mathbb{C}}, \underline{\omega}^{-k})$. Then*

$$\langle \omega_f, \eta_g \rangle = \frac{-i^k k!}{2^{k+2}\pi}(f, g).$$

2. Define η'_f to be the image in $H^1(X_{\mathbb{C}}, \underline{\omega}^{-k})$ of

$$\frac{G(\epsilon_f^{-1})}{\langle \omega_f, \bar{\omega}_f \rangle},$$

where

$$G(\epsilon_f^{-1}) =$$

is a certain Gauss sum that we don't specify here. Let $L \subset \mathbb{C}$ be the field of definition of f. Then η'_f is algebraic, in the sense that it comes from a class in $H^1(X_L, \underline{\omega}^{-k})$.

2 Modular forms and rigid cohomology

In this section we will need to superficially re-cap on the theory of overconvergent isocrystals (see [Ber96]) and the relation to the the theory of p-adic modular forms (see [Kat73]). For a subscheme Z of $X_{\mathbb{F}_p}$, we define $]Z[$ to be the affinoid of points in X^{an} reducing to Z.

Consider the open and closed modular curves Y and X, thought of as defined over $\mathbb{Q}_p$. They have Néron models $\mathcal{Y}$ and $\mathcal{X}$, over $\mathbb{Z}_p$. Let Y^{an} and X^{an} be their associated rigid analytic spaces, in the sense of Raynaud, and define the ordinary locus $X_{\mathrm{ord},\mathbb{F}_p} \subset X_{\mathbb{F}_p}$ as the affine obtained by removing the supersingular points. Define $Y_{\mathrm{ord},\mathbb{F}_p}$ in an analogous way. We denote by $SS \subset X_{\mathbb{F}_p}$ the zero dimensional subscheme of supersingular points, and by $]SS[$ the corresponding rigid analytic spaces.

We obtain rigid analytic spaces $\mathcal{Y}_{\mathrm{ord}} =]Y_{\mathrm{ord}}[$ and $\mathcal{X}_{\mathrm{ord}} =]X_{\mathrm{ord}}[$, which are the preimages under the reduction map of the ordinary loci. Let $j^\dagger$ denote the functor which sends a sheaf $\mathcal{F}$ on some strict open neighbourhood of X_{ord} to

$$\varinjlim \iota_{U*}\iota_U^*\mathcal{F}$$

where the limit is over all strict neighbourhoods $U \overset{\iota}{\hookrightarrow} X$ of X_{ord} in X.

Abusing notation (well I suppose abuse of notation is a relative term) we shall denote the overconvergent isocrystal corresponding to $\mathcal{H}$ by $\mathcal{H}$. Just as in the algebraic setting, the analytic bundle $\mathcal{H}$ is an extension of ω^{-1} by ω.

Definition 10.2. We refer to elements of $H^0(X_{\mathrm{ord}}, j^\dagger\omega^k)$ as overconvergent modular forms of weight k.

Using Kodaira-Spencer we can think of overconvergent modular forms of weight k as element of $H^1_{\mathrm{rig}}(X, j^\dagger\omega^{k-2} \otimes \Omega^1)$. As may be unclear from the above, in contrast to the parabolic conditions at the cusps, for an overconvergent modular form the only condition we impose at supersingular residue discs is the overconvergence condition, so an overconvergent modular forms is allowed to have arbitrarily bad poles at supersingular point (or even to not be defined).

There is an isomorphism

$$H^1_{\mathrm{dR}}(X, \mathrm{Sym}^k \mathcal{H}) \simeq H^1_{\mathrm{rig}}(X_{\mathbb{F}_p}, \mathrm{Sym}^k \mathcal{H})$$

and an exact sequence

$$0 \to H^1_{\mathrm{rig}}(X_{\mathbb{F}_p}, \mathrm{Sym}^k \mathcal{H}) \to H^1_{\mathrm{rig}}(X_{\mathrm{ord}}, \mathrm{Sym}^k \mathcal{H})$$

$$\to H^2_{SS}(X_{\mathbb{F}_p}, \mathrm{Sym}^k \mathcal{H}) \to H^2_{\mathrm{rig}}(X_{\mathbb{F}_p}, \mathrm{Sym}^k \mathcal{H}) \to 0.$$

The Frobenius endomorphism can be lifted to a map $\phi\colon \mathcal{Y}_{\mathrm{ord}} \to \mathcal{Y}_{\mathrm{ord}}$, via the theory of the canonical subgroup (see below)

Theorem 10.3 (Katz)**.** *The lift ϕ overconverges. That is, it extends to strict neighborhoods of $\mathcal{Y}_{ord}$ and $\mathcal{X}_{ord}$.*

Definition 10.4. An F-isocrystal on Y_{ord} is an overconvergent isocrystal $\mathcal{F}$ together with an isomorphism
$$\Phi : \phi^*\mathcal{F} \xrightarrow{\simeq} \mathcal{F}$$
of overconvergent isocrystals.

For any overconvergent isocrystal $(\mathcal{F}, \Phi)$ we have an action of Frobenius on $H^i(Y_{\text{ord}}, \mathcal{F})$ via

$$H^i(X_{\text{ord}}, \mathcal{F}) \xrightarrow{\phi^*} H^i(X_{\text{ord}}, \phi^*\mathcal{F}) \xrightarrow{\Phi} H^i(X, \mathcal{F}).$$

Therefore we obtain vector spaces $M_{\text{rig}}(f)$ and $M_{\text{rig}}(g)$ endowed with an action of ϕ. The characteristic polynomial of ϕ on $M_{\text{rig}}(f)$ is $X^2 - a_p(f)X + p^{k+1}\epsilon_f(p)$ (and similarly for g). Let α and β be the roots of the polynomial for f. Since $v_p(\alpha) + v_p(\beta) = k+1$, we can order them so that $v_p(\alpha) < k+1$. This allows us to define $\eta_f^* \in \mathrm{M}_{\text{dR}}(f)$ by using the isomorphism $\mathrm{M}_{\text{dR}}(f) \cong M_{\text{rig}}(f)$, and lifting $\eta'_f \in \mathrm{M}_{\text{dR}}(f)/F^1$ to the $\phi = \alpha$-eigenspace in $\mathrm{M}_{\text{dR}}(f)$.

2.1 Canonical subgroups and the cohomology of X_{ord}

In this section we explore the relationship between $H^1(X_{\text{ord}}, \mathrm{Sym}^k \mathcal{H})$ and $H^1(X(N(p)), \mathrm{Sym}^k \mathcal{H})$, following [Col94], where $X(N(p))$ denotes the modular curve with level structure $\Gamma_1(N) \cap \Gamma_0(p)$. Recall that $X(N(p))$ has a semistable model $\mathcal{X}(N(p))$ over $\mathbb{Z}_p$, with special fibre isomorphic to two copies of $X(N)_{\mathbb{F}_p}$ glued together along the supersingular locus [KM85]. Let X_0 and X_∞ denote the two irreducible components, and let $\widetilde{X}_1$ and $\widetilde{X}_2$ denote the open affines obtained by removing their points of intersection. Hence
$$X_0 \simeq X_\infty \simeq X_{\mathbb{F}_p}$$
and
$$\widetilde{X}_0 \simeq \widetilde{X}_\infty \simeq X_{\text{ord},\mathbb{F}_p}.$$
The construction of the overconvergent lift of Frobenius depends on the fact that there is a strict neighbourhood W_0 of $]X_0[$ in $X(N(p))$ which isomorphic as a rigid analytic space to a strict neighbourhood of X_{ord} in X. Similarly there is a strict neighbourhood W_∞ of $]X_\infty[$ in $X(N(p))$ isomorphic to a strict neighbourhood of X_{ord} in X. Let

$$h : X(N(p))_{\text{ord}} \to X_{\text{ord}}$$

be the cover obtained from forgetting level structure, and let

$$s : X_{\text{ord}} \to X(N(p))_{\text{ord}}$$

be Katz's section. Then the Frobeniuss lift is defined by

$$E \overset{s}{\mapsto} (E, C) \mapsto (E/C, E[p]/C) \overset{h}{\mapsto}$$

For each supersingular point $z \in X_{\mathbb{F}_p}$, let A_z be the annulus $]z[\cap W_0 \cap W_\infty$.
Then by the Mayer Vietoris exact sequence there is an exact sequence

$$0 \to H^0(X(N(p)), \mathrm{Sym}^k \mathcal{H}) \to H^0(X_{\text{ord}}, \mathrm{Sym}^k \mathcal{H})^{\oplus 2} \to \oplus_z H^0(A_z, \mathrm{Sym}^k \mathcal{H})$$

$$\to H^1(X(N(p)), \mathrm{Sym}^k \mathcal{H}) \to H^1(X_{\text{ord}}, \mathrm{Sym}^k \mathcal{H})^{\oplus 2}) \to \oplus_z H^1(A_z, \mathrm{Sym}^k \mathcal{H}) \to H^2(X(N(p)), \mathrm{Sym}^k \mathcal{H}) \to 0$$

(see [Col94]). In this way we may realise $H^1(X_{\text{ord}}, \mathrm{Sym}^k \mathcal{H})$ as a quotient of $H^1(X(N(p)), \mathrm{Sym}^k \mathcal{H})$.

Another somewhat more general perspective on the relation between the cohomology of $H^1(X(N(p)), \mathrm{Sym}^k \mathcal{H})$ and $H^1(X_{\text{ord}}, \mathrm{Sym}^k \mathcal{H})$ is to work with the Mokrane-Rapoport-Zink weight spectral sequence for the curve $\mathcal{X}(N(p))$.

Note this tells us that the extension of filtered ϕ-modules

$$0 \to H^0(SS, \mathrm{Sym}^k \mathcal{H})/\mathbb{Q}_p \to H^1_{\text{rig},c}(X_{\text{ord}}, \mathrm{Sym}^k \mathcal{H}) \to H^1_{\text{rig}}(X, \mathrm{Sym}^k \mathcal{H} \to 0$$

has a canonical splitting, coming from the canonical splitting of $H^1_{\text{dR}}(X(N(p)), \mathrm{Sym}^k \mathcal{H})$ into its old and new parts.

Lemma 10.5. *Let ω and η be differential forms of the second kind on X and $X(N(p))$ respectively. Let ξ denote the unique ϕ-equivariant lift of ξ to $H^1_{rig,c}(X_{ord},\mathbb{Q}_p)$. Then*

$$\langle h^*\omega,\eta\rangle_{X(N(p))}=\langle\omega,s^*\eta\rangle_X$$

In this way we can relate cup products of cohomology classes in $H^1_{\mathrm{rig}}(X_{\mathrm{ord}},\mathrm{Sym}^k\mathcal{H})$ and $H^1_{\mathrm{rig},c}(X_{\mathrm{ord}},\mathrm{Sym}^k\mathcal{H})$ to cup products of cohomology classes in $H^1_{\mathrm{dR}}(X(N(p)),\mathrm{Sym}^k\mathcal{H})$, and hence to Petersson inner products of modular forms on $X(N(p))$.

3 Nearly overconvergent modular forms

Suppose we have a cohomology class in $H^1(X,\mathrm{Sym}^k\mathcal{H})$. We want away of describing it (and specifically the value of the cup product of it with η_f^{α}) which is amenable to relating to p-adic L-functions.

Recall that we have an isomorphism

$$H^1(X_{\mathrm{ord}},j^\dagger\,\mathrm{Sym}^k\mathcal{H})\simeq H^0(X_{\mathrm{ord}},j^\dagger\,\mathrm{Sym}^k\mathcal{H}\otimes\Omega^1)/\nabla H^0(X_{\mathrm{ord}},j^\dagger\,\mathrm{Sym}^k\mathcal{H}).$$

Recall we also have a basis of sections of $\mathcal{H}$, and hence of $\mathrm{Sym}^k\mathcal{H}$. We write an element of $H^0(X_{\mathrm{ord}},\mathrm{Sym}^k\mathcal{H})$ as $\sum G_i v^{[k-i,i]}$.

Recall

$$\nabla(\sum G_i v^{[i,k-i]})=\sum(\theta(G_i)+(i-1)G_{i-1})v^{[i,k-i]}\otimes\frac{dq}{q}.$$

The unit root splitting induces an isomorphism

$$H^0(X_{\mathrm{ord}},j^\dagger(\mathrm{Sym}^k\mathcal{H})\otimes\Omega^1)\simeq S_k^{n-oc}(N,K).$$

Conversely, given a nearly overconvergent modular form f, we may wonder how to represent its class in cohomology.

To resolve this we need the following theorem of Coleman [Col96].

Theorem 10.6 (Coleman)**.** *The map*

$$H^0(X_{ord},j^\dagger(\mathrm{TSym}^k\mathcal{H}\otimes\Omega^1))\to H^1(X_{ord},\mathrm{Sym}^k\mathcal{H})$$

induces an isomorphism

$$H^1(X_{ord},\mathrm{Sym}^k\mathcal{H})\simeq S^\dagger_{k+2}(N)/\theta^{k+1}S^\dagger_{-k}(N)$$

Recall

$$\theta=q\frac{d}{dq}.$$

Recall the exact sequence

$$0\to H^0_{\mathrm{dR}}(X,\omega^k\otimes\Omega^1)\to H^1_{\mathrm{dR}}(X,\mathrm{Sym}^k\mathcal{H})\to H^1(X,\omega^{-k})\to 0$$

Recall

$$H^0(X,\mathrm{Sym}^k\mathcal{H}\otimes\Omega^1)\simeq H^0(X,\omega^k\otimes\Omega^1).$$

Coleman proves

$$\nabla H^0(X_{\mathrm{ord}},\mathrm{Sym}^k\mathcal{H}\otimes\Omega^1)\cap H^0(X_{\mathrm{ord}},\omega^k\otimes\Omega^1)\simeq\theta^{k+1}S_{-k}(N;K).$$

Definition 10.7. Define

$$\Pi^{o-c}:S^{n-oc}_{k+2}(N;K)\to S^\dagger_{k+2}(N;K)/\theta^{k+1}S_{-k}(N;K)$$

to be the map sending F to the class of $(\mathrm{spl}^{u-r})^{-1}(F)$ in $S^\dagger_{k+2}(N;K)/\theta^{k+1}S_{-k}(N;K)$ under the isomorphism

$$H^1(X_{\mathrm{ord}},\mathrm{Sym}^k\mathcal{H})\simeq S^\dagger_{k+2}(N;K)/\theta^{k+1}S^\dagger_{-k}(N;K).$$

$$\Phi : H^0(X_{\mathrm{ord}}, j^\dagger\Omega^1) \to H^0(X_{\mathrm{ord}}, j^\dagger\Omega^1)$$

$$K[\![q]\!] \to K[\![q]\!]$$
$$U : \sum a_n q^n \to \sum a_{np} q^n$$
$$V : \sum a_n q^n \to \sum a_n q^{np}$$

These operators give a description of the action of Frobenius on $M_{\mathrm{rig}}(f)$ in terms of operators on q-expansions:

Theorem 10.8. *For any f in $H^0(X_{ord}, j^\dagger\omega^k \otimes \Omega^1)$, let $[\omega_f]$ denote the corresponding class in $H^1(X_{ord}, j^\dagger \operatorname{Sym}^k \mathcal{H})$. Then*

$$\Phi[\omega_f] = p^{k+1}[V(\omega_f)]$$

and

$$\Phi^{-1}[\omega_f] = p^{-k-1}[U(\omega_f)].$$

For $f = \sum a_n q^n$, define the *p-depletion* of f to be

$$f^{[p]} = (1 - VU)f = \sum_{p|n} a_n q^n.$$

By the above theorem we obtain that for f as above therem is F in $H^0(X_{\mathrm{ord}}, \omega^k)$ such that

$$\omega_{f^{[p]}} = dF.$$

**** $S^\dagger_{k+2}(\Gamma_1(N), K) = H^0(\mathcal{X}_{\mathrm{ord}}, j^\dagger\Omega^1 \otimes \omega^k)$. Note that these sections are not allowed to have singularities at the cusps, but that no conditions are imposed at supersingular annuli.

$$H^1_{\mathrm{rig}}(\mathcal{X}_{\mathrm{ord}}, j^\dagger \operatorname{Sym}^k \mathcal{H}_{\mathrm{rig}}) = H^0(\mathcal{X}_{\mathrm{ord}}, j^\dagger(\operatorname{Sym}^k \mathcal{H}^\vee) \otimes \Omega^1)_{\mathrm{par}} / \nabla H^0(\mathcal{X}_{\mathrm{ord}}, j^\dagger(\operatorname{Sym}^k \mathcal{H}^\vee))$$

4 Syntomic Eisenstein class

Recall the de Rham Eisenstein class $\mathrm{Eis}^k_{\mathrm{dR},b,N}$ in $H^1_{\mathrm{dR}}(Y, \mathrm{TSym}^k \mathcal{H}(1))$ is the extension class which pulls back to

$$-N^k F_{k+2,b} v^{[0,k]} \otimes \frac{dq}{q}$$

where $F_{k+2,b}$ is the Eisenstein series

$$F_{k+2,b} = \zeta(-1-k) + \sum_{n>0} q^n \sum_{0<d|n} (\frac{n}{d})^{k+1} (\zeta_N^{bd} + (-1)^k \zeta_N^{-bd}).$$

We want to define a class $\mathrm{Eis}^k_{\mathrm{syn},b,N} \in H^1_{\mathrm{syn}}(\mathcal{Y}, \operatorname{Sym}^k \mathcal{H}(1))$. We can think of H^1_{syn} as classifying extensions of filtered F-isocrystals:

Definition 10.9. Given a connection $(\mathcal{V}, \nabla)$, a filtered F-isocrystal lifting $(\mathcal{F}, \nabla)$ is (morally) the additional data of an isomorphism of analytic connections

$$\Phi \colon \phi^*(\mathcal{V}^{\mathrm{an}}, \nabla^{\mathrm{an}}) \xrightarrow{\cong} (\mathcal{V}^{\mathrm{an}}, \nabla^{\mathrm{an}}), \quad \text{on } \mathcal{Y}_{\mathrm{ord}},$$

which overconverges.

More precisely, an overconvergent filtered F-isocrystal is a tuple $(M, \mathcal{M}, F, \alpha)$ where

- M is a vector bundle with connection ∇ on Y.

- $\mathcal{M}$ is an overconvergent isocrystal on $Y_{\mathbb{F}_p}$ equipped with an isomorphism with the analytification of M.
- A filtration $F^\bullet M$ satisfying the Griffiths transversality condition
$$\nabla(F^i M) \subset (F^{i-1}M) \otimes \Omega^1.$$
- An F-structure
$$\alpha : \phi^* \mathcal{M} \xrightarrow{\simeq} \mathcal{M}$$
on $\mathcal{M}$.

4.1 Lifting the de Rham class to a syntomic class

Recall that $\mathrm{TSym}^k \mathcal{H}$ has a canonical lift to a filtered F-isocrystal on $(\mathcal{Y}_{\mathrm{ord}}, \mathcal{X})$. Recall that lifting $\mathrm{Eis}^k_{\mathrm{dR},b,N}$ to an element of $H^1_{\mathrm{syn}}(\mathcal{Y}_{\mathrm{ord}}, \mathrm{TSym}^k \mathcal{H}_{\mathbb{Q}_p}(1))$ is the same as lifting an extension of connections to an extensions of filtered F-isocrystals, which is the same as finding α_{rig} such that
$$\nabla(\alpha_{\mathrm{rig}}) = (1-\varphi)\alpha_{\mathrm{dR}}.$$
To write this down we need to describe how ϕ acts on $\mathrm{TSym}^k \mathcal{H}_{\mathbb{Q}_p}(1)$. So if $\alpha_{\mathrm{rig}} = \sum H_j v^{[j,k]}$ then
$$dF_0 = -N^k(1-\phi)F_{k+2,b}\frac{dq}{q}$$
and for $j > 0$
$$dF_j = -jF_{j-1}\frac{dq}{q}.$$
Write
$$H_j = -N^k(-1)^{k-j}(k-j)!F^{(p)}_{2j-k,k+1-k,b}$$

In the case of the sheaf $\mathcal{H}^\vee$, this can be described more explicitly. By pulling back to the Tate curve, we compute
$$\nabla\omega_{\mathrm{can}} = \eta_{\mathrm{can}}\frac{dq}{q}.$$
This defines a basis of sections of $\mathcal{H}$ near infinity, such that $\nabla\eta_{\mathrm{can}} = 0$. The F-structure on $\mathcal{H}^\vee$ amounts to a map Φ, determined by
$$\Phi(\omega_{\mathrm{can}}) = p\omega_{\mathrm{can}}, \quad \Phi(\eta_{\mathrm{can}}) = \eta_{\mathrm{can}}.$$
Consider now the previously defined class $\mathrm{Eis}^k_{\mathrm{dR},b,N} \in {\mathrm{H}_{\mathrm{dR}}}^1(Y, \mathrm{TSym}^k \mathcal{H}(1))$. On a cusp, this is given by
$$-N^k F_{k+2,b}(\omega^\vee_{\mathrm{can}})^k \otimes \frac{dq}{q},$$
where
$$F_{k+2,b} = \zeta(-1-k) + \sum_{n>0} q^n \sum_{0<d|n} (\frac{n}{d})^{k+1}(\zeta^{bd} + (-1)^k \zeta_N^{-bd}) \tag{10.1}$$
See [KLZ15b], section 4.3, for the relation to the Eisenstein series considered by Katz and Kato. Recall that when $k = k' = 0$, we had $F_{2,b} = \mathrm{dlog}\, g_{0,b/N})$. Denote the corresponding extension class by $[\mathcal{V}]$. Then we are looking for an isomorphism $\phi^*\mathcal{V} \cong \mathcal{V}$, such that
$$(1-\phi)\alpha_{\mathrm{dR}} = \nabla\alpha_{\mathrm{rig}} = \sum H^j(\omega^\vee_{\mathrm{can}})^{k-1-j}(\eta^\vee_{\mathrm{can}})^j.$$
This in turn amounts to solving a system of differential equations:
$$dH_0 = (1-\phi)(\cdots)F_{k+2,b}$$
$$dH_j = H_{j-1}\frac{dq}{q}$$
$$\theta H_j = H_{j-1}.$$

In the case $k=0$, the class of $\mathrm{Eis}^0_{\mathrm{dR},b,N}$ in $H^1_{\mathrm{dR}}(Y,1)$ is given by $d\log(g_{0,b/N})$.

Let

$$v^{[r,s]} = (\omega_{\mathrm{can}}^\vee)^{[r]}(\eta_{\mathrm{can}}^\vee)^{[s]}$$

Then

$$\nabla v^{[r,s]} = (r+1)v^{[r+1,s-1]}\frac{dq}{q}$$

and

$$\phi v^{[r,s]} = p^{-r}v^{[r,s]}.$$

$$(1-\phi)(\alpha_{\mathrm{dR}}) =: -N^k F^{(p)}_{k+2,0,b}v^{[0,k]}$$

so

$$\nabla(Fv^{[r,s]}) = v^{[r,s]}dF + Fv^{[r,s]}\frac{dq}{q}$$

where

$$F^{(p)}_{t,s,b} = \sum_{n>0}\sum_{0<d|n,(p,d)=1}(\frac{n}{d})^{t+s-1}d^{-s}(\zeta_N^{bd}+(-1)^t\zeta_N^{-bd})$$

(recall that the q expansions have coefficients in $\mathbb{Z}_p[\zeta_N]$ rather than $\mathbb{Z}_p$, because of choice of a model of X). It follows from the definition of $F^{(p)}$ that

Lemma 10.10.

$$U(F^{(p)}_{t,s,b}) = p^{t+s-1}F^{(p)}_{t,s,b}$$

Proof. The q^n coefficient of $U(F^{(p)}_{t,s,b})$ is

$$\sum_{0<d|n,(p,d)=1}(\frac{pn}{d})^{t+s-1}d^{-s}(\zeta_N^{bd}+(-1)^t\zeta_N^{-bd})$$

□

We define $H^1_{\mathrm{syn}}(X,\mathcal{F})$ to be the group of extensions of the trivial filtered F-isocrystal by the filtered F-isocrystal.

It follows from the definition of a filtered F-isocrystal, (together with the interpretation of H^1_{dR} and H^1_{syn} in terms of extension classes of flat connections and overconvergent isocrystals respectively) that given overcovergent F-isocrystals $\mathcal{F}$ and $\mathcal{G}$, and an extension of the underlying connections $[E]\in H^1_{\mathrm{dR}}(X,\mathcal{F}^\vee\otimes\mathcal{G})$, $[E]$ lifts to an extension of filtered F-isocrystals if and only if it is in the $\phi=1$ eigenspace of $H^1_{\mathrm{dR}}(X,\mathcal{F}^\vee\otimes\mathcal{G})$. If it does lift, any two different choices of F-structure can only differ by an element of

$$\mathrm{Ext}^1_{\mathrm{fil},\phi}(\mathbb{Q}_p,H^0_{\mathrm{dR}}(X,\mathcal{F}^\vee\otimes\mathcal{G}))\simeq H^0(X,\mathcal{F}^\vee\otimes\mathcal{G})/(1-\phi)F^0,$$

giving the following Lemma, which is a special case of [BK10], Corollary A.14:

Lemma 10.11. *For any overconvergent filtered F-isocrystal $\mathcal{F}$, there is a short exact sequence*

$$0\to Ext^1_{\mathrm{fil},\phi}(\mathbb{Q}_p,H^0_{dR}(X,\mathcal{F}))\to\mathrm{Ext}^1(\mathcal{F},\mathcal{G})\to H^1_{dR}(X,\mathcal{F})^{\phi=1}\to 0$$

We can use this to develop a cohomology theory $H^i_{\mathrm{syn}}(X,\mathcal{F})$ such that H^1 corresponds to an extension of the trivial filtered F-isocrystal by $\mathcal{F}$. The details may be found in the appendix of [BK10]. The idea is to take the de Rham complex $R\Gamma_{\mathrm{dR}}(X,\mathcal{F})$, the rigid complex $R\Gamma_{\mathrm{rig}}(X,\mathcal{F})$, and define the ith syntomic cohomology group of X with coefficients in $\mathcal{F}$ to be the ith cohomology group of the complex

$$R_{\mathrm{syn}}(X,\mathcal{F}) = Cone(F^0R_{\mathrm{dR}}(X,\mathcal{F})\to R_{\mathrm{rig}}(X,\mathcal{F}).$$

By construction there is a long exact sequence in cohomology

$$\ldots\to H^i_{\mathrm{syn}}(X,\mathcal{F})\to F^0H^i_{\mathrm{dR}}(X,\mathcal{F})\overset{1-\phi^i}{H}_{\mathrm{rig}}(X,\mathcal{F})\to\ldots$$

5 Recap on finite polynomial cohomology

In this section we recap some results on finite polynomial cohomology from Lennart's lecture.

Recall that we saw that the F-structure on the de Rham Eisenstein class amounted to lifting it to an element of $H^1_{\text{syn}}(X, \mathrm{TSym}^k \mathcal{H})$, where the cohomology is cohomology of the syntomic complex

$$Cone(F^0 R\Gamma_{\text{dR}}(X, \mathcal{F}) \xrightarrow{1-\phi} R\Gamma_{\text{rig}}(X, \mathcal{F}))$$

Suppose that $\omega \otimes \eta$ was also in the $\phi = 1$ eigenspace. Then it could be lifted to an element of $H^2_{\text{syn}}(X, \mathrm{Sym}^{k+k'} \mathcal{H}^\vee(1+j))$, and we could view the pairing with the Rankin-Eisenstein class as a cup product in syntomic cohomology (see below), which gets us nearer to the Petersson inner products we are interested in. This is impossible, but the formal construction still makes if instead of thinking in terms of extension classes of filtered F isocrystals, we think in terms of cohomology of the above complex, and replace the polynomial $1-\phi$ above with a polynomial in ϕ chosen so that it annihilates $\eta \otimes \omega$.

For a polynomial $P(X)$ in $1 + t\mathbb{Q}_p[T]$ and a filtered F-isocrystal $\mathcal{F}$, define the finite polynomial complex to be the cone of the morphism of complexes

$$F^0 R\Gamma_{\text{rig}}(X, \mathcal{F}) \xrightarrow{P(\phi)} R\Gamma_{\text{dR}}(X, \mathcal{F}).$$

Define $H^i(X, \mathcal{F}, P)$ to be the i-th cohomology group of this complex.

From the construction of finite polynomial cohomology, there is a long exact sequence

$$\ldots \to H^i(X, \mathcal{F}, P) \to F^0 H^i_{\text{dR}}(X, F) \xrightarrow{P(\phi)} H^i_{\text{rig}}(X, \mathcal{F}) \to \ldots.$$

Hence any cohomology class ω in $H^1_{\text{dR}}(X, \mathcal{F})$ which lies in the $P(\phi) = 0$ eigenspace lifts to an element $\widetilde{\omega}$ in $H^1(X, \mathcal{F}, P)$, and the space of such lifts is an

$H^0_{\text{dR}}(X, \mathcal{F})/P(\phi)F^0(X, \mathcal{F})$-torsor. Let $\mathcal{F}$ and $\mathcal{G}$ be filtered F-isocrystals.

5.1 Cup products in finite polynomial cohomology

Finite polynomial cohomology naturally admits cup products. The basic motivation for a flexible choice of a frobenius polynomial is to be able to lift de Rham cohomology classes to finite polynomial cohomology classes by choosing a polynomial annihilating them. Hence the cup product of $H^i(X, \mathcal{F}, P)$ and $H^i(X, \mathcal{G}, Q)$ should be a finite polynomial cohomology class with respect to a polynomial whose roots are products of the roots of P and Q. With this in mind we introduce the following notation:

Definition 10.12. Given polynomials $P(T) = \prod(1 - \alpha_i T)$ and $Q(T) = \prod(1 - \beta_j T)$, define $P \star Q$ to be the polynomial $\prod(1 - \alpha_i \beta_j T)$.

For filtered F-isocrystals $\mathcal{F}$ and $\mathcal{G}$, and polynomial P and Q, there is a cup product

$$H^i(X, \mathcal{F}, P) \times H^i(X, \mathcal{G}, Q) \to H^{i+j}(X, \mathcal{F} \otimes \mathcal{G}, P \star Q)$$

Let ξ be an element of $H^i(X, \mathcal{F}, P)$ whose image in $F^0 H^i_{\text{dR}}(X, \mathcal{F})$ is zero. Let η be an element of $F^0 H^j_{\text{dR}}(X, \mathcal{G})$ such that $Q(\phi)\eta = 0$ in $H^j_{\text{rig}}(X, \mathcal{G})$.
Let $\widetilde{\eta}$ be a lift of η to $H^j(X, \mathcal{G}, Q)$, and let $[\xi]$ be a lift of ξ to $H^{i-1}_{\text{rig}}(X, \mathcal{F})$.

Let X have relative dimension d. Suppose $\mathcal{G} = \mathcal{F}^\vee(d+1)$, and that p^{-1} is not a root of $P \star Q$. Then via the canonical morphism of filtered F-isocrystals

$$\mathcal{F} \otimes \mathcal{F}^\vee(d+1) \to \mathbb{Q}_p(d+1)$$

we have $\mathbb{Q}_p$-valued pairings

$$\langle , \rangle_{\text{fp},X} : H^i(X, \mathcal{F}, P) \times H^j(X, \mathcal{F}^\vee, Q) \to \mathrm{Tr}_{P\star Q, X} : H^{2d+1}(X, \mathbb{Q}_p(d+1), P \star Q) \to$$

Lemma 10.13. *For ξ, η as above, $\langle \xi, \widetilde{\eta} \rangle_{\mathrm{fp},X}$ and $\langle [\xi], \eta \rangle_{rig,X}$ are independent of the choice of lift, and*

$$\langle \xi, \widetilde{\eta} \rangle_{\mathrm{fp},X} = \frac{1}{?} \langle [\xi], \eta \rangle_{rig,X}$$

Note that this may be viewed as an analoue in finite polynomial cohomology of the compatibility of cup products with the Leray spectral sequence (which was used in the construction of the motivic Eisenstein classes in Vivek's talk). Details of proof of the compatibility, which essentially holds by construction of cup products in finite polynomial cohomology, may be found in [Bes00a], section 2. Technically this is only stated for constant coefficients and polynomials P and Q of the form $1 - p^n T$, but as explained in [KLZ15b] the proof generalizes.

5.2 Action of Frobenius

Our next tool for calculating the pairing is the observation that the Poincaré cup product is Frobenius equivariant, hence if η is in the $\phi = \alpha$ eigenspace then

Recall the unit root splitting

$$\mathrm{spl}^{u-r} : \mathcal{H}_{\mathrm{rig}}|_{Y_{\mathrm{ord}}} \to F^0 \mathcal{H}_{\mathrm{rig}}|_{Y^{\mathrm{ord}}}.$$

This induces a splitting of $\mathrm{TSym}^k \mathcal{H}$.

With respect to our basis of sections of Proposition 6.5.6 of [KLZ15b] says

Theorem 10.14.

$$\mathrm{spl}^{u-r}(\omega'_g \cup \sigma_{rig}) = (*)(g.F^{(p)}_{k-k',k'-j+1,b})$$

Lemma 10.15. *Equivalence of pairing between elements of F^0 and extensions of filtered ϕ-modules and pairing on finite polynomial cohomology.*

6 The value of the pairing

In this section we set ourselves the goal of computing

$$(*) = \left\langle \mathrm{AJ}_{\mathrm{syn},f,g} \, \mathrm{Eis}^{[k,k',j]}_{\mathrm{syn},b,N}, \eta_f^\alpha \otimes \omega'_g \right\rangle.$$

Write $\lambda = \eta_f^\alpha \otimes \omega'_g$.

Define P_ω and P_η to be polynomials such that $P_\omega(\phi)\omega = 0$ and $P_\eta(\eta) = 0$.As remarked above, this is equal to

$$\left\langle \mathrm{Eis}^{[k,k',j]}_{\mathrm{syn},b,N}, \tilde{\lambda} \right\rangle_{\mathrm{fP},\mathcal{Y}\times\mathcal{Y}},$$

where $\mathrm{Eis}^{[k,k',j]}_{\mathrm{syn},b,N}$ is the image under $\Delta_* \circ \mathrm{CG}$ of $\mathrm{Eis}^k_{\mathrm{syn},b,N}$, and $\tilde{\lambda} = (\lambda, \lambda_{\mathrm{rig}})$. Using functorial properties of finite polynomial cohomology, this equals

$$\langle \sigma, \Delta^*(\tilde{\lambda}) \rangle_{\mathrm{fP},\mathcal{Y}},$$

where $\sigma = (\sigma_{\mathrm{dR}}, \sigma_{\mathrm{rig}}) = \mathrm{CG}^{[k+k',j]}(\mathrm{Eis}^{k+k'-2j}_{\mathrm{syn},b,N})$. Note that when $k = k' = 0$, the class σ is just

$$(\mathrm{dlog}\, g_{0,b/N}, (1-\phi)\log g_{0,b/N}).$$

This pairing can be related to the cup product

$$\langle \sigma, \Delta^*(\tilde{\lambda}) \rangle = \tilde{\eta} \cup \tilde{\omega},$$

where $\tilde{\eta} = (\eta_f^\alpha, F_\eta)$ and $\tilde{\omega} = (\omega'_g, F_\omega)$. In order to compute this cup product we need an explicit formula due to Besser.

Theorem 10.16 (Besser). *Let $P(T) = \prod(1-\alpha_i T)$ and $Q(T) = \prod(1-\beta_i T)$ be two polynomials. Then*

$$(x_{dR}, x_{rig})_P \cup (y_{dR}, y_{rig})_Q = (x_{dR} \cup y_{dR}, z_{rig}),$$

where

$$z_{rig} = (-1)^{\deg x_{dR}} \bigcup \left(b(\phi_1,\phi_2)(x_{dR} \otimes y_{rig}) + a(\phi_1,\phi_2)(y_{dR} \otimes x_{rig})\right),$$

and where $a(T_1,T_2)$ and $b(T_1,T_2)$ are defined by

$$\prod(1-\alpha_i\beta_j T_1T_2) = a(T_1,T_2)P(T_1) + b(T_1,T_2)Q(T_2).$$

Also, in the formula ϕ_1 and ϕ_2 are the Frobenii acting on each of the components.

The definition of the pairing on syntomic cohomology comes from the trace map $\mathrm{Tr}\colon H^3_{\mathrm{fP}}(\mathcal{Y}) \cong \mathbb{Q}_p$, so we are computing $\mathrm{Tr}(\sigma \cup \tilde{\eta} \cup \tilde{\omega})$. Set $P_g(T)$ to be

$$P_g(T) = \left(1 - \frac{p^{1+j}}{\alpha_j}T\right)\left(1 - \frac{p^{1+j}}{\beta_f}T\right).$$

Since $(*)$ is the same as $\langle \eta, \Xi\rangle$, where Ξ comes from $\sigma \cup \tilde{\omega}$, by writing this explicitly and using Besser's formula with the polynomials $P_\sigma = 1 - T$ and $P_{\tilde{\omega}} = P_\theta$ we get

$$(*) = -\left(1 - \frac{p^{k-2}\phi}{\langle p\rangle}\right)\omega'_g \otimes \sigma_{\mathrm{rig}},$$

where $\langle p\rangle$ is the diamond operator. At the end, we need to evaluate the Φ-equivariant pairing

$$\left\langle \eta_f^*, \sum H_j \omega'_g \right\rangle_{\mathrm{rig}}.$$

We now use the fact that cup product in rigid cohomology is ϕ-equivariant, hence the map

$$\left\langle \eta_f^*, . \right\rangle : H^1(X, \mathrm{TSym}^k \mathcal{H}(2)) \to \mathbb{Q}_p$$

factors through projection onto the slope $1 + k - v_p(\alpha)$ subspace. By section **** , the class of Ξ is equal to $(.)\Pi^{oc}(g.F^{(p)}_{k-k',k'-j+1,b})$.

Theorem 10.17. *We have*

$$\left\langle \mathrm{AJ}_{syn}\,\mathrm{Eis}^{[k,k',j]}_{syn,b,N}, \eta \otimes \omega \right\rangle = (.)\left(f, \Pi^{oc}(g.F^{(p)}_{k-k',k'-j+1,b})\right),$$

where the pairing on the right is the Petersson inner product on $\Gamma_1(N) \cap \Gamma_0(p)$.

It should be remarked that the link with p-adic L-functions, proved in [LLZ14], doesn't use $\Pi^{o-c})(gF^{[p]})$, but rather $e_{\mathrm{ord}}(g.F^{[p]})$ where e_{ord} is Hida's ordinary projector on q-expansions as defined in Chris's talk. However as proved in [DR14]

Corollary 10.17.1. *Suppose f is ordinary. Then*

$$\left\langle \mathrm{AJ}_{syn}\,\mathrm{Eis}^{[k,k',j]}_{syn,b,N}, \eta \otimes \omega \right\rangle = (.)\left\langle \eta_f^{\alpha,ord}, e_{ord}(g.F^{[p]}_{k-k',k'-j+1,1}) \right\rangle_{N(p)}$$

Proofs of the explicit reciprocity laws

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The aim of this section is to prove that the Euler system of Beilinson–Flach elements we have constructed is not zero by the usual method of relating it to values of a p-adic L function.

Let's recall that we have the equality (up to non-zero p-adic fators)

$$\langle \mathrm{AJ}_{\mathrm{syn},f,g}(\mathrm{Eis}^{k,k',j}_{\mathrm{syn}}, \eta'_f \otimes \omega_g)\rangle \sim (f, gH^{[p]}_{k-j})_{p\text{-adic}}.$$

We shall show that this p-adic Petersson product is related to the value of Hida's three variables p-adic L-function $L_p(f,g,j)$ which is constructed interpolating the algebraic part of the (generally non-zero) critical values $L^{\mathrm{alg}}(f,g,j')$.

1 Hida's three variables p-adic L function

1.1 Introduction

Let $f \in S_{k+2}(\Gamma_0(N_f))$, $g \in S_{k'+2}(\Gamma_0(N_g))$ be two cuspidal forms of respective weights $k+2$ and $k'+2$ with $k > k'$, Nebentypus ϵ_f and ϵ_g and levels N_f, N_g. Write $N = lcm(N_f, N_g)$. We can associate with these two forms a complex L-function

$$L(f,g,s) = L_N(\epsilon_f\epsilon_g, 2s+2-k-k') \times (\sum_{n\geq 1} a_n(f)a_n(g)n^{-s}), \tag{11.1}$$

where

$$L_N(\epsilon_f\epsilon_g, s) = (\prod_{p|N}(1-\epsilon_f\epsilon_g(p)p^{-s}))(\sum_{n\geq 1}\epsilon_f\epsilon_g(n)n^{-s}) = \sum_{\substack{n\geq 1\\(n,N)=1}} \epsilon_f\epsilon_g(n)n^{-s}$$

is the Dirichlet L function associated with the character $\epsilon_f\epsilon_g$ with no Euler factors at primes dividing N.

We explain in what follows, in the lines of [Hid88], Hida's p-adic interpolation of the L function associated with f and g. We proceed by reinterpreting the expression of $L(f,g,s)$ as a product of different terms, each of which we are going to p-adically interpolate to get a function in three variables (two Hida families variables, and one cyclotomic variable).

1.2 Notations

We first recall and fix some notations from Hida theory. As usual, we note $\Lambda = \mathbb{Z}_p[[\mathbf{Z}_p^*]]$ the Iwasawa algebra. Let $x \mapsto x^{\mathbf{k}}$ be the natural inclusion $\mathbf{Z}_p^* \to \mathbb{Z}_p[[\mathbf{Z}_p^*]]$. Let $\mathbf{k}$ denote a coordinate of the weight space $\mathcal{W}$ defined as the rigid space associated with $\mathrm{Spf}(\Lambda)$. It parametrizes continuous characters of $\mathbf{Z}_p^*$: if L is an extension of $\mathbb{Q}_p$, it's L points $\mathcal{W}(L)$ are $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^*, L^*)$. If $\tau \in \mathcal{W}$, we shall speak of "specialising at $\mathbf{k} = \tau$" when taking the homomorphism $\Lambda \to \overline{\mathbb{Q}_p}$ induced by τ.

Since we want to interpolate functions in three variables (two of them moving in Hida families and one cyclotomic variable), we will be working over the ring $\Lambda\hat{\otimes}_{\mathbb{Z}_p}\Lambda\hat{\otimes}_{\mathbb{Z}_p}\Lambda$. For the sake of clarity, we will write $\mathbf{k}, \mathbf{g} \in \Lambda_D$ and $\mathbf{j} \in \Lambda_\Gamma$ the canonical characters of each factor, with $\mathbf{k}$ and $\mathbf{g}$ denoting weight of families of modular forms and $\mathbf{j}$ denoting the cyclotomic variable. Both Λ_D and Λ_Γ are isomorphic to Λ but the notations enforces the fact that they encode different actions: D stands for diamond operator and Γ stands for the cyclotomic Galois group $\Gamma = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$.

Let N be a positive integer and p a prime not dividing N. For a Hida family $\mathbf{f}$ we mean a maximal ideal in the universal Hecke algebra $\mathbf{T}_{Np^\infty}$, and we denote by $\Lambda_{\mathbf{f}}$ the associated local Λ_D-algebra (the localisation of the Hecke algebra at $\mathbf{f}$), which is finite and projective as a Λ_D-module. If $\mathbf{f}$ is a Hida family, we say $\mathbf{f}$ is *non-Eisenstein* if the residual Galois representation $\rho_{\mathbf{f}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbf{F})$ associated with $\mathbf{f}$ is irreducible, where $\mathbf{F}$ is the residue field of $\Lambda_{\mathbf{f}}$.

Let $\mathbf{f}$ be a Hida family. The algebra $\Lambda_{\mathbf{f}}$ has a finite number of minimal primes, and we call one such prime a *branch* of $\mathbf{f}$. They are in bijection with the simple direct summands of the artinian ring $\Lambda_{\mathbf{f}} \otimes_{\Lambda_D} \mathrm{Frac}(\Lambda_D)$. If $\mathbf{a}$ is a branch of $\mathbf{f}$, then $\Lambda_{\mathbf{a}} = \Lambda_{\mathbf{f}}/\mathbf{a}$ is an integral domain whose field of fractions is a finite field extension of $\mathrm{Frac}(\mathbb{Z}_p[[1 + p\mathbb{Z}_p]])$.

1.3 Eisenstein series

For $\lambda \in \mathbb{Z}_{\geq 1}$, we have the usual analytic Eisenstein series

$$E_\lambda(\epsilon_f\epsilon_g) = a_0 + \sum_n (\sum_{d|n} \epsilon_f\epsilon_g(d)d^{\lambda-1})q^n.$$

We have the following theorem which expresses the special values of the L-function of f and g in terms of Gamma values and the Petersson inner product;

Theorem 11.1. *Let $j \in \mathbb{Z}$ be such that $k'+1 \leq j \leq \frac{k+k'}{2}$, then*

$$\Gamma(j+1)\Gamma(j-k')L(f,g,j+1) = (*)\langle f \mid \tau_N, (g|\tau_N)\delta^{j-k'-1}E_{k-k'-2(j-k'-1)}(\epsilon_f\epsilon_g)\rangle_{\Gamma_1(N)},$$

where τ_N is the Atkin–Lehner involution and $\delta = \frac{1}{2\pi i}(\frac{1}{2iy} + \frac{d}{dz})$ is the Maaß-Shimura operator and, for $f \in S_k(\Gamma_1(N))$ and $g \in M_k(\Gamma_1(N))$, $\langle f, g\rangle_{\Gamma_1(N)} = \int_{\Gamma_1(N)\backslash\mathcal{H}} \overline{f(z)}g(z)y^{k-2}dxdy$ denotes the Petersson scalar product.

It is the expression on the right that we are going to exploit in order to interpolate $L(f,g,s)$ p-adically.

1.4 Interpolation

Define, for $\mathbf{k}, \mathbf{k}', \mathbf{j} \in \mathcal{W}^3$,

$$\mathcal{E}(\mathbf{k}, \mathbf{k}', \mathbf{j}) = \sum_{(n,p)=1} (\sum_{d|n} \epsilon_f\epsilon_g(d)(\frac{n}{d})^{\mathbf{k}-\mathbf{j}}d^{\mathbf{j}-\mathbf{k}'-1})q^n \in \Lambda^{\otimes 3}[[q]],$$

where, for $m \in \mathbf{Z}_p^*$ and $\mathbf{l} \in \mathcal{W}$, $m^{\mathbf{l}}$ denotes $\mathbf{l}(m)$.

For $h = \sum_{n\geq 1} a_n(h)q^n$ a cusp form, denote by

$$h^{[p]} = \sum_{(n,p)=1} a_n(h)q^n$$

the series we obtain by removing the Euler factor at p. The following lemma shows that $\mathcal{E}$ interpolates (as p-adic modular forms) one of the terms appearing in the expression for the L-function of f and g.

Lemma 11.2. *Restriction to the ordinary locus gives*

$$(\delta^{j-k'-1}E_{k-k'-2(j-k'-1)}(\epsilon_f\epsilon_g))^{[p]} \mid_{\mathcal{X}^{\mathrm{ord}}} = \mathcal{E}(k,k',j).$$

We next take care of the Petersson scalar product. Suppose from now on that f, g are ordinary forms of weights k and k' as before, and let $\mathbf{f}, \mathbf{g}$ be two Hida families passing through them (that is, their specialization at the respective weights give f and g, respectively). We assume $\mathbf{f}$ and $\mathbf{g}$ are non-Eisenstein. Choose a branch $\mathbf{a}$ of $\mathbf{f}$ and suppose that $\mathbf{a}$ is N_f-new (that is, for all $k \geq 0$, the specialisation $\mathbf{a}_k$ at k is new at N_f) and non-Eisenstein.

Consider the Hecke algebra $\mathbf{T}_{N_f p^\infty}$ and let $\Lambda_\mathbf{a} = \Lambda_\mathbf{f}/\mathbf{a}$. Write $\tilde{\Lambda}_\mathbf{a}$ for the integral closure of $\Lambda_\mathbf{a}$ in its fraction field $\mathrm{Frac}(\Lambda_\mathbf{a})$. This defines a splitting

$$\mathbf{T}_{N_f p^\infty} \otimes \mathrm{Frac}(\Lambda_\mathbf{a}) \cong \mathrm{Frac}(\Lambda_\mathbf{a}) \oplus \mathcal{C}$$

for some complement $\mathcal{C}$. Let $\eta_\mathbf{a}$ be the idempotent corresponding to the splitting:

$$\eta_\mathbf{a} : \mathbf{S}^{\mathrm{ord}}_{\mathbf{k}+2}(N_f, \Lambda) \otimes \Lambda_\mathbf{a} \to I_\mathbf{a}$$

where $I_\mathbf{a}$ is a fractional $\tilde{\Lambda}_\mathbf{a}$-ideal.

If $\epsilon : \mathbf{Z}_p^* \to \overline{\mathbb{Q}}^\times$ is a finite order character and $k \in \mathbb{Z}$, we denote by $k + \epsilon : \mathbf{Z}_p^* \to \overline{\mathbb{Q}}^\times$ the character that sends $z \in \mathbf{Z}_p^*$ to $\epsilon(z) z^k$.

Proposition 11.3. *If $\mathbf{h} \in \mathbf{S}^{\mathrm{ord}}_{\mathbf{k}+2}(N_f, \Lambda) \otimes \Lambda_\mathbf{a}$ and $r = \min(1, \log_p(\mathrm{cond}(\epsilon))$, then we have*

$$\eta_\mathbf{a}(\mathbf{h}) = a(1, \mathbf{h} \mid_{\eta_\mathbf{a}}) = \frac{\langle f_{k+\epsilon} \mid \tau_{N_f p^r}, h\rangle}{\langle f_{k+\epsilon} \mid \tau_{N_f p^r}, f_{k+\epsilon}\rangle}.$$

Next, we define

$$(\mathbf{g} \cdot \mathcal{E}) = \mathrm{Tr}_{N/N_f} e^{\mathrm{ord}}(\mathbf{g} \cdot \mathcal{E}(\mathbf{k}, \mathbf{k}', \mathbf{j})) \in \mathbf{S}_{\mathbf{k}+2}(N_f, \Lambda) \otimes (\Lambda_\mathbf{a} \otimes \Lambda_\mathbf{g} \otimes \Lambda_D)$$

and note that the weight of any specialization of $\mathbf{g} \cdot \mathcal{E}(k, k', j)$ is always k.

We have now interpolated every term of the expression 11.1 for the L function associated with f and g. To put all things together, we make the following definition

Definition 11.4. Let $\mathbf{f}$ and $\mathbf{g}$ two Hida families as before. We define the *p-adic L function of* $\mathbf{f}$ *and* $\mathbf{g}$ as

$$L_p(\mathbf{a}, \mathbf{g}, \mathbf{j}) = \eta_\mathbf{a}(\mathbf{g} \cdot \mathcal{E}) \in I_\mathbf{a} \otimes \Lambda_\mathbf{g} \otimes \Lambda_\Gamma.$$

The following theorem shows that the p-adic L function interpolates, up to a transcendental factor, the classical L function.

Theorem 11.5. *We have, for every $j \in \mathbb{Z}$ such that $k' + 1 \leq j \leq \frac{k+k'}{2}$,*

$$L_p(\mathbf{a}, \mathbf{g}, \mathbf{j}) \mid_{(k,k',j)} = (*)\Gamma(j+1)\Gamma(j-k')L^{\mathrm{alg}}(f, g, j+1),$$

where $(*)$ is a certain product of Euler factors and some algebraic factor.

Remark 11.6.

- In contrast to the complex L function of f and g, the p-adic L function is not symmetric for the two Hida variables.
- From Netan's talk we know that for $0 \leq j \leq \min(k', k) = k'$ we have

$$\mathcal{E}(k, k', j) = (*) H^{[p]}_{k-j},$$
$$L_p(\mathbf{a}, \mathbf{g}, \mathbf{j}) \mid_{(k,k',j)} = (*)\langle \mathrm{AJ}_{\mathrm{syn,f,g}}(\mathrm{Eis}^{[k,k',j]}_{\mathrm{syn}}), \eta_f^* \otimes \omega_g\rangle.$$

- $L_p(\mathbf{a}, \mathbf{g}, \mathbf{j})$ is not zero (indeed, for $k \geq k' + 2$ and $k' + 1 \leq j \leq \frac{k+k'}{2}$, L_p interpolates, modulo some non-zero factors, the special values $L(f, g, j+1)$, which are non-zero).

2 Motivic p-adic L functions

We show next that we can re obtain Hida's p-adic L-function associated with f and g by applying (a generalized version of) Perrin-Riou's big logarithm to the specialization of Beilinson–Flach elements.

2.1 Preliminaries and notations

We recall briefly the construction of the motive associated with a modular form. The module

$$H^1_{\mathrm{ord}}(Np^\infty) = \varprojlim_r e'_{\mathrm{ord}} H^1_{\text{ét}}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$$

is a finitely generated and projective module over the algebra Λ_D (acting via inverse diamond operators at p: $u \in \mathbf{Z}_p^*$ acts on the r-th layer of the inverse limit as $\langle u^{-1} \rangle_{p^r}$). Moreover, $H^1_{\mathrm{ord}}(Np^\infty)$ has commuting Λ_D-linear actions of $G_{\mathbb{Q},S}$ (where S is the set of primes dividing Np) and of the Hecke operators T'_n, $n \geq 1$. For a Hida family $\mathbf{f}$, its associated motive $M(\mathbf{f})^*$ is defined as the localisation

$$M(\mathbf{f})^* = H^1_{\mathrm{ord}}(Np^\infty)_{\mathbf{f}}.$$

Let $\mathbf{f}$ be a Hida family. We have a decomposition

$$0 \to F^+M(\mathbf{f})^* \to M(\mathbf{f})^* \to F^-M(\mathbf{f})^* \to 0,$$

where $F^\pm M(\mathbf{f})^*$ are projective modules of finite rank over Λ_D. The Galois group $\mathcal{G}_{\mathbb{Q}_p}$ acts on the submodule $F^+M(\mathbf{f})^*$ via the the $\mathbf{T}_{Np^\infty}$-valued product of the unramified character that sends the Frobenius Fr_p to $\langle p \rangle_N^{-1}(U'_p)^{-1}$ with the $\mathbf{k}+1$-th power of the cyclotomic character (i.e $x \in \mathbf{Z}_p^* \mapsto x x^{\mathbf{k}} \in \Lambda_D$), and on the quotient $F^-M(\mathbf{f})^*$ via the unramified character corresponding to $\mathrm{Fr}_p \mapsto U'_p \in \mathbf{T}_{Np^\infty}$.

2.2 Some results on Beilinson–Flach elements

Recall that we have defined, for c coprime to $6N$, the Beilinson–Flach Euler system

$$({}_c\mathcal{BF})^{\mathbf{f},\mathbf{g}} \in H^1(\mathbb{Z}[1/Np], M(\mathbf{f})^* \otimes M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j})),$$

where $\Lambda_\Gamma(-\mathbf{j})$ is a $\mathcal{G}_{\mathbb{Q},\{p\}}$-module that is free of rank one over $\Lambda_\Gamma = \mathbb{Z}_p[[\Gamma]]$ (with a basis determined by a choice of a compatible system $(\zeta_{p^n})_n$ of roots of unity), the group $\mathcal{G}_{\mathbb{Q},\{p\}}$ acting on it via the inverse of the canonical character $\mathbf{j} : \Gamma \to \mathbb{Z}_p[[\Gamma]]^\times$. We remark that for any finite set of primes S with $p \in S$, and any profinite $\mathbb{Z}_p[\mathcal{G}_{\mathbb{Q},S}]$-module A, we have a canonical isomorphism

$$H^1(\mathbb{Z}[1/S], A \otimes \Lambda_\Gamma(-\mathbf{j})) = \varprojlim_r H^1(\mathbb{Z}[1/S, \zeta_{p^r}], A).$$

The right hand side is the usual definition of the Iwasawa cohomology groups. The left hand side is usually seen as cohomology group taking values in an algebra of distributions in the sense of p-adic analysis.

The strategy is as follow. We will construct a big logarithm map for families that we also call $\mathcal{L}$ such that the following diagram, applied to Beilinson–Flach elements, commutes

$$\begin{array}{ccc} & & H^1(\mathbb{Z}[1/Np], M(\mathbf{f} \otimes \mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j})) \\ & \swarrow\!\!\cdot\!\cdot\!\cdot & \downarrow{\scriptstyle \text{res+proj}} \\ H^1(\mathbb{Q}_p, F^-M(\mathbf{f}) \otimes F^+M(\mathbf{g}) \otimes \Lambda_\Gamma(-\mathbf{j})) & \longrightarrow & H^1(\mathbb{Q}_p, F^-M(\mathbf{f})^* \otimes M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j})) \\ {\scriptstyle \mathcal{L}}\downarrow & & \\ \mathbf{D}(F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*) \otimes \Lambda_\Gamma & \longrightarrow & I_{\mathbf{a}} \otimes \Lambda_{\mathbf{g}} \otimes \Lambda_\Gamma. \end{array}$$

For the dotted arrow, we show that $({}_c\mathcal{BF})^{\mathbf{f},\mathbf{g}}$ lives in $H^1(\mathbb{Z}[1/Np], F^-M(\mathbf{f}) \otimes F^+M(\mathbf{g}) \otimes \Lambda_\Gamma(-\mathbf{j}))$. We will need the following lemma:

Lemma 11.7. *Let T be a free $\mathbb{Z}_p$-module with a continuous $\mathcal{G}_{\mathbb{Q}_p}$-action and write $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We have*

1. *If V is de Rham, then*

$$\dim_{\mathbb{Q}_p} H^1_{\mathrm{g}}(\mathbb{Q}_p, V) = \dim_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}^+_{\mathrm{dR}}(V) + \dim_{\mathbb{Q}_p} H^0(\mathbb{Q}_p, V) + \dim_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1}.$$

2. *$H^1(\mathbb{Q}_p, T \otimes \Lambda(-\mathbf{j}))_{\mathrm{tor}} = H^0(\mathbb{Q}_p(\mu_{p^\infty}), T)$ and, in particular, Iwasawa cohomology has no p-torsion.*

3. *If $z \in H^1(\mathbb{Q}_p, T \otimes \Lambda(-\mathbf{j}))$ is sent to 0 in $H^1(\mathbb{Q}_p, V(\chi))$ for all but a finite number of χ, then z is torsion.*

Proof. 1. This an immediate consequence of the long exact sequence associated with the fundamental exact sequence (tensored by V), the Euler–Poincaré characteristic formula and the duality between H^1_{g} and H^1_{f}.

2. This is an integral version of a properties of Iwasawa cohomology that we have seen in Bruno's talk.

3. As Iwasawa cohomology is p-torsion free, we can check the statement over $\Lambda[1/p]$. We reduce to the case of a module over $\mathbb{Z}_p[[T]][1/p]$ and it follows from Weirstraß preparation theorem. □

Proposition 11.8 ([KLZ15a], lemma 8.1.8)**.** *The inclusion $F^+M(\mathbf{g})^* \hookrightarrow M(\mathbf{g})^*$ induces an injection*

$$H^1(\mathbb{Q}_p, F^-M(\mathbf{f}) \otimes F^+M(\mathbf{g}) \otimes \Lambda_\Gamma(-\mathbf{j})) \to H^1(\mathbb{Q}_p, F^-M(\mathbf{f}) \otimes M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j})).$$

Moreover, the image of ${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}$ in the right hand side lies in the image of this injection.

Proof. The first point follows by using the long exact sequence induced by $0 \to F^+M(\mathbf{g})^* \to M(\mathbf{g})^* \to F^-M(\mathbf{g})^* \to 0$ and the fact that the 0-th Iwasawa cohomology group $H^0(\mathbb{Q}_p, F^-M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$ vanishes.

For the second point, we show that the image of the Beilinson–Flach element in $H^1(\mathbb{Q}_p, F^-M(\mathbf{f})^* \otimes F^-M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$ is zero. First, we show that this module has no torsion: by (2) of the previous lemma, it is $H^0(\mathbb{Q}_p(\mu_{p^\infty}), F^-M(\mathbf{f})^* \otimes F^-M(\mathbf{g})^*)$. Define, for any $k \geq 0$, $M(\mathbf{f})^*_{k,r}$ (resp. $M(\mathbf{g})^*_{k,r}$) as the localisation at $\mathbf{f}$ (resp. at $\mathbf{g}$) of the module $e'_{\mathrm{ord}} H^1_{\text{ét}}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k(\mathcal{H}_{\mathbb{Z}_p})(1))$. It can be shown that the modules $M(\mathbf{f})^*$ and $M(\mathbf{g})^*$ are equal to the inverse limit of the modules $M(\mathbf{f})^*_{k,r}$ and $M(\mathbf{g})^*_{k,r}$ (this is a consequence of the control theorem, see [KLZ15a], prop. 7.2.1.(4)). We have then

$$H^0(\mathbb{Q}_p(\mu_{p^\infty}), F^-M(\mathbf{f})^* \otimes F^-M(\mathbf{g})^*) = \varprojlim_r H^0(\mathbb{Q}_p(\mu_{p^\infty}), F^-M(\mathbf{f})^*_{k,r} \otimes F^-M(\mathbf{g})^*_{k,r}).$$

Note that $F^-M(\mathbf{f})^*_{k,r} \otimes F^-M(\mathbf{g})^*_{k,r}$ is a free $\mathbb{Z}_p$-module of finite rank with unramified action of $\mathcal{G}_{\mathbb{Q}_p}$. If we pick $k > 0$ then the U'_p-eigenvalues on $M(\mathbf{f})^*_{k,r}$ and $M(\mathbf{g})^*_{k,r}$ are Weil's number of weight $k-1$. This implies that $F^-M(\mathbf{f})^*_{k,r} \otimes F^-M(\mathbf{g})^*_{k,r}$ has no element fixed by Frobenius, hence $H^0(\mathbb{Q}_p(\mu_{p^\infty}), F^-M(\mathbf{f})^* \otimes F^-M(\mathbf{g})^*)$ vanishes and thus $H^1(\mathbb{Q}_p, F^-M(\mathbf{f})^* \otimes F^-M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$ has no torsion.

By a result of Nekovar and Niziol, the image of motivic classes by the étale regulator lands in H^1_{g}. Hence the image of the Beilison–Flach elements modulo a finite order character χ of $\mathbf{Z}_p^*$ lies in $H^1_{\mathrm{g}}(\mathbb{Q}_p, F^-M(\mathbf{f})^*_{k,r} \otimes F^-M(\mathbf{g})^*_{k,r}(\chi))$. We apply the first point of the previous lemma to see that this space is zero. By the third point of the previous lemma, the Beilison–Flach element is torsion in $H^1(\mathbb{Q}_p, F^-M(\mathbf{f})^* \otimes F^-M(\mathbf{g})^*)$ and so it's zero. Hence the Beilison–Flach elements lies in $H^1(\mathbb{Q}_p, F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$. □

2.3 Perrin-Riou's big Logarithm in families

Let M be an unramified, p-adically complete $\mathbb{Z}_p[\mathcal{G}_{\mathbb{Q}_p}]$-module (not necessarily free of finite rank!). Define

$$\mathbf{D}(M) = (M \otimes_{\mathbb{Q}_p} \widehat{\mathbf{Z}}_p^{\mathrm{nr}})^{\mathcal{G}_{\mathbb{Q}_p}},$$

which inherits an action of the arithmetic Frobenius on $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$. Observe that in the usual case where $M = T \subseteq V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a lattice in a p-adic representation V, then $\mathbf{D}(M)$ is a lattice in $\mathbf{D}_{\mathrm{cris}}(V)$.

Theorem 11.9 ([KLZ15a], thm. 8.2.3)**.** *Let I be the ideal of Λ_Γ that is the kernel of specialization at $\mathbf{j} = 1$. There exists a map*

$$\mathcal{L} : H^1(\mathbb{Q}_p, M \otimes \Lambda_\Gamma(-\mathbf{j})) \to \mathbf{D}(M) \otimes \Lambda_\Gamma I^{-1},$$

satisfying the following properties:

- *$\mathcal{L}$ is functorial in M and commutes with the action of $\mathrm{End}_{\mathbb{Z}_p[\mathcal{G}_{\mathbb{Q}_p}]}(M)$ on both sides,*
- *if M is free of finite rank, then $\mathcal{L}$ coincides with Perrin-Riou's big logarithm map for the unramified representation $V = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,*
- *the kernel of $\mathcal{L}$ is $H^0(\mathbb{Q}_p, M)$,*
- *the image of $\mathcal{L}$ in $\mathbf{D}(M) \otimes \frac{I^{-1}\Lambda_\Gamma}{\Lambda_\Gamma}$ is contained in the submodule $\mathbf{D}(M)^{\varphi=1} \cong H^0(\mathbb{Q}_p, M)$.*

Let now M be $F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*(1-\mathbf{k}')$, which is (because of the twisting) an unramified $\mathbb{Z}_p[\mathcal{G}_{\mathbb{Q}_p}]$-module. Let

$$\mathbf{D}(F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*)$$

be $\mathbf{D}(M)$ with the twisted action of $\mathbf{Z}_p^*$ by $1+\mathbf{k}'$. As a consequence of the last theorem we have

Theorem 11.10 ([KLZ15a], thm. 8.2.8)**.** *Suppose at least one of $\mathbf{f}$ or $\mathbf{g}$ is non-Eisenstein. Then there is an injective morphism of $(\Lambda_{\mathbf{f}} \otimes \Lambda_{\mathbf{g}} \otimes \Lambda_\Gamma)$-modules*

$$\mathcal{L} : H^1(\mathbb{Q}_p, F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*) \otimes \Lambda_\Gamma(-\mathbf{j}) \to \mathbf{D}(F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*) \otimes \Lambda_\Gamma$$

such that for all classical specialisation f, g of $\mathbf{f}$ and $\mathbf{g}$ it recovers Perrin-Riou's log *and* $\exp^*$ *maps.*

By proposition 11.8, we know that the Beilinson–Flach elements ${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}$ live in $H^1(\mathbb{Q}_p, F^-M(\mathbf{f}) \otimes M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$, so we can apply Perrin-Riou's Logarithm map in families to those elements to get an element in $\mathbf{D}(F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*) \otimes \Lambda_\Gamma$ that, as we have already noted, should be seen (via the Amice isomorphism) as a distribution taking values in $\mathbf{D}(F^-M(\mathbf{f})^* \otimes F^+M(\mathbf{g})^*)$, whose values at special characters have arithmetic importance. This is the purpose of the next section.

2.4 Recovering the p-adic L function

The following result is a consequence of the comparison of Eichler–Shimura isomorphisms and should not surprise the reader

Proposition 11.11 ([KLZ15a], prop. 10.1.2)**.** *Let $\mathbf{f}$ and $\mathbf{g}$ be two Hida families and $\mathbf{a}$ be a cuspidal branch of $\mathbf{f}$ that is new. Then:*

- *There exist canonical linear map*

  $$\omega_{\mathbf{g}} : \mathbf{D}(F^+M(\mathbf{g})^*)(1-\mathbf{k}-\epsilon_{\mathbf{g}}) \to \Lambda_{\mathbf{g}}$$

  which interpolates for every specialisation g of $\mathbf{g}$ the pairing with ω_g.
- *There exist a canonical linear map*

  $$\tilde{\eta}_{\mathbf{a}} : \mathbf{D}(F^-M(\mathbf{f})^*) \otimes_{\Lambda_{\mathbf{f}}} \tilde{\Lambda}_{\mathbf{a}} \to I_{\mathbf{a}}$$

  which interpolates the pairing with η_f for f a specialisation of $\mathbf{f}$ which belongs to $\mathbf{a}$.

Define now

$$\mathcal{L}(\mathbf{a}, \mathbf{g}, \mathbf{j}) = \langle \mathcal{L}({}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}), \tilde{\eta}_{\mathbf{a}} \otimes \omega_{\mathbf{g}} \rangle \in I_{\mathbf{a}} \otimes \Lambda_{\mathbf{g}} \otimes \Lambda_\Gamma.$$

We can now state the main theorem:

Theorem 11.12 ([KLZ15a], thm. 10.2.2). *We have*

$$\mathcal{L}(\mathbf{a}, \mathbf{g}, \mathbf{j}) = (*)\ L_p(\mathbf{a}, \mathbf{g}, \mathbf{j}),$$

where $(*)$ *is an explicit factor.*

Proof. We evaluate the left hand side and show that it coincides with the right hand side at all points of type $0 < j < k < k'$. This will be enough to conclude as these points are Zariski dense and $I_{\mathbf{a}} \otimes \Lambda_{\mathbf{g}} \otimes \Lambda_{\Gamma}$ is torsion-free as $\Lambda_{\mathbf{f}} \otimes \Lambda_{\mathbf{g}} \otimes \Lambda_{\Gamma}$-module.

By Theorem 11.10 we have that $\mathcal{L}$ at (k, k', j), coincides, up to some explicit non vanishing factors, with the Bloch–Kato logarithm. From Francesc's talk, we know that $({}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}})$ is, up to some explicit non vanishing factor, the $(f_k, g_{k'})$-isotipical component of the Eisenstein class.

From Lennart's talk we know the compatibility between syntomic and étale regulator: $r_{\mathrm{syn}} = \log r_{\mathrm{et}}$. We can conclude as the pairing in de Rham cohomology coincides with the p-adic Petersson product and this is exactly the value of the right hand side at (k, k', j) (see Remark 11.6).

□

Applications: bounding Selmer and Sha

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1 Introduction

In this talk we will explain how Euler systems can be used to bound Selmer groups of p-adic Galois representations of $G_{\mathbb{Q}}$. Then as an application we will show that under certain technical assumptions, the Euler systems constructed in this workshop can be used to bound the Selmer group attached to an elliptic curve $E/\mathbb{Q}$ twisted by a two dimensional odd irreducible Artin representation ρ of $G_{\mathbb{Q}}$ whenever the Hasse–Weil–Artin L-function $L(E, \rho, s)$ is non-vanishing at $s = 1$. Let us start with some notations.

Let $p > 2$ be a rational prime number and F a finite extension of $\mathbb{Q}_p$. Write $\mathcal{O}$ for the ring of integers of F and let $\pi \in \mathcal{O}$ be an uniformizer. T will be a p-adic global Galois representation, i.e. a free $\mathcal{O}$-module of finite rank with a continious action of $G_{\mathbb{Q}}$. We define $V = T \otimes_{\mathcal{O}} F$, $W = V/T$ and let $T^* = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O}(1)), V^* = \mathrm{Hom}_F(V, F(1))$ and $W^* = V^*/T^*$ be the dual representations of T, V and W. Further, for $m \geq 0$ we set $W_m = \pi^{-m}T/T$.

2 Definition of Selmer groups

In this section we are going to attach Selmer groups to the above representations. These are given by subspaces of $H^1(\mathbb{Q}, V)$ (resp. $H^1(\mathbb{Q}, T)$, $H^1(\mathbb{Q}, W)$) cut out by local conditions which are given as follows. If $\ell \neq p$ is a finite place we let

$$H^1_f(\mathbb{Q}_\ell, V) = \ker(H^1(\mathbb{Q}_\ell, V) \to H^1(I_\ell, V))$$

be the unramified cocycles. Here $I_\ell = \mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell^{\mathrm{ur}}) \subseteq \mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ is the inertia subgroup. Define $H^1_f(\mathbb{Q}_\ell, T)$ (resp. $H^1_f(\mathbb{Q}_\ell, W)$) as the preimage (resp. image) of $H^1_f(\mathbb{Q}_\ell, V)$ with respect to the canonical maps $T \hookrightarrow V \twoheadrightarrow W$. In the same way, we let $H^1_f(\mathbb{Q}_\ell, W_m)$ be the preimage of $H^1_f(\mathbb{Q}_\ell, W)$ with respect to the inclusion $W_m \hookrightarrow W$. For $\ell = \infty$ we have $H^1(\mathbb{R}, M) = 0$ for $M = V, T, W$ or W_m since $p > 2$, so there is nothing to do.

For $\ell = p$ the choice of the local condition is much more subtle. It turns out that the unramified condition is not the correct choice when $\ell = p$ if we want arithmetically interesting Selmer groups. In this talk we consider the Bloch-Kato condition defined by

$$H^1_f(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes_{\mathbb{Q}_p} B_{cris})).$$

The definitions of the local conditions for T, W, W_m are given by preimages and images as above. We also make the analogous definitions for the dual representations.

In every case, we write $H^1_s(\mathbb{Q}_\ell, V) = H^1(\mathbb{Q}_\ell, V)/H^1_f(\mathbb{Q}_\ell, V)$ and likewise for T,W and W_m. We are now ready to define the Selmer groups.

Definition 12.1. Let Σ be a finite set of primes. The relaxed Selmer group is given by

$$\mathrm{Sel}^{\Sigma}(\mathbb{Q}, V) = \ker(H^1(\mathbb{Q}, V) \longrightarrow \prod_{l \notin \Sigma} H^1_s(\mathbb{Q}_\ell, V)).$$

The strict Selmer group is given by

$$\mathrm{Sel}_{\Sigma}(\mathbb{Q}, V) = \ker(\mathrm{Sel}^{\Sigma}(\mathbb{Q}, V) \longrightarrow \bigoplus_{l \in \Sigma} H^1(\mathbb{Q}_\ell, V)).$$

When Σ is empty set of primes, we omit it from the notation and just write $\mathrm{Sel}(\mathbb{Q}, V)$. In both cases, we have similar definitions for T, W, W_m and their duals.

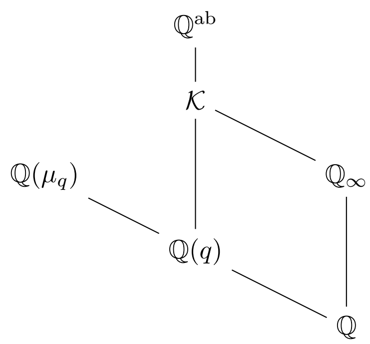
One motivation for the choosing the Bloch–Kato local condition at $\ell = p$ is that if $E/\mathbb{Q}$ is an elliptic curve and $T = T_pE$ is the p-adic Tate module attached to E, then the Selmer groups $\mathrm{Sel}(\mathbb{Q}, W_m)$ agree with the classical p^m-Selmer groups of $E/\mathbb{Q}$ as defined in [Sil09, Chapter X.4], where the local conditions are determined by the local Kummer maps. A good exposition of this can be found in the online notes [Bel09].

3 Rough definition of Euler systems

First we give a rather imprecise definition of an Euler system. For full details see [Rub00, Chapter II]. Let us assume that $\rho\colon G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathcal{O}}(T)$ is unramified at almost all places. Suppose that we are given a field $\mathcal{K}$ with the following properties:

- The cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ is contained in $\mathcal{K}$, so in particular $\mathcal{K}$ is an infinite extension over $\mathbb{Q}$.
- $\mathcal{K} \subseteq \mathbb{Q}^{\mathrm{ab}}$
- For almost all primes $q \equiv 1 \bmod p$ the following holds:
  Let $\mathbb{Q}(q)$ be the subextension of $\mathbb{Q}(\mu_q)$ s.t. $\mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q}(q))$ is the prime to p-part of $\mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q}) \cong (\mathbb{Z}/q\mathbb{Z})^*$. Then $\mathbb{Q}(q)$ is contained in $\mathcal{K}$.

We can summarize the properties in the following diagram:



Definition 12.2. An Euler system for $(T, \mathcal{K})$ is a collection of cohomology classes

$$\underline{c} = \{c_K \in H^1(K, T) \text{ s.t. } K \subseteq \mathcal{K} \text{ is a finite extension of } \mathbb{Q}\ \}$$

satisfying certain norm relations.

We will not specify the norm relations since they are not central for this talk. We will need the following hypothesis, called Hyp($\mathbb{Q}$,V):

(i) There exists a $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty}))$ s.t. $V/(\tau - 1)V$ is one dimensional over F.

(ii) V is an irreducible $F[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module.

The following theorem is due to Rubin, see [Rub00, II.2.3].

Theorem 12.3 (Rubin)**.** *Suppose that $\underline{c}$ is an Euler system for $(T, \mathcal{K})$ and assume that* $\mathrm{Hyp}(\mathbb{Q}, V)$ *is satisfied. If $c_{\mathbb{Q}} \notin H^1(\mathbb{Q}, T)_{\mathrm{tors}}$ then $\mathrm{Sel}_{\{p\}}(\mathbb{Q}, W^*)$ is finite.*

4 Idea of proof

In this section we give some ideas of the proof of Rubin's theorem. The strategy is to bound the finite sets $\mathrm{Sel}_p(\mathbb{Q}, W_m^*)$ independently of m.

By local duality we have $H^1_f(\mathbb{Q}_\ell, W_m)^\perp = H^1_f(\mathbb{Q}_\ell, W_m^*)$ under the local pairing

$$\langle -, - \rangle : H^1(\mathbb{Q}_\ell, W_m) \times H^1(\mathbb{Q}_\ell, W_m^*) \longrightarrow \mathcal{O}/\pi^m\mathcal{O}.$$

Let us fix a finite set of places Σ s.t. $p \notin \Sigma$. Consider the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Sel}^{\{p\}}(\mathbb{Q}, W_m) & \longrightarrow & \mathrm{Sel}^{\Sigma\cup\{p\}}(\mathbb{Q}, W_m) & \xrightarrow{\mathrm{loc}^s_\Sigma} & \bigoplus_{l\in\Sigma} H^1_s(\mathbb{Q}_\ell, W_m) \\
 & & & & & \times \\
0 \to & \mathrm{Sel}_{\{p\}\cup\Sigma}(\mathbb{Q}, W_m^*) & \longrightarrow & \mathrm{Sel}_{\{p\}}(\mathbb{Q}, W_m^*) & \xrightarrow{\mathrm{loc}^f_\Sigma} & \bigoplus_{l\in\Sigma} H^1_f(\mathbb{Q}_\ell, W_m^*), \\
 & & & & & \big\downarrow \\
 & & & & & \mathcal{O}/\pi^m\mathcal{O}
\end{array}$$

where the rows are exact by definition and the vertical map is given by the sum of the local pairings. Now global duality gives us $\mathrm{Im}(\mathrm{loc}^s_\Sigma)^\perp = \mathrm{Im}(\mathrm{loc}^f_\Sigma)$ (for an arbitrary Σ with $p \notin \Sigma$).

One proceeds by inductively constructing Σ and elements ("Kolyvagin derivatives") $\kappa \in \mathrm{Sel}^{\Sigma\cup\{p\}}(\mathbb{Q}, W_m)$ from the Euler system $\underline{c}$ in order to show that $\#\,\mathrm{coker}(\mathrm{loc}^s_\Sigma)$ and $\#\,\mathrm{Sel}_{\Sigma\cup\{p\}}(\mathbb{Q}, W_m^*)$ are both bounded independently of m.

Remark 12.4.

- For a fixed $m \geq 1$, one constructs Σ with primes ℓ s.t. the Frobenius Fr_ℓ is conjugate to $\tau \in \mathrm{Gal}(\mathbb{Q}(W_m)(\mu_{p^m})/\mathbb{Q})$, where $\mathbb{Q}(W_m)$ is the fixed field under the kernel of $\rho \bmod \pi^m$. Furthermore, one only make use of the norm relations modulo p^m.
- We use the assumption $\mathbb{Q}_\infty \subseteq \mathcal{K}$ to prove that the Kolyvagin derivative classes κ belong to $\mathrm{Sel}^{\Sigma\cup\{p\}}(\mathbb{Q}, W_m)$. If however there is another proof that these classes satisfy the correct local conditions then we can remove the assertion that $\mathbb{Q}_\infty \subseteq \mathcal{K}$. For instance one can prove good local properties of the Euler system classes studied in [LLZ14], which follow from the geometric construction of the classes. This leads to other variants of the definition of an Euler system (see [LLZ14, §7.1]).

5 Application to BSD

We will apply the Euler system machinery to the following case: Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication and ρ a two dimensional odd irreducible Artin representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The full details of this can be found in [LZ15, Chapter 8]. Suppose that

(0) $\mathrm{L}(E, \rho, 1) \neq 0$

and that the following technical assumptions are satisfied:

(i) $p \geq 5$

(ii) The conductors N_ρ of ρ and N_E of E are coprime and $p \nmid N_\rho N_E$.

(iii) E has good ordinary reduction at p.

(iv) $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{Z}_p}(T_p E)$ is surjective ("big image assumption").

(v) $\rho(\mathrm{Fr}_p)$ has distinct eigenvalues.

Theorem 12.5 ([KLZ15a, Theorem 11.7.4]). *Let K be the splitting field of ρ. Under the above assumptions we have that $Hom_{\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q}_p)]}(\rho, \mathrm{Sel}_{p^\infty}(E/K))$ is finite.*

The important steps in the proof are the following:

Step 1. By modularity theorems due to [Wil95], [TW95], [BCDT01] (resp. [KW09]) it is known that E corresponds to a modular form f of weight 2 (resp. ρ corresponds to a modular form g of weight 1). We also remark that by a theorem of Serre [Ser72], since E does not have complex multiplication, condition (iv) holds for all but finitely many primes p.

Step 2. Suppose that α and β are the roots of the Hecke polynomial of f s.t. $v_p(\alpha) = 0$, which is possible by the ordinary assumption on f. Similarly let γ and δ be the roots of the Hecke polynomial of g. Note that $\gamma \neq \delta$ by assumption (v). Let f_α be the p-stabilization of f with U_p eigenvalue α. Define f_β, g_γ and g_δ similarly. Further, let $\underline{f_\alpha}$ (resp. $\underline{g_\gamma}$) be a Hida family through f_α (resp. g_γ). In the preceding talks of this workshop we constructed elements

$$_c\mathrm{BF}_{\underline{m}}^{\underline{\mathrm{f}_\alpha},\underline{\mathrm{g}_\gamma}} \in H^1(\mathbb{Q}(\mu_m), M(\underline{f_\alpha} \otimes \underline{g_\gamma})^* \otimes \Lambda_\Gamma(-\boldsymbol{j})).$$

We can then specialise these classes at weights $(2, 1)$ and $j = 0$ to obtain classes

$$_c\mathrm{BF}_m^{\mathrm{f}_\alpha,\mathrm{g}_\gamma} \in H^1(\mathbb{Q}(\mu_m), M(f_\alpha \otimes g_\gamma)^*).$$

They form an Euler system for the Galois representation $V = M(f_\alpha \otimes g_\gamma)^* \cong M(f \otimes g)^*$. (To be exact, the norm relations for an Euler system hold only up to congruence, but this is all we need.) We denote this Euler system $\underline{c}^{\alpha\gamma}$.

Step 3. We can use the explicit reciprocity laws from the previous talks to show that the bottom class $c_{\mathbb{Q}}^{\alpha\gamma}$ of the Euler system $\underline{c}^{\alpha\gamma}$ is non-torsion and hence we get a bound for the strict Selmer group $\mathrm{Sel}_{\{p\}}(\mathbb{Q}, W^*)$ by Rubin's theorem. Here we need to use the "big image assumption" (*iv*) to verify that Hyp($\mathbb{Q}$,V) is satisfied.

Using the boundedness of $\mathrm{Sel}_{\{p\}}(\mathbb{Q}, W^*)$ we can bound $\mathrm{Sel}(\mathbb{Q}, W^*)$. In the following we are going to study the difference between $\mathrm{Sel}_{\{p\}}(\mathbb{Q}, W^*)$ and $\mathrm{Sel}(\mathbb{Q}, W^*)$. In [Rub00, II.2.10], Rubin explains how to extend his theorem to obtain a bound for $\mathrm{Sel}(\mathbb{Q}, W^*)$, in the case where $H^1_s(\mathbb{Q}_p, V)$ and $H^1_f(\mathbb{Q}_p, V^*)$ are one-dimensional over F. In our case, the Bloch-Kato subspace is two-dimensional and we use an idea due to Darmon and Rotger (see [DR]) to overcome this difficulty. We remark that for the rest of this argument we only the bottom class of our Euler system. Consider the following diagram,

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Sel}(\mathbb{Q},T) & \longrightarrow & \mathrm{Sel}^{\{p\}}(\mathbb{Q},T) & \xrightarrow{\mathrm{loc}^s} & H^1_s(\mathbb{Q}_p,T) \\
 & & & & & \times \\
0 \longrightarrow & \mathrm{Sel}_{\{p\}}(\mathbb{Q},W^*) & \longrightarrow & \mathrm{Sel}(\mathbb{Q},W^*) & \xrightarrow{\mathrm{loc}^f} & H^1_f(\mathbb{Q}_p,W^*) \\
 & & & & & \downarrow \\
 & & & & & F/\mathcal{O}
\end{array}$$

By global duality we have $\mathrm{Im}(\mathrm{loc}^s)^\perp = \mathrm{Im}(\mathrm{loc}^f)$ and $H^1_f(\mathbb{Q}_p,T)^\perp = H^1_f(\mathbb{Q}_p,W^*)$. Here we are implicitly using the fact that the Bloch–Kato condition on V and V^* give subspaces of $H^1(\mathbb{Q}_p,V)$ and $H^1(\mathbb{Q}_p,V^*)$ respectively, which are orthogonal complements under the local duality pairing. To show that $\mathrm{Sel}(\mathbb{Q},W^*)$ is finite it suffices to show that loc^s is surjective.

We consider the (F-linear) dual exponential map

$$\exp^*\colon H^1_s(\mathbb{Q}_p,V) \longrightarrow \mathrm{Fil}^0 D_{cris}(V).$$

Using the explicit reciprocity laws one can show that the image of $c^{\alpha\gamma}_{\mathbb{Q}} \in \mathrm{Sel}^{\{p\}}(\mathbb{Q},V)$ is non-trivial. Unfortunately, $H^1_s(\mathbb{Q}_p,V)$ is two dimensional. As a solution to show the surjectivity we will additionally use $c^{\alpha\delta}_{\mathbb{Q}_p}$.

Let us write

$$D_{cris}(M(f)^*) = D_{cris}(M(f)^*)^\alpha \oplus D_{cris}(M(f)^*)^\beta$$

and

$$D_{cris}(M(g)^*) = D_{cris}(M(g)^*)^\gamma \oplus D_{cris}(M(g)^*)^\delta$$

for the φ-eigenspace decompositions. For $V = M(f)^* \otimes M(g)^*$ as above we get

$$D_{cris}(V) = D^{\alpha\gamma} \oplus D^{\alpha\delta} \oplus D^{\beta\gamma} \oplus D^{\beta\delta}.$$

Finally, define $D_{cris}(V)^\alpha = D^{\alpha\gamma} \oplus D^{\alpha\delta}$. In this situation the projection

$$\mathrm{pr}_\alpha\colon \mathrm{Fil}^0(D_{cris}(V)) \longrightarrow D_{cris}(V)^\alpha$$

is an isomorphism and the kernels of the two maps

$$\begin{array}{ccccc}
 & & & \overset{\mathrm{pr}_\gamma}{\nearrow} & D^{\alpha\gamma} \\
H^1_s(\mathbb{Q}_p,V) & \xrightarrow{\exp^*} & \mathrm{Fil}^0(D_{cris}(V)) & & \\
 & & & \underset{\mathrm{pr}_\delta}{\searrow} & D^{\alpha\delta}
\end{array}$$

span $H^1_s(\mathbb{Q}_p,V)$. Further, we have $\mathrm{pr}_\gamma \circ \exp^*(c^{\alpha\gamma}_{\mathbb{Q}_p}) = 0$, so $\mathrm{pr}_\alpha \circ \exp^*(c^{\alpha\gamma}_{\mathbb{Q}_p}) \in D^{\alpha\delta}$. In the same way we get $\mathrm{pr}_\alpha \circ \exp^*(c^{\alpha\delta}_{\mathbb{Q}_p}) \in D^{\alpha\gamma}$ and each of these elements are non-zero if and only if $\mathrm{L}(f,g,1) \neq 0$ by the explicit reciprocity laws. This proves the surjectivity.

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Program

Workshop on The Arithmetic of Euler Systems

	Saturday 22	Sunday 23	Monday 24	Tuesday 25
9h-10.30h		Introduction to modular curves (Y. Kezuka)	Siegel units and Eisenstein classes (A. Cauchi)	Compatibility in p-adic families (F. Fite)
10.30h-11h		Coffee break	Coffee break	Coffee break
11h-12.30h		Hida theory (C. Williams)	Definition of the Euler systems (H. Guhanvenkat)	Norm-compatibility relations (V. Pal)
17h-18h			(2a) S. Molina	(3a) TBA
18h-18.30h			Break	Break
18.30h-19.30h		Overview (D. Loeffler)	(2b) S. Zerbes	(3b) F. Castella
19.30h-22h	Wine and cheese reception			

	Wednesday 26	Thursday 27	Friday 28	Saturday 29
9h-10.30h	Rest day/ Hike	Bloch–Kato / Perrin-Riou theory (B. Joyal)	Syntomic and FP cohomology (L. Gehrmann)	Proofs of the explicit reciprocity laws (G. Rosso)
10.30h-11h		Coffee break	Coffee break	Coffee break
11h-12.30h		P-adic Eichler–Shimura isomorphisms (Y. Wu)	Evaluation of the regulators (N. Dogra)	Applications: bounding Selmer and Sha (J. Lamplugh)
17h-18h		(5a) C.-H. Kim	(6a) H. Darmon	
18h-18.30h				
18.30h-19.30h		(5b) C. Skinner	(6b) V. Rotger	
20h-23h		Conference dinner		

Participants

List of Participants

AGBOOLA, Adebisi	University of California, Santa Barbara
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BERGUNDE, Felix	University Bielefeld
BIRKBECK, Christopher	University of Warwick
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CAUCHI, Antonio	University College London
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DE VERA PIQUERO, Carlos	Universität Duisburg-Essen
DEMBELE, Lassina	Warwick University
DOGRA, Netan	Oxford
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GEHRMANN, Lennart	Universität Duisburg-Essen
GUHANVENKAT, Harikumar	Universita di Padova
GUITART, Xavier	Universitat de Barcelona
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PURKAIT, Soma	Kyushu University
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WILLIAMS, Chris	University of Warwick
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