Grothendieck topologies and their application to rigid geometry

Cameron Franc, Marc Masdeu

October 8, 2009

Abstract

This short note is the rough draft of the material presented in the Rigid Seminar, Fall 2009. In the first part we have used as references the notes by Ghitza and Archibald ([AG03]), notes on étale cohomology from a course given by Iovita ([Iov08]), and some unreliable sources (namely, the Wikipedia). For the second part we have used lecture notes by S. Bosch ([Bos]), the notes of B. Conrad ([Con08]) and the paper of J. Tate ([Tat71]).

1 Grothendieck Topologies

1.1 Abstracting sheaf theory

Let X be a topological space. That is, a giving of a collection of **open sets**, satisfying:

1. $\emptyset$ and X are opens.
2. The **intersection** of two opens is open.
3. The arbitrary union of opens is open.

Recall the definition of a presheaf on X (say, of sets). It is a rule F which assigns to every open set U a set $F(U)$, and given two open sets $U \subseteq V$, one has a *restriction* map of sets $\rho_{V,U}\colon F(V) \to F(U)$, satisfying the obvious transitivity relations (that is, $\rho_{U,U} = \mathrm{Id}_{F(U)}$, and $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$ for all $W \subseteq V \subseteq U$. If we want to be more pedantic –and in this case we definitely want–, we can define a presheaf as a *contravariant* functor F from the category X_{top} to **Sets**, where X_{top} is the category having as objects the open sets $U \subseteq X$, and for $U, V \subseteq X$, we have:

$$\mathrm{Hom}(U,V) = \begin{cases} \emptyset & U \nsubseteq V \\ \{*\} & U \subseteq V. \end{cases}$$

A presheaf F is called a *sheaf* if it satisfies the following property: for every **open** U and every **open covering** $U = \cup_i U_i$, the following sequence of sets is exact:

$$F(U) \xrightarrow{\alpha} \prod_i F(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} F(U_i \cap U_j).$$

The map α is the product of the restrictions $F(U) \to F(U_i)$, and the maps β, γ are defined as follows: $\beta((f_i)_i) = (g_{ij})_{i,j}$, where:

$$(g_{ij})_{i,j} := f_i|_{U_i \cap U_j},$$

and $\gamma((f_i)_i) = (h_{ij})_{i,j}$, with:

$$(h_{ij})_{i,j} := f_j|_{U_i \cap U_j}.$$

By *exact* we mean that:

1. The map α is injective, and
2. The image of α is the *equalizer* of β and γ.

Again in more informal terms, this means:

1. If two sections $f, g \in F(U)$ agree on U_i for all i, then $f = g$.
2. If we are given elements $f_i \in F(U_i)$ such that they agree on the intersections, then there is an element $f \in F(U)$ (which is necessarily unique, because of the previous condition) that gives all the f_i by restriction.

These two properties are together called the sheaf axiom, and can be summarized as follows: the sections of a sheaf are determined by their restrictions to coverings.

We want to be able to do sheaf theory without necessarily having a topology, or allowing more or less opens, and more or less coverings.

1.2 Grothendieck's idea

The most general definition of Grothendieck topology uses the concept of *sieves*. Since the author didn't know of this definition before preparing the lecture, this is a good excuse to introduce it:

Definition 1.1. Let $\mathbf{C}$ be a category. A *sieve* S on an object X of $\mathbf{C}$ is a subfunctor of $\mathrm{Hom}(-, X)$.

In more down-to-earth words, it is a rule that selects, for every object X' of $\mathbf{C}$, a subset $S(X')$ of $\mathrm{Hom}(X', X)$.

Let S be a sieve on X, and let $f\colon Y \to X$ be a morphism in $\mathbf{C}$. Then we can construct a sieve f^*S on Y, by:

$$(f^*S) := S \times_{\mathrm{Hom}(-,X)} \mathrm{Hom}(-, Y).$$

That is, given an object Y',

$$(f^*S)(Y') = \{g \in \mathrm{Hom}(Y', Y) \mid f \circ g \in S(Y')\}.$$

Here is the definition of Grothendieck topology:

Definition 1.2. A **Grothendieck topology** T consists of the following data: a category, denoted $\mathrm{Cat}\, T$, along with a collection of *covering sieves*, denoted $\mathrm{Cov}\, T$. This means that, for each object X of $\mathrm{Cat}\, T$, there is a distinguished collection of sieves on X. These are subject to the following axioms:

1. (identity): The functor $\mathrm{Hom}(-, X)$ is a covering sieve on X, for each object X.

2. (base change): If S is a covering sieve on X and $f\colon Y \to X$ is a morphism, then f^*S is a covering sieve on Y.

3. (local character): Let S be a covering sieve on X, and let R be any sieve. If for each object Y and each morphism $f \in S(Y)$ the pullback f^*R is a covering sieve on Y, then R is a covering sieve on X.

Assume now that the category $\mathbf{C}$ has fiber products. Then

Definition 1.3. A **Grothendieck topology** T consists of the following data: a category, denoted $\operatorname{Cat} T$, along with a collection of *coverings*, denoted $\operatorname{Cov} T$. By a collection of coverings we mean that $\operatorname{Cov} T$ contains families of morphisms:

$$\{\phi_i\colon U_i \to U\}_{i\in I},$$

where U, U_i are objects in $\operatorname{Cat} T$ and ϕ_i is a morphism in $\operatorname{Cat} T$. These families must satisfy the following axioms:

1. (trivial covering):If $V \to U$ is an isomorphism in $\operatorname{Cat} T$, then $\{V \to U\} \in \operatorname{Cov} T$.

2. (base change): If $\{\phi_i\colon U_i \to U\}_{i\in I} \in \operatorname{Cov} T$ and $V \to U$ is any morphism in $\operatorname{Cat} T$, then $U_i \times_U V$ exists for each $i \in I$ and:

$$\{U_i \times_U V \to V\}_{i\in I} \in \operatorname{Cov} T.$$

3. (local character): If $\{\phi_i\colon U_i \to U\}_{i\in I} \in \operatorname{Cov} T$ is such that for each $i \in I$, there exists $\{\phi_{ij}\colon V_{ij} \to U_i\}_{j\in J_i} \in \operatorname{Cov} T$, then:

$$\{\phi_i \circ \phi_{ij}\colon V_{ij} \to U\}_{i\in I,\ j\in J_i} \in \operatorname{Cov} T.$$

Definition 1.4. A **presheaf** of sets on T is a contravariant functor $F : \operatorname{Cat} T \to \mathbf{Sets}$.

Definition 1.5. A **sheaf** of sets on T is a presheaf F of sets such that for every $\{\phi_i\colon U_i \to U\}_{i\in I} \in \operatorname{Cov} T$, the sequence:

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \times_U U_j)$$

is exact in the sense discussed above. Note that now β and γ are induced by F applied to the natural maps $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$.

1.3 Examples

We discuss some examples of Grothendieck topologies. In the next talk we will see the particular examples in which we are interested.

1. Let X be a topological space. Let $\operatorname{Cat} T$ be the category with objects the open subsets of X and morphisms inclusion. Let $\operatorname{Cov} T$ be the collection of open coverings of open subsets of X. Then T is a Grothendieck topology.

   The first axiom of a covering corresponds to the fact that U is itself a covering of an open subset $U \subset X$. The second says that if you have an open covering of U by U_i's, and an open covering of each U_i, then putting all of these together gives an open covering of U. The third axiom says that if $\{U_i\}$ is a covering of U, then $\{V \cap U_i\}$ is a covering of $U \cap V$.

2. Let X be a set. Let $\mathcal{U}$ be a collection of subsets of X, which is closed under finite intersection. Define a Grothendieck topology as before. Note that $\mathcal{U}$ can be a proper subcollection of the opens that make X into a topological space. The notion of Grothendieck topology still makes sense, and this is an example of a topology which is "slightly coarser" than that of the previous example: they will lead to the same sheaf theory.

3. Let G be a finite group. Let $\operatorname{Cat} T_G$ be the category of left G-sets. That is, finite sets with a left-action of G. Let $\operatorname{Cov} T_G$ denote the collection of surjective families of morphisms in $\operatorname{Cat} T_G$. Then T_G is a Grothendieck topology. Let's work out what is a sheaf of sets on T_G. Consider G as a G-set, with left-action given by multiplication. Call this $\langle G\rangle$, to distinguish it from the group G. Let $M := F(\langle G\rangle)$. This is by definition a set, and we will describe an action of G on it. Given $\sigma \in G$, define $f_\sigma \colon \langle G\rangle \to \langle G\rangle$ by:

   $$f_\sigma(g) := g\sigma.$$

   Note that $f_\sigma(hg) = hg\sigma = h(g\sigma) = hf_\sigma(g)$, so that f_σ is G-equivariant. This gives a map $F(f_\sigma)\colon M \to M$, and we define:

   $$\sigma \cdot m := F(f_\sigma)(m).$$

   Given $\sigma, \tau \in G$, note that $f_{\sigma\tau} = f_\tau \circ f_\sigma$. Then compute:

   $$\sigma \cdot (\tau \cdot m) = F(f_\sigma) \circ F(f_\tau)(m) = F(f_\tau \circ f_\sigma)(m) = F(f_{\sigma\tau})(m) = (\sigma\tau) \cdot m.$$

   So we have defined a left action of G on M.

   Claim. *The functor F is naturally isomorphic to* $\operatorname{Hom}_G(-, M)$.

   Proof. Let S be any G-set. Suppose that we write S as a disjoint union, $S = \cup_i S_i$. Then there is a covering $j_i\colon S_i \to S$. Applying the sheaf axiom and noticing that $S_i \times_S S_j$ is nonempty only when $i = j$ and that in this case the two projections coincide, we obtain that $F(S) \simeq \prod_i \operatorname{Hom}_G(S_i, M) = \operatorname{Hom}_G(S, M)$. Hence we can assume without loss of generality that S is a *transitive* G-set.

   Fix a point $a \in S$. Then $p\colon \langle G\rangle \to S$, mapping $g \mapsto g \cdot a$, is a G-equivariant surjective map. Consider the sheaf axiom applied to this covering. It says that the sequence:

   $$F(S) \to M \to F(S \times_{S,p} S)$$

   is exact. In particular, $F(S)$ is a subset of M. Now, if $h \in H$, then note that $p \circ f_h = p$. Hence $F(p \circ f_h) = F(p)$, and so, given $x \in F(S)$ and $g \in G$:

   $$g \cdot (F(p)(x)) = F(f_g) \circ F(p)(x) = F(p \circ f_g)(x) = F(p)(x),$$

hence the image of $F(S)$ under the injective map α lies in M^H, the subset of elements of M fixed by H.

Claim. *The image of $F(S)$ under α* equals M^H.

Proof. For this, we need to analyze the map $F(\langle G\rangle \to F(\langle G\rangle \times_{S,p} \langle G\rangle)$.

Claim. *The map $\phi\colon \langle G\rangle \times_{S,p} \langle G\rangle \to \langle G\rangle \times H_0$ which sends (x,y) to $(x, x^{-1}y)$ is a G-equivariant isomorphism (here H_0 is the set H with trivial G-action).*

Proof. First, note that if (x,y) is an element in the fiber product, then by hypothesis $p(x) = p(y)$. That is $x \cdot a = y \cdot a$, and hence $x^{-1}y$ fixes a. So the map is well defined. Now, let $g \in G$. Then:

$$\phi(gx, gy) = (gx, (gx)^{-1}gy) = (gx, x^{-1}y),$$

so that with the given G-actions, ϕ is G-equivariant. □

The two projections translate to the two maps $\beta_1\colon \langle G\rangle \times H_0 \to \langle G\rangle$ and $\gamma_1\colon \langle G\rangle \times H_0 \to \langle G\rangle$ given by $\beta_1(g,h) = g$ and $\gamma_1(g,h) = gh$.

We apply again the sheaf action to the surjective family $\{\iota_h\colon \langle G\rangle \to \langle G\rangle \times H_0\}_{h\in H}$ given by $\iota_h(g) := (g,h)$. This gives an exact sequence:

$$F(\langle G\rangle \times H_0) \to \prod_{h\in H} F(\langle G\rangle) \to \prod_{h\in H} F(\langle G\rangle \times_{\iota_h} \langle G\rangle).$$

The last term is what we have written because for $h \neq h' \in H$, the corresponding fiber product is the empty set. But the two projections

$$\langle G\rangle \times_{\iota_h} \langle G\rangle \to \langle G\rangle$$

are equal, since elements in the left-hand side look like pairs (x,y) such that $(x,h) = (y,h)$. That is, pairs in the diagonal! This makes the two maps induced when applying F to be the same, and hence:

$$F(\langle G\rangle \times H_0) \cong \prod_{h\in H} F(\langle G\rangle).$$

Finally, we have obtained:

$$F(S) \to F(\langle G\rangle) \to \prod_{h\in H} F(\langle G\rangle).$$

Now, note that $p_1 \circ \iota_h = \mathrm{Id}_{\langle G\rangle}$ and $p_2 \circ \iota_h = f_h$, that is multiplication by h on the right. Also, $F(\iota_h) = \pi_h$, the projection in to the h-component. Hence:

$$F(p_1)(m) = (m, m, \ldots, m), \qquad F(p_2)(m) = (h_1 m, h_2 m, \ldots, h_t m). \tag{1}$$

Then, the values on which $F(p_1)$ agrees with $F(p_2)$ are the elements $m \in M$ such that $hm = m$ for all $h \in H$. That is, exactly M^H, as claimed. □

We have proven so far that $F(S) = M^H \cong \mathrm{Hom}_G(S, M)$, where the last isomorphism is given by $m \mapsto [a \mapsto m]$. The composition of the two identifications doesn't depend on which point a of S was chosen, and it is indeed functorial. □

4. Consider the previous example. Let $\mathbf{Sh}(T_G)$ be the category of *abelian* sheaves on T_G. Its objects are the sheaves described in the previous example, but taking values on $\mathbf{Ab}$, the category of abelian groups. The morphisms are morphisms of the underlying presheaves (that is, natural transformations of contravariant functors). Associated to an object F of $\mathbf{Sh}(T_G)$ one defines a G-set M as in the previous example. But since F takes values on $\mathbf{Ab}$, the G-set M has an abelian group structure, and thus becomes a G-module. One then proves that this defines an equivalence of categories $\mathbf{Sh}(T_G) \simeq \mathbf{Mod}_G$, where $\mathbf{Mod}_G$ is the category of G-modules. There is a "global sections" functor $\Gamma\colon \mathbf{Sh}(T_G) \to \mathbf{Ab}$ which on objects maps $F \mapsto F(\{*\})$. Also, there is a "G-invariants" functor $(-)^G\colon \mathbf{Mod}_G \to \mathbf{Ab}$ which to M it associates M^G. We have seen that these two functors naturally isomorphic using the equivalence above. Therefore the right derived functors of Γ and of $(-)^G$ are also naturally isomorphic. We conclude that the sheaf cohomology on this Grothendieck topology computes group cohomology in the usual sense. We thus recovered using sheaf theory, another notion which a priori has no connection to sheaf theory.

5. Let X be a Noetherian connected scheme. Let $\mathrm{Cat}\, T_{X^{\mathrm{et}}}$ denote the category of schemes Y of finite type and étale over X. Let $\mathrm{Cov}\, T_{X^{\mathrm{et}}}$ denote the collection of surjective families of morphisms indexed by a finite set. Then $T_{X^{\mathrm{et}}}$ is a Grothendieck topology.

1.4 Questions

Basic: Formulate the equivalence of the two definitions of Grothendieck topology, in the case of having fiber products.

Challenge: In Grothendieck topologies there is a notion of *slightly finer* topology. Find its definition, and understand why this notion doesn't appear in classical topology.

2 Rigid topology

2.1 Recall of some definitions

Let $\mathbb{K}$ be a nonarchimedean valued field which is complete for a nontrivial valuation, and write $\mathcal{O}_{\mathbb{K}}$ for the valuation ring. If the residue field of $\mathbb{K}$ is of characteristic p, we suppose that $|p| = 1/p$. Let A be a $\mathbb{K}$-affinoid algebra, and let $X = M(A)$ be the max-spectrum of A. Recall that Xander defined the canonical topology on X. We'd like to recall the definitions of some important subsets of $M(A)$ which are open in the canonical topology.

Definition 2.1. Let $f_1, \dots, f_r \in A$ and let $g_1, \dots, g_s \in A$. A *Laurent domain* in X is a subset of the form

$$X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X \ : \ |f_i(x)| \leq 1, \ |g_j(x)| \geq 1 \text{ for all } i, j\}.$$

This notation does not reflect particularly well that we allow no g_j's to appear in a Laurent domain. In such cases, the set is also called a *Weierstrass domain* and denoted

$$X(f_1, \dots, f_r) = \{x \in X \ : \ |f_i(x)| \leq 1 \text{ for all } i\}.$$

Finally let $f_0, f_1, \dots, f_r \in A$ generated the unit ideal of A. Then the set

$$X(f_1/f_0, \dots, f_r/f_0) = \{x \in X \ : \ |f_i(x)| \leq |f_0(x)|\}$$

is called a *rational domain* in X.

These sets can be shown to be open in the canonical topology, and in fact, they are admissible open subsets. We'll recall the definition of admissibility in a moment, but we'd first like to give some examples.

Example. For simplicity suppose that $\mathbb{K} = \mathbb{C}_p$. Take $A = \mathbb{K}\langle X \rangle$ and consider the functions $f = (X - a)/p^n$ for some $a \in \mathcal{O}_{\mathbb{K}}$ and $n \geq 0$. Let $g_j = (X - b_j)/p^m$ for some $b_1, \dots, b_s \in \mathcal{O}_{\mathbb{K}}$ and $m \geq 0$. Then if we identify X with the unit ball in $\mathbb{K}$, which we are free to do thanks to the assumption $\mathbb{K} = \overline{\mathbb{K}}$, then

$$X(f, g_1^{-1}, \dots, g_s^{-1}) = \{z \in \mathcal{O}_{\mathbb{K}} \ : \ |z - a| \leq 1/p^n, \ |z - b_j| \geq 1/p^m \text{ for all } j\}.$$

If $m \geq n$ and the points b_j are inside the ball $|z - a| \leq p^n$, then this set looks like a closed ball with disks removed about each b_j.

Example. This example shows why the assumption that $f_0, \dots, f_r \in A$ are relatively prime is necessary to ensure that the corresponding rational domain is open in the canonical topology. Take $\mathbb{K}$ to be algebraically closed and $A = \mathbb{K}\langle X \rangle$, and put $f_0 = X$ and $f_1 = X^2$. These are not relatively prime in A. Note that

$$\{z \in \mathcal{O}_{\mathbb{K}} \ : \ |z^2| \leq |z|\} = \{0\} \amalg \{z \in \mathcal{O}_{\mathbb{K}} \ : \ |z| = 1\}.$$

If this were an open subset, it would imply that $\{0\}$ is open, hence that the topology on $\mathbb{K}$ is discrete. This is contrary to our assumptions on $\mathbb{K}$.

We now wish to recall the important notion of an affinoid subdomain of X.

Definition 2.2. A subset $U \subseteq X$ is called an *affinoid subdomain* if there exists a morphism of affinoid $\mathbb{K}$-spaces $\iota\colon X' \to X$ such that $\iota(X') \subseteq U$ and such that the following universal property holds: every morphism of affinoid $\mathbb{K}$- spaces $\phi\colon Y \to X$ which lands in U admits a unique factorization through $\iota\colon X' \to X$.

It turns out that if U is an affinoid subdomain, and $\iota\colon X' \to X$ is the corresponding map of affinoid $\mathbb{K}$-spaces, then ι identifies X' with U. This identification is a homeomorphism with respect to the canonical topologies on X and X'. If B is the affinoid algebra of X', so that $X' = M(B)$, then B is uniquely determined up to unique isomorphism, thanks to the universal property defining U. We will thus identify U with X' and, following Conrad, let A_U denote the affinoid algebra defining U. We thus have an association $U \mapsto A_U$ which is in fact a sheaf on the "weak Grothendieck topology" of X; the next section will make this discussion precise.

One can show that Laurent and rational domains are examples of affinoid subdomains. Last time Xander wrote down the affinoid algebra of a Laurent or rational domain. It turns out that every affinoid subdomain is open in the canonical topology.

The following result is sometimes useful, but is not easy to prove. For details see section 1.8 of the Bosch notes on rigid geometry.

Theorem 2.3 (Gerritzen-Grauert)**.** *If X is an affinoid $\mathbb{K}$-space, then every affinoid subdomain of X can be expressed as a finite union of rational subdomains.*

We will see an application of this theorem in the proof of Tate's acyclicity theorem below.

2.2 Weak Grothendieck topology

Let $X = M(A)$ be an affinoid $\mathbb{K}$-space. We begin this section by using affinoid subdomains to define a Grothendieck topology on X.

Definition 2.4. The *weak Grothendieck topology* on X consists of the following data: the category underlying the weak topology has as objects all affinoid subdomains of X, and inclusions as morphisms. The coverings are finite families $(U_i \to U)_{i\in I}$ of affinoid subdomains such that $U = \bigcup_i U_i$.

This topology is sufficient for defining a sheaf of affinoid functions on an affinoid space. It is not sufficient, however, if one wishes to glue affinoid pieces to make general rigid spaces. This is why one calls it the *weak* Grothendieck topology. Below we will examine the example of the open unit ball in $\mathbb{K}$. It is not affinoid, hence not admissible open in the weak Grothendieck topology, but it will be seen to be an admissible open in the strong Grothendieck topology. Any theory of rigid geometry should be sufficiently flexible to handle as natural a space as the open unit ball. This gives one concrete reason why the weak topology is not good enough for our purposes.

This begs the questions of why one would consider the weak topology at all. Conrad skirts the issue entirely in his article. One reason is as follows: it is easy to define a presheaf of affinoid functions on X by setting $\mathcal{O}_X(U) = A_U$ for every affinoid subdomain U of X. It is not clear that this is a sheaf for the weak topology, but it turns out to be the case. This is the content of Tate's acyclicity theorem. One can then show, in an entirely formal manner, that since the "strong

Grothendieck" topology on X is not too different from the weak topology. More precisely, one can show that each sheaf on the weak topology extends uniquely to a sheaf on the strong topology. Thus, if one were to omit discussion of the weak topology in an exposition of rigid geometry, one would be forced to define the sheaf $\mathcal{O}_X$ directly on the strong topology, and then also argue that it is indeed a sheaf. I've never seen this done directly; in most expositions, for instance [BGR84] and [FVDP04], one travels a detour through the weak topology to obtain the sheaf of rigid analytic functions on an affinoid space.

This should not seem so strange. This mirrors one possible exposition of scheme theory. One of the first challenges of that theory is to define a sheaf of regular functions on an affine scheme. Let $Y = \operatorname{Spec}(B)$ be an affine scheme. Let $\mathcal{B}$ denote the basis for the Zariski topology consisting of the basic open subsets of Y,

$$\mathcal{B} = \{Y_f \ : \ f \in B\}.$$

One can then define a "$\mathcal{B}$-sheaf" on Y by putting $\mathcal{O}_Y(Y_f) = B_f$. This is nothing but a sheaf in the Grothendieck topology on Y given by the open subsets in $\mathcal{B}$. Then one can argue that such a sheaf extends uniquely to a sheaf on Y in the full Zariski topology. This is precisely the same argument that is used to prove that a sheaf on the weak Grothendieck topology of an affinoid space extends uniquely to a sheaf on the strong topology.

2.3 Tate's acyclicity theorem

The sole purpose of this section is to give an outline of the proof of the following theorem, first proved by Tate in his paper "Rigid analytic spaces".

Theorem 2.5 (Tate's acyclicity theorem)**.** *Let X be an affinoid $\mathbb{K}$-space. The presheaf $\mathcal{O}_X$ is a sheaf for the weak Grothendieck topology.*

Tate proved slightly more than what is stated above. He proved that the Čech complex of such a covering with values in the presheaf $\mathcal{O}_X$ is acyclic. Since the sheaf exact sequence consists of the first terms of the augmented Čech complex, this acyclicity implies that $\mathcal{O}_X$ is a sheaf. To be precise, Tate only proved this for Laurent domains, which he called special affine subsets of X, and finite coverings by Laurent domains. At the time he was lacking the technology of the Gerritzen-Grauert theorem, which allows one to extend Tate's proof to the weak Grothendieck topology. We outline the steps involved in the proof as presented in section 1.9 of the Bosch notes.

Outline of proof. Given an affinoid subdomain of $U \subseteq X$ and a finite covering by affinoid subdomains $U_i \to U$, such that $U = \bigcup_i U_i$, one must show that the sequence

$$0 \to \mathcal{O}_X(U) \to \prod_{i \in I} \mathcal{O}_X(U_i) \to \prod_{i,j \in I} \mathcal{O}_X(U_i \cap U_j)$$

is exact.

1. Bosch first defines a particular type of covering by rational subdomains, called a *rational covering.* If $f_0, \dots, f_r \in A_U$ is a collection of functions which generate the unit ideal, then the rational subdomains

$$U_i = U(f_0/f_i, \dots, f_r/f_i), \quad i = 0, \dots, r$$

give a finite covering $(U_i \to U)_{i=0}^r$ of U. Then using the Gerritzen-Grauert theorem, the problem is reduced to considering such a rational covering.

2. Reduce from considering rational coverings to considering Laurent coverings. From this point onward, the proof follows Tate's original argument.

3. Use induction to reduce to the case of considering a Laurent covering of the type $U(f), U(f^{-1})$ for a single $f \in A_U$.

4. In this case, carry out the computation by brute force. This means showing that the sequence

$$0 \to A_U \xrightarrow{\varepsilon} A_U\langle f\rangle \times A_U\langle f^{-1}\rangle \xrightarrow{\delta} A_U\langle f, f^{-1}\rangle,$$

where $\varepsilon(f) = (f|_{U(f)}, f|_{U(f^{-1})})$ and $\delta(f,g) = f|_{U(f,f^{-1})} - g|_{U(f,f^{-1})}$ is exact.

□

2.4 Strong Grothendieck topology

In this final section we define the strong Grothendieck topology, to pave the way for the construction of global rigid spaces by glueing.

Definition 2.6. Let $X = M(A)$ be an affinoid $\mathbb{K}$-space. Then the *strong Grothendieck topology* on X consists of the following data:

1. A subset $U \subseteq X$ is said to be *admissible open* if there is a covering, possibly infinite, $U = \bigcup_{i\in I} U_i$ such that each $U_i \subseteq X$ is an affinoid subdomain, and the covering satisfies the following property: for every morphism of affinoid $\mathbb{K}$-spaces $\phi\colon Z \to X$ such that $\phi(Z) \subseteq U$, the covering $(\phi^{-1}(U_i))_{i\in I}$ admits a refinement which is a *finite* covering of Z by affinoid subdomains. The underlying category of the strong Grothendieck topology is the collection of admissible open subsets of X with inclusions as morphisms.

2. A covering $V = \bigcup_{j\in J} V_j$ of an admissible open subset $V \subseteq X$ is said to be *admissible* if each V_j is an admissible open subset of X and if for each morphism of affinoid $\mathbb{K}$-spaces $\phi\colon Z \to X$ such that $\phi(Z) \subseteq V$, the covering $(\phi^{-1}(V_j))_{j\in J}$ of Z admits a refinement which is a finite covering of Z by affinoid subdomains. The coverings in the strong Grothendieck topology are the admissible coverings.

Note that since an arbitrary union of affinoid subdomains need not itself be affinoid, admissible subsets of X are not necessarily affinoid. The strong topology does have the property that a finite union of affinoid subdomains is at least admissible open. We close by stating some facts about the strong topology, and then a key example.

1. The strong Grothendieck topology is "finer" than the Zariski topology, in the sense that every open subset in the Zariski topology is admissible open, and every covering of a Zariski open by Zariski opens is an admissible covering.

2. Every sheaf on the weak Grothendieck topology extends uniquely to a sheaf on the strong Grothendieck topology. In particular, there is a sheaf of regular functions $\mathcal{O}_X$ on the strong Grothendieck topology such that for any affinoid subdomain of X, one has $\mathcal{O}_X(U) = A_U$.

3. The strong Grothendieck topology satisfies certain "completeness properties" which allow one to define global rigid spaces.

Example. Take $\mathbb{K} = \mathbb{C}_p$ and $A = \mathbb{K}\langle X\rangle$. Write $U = \{z \in M(A) \ : \ |z| = 1\}$ and $V = \{z \in M(A) \ : \ |z| < 1\}$. Note that U is a Laurent domain and is thus an admissible open in $X = M(A)$. The set V is open in the canonical topology of X, but it is not an affinoid subdomain. Otherwise we could write $V = M(B)$ and the inclusion $V \hookrightarrow X$ would correspond to a morphism of affinoid algebras $\phi\colon A \to B$. Now the element $\phi(X) \in B$ does not attain its maximum on $M(B)$, since its norm is < 1 yet may get arbitrarily close to 1 on $M(B)$. The set V is admissible open, though. To see this one notes that if $V_n = \{z \in M(A) \ : \ |z| \leq 1/p^{1/n}\}$ then $V = \bigcup_n V_n$ expresses V as a union of affinoid subdomains of X. Another application of the maximum modulus principle allows one to show that this satisfies the criteria of admissibility. One final application of the maximum modulus principle allows one to show that $\{U, V\}$ is not an admissible covering of X (Hint: apply the definition of admissible covering with the identity map $\phi\colon X \to X$).

References

[AG03] A. Archibald and A. Ghitza. Grothendieck topologies. *Available at http://www.math.mcgill.ca/goren/SeminarOnCohomology/notex.ps*, 2003.

[BGR84] S. Bosch, U. Guntzer, and R. Remmert. *Non-archimedean analysis*. Springer-Verlag New York, 1984.

[Bos] S. Bosch. Lectures on formal and rigid geometry. *preprint*, 378.

[Con08] B. Conrad. Several approaches to non-archimedean geometry. *P-Adic Geometry: Lectures from the 2007 Arizona Winter School*, page 9, 2008.

[FVDP04] J. Fresnel and M. Van Der Put. *Rigid analytic geometry and its applications*. Birkhauser, 2004.

[Iov08] Adrian Iovita. Etale fundamental groups and cohomology (notes from a course). 2008.

[Tat71] J. Tate. Rigid analytic spaces. *Inventiones mathematicae*, 12(4):257–289, 1971.