Geometric Modular Forms

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Contents

Disclaimer

This notes have been taken at a course given by Adrian Iovita in the framework of the AlGANT program in Padova (Italia), during the summer of 2008. The typos, innaccuracies and plain mistakes should be attributed to lack of understanding of the note-taker. Use at your own risk and please, send any suggestions/corrections to `marc.masdeu@gmail.com`.

1 Review

Recall what are modular forms of level 1: for them, we consider the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, acting on the complex upper-half plane $\mathfrak{h} = \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$.

If $k \in 2\mathbb{Z}_{\geq 0}$, then f is a modular form on Γ of weight k if:

1. $f \colon \mathfrak{h} \to \mathbb{C}$ is holomorphic.
2. $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$ for all $\tau \in \mathfrak{h}$ and for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$.
3. f is holomorphic at ∞: it has a Fourier expansion $f(\tau) = \sum_{n\geq 0} a_n q^n$, where $q = e^{2\pi i\tau}$.

We denote by $M_k(\mathbb{C})$ the set of modular forms of weight k on $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This turns out to be a finite-dimensional $\mathbb{C}$-vectorspace.

Let $M(\mathbb{C}) \overset{\text{def}}{=} \bigoplus_{k\in 2\mathbb{Z}_{\geq 0}} M_k(\mathbb{C}) \hookrightarrow \mathbb{C}[\![q]\!]$, which is a graded ring (the grading given by k).

Example. *For $k \in 2\mathbb{Z}_{\geq 4}$, define the* weight-$k$ Eisenstein series *as:*

$$G_k(\tau) \overset{def}{=} \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} (m\tau+n)^{-k}$$

which is a modular form of weight k. Also, $G_k(\infty) \neq 0$ (meaning that $a_0(G_k) \neq 0$).

Define also the normalized Eisenstein series *as:*

$$E_k(\tau) \overset{def}{=} \frac{k!}{2^{k-2}\pi^k B_k} G_k(\tau)$$

where B_k is the k^{th} Bernoulli number, given by the formula:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

This is done so that:

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Q}[\![q]\!]$$

where $\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$.

In particular, we give shorter names to some of these modular forms (this notation is due to Ramanujan):

$$Q \overset{def}{=} E_4 = 1 + 240 \sum \sigma_3(n) q^n \in \mathbb{Z}[\![q]\!]$$
$$R \overset{def}{=} E_6 = 1 - 504 \sum \sigma_5(n) q^n \in \mathbb{Z}[\![q]\!]$$

Theorem 1.1.

1. *The set $\{Q^i R^j\}_{4i+6j=k}$ (where i, j are taken always to be positive) is a basis for $M_k(\mathbb{C})$ (for all $k \geq 0$). In particular, $M_2(\mathbb{C}) = \{0\}$ and $M_0(\mathbb{C}) = \mathbb{C}$.*

2. $M(\mathbb{C}) \simeq \mathbb{C}[Q, R]$ *as graded* $\mathbb{C}$*-algebras, where* Q *has degree* 4 *and* R *has degree* 6*.*

We have a map:

$$M_k(\mathbb{C}) \to \mathbb{C} \quad f \mapsto f(\infty) = a_0(f)$$

and the space of cusp forms $S_k(\mathbb{C})$ is defined to be its kernel.

Example. *The smallest-weight nonzero cusp form is:*

$$\Delta(\tau) \stackrel{def}{=} q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\mathbb{C})$$

and one can write it in terms of Q *and* R *as:*

$$\Delta = \frac{Q^3 - R^2}{1728}$$

2 Modular Forms $\pmod p$ of Level 1

Fix a prime $p \geq 5$. Write $\mathcal{O} \stackrel{\text{def}}{=} \mathbb{Z}_p \cap \mathbb{Q} = \mathbb{Z}_{(p)}$. The ring $\mathcal{O}$ is local, with maximal ideal $p\mathcal{O}$. We have a canonical map $\mathcal{O} \to \mathbb{F}_p$, and the following commutes:

$$\begin{array}{ccc} \mathcal{O} & & \\ \downarrow & \searrow^{x \mapsto \tilde{x}} & \\ \mathbb{Z}_p & \longrightarrow & \mathbb{F}_p \end{array}$$

The kernel of the reduction map $\mathcal{O} \to \mathbb{F}_p$ is exactly $p\mathcal{O}$.

For $k \geq 0$ an even integer, define $M_k(\mathcal{O}) \stackrel{\text{def}}{=} \{f \in M_k(\mathbb{C}) \mid f = \sum a_n(f) q^n \in \mathcal{O}[\![q]\!]\}$, and again, let $M(\mathcal{O}) \stackrel{\text{def}}{=} \sum_{k \geq 0} M_k(\mathcal{O}) \hookrightarrow \mathcal{O}[\![q]\!]$. Note that $Q, R \in M(\mathcal{O})$.

Write $M_k(\mathbb{F}_p) \stackrel{\text{def}}{=} M_k(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}_p$, which is a finite-dimensional $\mathbb{F}_p$-vectorspace.

Note also that, for each $k \geq 0$, $M_k(\mathbb{F}_p) \hookrightarrow \mathbb{F}_p[\![q]\!]$. Again, write $M(\mathbb{F}_p)$ for the sum of all $M_k(\mathbb{F}_p)$, seen as subrings of $\mathbb{F}_p[\![q]\!]$. This is no longer equal to $\bigoplus_{k \geq 0} M_k(\mathbb{F}_p)$, as different modular forms (necessarily of distinct weight) may have congruent ($\pmod p$) q-expansions.

However, the ring $M(\mathbb{F}_p)$ still has an $\mathbb{F}_p$-algebra structure, and we'd like to investigate it. We will call the elements of $M(\mathbb{F}_p)$ **modular forms** $\pmod p$ **of level** 1.

2.1 The Algebra Structure of $M(\mathbb{F}_p)$

First, note that we have a commutative diagram:

$$\begin{array}{ccc} M(\mathbb{C}) & \xleftarrow{\simeq} & \mathbb{C}[Q, R] \\ \uparrow & & \uparrow \\ M(\mathcal{O}) & \xleftarrow{\varphi} & \mathcal{O}[Q, R] \end{array}$$

The map φ is defined by sending Q and R to their respective q-expansions (which have coefficients in $\mathcal{O}$, as we have observed earlier).

Lemma 2.1. *The map φ is an isomorphism of $\mathcal{O}$-graded algebras.*

Proof. We just need to prove surjectivity. The argument is quite standard, and we sketch it here. We want to prove that, if $f \in M_k(\mathcal{O})$ is

$$f = \sum_{4i+6j=k} a_{ij} Q^i R^j$$

then $a_{ij} \in \mathcal{O}\, \forall (i,j)$. This is clear for $k \leq 10$, and we proceed by induction to the case of weight k. Write $f = \sum_{n \geq 0} a_n q^n$. Then the form $g \stackrel{\text{def}}{=} f - a_0 Q^i R^j$ is a cusp form, where we pick i, j non-negative and such that $4i + 6j = k$. So $g \in S_k(\mathcal{O})$, and so

$$\frac{g}{\Delta} \in M_{k-12}(\mathcal{O})$$

By the induction hypothesis,

$$\frac{g(\tau)}{\Delta(\tau)} = \sum b_{ls} Q^l R^s, \quad b_{ls} \in \mathcal{O} \forall (l, s)$$

and then $g(\tau) = \Delta(\tau) \sum b_{ls} Q^l R^s \in \mathcal{O}[Q, R]$ because $1728 = 12^3$ is a unit in $\mathcal{O}$. So $f(\tau) = a_0 Q^i R^j + g(\tau)$ lies in $\mathcal{O}[Q, R]$ as well, as we wanted to prove. □

By reducing $\pmod p$, we get another commutative diagram:

$$\begin{array}{ccc} M(\mathcal{O}) & \xleftarrow{\ \simeq\ } & \mathcal{O}[Q, R] \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ M(\mathbb{F}_p) & \overset{\psi}{\twoheadleftarrow} & \mathbb{F}_p[Q, R] \end{array}$$

To determine $M(\mathbb{F}_p)$, it is thus enough to determine $\ker \psi$. For this, we introduce the main tool used in its calculation.

Define, on $\mathbb{C}[\![q]\!]$, the derivation θ, given by:

$$\theta \stackrel{\text{def}}{=} \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$$

In general, if $f = \sum a_n q^n \in M_k(\mathbb{C})$, then $\theta(f) = \sum n a_n q^n$ will not be a modular form.

Let $P \stackrel{\text{def}}{=} E_2$ (following Ramanujan), which has q-expansion given by:

$$P = 1 - 24 \sum_{n \geq 1} \sigma_1(q) q^n$$

This is not a modular form (although it transforms well with respect to $\tau \mapsto \tau + 1$, which translates into it having a q-expansion). However, the following formula holds:

$$P\left(\frac{-1}{\tau}\right) = \tau^2 P(\tau) + \frac{12\tau}{2\pi i}$$

Define then $\partial \stackrel{\text{def}}{=} 12\theta - kP$, $\partial \colon M(\mathbb{C}) \to \mathbb{C}[\![q]\!]$.

Lemma 2.2.

1. *If* $f \in M_k(\mathbb{C})$*, then* $\partial f \in M_{k+2}(\mathbb{C})$*.*
2. $\partial(M(\mathcal{O})) \subseteq M(\mathcal{O})$*.*
3. ∂ *is a derivation on* $M(\mathcal{O})$*.*

Proof. It's a simple calculation (differentiate the transformation formula for a modular form, and compare with how P transforms). □

Again, by reducing, we get a derivation $\partial\colon M(\mathbb{F}_p) \to M(\mathbb{F}_p)$, given by $\partial(\tilde{f}) = 12\theta(\tilde{f}) - k\tilde{f}\tilde{P}$.

We will use the classical fact that, for any prime p, $B_{p-1} \in \mathbb{Q}$ has p-adic valuation -1, so that $\frac{p-1}{B_{p-1}} \equiv 0 \pmod{p\mathcal{O}}$. This already implies that $E_{p-1} \equiv 1 \pmod{p\mathcal{O}[\![q]\!]}$, and hence $\widetilde{E_{p-1}} = 1 \in M(\mathbb{F}_p)$.

Write $A(Q,R) \in \mathcal{O}[Q,R]$ for the polynomial such that $A(Q,R) = E_{p-1}$. Then, if $\psi\colon \mathbb{F}_p[Q,R] \twoheadrightarrow M(\mathbb{F}_p)$ is the previously defined map, it is clear that $\tilde{A} - 1 \in \ker\psi$.

Theorem 2.3 (Swinnerton-Dyer). *The kernel of* ψ *is the principal ideal generated by* $\tilde{A} - 1$*.*

Proof. Ommitted. □

Corollary 2.4. *We get the structure of* $M(\mathbb{F}_p)$*:*

$$M(\mathbb{F}_p) \simeq \mathbb{F}_p[Q,R]/(\tilde{A}(Q,R) - 1)$$

In particular, it is the ring of regular functions on a normal (hence smooth) curve in the affine plane.

We have thus found a modular form "of weight $p-1$" whose q-expansion is the constant 1. It is called the *Hasse invariant* and written $A(Q,R) = E_{p-1}$.

One can show that $M(\mathbb{F}_p)$ has a grading by the group $\mathbb{Z}/(p-1)\mathbb{Z}$, by noting that $A-1$ is a homogeneous polynomial with respect to this group (because $\deg A = p-1$).

Consider now E_{p+1}. By Kummer's congruences, as $p+1 \equiv 2 \pmod{p-1}$ we have:

$$\frac{B_{p+1}}{2(p+1)} \equiv \frac{B_2}{2\cdot 2} \pmod{p\mathcal{O}}$$

and both of them are units in $\mathcal{O}$. Hence their inverses are also congruent modulo $p\mathcal{O}$. Moreover, $\sigma_p(n) \equiv \sigma_1(n) \pmod{p\mathcal{O}}$, as an easy argument shows, and therefore:

$$E_{p+1} \equiv E_2 \pmod{p\mathcal{O}[\![q]\!]}$$

This implies that $\tilde{P}$ is actually a $\pmod p$ modular form!

Also, 12 is invertible in $\mathcal{O}$, and hence the θ operator becomes a differential operator in $M(\mathbb{F}_p)$, because it can be expressed as a linear combination of ∂ and E_{p+1}.

In what follows, we will understand things more geometrically, as well as generalize the (mod p) modular forms to levels $N \geq 1$.

3 Reinterpreting Classical Modular Forms (on $\mathbb{C}$)

We consider again the level 1 modular group $\Gamma = \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. One can quotient out $\mathfrak{h}$ by Γ, and then compactify (by adding one point) both objects. We call $Y(1) \stackrel{\text{def}}{=} \Gamma(1)\backslash\mathfrak{h}$, and $X(1) \stackrel{\text{def}}{=} \overline{Y(1)} = \Gamma(1)\backslash\overline{\mathfrak{h}}$. Note then that both $Y(1)$ and $X(1)$ are Riemann surfaces (and thus smooth algebraic curves), and that $X(1)$ is also compact (so a projective curve), while $Y(1)$ is an affine curve.

Given $f\colon \mathfrak{h} \to \mathbb{C}$ a modular form of weight k, define:

$$\omega_f \stackrel{\text{def}}{=} f(z)dz^{\otimes\frac{k}{2}} \in \left(\Omega^1_{\mathfrak{h}/\mathbb{C}}\right)^{\otimes\frac{k}{2}}(\mathfrak{h})$$

If $\gamma \in \Gamma$, then we can compute:

$$\gamma^*\omega_f = f(\gamma z)d(\gamma z)^{\otimes\frac{k}{2}} = (cz+d)^k f(z)\left(\frac{d}{dz}\frac{az+b}{cz+d}\right)^{\frac{k}{2}}(dz)^{\otimes\frac{k}{2}} = \omega_f$$

and hence ω_f is Γ-invariant, so it can be seen as a differential on $Y(1)$:

$$\omega_f \in \left(\Omega^1_{Y(1)/\mathbb{C}}\right)^{\otimes\frac{k}{2}}(Y(1))$$

This module is actually the algebra of Kähler differentials on $Y(1)$ (recall that it's an affine curve!).

We now seek more ways of interpreting modular forms. For that, let $\mathcal{R}$ be the set of lattices in $\mathbb{C}$ (that is, $L \subseteq \mathbb{C}$ is a lattice if it is a free $\mathbb{Z}$-module of rank 2, such that $L \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}$).

If $L \in \mathcal{R}$, then $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, with ω_1, ω_2 two $\mathbb{R}$-linear-independent complex numbers. Then $\mathbb{C}/L$ is a compact torus, and we can make it an elliptic curve by decreeting $\mathcal{O} \stackrel{\text{def}}{=} \overline{0} \in \mathbb{C}/L$.

Let now $\mathcal{M} \stackrel{\text{def}}{=} \{(\alpha_1, \alpha_2) \in (\mathbb{C}^\times)^2 \mid \Im(\frac{\alpha_1}{\alpha_2}) > 0\}$. We have a map:

$$\begin{aligned} \varphi\colon \mathcal{M} &\to \mathcal{R} \\ (\alpha_1, \alpha_2) &\mapsto \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \end{aligned}$$

Note that φ is surjective. We have actions of various groups on each of the previously defined sets, as follows:

- On $\mathcal{M}$, Γ acts by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\alpha_1, \alpha_2) \stackrel{\text{def}}{=} (a\alpha_1 + b\alpha_2, c\alpha_1 + d\alpha_2)$, and $\mathbb{C}^\times$ acts by scaling.
- On $\mathcal{R}$, $\mathbb{C}$ acts by scaling.
- On $\mathfrak{h}$, the group Γ acts as described above.

Consider the map $\alpha\colon \mathcal{M} \to \mathfrak{h}$ sending $(\alpha_1, \alpha_2) \mapsto \frac{\alpha_1}{\alpha_2}$, and the map $\beta\colon \mathfrak{h} \to \mathcal{R}$ sending $\tau \mapsto L_\tau \stackrel{\text{def}}{=} \mathbb{Z}\tau \oplus \mathbb{Z}$. We have a **non-commutative** diagram:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \varphi\ } & \mathcal{R} \\ & {}_{\alpha}\searrow \quad \nearrow_{\beta} & \\ & \mathfrak{h} & \end{array}$$

One sees that α is Γ-invariant, and also that β induces an isomorphism $\mathfrak{h} \simeq \mathcal{R}/\mathbb{C}^\times$.

Define now another set $\mathcal{E}$, to be the set of isomorphism classes of elliptic curves over $\mathbb{C}$. We get then maps $\mathfrak{h} \to \mathcal{E}$ and $\mathcal{R} \to \mathcal{E}$, by respectively sending $\tau \mapsto E_\tau = (\mathbb{C}/L_\tau, \overline{0})$ and $L \mapsto (\mathbb{C}/L, \overline{0})$.

Proposition 3.1. *The map $u\colon \mathcal{R} \to \mathcal{E}$ factors through $\mathbb{C}^\times$ and induces an isomorphism $\mathcal{R}/\mathbb{C}^\times \simeq \mathcal{E}$.*

Proof. Surjectivity follows from the fact that, if $E/\mathbb{C}$ is any elliptic curve, then $E(\mathbb{C}) \simeq \mathbb{C}/L$ for some lattice L, which can be computed by fixing an invariant differential ω on E, and then:

$$L = \left\{ \int_\gamma \omega \mid \gamma \in H_1(E, \mathbb{Z}) \right\} \subseteq \mathbb{C}$$

For injectivity, if $L_1, L_2 \in \mathcal{R}$ are two lattices such that

$$\mathbb{C}/L_1 \overset{\psi}{\simeq} \mathbb{C}/L_2$$

then $\mathbb{C}$ is the universal covering space for $\mathbb{C}/L_i$, so ψ can be lifted to a holomorphic map $\overline{\psi}\colon \mathbb{C} \to \mathbb{C}$, such that $\overline{\psi}(\overline{0}) = \overline{0}$. We get then that, for each $z \in \mathbb{C}$ and $l_1 \in L_1$,

$$\overline{\psi}(z + l_1) - \overline{\psi}(z) \in L_2$$

and, as L_2 is discrete, this implies:

$$\overline{\psi}(z + l_1) - \overline{\psi}(z) = c \text{ (a constant).}$$

Taking its derivative, we see that $\overline{\psi}'$ is invariant under L_1 and that it is holomorphic, and so $\overline{\psi}' = b \in \mathbb{C}$ is a constant. Hence $\psi(z) = bz + c$ for some $c \in \mathbb{C}$ and, as $\psi(0) = 0$ we must have $c = 0$. So $\psi(z) = bz$, and $L_2 = bL_1$, as we wanted to show. $\square$

Also, the map β induces an isomorphism $\mathcal{R}/\mathbb{C}^\times \simeq \Gamma\backslash\mathfrak{h}$, because two lattices L_τ and $L_{\tau'}$ are homotethic if, and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Consider now the set $\mathfrak{h} \times \mathbb{C}$, with its canonical projection p_1 onto $\mathfrak{h}$. Then Γ and $\mathbb{Z}^2$ both act on $\mathfrak{h} \times \mathbb{C}$, the actions on $(\tau, v) \in \mathfrak{h} \times \mathbb{C}$ given by:

$$(\tau, v) \cdot (a, b) \overset{\text{def}}{=} (\tau, v + a\tau + b)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, v) \overset{\text{def}}{=} (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau, (c\tau + d)^{-1} v)$$

With these actions, the projection p_1 is Γ-equivariant. Consider then the quotient $\mathbb{E} \overset{\text{def}}{=} (\mathfrak{h} \times \mathbb{C})/\mathbb{Z}^2$, together with the induced map $p\colon \mathbb{E} \to \mathfrak{h}$. Then, if $\tau \in \mathfrak{h}$, we have that the fiber above it, $p^{-1}(\tau)$ is isomorphic to E_τ.

If, further, we divide by Γ on the left, we get $\mathbb{E}(1)$, which comes with a map to $Y(1)$, and which maps $[E_\tau] \mapsto [\tau]$.

Consider now a function $F\colon \mathcal{R} \to \mathbb{C}$ such that

$$F(\lambda L) = \lambda^{-k} F(L) \quad \forall \lambda \in \mathbb{C}^\times, \forall L \in \mathcal{R}$$

Given such a "homogeneous" function, define $f\colon \mathfrak{h} \to \mathbb{C}$ by

$$f(\tau) \overset{\text{def}}{=} F(L_\tau) = F(\tau\mathbb{Z} \oplus \mathbb{Z})$$

Note then, that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have:

$$\begin{aligned} f(\gamma\tau) = F(L_{\gamma\tau}) = \left(\frac{a\tau+b}{c\tau+d}\mathbb{Z} \oplus \mathbb{Z}\right) = \\ = F\left(\frac{1}{c\tau+d}((a\tau+b)\mathbb{Z} \oplus (c\tau+d)\mathbb{Z})\right) = F((c\tau+d)^{-1}L_\tau) = \\ = (c\tau+d)^k F(L_\tau) = (c\tau+d)^k f(\tau) \end{aligned}$$

so that if, in addition, f satisfies holomorphicity on $\mathfrak{h}$ and at ∞, then f is a weight-k modular form for Γ.

Note that the set $\mathcal{R}$ is a fibration over $\mathcal{R}/\mathbb{C}^\times$, and we would like to work instead on a fibration over $\mathcal{E}$, say $\mathcal{E}'$, such that:

$$\begin{array}{ccc} \mathcal{R} & \dashrightarrow & \mathcal{E}' \\ \downarrow & & \downarrow \\ \mathcal{R}/\mathbb{C}^\times & \longrightarrow & \mathcal{E} \end{array}$$

For this, define $\mathcal{E}'$ to be the set of isomorphism classes of pairs (E, ω), where $E/\mathbb{C}$ is an elliptic curve, and ω is a basis for $H^0(E, \Omega^1_{E/\mathbb{C}})$ (note that this has dimension 1, so ω is just any nonzero element of H^0. The isomorphism is defined by:

$$(E, \omega) \simeq (E', \omega') \iff \exists \varphi\colon E \overset{\simeq}{\to} E' \text{ an isomorphism such that } \varphi^*\omega' = \omega$$

We have an obvious map $\mathcal{E}' \to \mathcal{E}$ which forgets ω, and also there is a map $\mathcal{R} \to \mathcal{E}'$, which to L associates the pair $(\mathbb{C}/L, dz)$ (z is the coordinate function on $\mathbb{C}$). Note that, under this map, if $\lambda \in \mathbb{C}^\times$, then λL is mapped to:

$$[(\mathbb{C}/(\lambda L), dz)] = [(\mathbb{C}/L, \lambda dz)]$$

We thus can let $\mathbb{C}^\times$ act on $\mathcal{E}'$ compatibly, by setting

$$\lambda[(E, \omega)] \overset{\text{def}}{=} [(E, \lambda\omega)]$$

Again, for k an even integer, consider a function $G\colon \mathcal{E}' \to \mathbb{C}$ such that $G(E, \lambda\omega) = \lambda^{-k}G(E, \omega)$ for all $\lambda \in \mathbb{C}^\times$. A similar computation as before yields that the function $g\colon \mathfrak{h} \to \mathbb{C}$ defined by

$$g(\tau) \overset{\text{def}}{=} G([(\mathbb{C}/L_\tau, dz)])$$

satisfies a modular transformation, and so it is (provided also the right holomorphicity conditions) a weight-k modular form.

This is the right concept to generalize, and this is what we will do in the next section.

4 Geometric Modular Forms "à la Katz"

We start right away with a definition that we will take as a basis for a vast generalization:

Definition 4.1 (due to N.Katz). Fix an integer k, and let R_0 be a (commutative, unital) ring. A **modular form** of weight k and level 1, defined over R_0 is a rule, f, which assigns to every pair $(E/R, \omega)$ (where R is an R_0-algebra, E/R is an elliptic curve over R, and ω is a basis for $H^0(E, \Omega^1_{E/R})$) an element $f(E, \omega) \in R$, such that:

1. $f(E/R, \omega)$ depends only on the isomorphism class of (E, ω).

2. $f(E/R, \lambda\omega) = \lambda^{-k} f(E/R, \omega)$ for all $\lambda \in R^\times$.

3. f commutes with arbitrary base change. That is, if $\varphi\colon R \to R'$ is an R_0-algebra homomorphism, and $(E/R, \omega)$ is a pair over R, then we can consider the pair $E'/R' \stackrel{\text{def}}{=} E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \xrightarrow{p} E$ and $\omega' \stackrel{\text{def}}{=} p^*\omega$. We require then that:
$$f(E'/R', \omega') = \varphi\left(f(E/R, \omega)\right)$$

Remark. The last condition deals in some way with holomorphicity. For now, we still don't deal with the cusp at ∞, but we will do it shortly (actually, the last condition ensures meromorphicity at ∞, at least).

Here is the goals that we set ourselves for this course:

- Generalize this definition for $N \geq 1$.
- What about holomorphicity at ∞?
- Relate these notions to the classical definition ($R_0 = \mathbb{C}$).

5 Higher Levels (level-N Structures)

Fix now $N \geq 1$. We consider the congruence groups
$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$$
Suppose that $E/\mathbb{C}$ is an elliptic curve. Its N-torsion is
$$E[N] \stackrel{\text{def}}{=} \{P \in E(\mathbb{C}) \mid NP = 0\}$$

If $E \simeq \mathbb{C}/L$, then $E[N] \simeq \frac{1}{N}L/L \simeq L/(NL) \simeq (\mathbb{Z}/N\mathbb{Z})^2$, and so $E[N]$ is a free $\mathbb{Z}$-module of rank 2.

An isomorphism $\alpha\colon \mathbb{Z}/N\mathbb{Z}^2 \to E[N]$ is the same as fixing an (ordered) basis of $E[N]$, and such an isomorphism will be called a **level-N structure on** E.

Definition 5.1. The set $\mathcal{E}(N)$ is defined to be the set of isomorphism classes of pairs (E, α), where $E/\mathbb{C}$ is an elliptic curve, and α is a level-N structure on E. Isomorphisms of such pairs are defined to be those isomorphisms of elliptic curves that preserve the level-N structure, as one expects.

We have a natural map:

$$\begin{aligned}\mathfrak{h} &\to \mathcal{E}(N)\\ \tau &\mapsto (E_\tau, (\tau/N, 1/N))\end{aligned}$$

Note that, if $\gamma \in \Gamma(N)$, then $\gamma\tau$ is sent to the same as τ. We thus get a map $\Gamma(N)\backslash\mathfrak{h} \to \mathcal{E}(N)$.

This map is injective (but not surjective!). We will see that its image is. For this, recall the Weil pairing:

$$\langle\cdot,\cdot\rangle_{\mathrm{Weil}} \colon E[N] \times E[N] \to \mu_N$$

If ζ is a primitive N^{th} root of 1, define the ζ-component of $\mathcal{E}(N)$ as:

$$\mathcal{E}(N)_\zeta \stackrel{\mathrm{def}}{=} \{ \text{ iso classes } (E, (e_1, e_2)) \mid \langle e_1, e_2\rangle = \zeta\}$$

One can then see that

$$\mathcal{E}(N) = \coprod_{\zeta\in\mu_N} \mathcal{E}(N)_\zeta$$

(where we only consider those ζ's which are primitive). In some sense (very imprecise, because we don't have a topology on it so far), $\mathcal{E}(N)$ is "disconnected".

Also, note that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on $\mathcal{E}(N)$ by

$$\sigma \cdot (E, \alpha) \stackrel{\mathrm{def}}{=} (E, \alpha \circ \sigma)$$

This action induces an isomorphism:

$$\mathcal{E}(N)_\zeta \xrightarrow{\simeq} \mathcal{E}(N)_{\zeta^{\det\sigma}}$$

Claim. *The map $\tau \mapsto (E_\tau, (\tau/N, 1/N))$ defined above induces an isomorphism*

$$\Gamma(N)\backslash\mathfrak{h} \simeq \mathcal{E}(N)_{\zeta_N}$$

(where we take $\zeta_N \stackrel{def}{=} e^{2\pi i/N}$, and note that $\langle \tau/N, 1/N\rangle_{Weil} = \zeta_N$).

Remark also that, as all the components of $\mathcal{E}(N)$ are isomorphic, then $\mathcal{E}(N)$ can be thought of as a union of ($\phi(N)$ copies) of $\Gamma(N)\backslash\mathfrak{h}$.

Next, define $\mathcal{E}(N)'$ to be the set of isomorphism classes (E, ω, α), where $E/\mathbb{C}$ is an elliptic curve, $\omega \in H^0(E, \Omega^1_{E/\mathbb{C}}))$ is a basis, and α is a level-N structure. Again, we have an obvious map $\mathcal{E}(N)' \to \mathcal{E}(N)$.

Now, we may consider functions $F \colon \mathcal{E}(N)' \to \mathbb{C}$ such that, for all $\lambda \in \mathbb{C}^\times$,

$$F(E, \lambda\omega, \alpha) = \lambda^{-k} F(E, \omega, \alpha)$$

and again, the function $f \colon \mathfrak{h} \to \mathbb{C}$ defined by

$$f(\tau) \stackrel{\mathrm{def}}{=} F(E_\tau, dz, (\tau/N, 1/N))$$

is $\Gamma(N)$-modular of weight k.

Definition 5.2. Fix $N \geq 1$, and $k \in \mathbb{Z}$. Let R_0 be a $\mathbb{Z}[1/N]$-algebra (this means exactly that $N \in R_0^\times$). A **modular form of weight** k **and level** N **defined over** R_0 is a rule which assigns, to every triple $(E/R, \omega, \alpha)$ as before (we take α to be an isomorphism of group schemes, between the constant group scheme $\underline{(\mathbb{Z}/N\mathbb{Z})^2}$ and $E[N]$, and note that such an isomorphism may not exist for all R), an element $f(E/R, \omega, \alpha) \in R$ such that:

1. f depends only on the isomorphism class of the triple.
2. $f(E/R, \lambda\omega, \alpha) = \lambda^{-k} f(E/R, \omega, \alpha)$ for all $\lambda \in R^\times$.
3. Compatible with base change.

Remarks.

- Let E/R be an elliptic curve over R (so it comes with a morphism $\varphi\colon E \to \operatorname{Spec}(R)$). On E, we have the sheaf $\Omega^1_{E/R}$, and we can push it forward to $\underline{\omega}_{E/R} \stackrel{\text{def}}{=} \varphi_* \Omega^1_{E/R}$. Then $\underline{\omega}_{E/R}$ is a coherent, locally-free sheaf of rank 1 on $\operatorname{Spec} R$. Hence it is the sheaf associated to a certain projective, rank-1 module over R, which is $H^0(E, \Omega^1_{E/R})$. In fact, one can prove that, if R is noetherian, then $H^0(E, \Omega^1_{E/R})$ is actually free.

- Suppose that f is a modular form of weight k and level N over R_0. Let R be an R_0-algebra such that there exists a triple $(E/R, \omega, \alpha)$ (as we have remarked above, we may have to change R in order to be able to even consider a triple like that!). Define then:
$$f(E/R, \omega, \alpha)\omega^{\otimes k} \in H^0(\operatorname{Spec} R, \underline{\omega}^{\otimes k}_{E/R})$$
If ω' is another basis for $H^0(E, \Omega^1_{E/R})$, then $\omega' = \lambda\omega$ for some $\lambda \in R^\times$, and we can compute:
$$f(E/R, \omega', \alpha)\omega'^{\otimes k} = f(E/R, \lambda\omega, \alpha)(\lambda\omega)^{\otimes k} = f(E/R, \omega, \alpha)\omega^{\otimes k}$$
and so it doesn't depend on the choice of ω. We call this element $g(E/R, \alpha) \stackrel{\text{def}}{=} f(E/R, \omega, \alpha)\omega^{\otimes k}$, which is a global section of $\underline{\omega}^{\otimes k}_{E/R}$.

  It is easy to see that the giving of such a g is equivalent to the giving of the previous f.

We have another apparently more restrictive definition, which would admit any scheme S as the basis (and not only affine schemes). However, this new definition is equivalent to the one we have. This is proven using the base change property, which allows one to glue the sections on affines into a global section.

In the following sections, we will study in more detail the families of elliptic curves (that is, elliptic curves over an arbitrary scheme), so to make these concepts more clear.

6 Families of Elliptic Curves

Let S be a noetherian scheme. Recall that a geometric point of S is a morphism $s \to S$, where $s = \operatorname{Spec}(k)$ for some algebraically-closed field k. It is the same as the giving of a point $x \in S$, together with the choice of an embedding of the residue field $\kappa(x)$ into an algebraic closure.

Definition 6.1. An **elliptic curve over** S is a pair (E, e) where $E \xrightarrow{f} S$ is a smooth, proper morphism of schemes and $e\colon S \to E$ is a section ($f \circ e = \mathrm{Id}_S$) such that for all geometric points $s \to S$, the fiber $E_s \overset{\text{def}}{=} E \times_S s$ is a (smooth, proper) *connected* curve of genus 1 over $\kappa(s)$.

We want to study these objects. In particular, we want to find a cubic equation which they satisfy, study their deRham cohomology, and see the connection they have with modular forms.

7 Cohomology of Sheaves

Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and suppose that $\mathcal{A}$ has enough injectives (e.g. $\mathcal{A} = \mathbf{Ab}, A-\mathbf{Mod}, \mathbf{Sh}(X), \ldots$).

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a (covariant) functor, which is left-exact. We can then consider the sequence of right-derived functors $\{R^n F\}_{n \geq 0}$

$$R^n F\colon \mathcal{A} \to \mathcal{B}$$

For the definition of these functors, one can refer to any book on categories.

Examples.

- *Let X be a topological space, $\mathcal{A} = \boldsymbol{Sh}(X)$ (the category of sheaves of abelian groups on X), and $\mathcal{B} = \boldsymbol{Ab}$ (the category of abelian groups). As $F\colon \mathcal{A} \to \mathcal{B}$ we take the global sections functor $F \overset{def}{=} \Gamma(X, -)$. One can check that F is left exact. In this way we obtain the* sheaf cohomology*:*

$$H^n(X, \mathcal{F}) \overset{def}{=} R^n\Gamma(X, \mathcal{F})$$

- *Let X be a topological space, let $Z \hookrightarrow X$ be a closed subspace. Let $\mathcal{A}, \mathcal{B}$ be as before, and consider the functor $F\colon \mathcal{A} \to \mathcal{B}$ defined by $F \overset{def}{=} \Gamma_Z(X, -)$, where for any sheaf $\mathcal{F}$ on X,*

$$\Gamma_Z(X, \mathcal{F}) \overset{def}{=} \{s \in \Gamma(X, \mathcal{F}) \mid \operatorname{Supp} s \subseteq Z\}$$

where $\operatorname{Supp} s = \{x \in X \mid s_x \neq 0\}$. *The group $\Gamma_Z(X, \mathcal{F})$ is called* ***sections with support on*** *Z. Again, one can check that F is left-exact, and we obtain the* local cohomology with support on Z*:*

$$H^n_Z(X, \mathcal{F}) \overset{def}{=} R^n\Gamma_Z(X, \mathcal{F})$$

- *Let X be a locally-compact topological space, and let $\mathcal{A}, \mathcal{B}$ be as before. Define in this case*

$$F(\mathcal{F}) \overset{def}{=} \Gamma_c(X, \mathcal{F}) \overset{def}{=} \{s \in \Gamma(X, \mathcal{F}) \mid \operatorname{Supp} s \text{ is compact}\}$$

This leads to cohomology with compact support, *written $H^n_c(X, \mathcal{F})$.*

- *Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous map. Let $\mathcal{A} = \boldsymbol{Sh}(X), \mathcal{B} = \boldsymbol{Sh}(Y)$. Consider the functor $F \overset{def}{=} f_*\colon \mathcal{A} \to \mathcal{B}$, defined by $(f_*\mathcal{F})(V) \overset{def}{=} \mathcal{F}(f^{-1}(V))$. Its right derived functors will be important in the sequel, but we will simply denote them by $R^n f_*$.*

7.1 Relationship between $H^n(X, \mathcal{F})$ and $R^n f_*$

First, suppose that $S = \{s\}$ is a one-point space. Then the category $\boldsymbol{Sh}(S)$ is canonically identified with $\boldsymbol{Ab}$, by taking global sections. There is also a unique map $f\colon X \to S$, and $f_*\mathcal{F}$ corresponds, under the mentioned identification, to $\Gamma(X, \mathcal{F})$. Moreover, the right derived functors are just the sheaf cohology of $\mathcal{F}$ on X: $(R^n f_*)(\mathcal{F}) \leftrightarrow H^n(X, \mathcal{F})$.

Next, suppose that X and S are noetherian schemes, and let $f\colon X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. We have then:

Theorem 7.1 (Serre). *For each $n \geq 1$, the right derived functor $R^n f_*\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_S$-modules on S.*

Theorem 7.2 (Mumford). *Suppose further that S is connected and reduced. Fix $i \geq 1$, and let $\alpha\colon S \to \mathbb{Z}$ be defined by:*

$$\alpha(s) \overset{def}{=} \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s) \in \mathbb{Z}$$

(the sheaf $\mathcal{F}_s$ on X_s is defined to be the pullback of $\mathcal{F}$ along the canonical projection $X_s = X \times_s S \to X$). Assume that α is constant, say with value d_i. Then $R^i f_*\mathcal{F}$ is locally-free of rank d_i, and both $R^i f_*\mathcal{F}$ and $R^{i-1} f_*\mathcal{F}$ commute with base change.

Proof. Look it up in [Mum70]. □

Remark. If α is constant, then for each $s \in S$ we can consider the fiber $(R^i f_*\mathcal{F})_s \simeq H^i(X_s, \mathcal{F}_s)$. Hence we can think of the sheaf $R^i f_*\mathcal{F}$ as "putting together" all the cohomology groups $\{H^i(X_s, \mathcal{F}_s)\}_{s\in S}$.

8 Equations For Elliptic Curves

In this section we find the equation of an elliptic curve. We will start with the simplest case, and move up to the relative case (that is, the case of families of elliptic curves). Do they have a "global" equation? Wait and see!

8.1 Complex case

Let $S = \operatorname{Spec}(\mathbb{C})$, and consider an elliptic curve $(E/\mathbb{C}, \infty)$. Then $E(\mathbb{C})$ is a Riemann surface of genus 1, $E(\mathbb{C}) \simeq T = \mathbb{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}$, with ∞ corresponding to the class of 0. Define the *Weierstrass $\wp$-function* (which depends on Λ, although we don't write it explicitly):

$$\wp(z) \overset{\text{def}}{=} \frac{1}{z^2} + \sum_{\lambda\in\Lambda}{}' \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

(the notation $\sum'$ is rather standard, and it means summing over all values of λ that actually make sense. Here we mean that we ommit $\lambda = 0$).

The function $\wp(z)$ is absolutely convergent, and so we can differentiate term by term, and obtain $\wp'(z)$, which has a formula:

$$\wp'(z) = \frac{d}{dz}\wp(z) = -\frac{2}{z^3} - 2\sum_{\lambda\in\Lambda}{}' \frac{1}{(z-\lambda)^3}$$

It becomes then obvious that $\wp'(z)$ is Λ-periodic, and one deduces then that so is $\wp(z)$. So we have two meromorphic functions on the torus $T = \mathbb{C}/\Lambda$.

One can then show that $\wp$ and $\wp'$ satisfy the following equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$$

with

$$g_2(\Lambda) = 60\sum_{\lambda\in\Lambda}{}' \lambda^{-4} \qquad\qquad g_3(\Lambda) = 140\sum_{\lambda\in\Lambda}{}' \lambda^{-6}$$

Moreover, this is the only relation that $\wp$ and $\wp'$ satisfy: if we note $\mathcal{M}(T)$ as the field of meromorphic functions on T, we have:

$$\mathcal{M}(T) = \frac{\mathbb{C}[X,Y]}{(Y^2 - 4X^3 + g_2(\Lambda)X + g_3(\Lambda))}$$

and we get an analytic isomorphism $T \setminus \{0\} \to \{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2x - g_3\}$, which extends to T after projectivizing the right hand side. So we obtain:

$$T \simeq \operatorname{Proj}\left(\frac{\mathbb{C}[X,Y,Z]}{(Y^2Z - 4X^3 + g_2(\Lambda)XZ^2 + g_3(\Lambda)Z^3)}\right)$$

8.2 Absolute case

Let now $S = \operatorname{Spec}(k)$ where $k = \overline{k}$ is an algebraically-closed field, with $\operatorname{char} k \neq 2, 3$. Recall that we have a section $e\colon S \to E$, so that in this situation it gives a point $\infty \in E(k)$. Also, by hypothesis, E/k is a smooth, proper curve of genus 1.

8.2.1 Riemann-Roch

In general, if C is a smooth proper connected curve of genus $g \geq 0$, we can consider the *divisor group* $\operatorname{Div}(C)$, which is the free abelian group generated by $C(k)$ (the closed points of C). There is the *degree* map:

$$\begin{aligned}\deg\colon \operatorname{Div}(C) &\twoheadrightarrow \mathbb{Z}\\ \sum n_i P_i &\mapsto \sum n_i\end{aligned}$$

Let also $\operatorname{Div}^0(C) \overset{\text{def}}{=} \ker(\deg) \subseteq \operatorname{Div}(C)$ be the group of degree-0 divisors.

Let $k(C)$ be the function field of C ($k(C) = \mathcal{O}_{C,\eta}$ where η is the generic point of C). We have a group homomorphism:

$$\begin{aligned} \operatorname{div}\colon k(C)^\times &\to \operatorname{Div}(C) \\ f &\mapsto \sum_{P\in C(k)} \operatorname{ord}_P(f)P \end{aligned}$$

where $\operatorname{ord}_P(f)$ is the order of the image of f in the stalk $\mathcal{O}_{C,P}$ (which is a DVR). One shows that $\operatorname{ord}_P(f)$ is nonzero for finitely-many points, so that the map is well defined. Also, if $n_P = \operatorname{ord}_P(f) > 0$ we say that f has a zero of order n_P at P, and if $n_p < 0$ we say that f has a pole of order $-n_P$ at P.

One shows, furthermore, that $\deg(\operatorname{div} f) = 0$, so that the divisor map has image inside $\operatorname{Div}^0(C)$. If we denote by $P(C)$ the image of the divisor map $\div$ (this is the group of *principal divisors*) we can then define the *Picard group* and the 0-*Picard group* as $\operatorname{Pic}(C) \overset{\text{def}}{=} \operatorname{Div}(C)/P(C)$, and $\operatorname{Pic}^0(C) \overset{\text{def}}{=} \operatorname{Div}^0(C)/P(C)$.

We develop now another point of view from which these objects can be studied, that allows for a better generalization.

Let $P \in C(k)$ be a closed point, so that $P \overset{i_p}{\hookrightarrow} C$ is a closed immersion. It has a corresponding sheaf of $\mathcal{O}_C$-ideals, $\mathcal{I}(P)$. This fits in a short exact sequence:

$$0 \to \mathcal{I}(P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$$

where $\mathcal{O}_P = (i_P)_*(\underline{k})$ is the push-forward of the constant sheaf with value k, on the scheme $\{P\}$. Note also that $\mathcal{I}(P)$ is an invertible sheaf (by definition, this means a coherent, locally-free sheaf of rank 1). Denote also by $\mathcal{L}(P) \overset{\text{def}}{=} \mathcal{I}(P)^{-1} \overset{\text{def}}{=} \mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}(P), \mathcal{O}_C)$ (this is the sheaf-Hom, see [Har77] for some more details). For us, it is enough to know that $\mathcal{L}(P) \otimes_{\mathcal{O}_C} \mathcal{I}(P) \simeq \mathcal{O}_C$, and that, if $U \subseteq C$ is any open, then the sections of $\mathcal{L}(P)$ on U can be thought of as the "meromorphic" functions $f \in k(C)$ such that $\operatorname{ord}_Q(f) \geq 0$ for all $Q \neq P$, and $\operatorname{ord}_P(f) \geq -1$.

To a divisor $D = \sum_i n_i P_i$ we can associate an invertible sheaf on C,

$$\mathcal{L}(D) \overset{\text{def}}{=} \mathcal{L}(P_1)^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}(P_s)^{\otimes n_s}$$

Claim. *The previous map induces an isomorphism between* $\operatorname{Pic}(C)$ *and the set of isomorphism classes of invertible sheaves.*

Proof. First, note that, if $f \in k(C)^\times$, then $\mathcal{L}(\operatorname{div}(f)) \simeq f^{-1}\mathcal{O}_C \simeq \mathcal{O}_C$, so the given map factors through $P(C)$. We need to prove bijectivity, and for this one can see [Har77]. □

Recall that, if C/k is a smooth, proper, connected curve, then its genus g can be defined as the dimension of $H^1(C, \mathcal{O}_C)$ (as a k-vectorspace) (this gives all the information we need, as $H^0(C, \mathcal{O}_C) \simeq k$, and the higher cohomology vanishes (actually, it does so for any coherent sheaf, not only for $\mathcal{O}_C$).

Consider the sheaf of regular differentials $\Omega^1_{C/K}$, which comes from sheafifying the module of Kähler differentials on a ring.

Example. *Say somebody stops us asking for how many functions are there with a pole of order 3 at P, and a zero of order 2 at Q ($P \neq Q \in C(k)$), and such that it is regular everywhere else. We would –very slowly– pull out our Riemann-Roch machinery (which we will introduce in the following) and try to use it to find the dimension of $H^0(C, \mathcal{L}(D))$, with $D \overset{def}{=} 3P - 2Q$. This would give us how many functions are there with at worst a pole of order 3 at P, and a zero at least of order 2 at Q. We would have to consider other spaces to get equalities instead of inequalities, but this would at least be a first step to save our day!*

Consider again the invertible sheaf $\Omega^1_{C/k}$. There is a canonical map (called the *trace map*):

$$\text{Tr}\colon H^1(C, \Omega^1_{C/k}) \xrightarrow{\simeq} k$$

So for any invertible sheaf $\mathcal{L}$, we get a pairing:

$$H^0(C, \mathcal{L}) \times H^1(C, \mathcal{L}^{-1} \otimes \Omega^1_{C/k}) \to H^1(C, \Omega^1_{C/k}) \simeq k$$

By Serre duality (a nontrivial result), this is a perfect pairing, and so we get an identification:

$$H^0(C, \Omega^1_{C/k}) \simeq H^1(C, \mathcal{O}_C)^\vee$$

For $\mathcal{L} = \Omega^1_{C/k}$, this already implies that $H^0(C, \Omega^1_{C/k})$ has dimension g (this is another possible definition of the genus of C).

As Ω^1_C is an invertible sheaf, then by the previous theory there has to exist some divisor K such that $\Omega^1_C \simeq \mathcal{L}(K)$. Then $\deg K = 2g - 2$, as we will see in the following important theorem.

Theorem 8.1 (Riemann-Roch). *Let $D \in \text{Div}(C)$. Then:*

$$\dim_k H^0(C, \mathcal{L}(D)) - \dim_k H^1(C, \mathcal{L}(D)) = \deg D + 1 - g$$

Note also that, by Serre duality,

$$\dim_k H^1\left(C, \mathcal{L}(D)\right) = \dim_k H^0\left(C, \mathcal{L}(-D) \otimes \Omega^1_C\right)$$

Remark. Suppose that $\deg D > 2g - 2$. Then $\deg(K - D) < 0$, and so $H^0(C, \mathcal{L}(-D) \otimes \Omega^1_C) = \{0\}$. Hence in this case,

$$\dim_k H^0\left(C, \mathcal{L}(D)\right) = \deg D + 1 - g$$

8.2.2 Application to our situation

Suppose now that E/k is an elliptic curve over an algebraically-closed field k. Let $\infty \in E(k)$ be the image of the section $e\colon \text{Spec}(k) \to E$ and consider, for each $n \geq 1$, the divisor $D_n \overset{\text{def}}{=} n \cdot \infty$. Then (recall $g = 1$), $\deg D_n = n > 0 = 2g - 2$, and so we can use the previous remark to conclude that, if we denote $H_n \overset{\text{def}}{=} H^0(E, \mathcal{L}(n \cdot \infty))$ (not to be confused with any kind of homology!) and $l_n \overset{\text{def}}{=} \dim_k H_n$, then $l_n = n$. This will allow us to find an equation for the elliptic curve.

For $n = 1, l_n = 1$, then $H_n \simeq k$, and a basis for it is given by $1 \in k$. For $n = 2$, as $l_n = 2$ there is a nonconstant element $x \in H_2$ such that $\{1, x\}$ form a basis for H_2. We want to choose x in a "canonical way". For this, note that $\dim_k H^0(e, \Omega^1_E) = 1$, and so fix once and for all a basis ω for it.

Consider now the formal group of E at ∞,

$$\hat{E} = \operatorname{Spf} k[\![T]\!]$$

and the restriction of ω to $\hat{E}$:

$$\omega|_{\hat{E}} = (a_0 + a_1 T + \cdots)dT \quad a_0 \neq 0$$

Choose the local parameter T such that a_0 is 1. This will allow us to normalize the choice of x, by imposing that $x|_{\hat{E}}$ has a Laurent expansion of the form:

$$x|_{\hat{E}} = \frac{1}{T^2}(1 + b_1 T + \cdots)$$

We continue now with $n = 3, l_3 = 3$. Then we can choose some $y \in H_3 \setminus H_2$, such that:

$$y|_{\hat{E}} = \frac{1}{T^3}(1 + c_1 T + \cdots)$$

(we could take -1 or 2, instead of 1, and in other instances this is important).

We now consider $n = 4, 5$, and note that $y^2 - x^3 \in H_5$, which was spanned by $\{1, x, y, x^2, xy\}$, and so we get an equation:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

By making a change of variables $y \mapsto y - \frac{a_1}{2}x - \frac{a_3}{2}$, we get another equation:

$$y^2 = x^3 + b_2 x^2 + b_4 x + b_6$$

Finally, a change $x \mapsto x - \frac{b_2}{3}, y \mapsto y/2$ leaves us with the equation:

$$y^2 = 4x^3 - g_2 x - g_3 \quad g_2, g_3 \in k$$

One can prove then that

$$E \simeq \operatorname{Proj}\left(\frac{k[x, y, z]}{(y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3)}\right)$$

8.3 Relative case

Consider now S any noetherian scheme, and assume that 6 is invertible in S (what we really mean is that S is a scheme over $\operatorname{Spec}(\mathbb{Z}[1/6])$, or that all the residue fields of S have characteristic not dividing 6). Consider E/S an elliptic curve, that is:

$$\begin{array}{c} E \\ f \downarrow \uparrow e \\ S \end{array}$$

where f is the structure map, and e is the section that is part of the definition. Also, recall that, by definition, for any geometric point $s \to S$, the fiber $E_s/\kappa(s)$ is a smooth, proper, connected curve of genus 1. Together with the image of s under e, we obtain an elliptic curve over $\kappa(s)$, and $\kappa(s)$ has characteristic not dividing 6. By the absolute case, E_s has an equation:

$$E_s \simeq \operatorname{Proj}\left(\frac{\kappa(s)[x,y,z]}{(y^2z - 4x^3 + g_{2,s}xz^2 + g_{3,s}z^3)}\right)$$

We would like to put all these equations into a single one, given by global sections $g_2, g_3 \in \mathcal{O}_S(S)$. For this, we will need a relative version of the Riemann-Roch theorem.

8.3.1 Relative Riemann-Roch

Let C be a family of curves defined over a scheme S, say $f\colon C \to S$, where C/S has genus g (that is, f is smooth and proper, and for each geometric point $s \to S$, the fiber $C_s = C \times_S s$ is a curve of genus g over $\kappa(s)$).

Definition 8.2. An **effective cartier divisor** on C is a closed subscheme $D \subseteq C$,

$$\begin{array}{ccc} D & \xrightarrow{i} & C \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

such that $g\colon D \to S$ is finite-flat.

Remark. This readily implies that the push-forward of the structure sheaf of D, $g_*\mathcal{O}_D$, is a locally-free $\mathcal{O}_S$-module of finite rank (if M is a finitely-generated module over a noetherian ring A, then M is A-flat if, and only if, M is projective over A).

Example. *The divisors D that we allow should look like families of hypersurfaces over S, and we don't allow crossings. This statement is probably quite wrong, and will have to be corrected eventually, and maybe some pictures will have to be drawn.*

Let now $\mathcal{I}(D)$ be the ideal sheaf defined by the inclusion $D \subseteq C$ (again, it is a locally-free rank 1 sheaf of ideals of $\mathcal{O}_C$). So we have an exact sequence:

$$0 \to \mathcal{I}(D) \to \mathcal{O}_C \to i_*\mathcal{O}_D \to 0$$

As before, let $\mathcal{L}(D) \overset{\text{def}}{=} \mathcal{I}(D)^{-1} \overset{\text{def}}{=} \mathcal{H}\text{om}_{\mathcal{O}_C}(\mathcal{I}(D), \mathcal{O}_C)$. As $\mathcal{L}(D)$ is flat, we get an exact sequence:

$$0 \to \mathcal{I}(D) \otimes \mathcal{L}(D) \to \mathcal{O}_C \otimes \mathcal{L}(D) \to i_*\mathcal{O}_D \otimes \mathcal{L}(D) \to 0$$

which is isomorphic to:

$$0 \to \mathcal{O}_C \to \mathcal{L}(D) \to i_*\mathcal{O}_D \otimes \mathcal{L}(D) \to 0$$

The map $\mathcal{O}_C \hookrightarrow \mathcal{L}(D)$ is given by a global section l of $\mathcal{L}(D)$ (the image of $1 \in \mathcal{O}_C$). Then multiplication by l induces an exact sequence:

$$0 \to \mathcal{O}_C \xrightarrow{\cdot l} \mathcal{L}(D) \to \mathcal{L}(D)/(l\mathcal{O}_C) \to 0$$

and so $\mathcal{L}(D)/(l\mathcal{O}_C) \simeq i_*\mathcal{O}_D \otimes \mathcal{L}(D)$. This sheaf is supported on D, and it is flat over $\mathcal{O}_S$.

Claim. *The pair $(\mathcal{L}(D), l)$ determines D.*

Proof. Given a pair $(\mathcal{L}, l)$, then we can let $\mathcal{I} \overset{\text{def}}{=} \mathcal{L}^{-1}$, which is a sheaf of ideals of $\mathcal{O}_C$. Then $D = \operatorname{Supp}(\mathcal{L}/l\mathcal{O}_C)$, and $\mathcal{O}_D = \mathcal{O}_C/\mathcal{I}(D)$. □

We want to define also the degree of the divisor D. Note that $f_*(\mathcal{L}(D)/l\mathcal{O}_C)$ is locally-free over $\mathcal{O}_S$, and the degree of D will be defined as its rank.

Theorem 8.3 (Relative Riemann-Roch). *With the given setup, assume that $f_*\mathcal{L}(D)$ and $R^1 f_*\mathcal{L}(D)$ are locally-free of finite rank. Then,*

$$\operatorname{rk}_{\mathcal{O}_S} f_*\mathcal{L}(D) - \operatorname{rk}_{\mathcal{O}_S} R^1 f_*\mathcal{L}(D) = \deg D + 1 - g$$

Proof. Starting from the exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{\cdot l} \mathcal{L}(D) \to \mathcal{L}(D)/(l\mathcal{O}_C) \to 0$$

we get, by aplying the functor f_*, a long exact sequence:

$$0 \to f_*\mathcal{O}_C \xrightarrow{\cdot l} f_*\mathcal{L}(D) \to f_*(\mathcal{L}(D)/(l\mathcal{O}_C)) \xrightarrow{\delta} R^1 f_*\mathcal{O}_C \to R^1 f_*\mathcal{L}(D) \to R^1 f_*(\mathcal{L}(D)/l\mathcal{O}_C) \xrightarrow{\delta} R^2 f_*\mathcal{O}_C \to$$

We will study the terms in this sequence and then look at the ranks.

First, note that $R^2 f_*\mathcal{O}_C = 0$: if $s \to S$ is a geometric point, then C_s is smooth, proper of dimension 1 over $\kappa(s)$, so that $H^2(C_s, \mathcal{O}_{C_s}) - 0$. Then, by Mumford's theorem, $R^2 f_*\mathcal{O}_C$ is locally free of rank 0, so it is zero.

Next, by definition $f_*(\mathcal{L}(D)/l\mathcal{O}_C)$ is locally free of rank $\deg D$.

As D has relative dimension 0 over S, and $\mathcal{L}(D)/l\mathcal{O}_C$ is supported on D, its first right-derived functor is 0: $R^1 f_*(\mathcal{L}(D)/l\mathcal{O}_C) = 0$.

If $s \to S$ is a geometric point, then $H^0(C_s, \kappa(s)) \simeq \kappa(s)$, so $f_*\mathcal{O}_C$ is locally-free of rank 1. Also, $H^1(C_s, \mathcal{O}_{C_s})$ has dimension $g(C_s) = g$, so that $R^1 f_*\mathcal{O}_C$ is locally-free of rank g.

Now as $f_*\mathcal{L}(D)$ and $R^1 f_*\mathcal{L}(D)$ are locally-free of finite rank, then we can look at the alternating sum of the rank of the terms in the exact sequence, and it has to be 0, thus giving the result. □

8.3.2 Application to our situation

Let now E/S be a family of elliptic curves.

$$\begin{array}{c} E \\ f \downarrow \uparrow e \\ S \end{array}$$

Let $\infty \overset{\text{def}}{=} e(S) \hookrightarrow E$. As $f \circ e = \mathrm{Id}_S$, then the structure sheaf $\mathcal{O}_\infty$ satisfies $f_*\mathcal{O}_\infty = \mathcal{O}_S$, so that $\deg \infty = 1$.

We consider, as in the absolute case, the divisors $n \cdot \infty$, which are associated to $\mathcal{L}_n = \infty^{\otimes n}$ and to global sections l_n. To calculat the rank of $f_*\mathcal{L}_n$ we could use the relative Riemann-Roch.

Consider now the sheaf $\underline{\omega}_{E/S} \stackrel{\text{def}}{=} f_*\Omega^1_{E/S}$, which is locally-free of rank 1 (we check this on the stalks, as usual). Fix a trivializing affine cover $\{U_i\}_{i\in I}$ for both $\underline{\omega}_{E/S}$ and $f_*\mathcal{L}_1$ (that is, so that $U_i = \operatorname{Spec}(A_i)$, and both $\underline{\omega}_{E/S}|_{U_i}$ and $f_*\mathcal{L}_1|_{U_i}$ are free of rank 1. As $\mathcal{L}_n$ is a power of $\mathcal{L}_1$, it follows that the cover $\{U_i\}$ trivializes all the $\mathcal{L}_n$ as well.

Fix one of the opens in the cover, say $U = \operatorname{Spec}(A)$. Choose a basis $\omega \in \underline{\omega}_{E/S}(U)$, and consider the formal completion of E_U along ∞_U, denoted $\hat{E_U} \stackrel{\text{def}}{=} \operatorname{Spf}(A[\![T]\!])$. We choose T as before, using the restriction of ω to $\hat{E_U}$.

For $n = 1$, the sheaf $f_*\mathcal{L}_1(U)$ is free of rank 1, so choose a basis and call it 1. Note now that, for any $n \geq 1$, if $\alpha \in (f_*\mathcal{L}_n)(U)$, then we can think of it in $f_*\mathcal{L}_{n+1}(U)$ by mapping $\alpha \mapsto \alpha \otimes 1$. As these sections are free, this map is injective. This allows us to repeat the same argument as before, to get an equation:

$$y^2 = 4x^3 - g_{2,U}x - g_{3,U}$$

where $g_{i,U} \in A$, and they are uniquely determined by $f_*\mathcal{L}(\infty)$. So if U, U' are two affines in the covering, the corresponding restrictions to the intersections agree, and hence they glue to a global section $g_i \in \mathcal{O}_S(S)$, such that:

$$E/S \simeq \underline{\operatorname{Proj}}\left(\frac{\mathcal{O}_S[x,y,z]}{(y^2z - 4x^3 + g_2xz^2 + g_3z^3)}\right)$$

Remark. Let R be a reduced, noetherian ring without idempotents, and such that $6 \in R^\times$. We can then consider E/R an elliptic curve, given by an equation

$$y^2 = 4x^3 - g_2x - g_3$$

The sheaf $\underline{\omega}_{E/R} = f_*\Omega^1_{E/R}$ is locally-free, so it corresponds to a projective rank-1 module. This is actually free. In other words, there is a globally-defined nowhere vanishing section of $\underline{\omega}_{E/R}$. To see it, define:

$$\omega \stackrel{\text{def}}{=} \frac{dx}{2y}$$

which is defined for $y \neq 0$. But as

$$2ydy = (12x^2 - g_2)dx$$

we can see that ω can also be given, in an open neighborhood of $\{y = 0\}$, by:

$$\omega - \frac{dy}{12x^2 - g_2}$$

9 Explicit Description of the Sheaf Cohomology Groups

Let X be a noetherian scheme. Let $\mathcal{F}$ be any coherent sheaf on X (we could do it for quasicoherent).

Theorem 9.1. *If X is affine, then the sheaf cohomology vanishes:*

$$H^i(X, \mathcal{F}) = 0 \quad \forall i > 0$$

9.1 The Čech cohomology

Assume now that X is separated (that is, the diagonal is a closed subscheme of $X \times X$). This implies, in particular, that the intersection of a finite collection of affine opens of X is again affine.

We define the **Čech complex**, which will allow us to explicitly compute the cohomology of general noetherian separated schemes. First, fix an affine open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X. Give I a well-ordering (we will usually deal with finite covers, so this won't be a problem) and, for every finite subset $J \subseteq I$, denote by U_J:

$$U_J \overset{\text{def}}{=} \cap_{j \in J} U_j$$

(which is an affine open, because X is separated). The Čech complex is then:

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}): \quad \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d_0} \check{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d_1} \check{C}^2(\mathcal{U}, \mathcal{F}) \to \cdots$$

where:

$$\check{C}^p(\mathcal{U}, \mathcal{F}) \overset{\text{def}}{=} \prod_{\substack{J \subseteq I \\ |J| = p+1}} \mathcal{F}(U_J)$$

and $d_p \colon \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ is defined by:

$$d_p\left((x_J)_J\right) = \left(d_p(x_J)\right)_K$$

where, if $K \subseteq I$ and $|K| = p + 2$, then one can order $K = \{k_0 < k_1 < \cdots < k_{p+1}\}$, let $K_h \overset{\text{def}}{=} \{k_0 < \cdots < \hat{k_h} < \cdots < k_{p+1}\}$ be the ordered set obtained by removing the h^{th} term, and:

$$\left(d_p(x_J)\right)_K = \sum_{h=0}^{p+1} (-1)^h x_{K_h}|_{U_K} \in \mathcal{F}(U_K)$$

As an exercise, one can check that $d_{p+1} \circ d_p = 0$, so that this is a complex. Also, note that, as $\mathcal{F}$ is a sheaf,

$$H^0\left(\check{C}^\bullet(\mathcal{U}, \mathcal{F})\right) = \ker d^0 = H^0(X, \mathcal{F})$$

Theorem 9.2. *If X is noetherian and separated, $\mathcal{F}$ is a coherent sheaf, and $\mathcal{U}$ is an open affine cover, then:*

$$H^i\left(\check{C}^\bullet(\mathcal{U}, \mathcal{F})\right) \simeq H^i(X, \mathcal{F})$$

Example (Computation of $H^1(X, \mathcal{F})$ using Čech cohomology). *If we follow the definitions, we get that:*

$$H^1\left(C^\bullet(\mathcal{U}, \mathcal{F})\right) = \frac{Z^1}{B^1}$$

where, if we denote $U_{ij} \overset{def}{=} U_i \cap U_j$ and $U_{ijk} \overset{def}{=} U_i \cap U_j \cap U_k$,

$$Z^1 = \left\{(f_{ij})_{i<j} \mid f_{ij} \in \mathcal{F}(U_{ij}) \text{ and } f_{ij}|_{U_{ijk}} - f_{ik}|_{U_{ijk}} - f_{jk}|_{U_{ijk}} = 0 \,\forall i < j < k\right\}$$
$$B^1 = \left\{(g_i|_{U_{ij}} - g_j|_{U_{ij}})_{i<j} \mid (g_i)_{i \in I} \text{ with } g_i \in \mathcal{F}(U_i)\right\}$$

The elements in Z^1 are called 1-cocycles, *and those in B^1 are called* 1-coboundaries.

Example (For the case of Curves). *Supppose that* $C = X$ *is a curve (smooth, proper, connected) over* k*, of genus* g *(for simplicity, assume* k *is an algebraically closed field). let* P, Q *be two closed points. Consider the cover given by* $U_P \overset{def}{=} C \setminus \{P\}$ *and* $U_Q \overset{def}{=} C \setminus \{Q\}$*, and denote by* $U_{PQ} \overset{def}{=} U_P \cap U_Q$ *their intersection. The Čech complex, which has only two terms, fits then in an exact sequence:*

$$
\begin{array}{ccccccccc}
0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow \mathcal{F}(U_P) \oplus \mathcal{F}(U_Q) \xrightarrow{d_0} \mathcal{F}(U_{PQ}) \longrightarrow H^1(X,\mathcal{F}) \to 0 \\
x \longmapsto (x|_{U_P}, x|_{U_Q}) \\
(a,b) \longmapsto a|_{U_{PQ}} - b|_{U_{PQ}}
\end{array}
$$

This is precisely the Mayer-Vietoris *sequence for the cover* $\{U_P, U_Q\}$*.*

Example (Cohomology of an elliptic curve). *Let now* E/k *be an elliptic curve. We follow the previous example, and choose* $P = \infty$*, and* $Q \neq P$ *any other point. Then:*

$$H^1(E, \mathcal{O}_E) \simeq \frac{\mathcal{O}_E(U_{\infty,Q})}{\textit{image of } (\mathcal{O}_E(U_\infty) \oplus \mathcal{O}_E(U_Q))}$$

Consider the divisor $D \overset{def}{=} \infty + Q$*. By Riemann-Roch,*

$$\dim_k H^0(E, \mathcal{L}(D)) = 2$$

So there is some $z \in H^0(E, \mathcal{L}(D))$ *which is nonconstant. Note that, as there are no functions with only one simple pole, the function* z *must have simple poles at both* ∞ *and* Q *(and is regular everywhere else). It is a fun and easy exercise now to check that* $[z]$ *is a generator for* $H^1(E, \mathcal{O}_E)$*.*

10 The deRham Cohomology

Let X be a smooth, separated scheme over a noetherian ring k. We have already introduced the k-derivation:

$$d\colon \mathcal{O}_X \to \Omega^1_{X/k}$$

We can then define, for each $i \geq 2$,

$$\Omega^i \overset{\text{def}}{=} \Omega^{i-1} \wedge \Omega^1$$

and the derivation d can be extended to:

$$d\colon \Omega^i_{X/k} \to \Omega^{i+1}_{X/k}$$

by sheafifying the Kähler construction. That is, locally, and for $i = 1$, if $\omega = \sum f_i da_i$, then $d\omega = \sum df_i \wedge da_i$. In this way, we get a complex *of sheaves*, called the **deRham complex**. Its cohomology groups (that is, the *hypercohomology*) is the **deRham cohomology** (which are k-modules):

$$H^1_{\text{dR}}(X/k) \overset{\text{def}}{=} \mathbb{H}^i(X, \Omega^\bullet_{X/k})$$

(this can be defined in terms of injective resolutions. Here we will just see how to compute them, and will leave out the theorems that allow this to be done).

10.1 How to calculate hypercohomology

In general, let

$$\mathcal{F}^\bullet\colon \quad \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \cdots$$

be a complex of coherent, locally-free sheaves on X (a smooth, separated scheme over a noetherian ring k). Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open affine cover of X, with I well-ordered. We define then a double complex of k-modules $(C^{\bullet,\bullet}, d, \delta)$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \delta=\check{\delta}} & & \uparrow{\scriptstyle \delta=-\check{\delta}} & & \uparrow{\scriptstyle \delta=\check{\delta}} & & \\
C^{2,0} = \prod_{i<j<k} \mathcal{F}^0(U_{ijk}) & \xrightarrow{d} & C^{2,1} = \prod_{i<j<k} \mathcal{F}^1(U_{ijk}) & \xrightarrow{d} & C^{2,2} = \prod_{i<j<k} \mathcal{F}^2(U_{ijk}) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle \delta=\check{\delta}} & & \uparrow{\scriptstyle \delta=-\check{\delta}} & & \uparrow{\scriptstyle \delta=\check{\delta}} & & \\
C^{1,0} = \prod_{i<j} \mathcal{F}^0(U_{ij}) & \xrightarrow{d} & C^{1,1} = \prod_{i<j} \mathcal{F}^1(U_{ij}) & \xrightarrow{d} & C^{1,2} = \prod_{i<j} \mathcal{F}^2(U_{ij}) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle \delta=\check{\delta}} & & \uparrow{\scriptstyle \delta=-\check{\delta}} & & \uparrow{\scriptstyle \delta=\check{\delta}} & & \\
C^{0,0} = \prod_{i} \mathcal{F}^0(U_{i}) & \xrightarrow{d} & C^{0,1} = \prod_{i} \mathcal{F}^1(U_{i}) & \xrightarrow{d} & C^{0,2} = \prod_{i} \mathcal{F}^2(U_{i}) & \xrightarrow{d} & \cdots
\end{array}$$

Note that, in the *odd columns*, we take $-\check{\delta}$ instead of $\check{\delta}$. Note also that $d^2 = \delta^2 = d\delta + \delta d = 0$ (the last is an *anticommutativity relation*).

We can then make a single complex of this double complex, which we will call $(K^\bullet, D)$, where

$$K^n \overset{\text{def}}{=} \oplus_{p+q=n} C^{p,q} \qquad D \overset{\text{def}}{=} d + \delta\colon K^n \to K^{n+1}$$

(note that $D^2 = (d+\delta)\circ(d+\delta) = d^2 + (d\delta + \delta d) + \delta^2 = 0 + 0 + 0 = 0$).

Theorem 10.1. *With the previous notation,*

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) = H^i(K^\bullet)$$

Theorem 10.2. *If* $0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathfrak{h}^\bullet \to 0$ *is a short exact sequence of complexes of coherent, locally free sheaves, then we get a long exact sequence in hypercohomology:*

$$0 \to \mathbb{H}^0(X, \mathcal{F}^\bullet) \to \mathbb{H}^0(X, \mathcal{G}^\bullet) \to \mathbb{H}^0(X, \mathfrak{h}^\bullet) \to \mathbb{H}^1(X, \mathcal{F}^\bullet) \to \mathbb{H}^1(X, \mathcal{G}^\bullet) \to \cdots$$

Example. *Consider the deRham complex:*

$$\Omega^\bullet_{X/k}\colon \quad \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \to \cdots$$

Let $\mathcal{U} = \{U_i\}_{i\in I}$ *be an open affine covering of* X*. We will write explicitly the first two deRham cohomology groups. Again, it's a matter of tracing the definitions, to get:*

1. $H^0_{dR}(X/k) = \ker D^0 = \ker\left(d\colon H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X)\right)$.

2. *For H^1_{dR}, we will write again as Z^1/B^1, where Z^1 are hypercocycles, and B^1 are hypercoboundaries:*

$$Z^1 = \left\{(\omega_i, f_{ij}) \mid \omega_i \in \Omega_X(U_i), f_{ij} \in \mathcal{O}_X(U_{ij}) \mid d\omega_i = 0, \omega_i|_{U_{ij}} - \omega_j|_{U_{ij}} = df_{ij}, f_{ij}| - f_{ik} + f_{jk}| = 0\right\}$$

$$B^1 = \left\{(dx_i, x_i|_{U_{ij}} - x_j|_{U_{ij}}) \mid (x_i)_i \in \prod_i \mathcal{O}_X(U_i)\right\}$$

Example (Curves). *Let C/k be a curve. Then, we can take a covering with only two opens: $\mathcal{U} = \{U, V\}$, and the previous example is simplified ($\Omega^2 = 0$, and there is only one intersection). We can write then:*

$$H^1_{dR}(C/k) = \frac{\{(\omega_U, \omega_V, f_{UV}) \mid \omega_U| - \omega_V| = df_{UV}\}}{\{(dx_U, dx_V, x_U| - x_V|) \mid x_U \in \mathcal{O}_X(U), x_V \in \mathcal{O}_X(V)\}}$$

We can fit this group in an exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(C, \Omega^1_{X/k}) & \longrightarrow & H^1_{dR}(C/k) & \longrightarrow & H^1(C, \mathcal{O}_C) & \longrightarrow 0 \\ & \omega & \longmapsto & [(\omega_U, \omega_V, 0)] & & & \\ & & & [(\omega_U, \omega_V, f_{UV})] & \longmapsto & [f_{UV}] & \end{array}$$

which is called the ***Hodge filtration*** *exact sequence.*

Remark. If X is a smooth, projective, algebraic variety over $\mathbb{C}$, and we consier its analyfication X^{an}, then there is the Hodge-deRham spectral sequence:

$$E_1^{p,q} = H^q(X^{\mathrm{an}}, \Omega^p_{X/\mathbb{C}}) \implies H^{p+q}_{\mathrm{dR}}(X/k)$$

which collapses already at E_1 (for instance, using the theory of harmonic forms), and thus gives the so-called **Hodge decomposition**:

$$H^n_{\mathrm{dR}}(X/k) = \oplus_{p+q=n} H^q(X, \Omega^p_X)$$

Hence then name for the previous exact sequence. However, in the algebraic realm we don't get a decomposition, just an exact sequence (or, in higher dimension, a spectral sequence) and hence just a filtration. So the splitting (which exists if we work over k a field) is not canonical (although there are good choices that one can make, in some instances). At least, we can compute the dimension:

$$\dim_k H^1_{\mathrm{dR}}(C/k) = 2g$$

because the two spaces in the extremes of the s.e.s. are dual to each other (thanks to Serre duality).

Also, for X proper and connected, we know the dimensions of the other deRham cohomology groups:

$$\dim_k H^0_{\mathrm{dR}}(C/k) = 1 = \dim_k H^2_{\mathrm{dR}}(C/k)$$

The dimension of H^2_{dR} can be computed explicitly (by using its definition, which is simple because we only have two open sets in the covering). Alternatively, we can wait and this will follow from Poincaré duality.

11 Relative deRham Cohomology

Let $f\colon X \to Y$ be a smooth, proper morphism. We consider then the **relative deRham complex**:

$$\Omega^\bullet_{X/Y}\colon \quad \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \to \cdots$$

We want to compute the right-derived functors of the push-forward f_*, which will be sheaves on Y:

$$\mathbb{R}^i f_*(\Omega^\bullet_{X/Y}) \stackrel{\text{def}}{=} H^i(X/Y)$$

Let $U = \operatorname{Spec} R \subseteq Y$ be an open affine. Let $X_U \stackrel{\text{def}}{=} X \times_Y U$. For each $i \geq 0$, we can compute $H^i_{\text{dR}}(X_U/R)$, which is an R-module. The mapping that sends an affine open U to $H^i_{\text{dR}}(X_U/R)$ gives a presheaf on Y, and one can show that $H^i(X/Y)$ is the sheafification of this presheaf.

Let's now look at a more particular setting: assume that S is noetherian, reduced, and connected, and that $f\colon C \to S$ is a proper family of curves of genus g. If $s \to S$ is a geometric point, we have seen that:

$$\dim_{\kappa(s)} H^i_{\text{dR}}\left(C_s/\kappa(s)\right) = \begin{cases} 1 & \text{if } i = 0, 2 \\ 2g & \text{if } i = 1 \\ 0 & \text{if } i > 2 \end{cases}$$

(in particular, the dimensions do not depend on s).

By Mumford's theorem, it follows that $H^i_{\text{dR}}(C/S)$ are coherent, locally-free $\mathcal{O}_S$-modules (of ranks $1, 2g, 0$, respectively).

Moreover, we have an exact sequence of sheaves on S:

$$0 \to f_*\Omega^1_{C/S} \to H^1_{\text{dR}}(C/S) \to R^1 f_*\mathcal{O}_C \to 0$$

such that, if $s \to S$ is a geometric point, and we look at the fibers on s (that is, we pull-back the sheaves along the map $s \to S$, which is not exactly the same as looking at the stalks of the image point $\overline{s} \in S$) we recover the Hodge filtration.

12 Application to Elliptic Curves

Let now $(E/k, \infty)$ be an elliptic curve over a *noetherian ring* k. We will see that, provided that $6 \in k^\times$, we get canonical bases for the deRham cohomology.

Let $\mathcal{O}(\infty) \stackrel{\text{def}}{=} \mathcal{L}(\infty)$ denote the sections with at most a simple pole at ∞, and by

$$\Omega^1(2\infty) \stackrel{\text{def}}{=} \Omega^1_{E/k} \otimes_{\mathcal{O}_E} \mathcal{L}(2\infty)$$

We have a differential $d\colon \mathcal{O}(\infty) \to \Omega^1(2\infty)$, which induces an inclusion of complexes:

$$\begin{array}{cccc} \Omega^\bullet(2\infty)\colon & \mathcal{O}(\infty) & \xrightarrow{d} & \Omega^1(2\infty) \\ \uparrow & \uparrow & & \uparrow \\ \Omega^\bullet_{E/k}\colon & \mathcal{O}_E & \xrightarrow{d} & \Omega^1_{E/k} \end{array}$$

Let $\underline{k}$ be the constant sheaf on ∞, and consider the skyscrapper sheaf supported on ∞, $K \overset{\text{def}}{=} j_*\underline{k}$ (where $j\colon \infty \hookrightarrow E$ is the inclusion).

We can define a morphism

$$\text{res}_\infty\colon \Omega^1(2\infty) \twoheadrightarrow K$$

given by, if $U \subseteq E$ is an open and $\infty \in U$, then given $\omega \in \Omega^1(2\infty)(U)$, we can write it as:

$$\omega_\infty = \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + \omega'$$

(with t a local parameter at ∞, and ω' without poles), and we define $res_\infty(\omega) \overset{\text{def}}{=} a_{-1}$ (one needs to check that this is indeed well defined).

Next, we define the **differentials of the second kind**:

$$\Omega_{\text{II}} \overset{\text{def}}{=} \ker \text{res}_\infty$$

We get an exact sequence of complexes (because $d\colon \mathcal{O}(\infty) \to \Omega_{\text{II}}$ is a well defined differential):

$$0 \to \Omega^\bullet_{\text{II}} \to \Omega^\bullet(2\infty) \overset{\text{res}_\infty}{\to} K^\bullet \to 0$$

which yields a long exact sequence in hypercohomology:

$$\cdots \to \mathbb{H}^0(E, K^\bullet) \to \mathbb{H}^1(E, \Omega^\bullet_{\text{II}}) \to \mathbb{H}^1(E, \Omega^\bullet(2\infty)) \to \mathbb{H}^1(E, K^\bullet) \to \mathbb{H}^2(E, \Omega^\bullet_{\text{II}}) \to \cdots$$

Note that $\mathbb{H}^0(E, K^\bullet) = 0$, and that $\mathbb{H}^1(E, K^\bullet) \simeq K$. Moreover, $\mathbb{H}^2(E, \Omega^\bullet_{\text{II}}) = H^2_{\text{dR}}(E/k) \simeq K$, and the map is an isomorphism. This implies then that:

$$\mathbb{H}^1(E, \Omega^\bullet_{\text{II}}) \simeq \mathbb{H}^1(E, \Omega^\bullet(2\infty))$$

Next, note that the following is a commutative diagram (with exact rows, by definition):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker d & \longrightarrow & \mathcal{O}(\infty) & \overset{d}{\longrightarrow} & \Omega_{\text{II}} & \longrightarrow & \operatorname{coker} d & \longrightarrow & 0 \\
 & & \uparrow \simeq & & \uparrow & & \uparrow & & \uparrow \simeq & & \\
0 & \longrightarrow & \ker d_E & \longrightarrow & \mathcal{O}_E & \overset{d_E}{\longrightarrow} & \Omega^1_{E/k} & \longrightarrow & \operatorname{coker} d_E & \longrightarrow & 0
\end{array}$$

The induced maps on the kernel and cokernel are isomorphisms, as can be checked directly on the stalks (and use that $\frac{1}{t^2}$ is integrable). This implies that the two complexes have the same cohomology, and so we conclude finally that (this is what we will use to calculate the deRham cohomology of E):

$$H^1_{\text{dR}}(E/k) \simeq \mathbb{H}^1(E, \Omega^\bullet(2\infty))$$

Recall now that $H^1(E, \mathcal{O}(\infty)) = 0 = H^1(E, \Omega^1(2\infty))$, and so:

$$\mathbb{H}^1(E, \Omega^\bullet(2\infty)) = \operatorname{coker}\big(\mathcal{O}(\infty)(E) \to \Omega^1(2\infty)(E)\big)$$

but the map is the 0 map (because $\mathcal{O}(\infty)$ contains only the constants, which are killed by d), so that:

$$H^1_{\text{dR}}(E/k) \simeq H^0(E, \Omega^1(2\infty))$$

By Riemann-Roch, $\dim_k H^0(E,\Omega^1(2\infty)) = 2$, and we will find a basis for this (again, provided that $6 \in k^\times$).

We have an equation:

$$E\colon y^2 = 4x^3 - g_2 x - g_3$$

and a regular differential $\omega = dx/y \in H^0(E,\Omega^1_{E/k})$, which we fixed a priory. This is of course in $H^0(E,\Omega^1(2\infty))$ and is nonzero there, so it's an element of a basis. The other element will be:

$$\eta \overset{\text{def}}{=} \frac{xdx}{y} \in H^0\left(E,\Omega^1(2\infty)\right) \setminus H^0\left(E,\Omega^1_{E/k}\right)$$

(note that, as η is not regular, it can't be proportional to ω). So we get a "canonical" basis $\{\omega,\eta\}$ (canonical in the sense that it is uniquely determined, once we fix a regular differencial ω!).

Note in particular that, even if k is not a field, the first deRham cohomology of E/k is a free k-module of rank 2 (a priori we just knew that it was projective).

Recall the Hodge filtration for an elliptic curve:

$$0 \to \underline{\omega}_{E/k} \to H^1_{\mathrm{dR}}(E/k) \to R^1 f_* \mathcal{O}_E \to 0$$

and let $\overline{\eta}$ be the image of η in $R^1 f_*\mathcal{O}_E$. As this group is isomorphic to $\underline{\omega}_{E/k}^{-1}$, we have a canonical splitting $\overline{\eta} \mapsto \eta$ which gives an isomorphism of k-modules:

$$H^1_{\mathrm{dR}}(E/k) \simeq \underline{\omega}_{E/k} \oplus \underline{\omega}_{E/k}{}^{-1}$$

To understand η a little more, we will explicitly find a hypercocycle that represents it. So consider the usual cover $U_P = E \setminus P$, and $U_\infty = E \setminus \infty$ (with $P \neq \infty$), and $U_{P\infty} = U_P \cap U_\infty$. Let also $\eta_\infty \overset{\text{def}}{=} \eta|_{U_\infty}$ (and note that $\eta_\infty \in \Omega^1_{E/k}(U_\infty)$).

Consider the divisor $D \overset{\text{def}}{=} \infty + P$. By Riemann-Roch, $\dim_k H^0(E,\mathcal{L}(D)) = 2$, and so there exists $f \in H^0(E,\mathcal{L}(D))$ which is not a constant. By what we have previously seen, f must have poles of exact order 1 at both P and ∞, and no residues there. Hence, there exists some constant $a \in k$ such that $\eta - adf$ is regular at ∞. Denote then by $\eta_P \overset{\text{def}}{=} (\eta - adf)|_{U_P} \in \Omega^1_{E/k}(U_P)$. As $f|_{U_{P\infty}} \in \mathcal{O}_E(U_{P\infty})$, it follows that the triple:

$$(\eta_\infty, \eta_P, af|_{U_{P\infty}})$$

is a 1-hypercocycle corresponding to η. Hence we conclude that, under the natural map:

$$H^1_{\mathrm{dR}}(E/k) \xrightarrow{\beta} H^1(E,\mathcal{O}_E) \to 0$$

the basis element η is sent to $\beta(\eta) \overset{\text{def}}{=} [af|_{U_{P\infty}}] \overset{\text{def}}{=} \overline{\eta}$.

13 Connections

Let Y/k be a smooth scheme, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Y$-modules on Y.

Definition 13.1. A **connection** on $\mathcal{F}$ is a morpshism of sheaves of $\mathcal{O}_Y$-modules on Y,

$$\nabla\colon \mathcal{F} \to \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{F}$$

such that:

- It is k-linear: for each $U \subseteq Y$ open subset, $\nabla_U\colon \mathcal{F}(U) \to \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y(U)} \mathcal{F}(U)$ is a k-linear map.
- If $U \subseteq Y$ is an open subset, and $s \in \mathcal{O}_Y(U), x \in \mathcal{F}(U)$, then it satisfies the Leibniz rule:
$$\nabla_U(sx) = (ds) \otimes x + s\nabla_U(x)$$

Example. *Let* $Y = \operatorname{Spec}(k[x,y]/(xy-1)) = \mathbb{G}_{m,k}$. *Let* $A = k[x,y]/(xy-1)$. *Then* $\Omega^1_{Y/k} = \Omega^1_{A/k} \simeq A\frac{dx}{x}$.

If $\mathcal{F}$ *is a free* A*-module of rank* 1*, say* $\mathcal{F} = Ae$*, then to give a connection* $\nabla\colon \mathcal{F} \to \Omega^1_{A/k} \otimes \mathcal{F} \simeq \Omega^1_{A/k} \otimes e$ *is the same as the giving of* $\nabla(e) = \omega_0 \otimes e$*. Once this choice is made, then if* $a \in A$*,*

$$\nabla(ae) = da \otimes e + a\nabla(e) = (da + a\omega_0) \otimes e$$

and we can even write $\omega_0 = a_0\frac{dx}{x}$*, so that the previous equality becomes:*

$$\nabla(ae) = \left(\frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + aa_0\frac{dx}{x}\right) \otimes e = \left(x\frac{\partial a}{\partial x} - y\frac{\partial a}{\partial y} + aa_0\right)\frac{dx}{x} \otimes e$$

If $\mathcal{F}$ *is a free* A*-module of rank* 2*, then the giving of* ∇ *is equivalent (after choosing a basis for* $\mathcal{F}$*) to the giving of a* 2×2 *matrix with entries in* $\Omega^1_{A/k}$*.*

Once a connection ∇ is given, we can construct a map:

$$\nabla\colon \Omega^1_{Y/k} \otimes \mathcal{F} \to \Omega^2_{Y/k} \otimes \mathcal{F}$$

by defining (if $\nabla(f) = \sum \omega_i \otimes f_i$):

$$\nabla(\omega \otimes f) \stackrel{\text{def}}{=} \sum (\omega \wedge \omega_i) \otimes f_i$$

The compositum $\nabla^2 = \nabla \circ \nabla$ is to be interpreted as a curvature.

Definition 13.2. We say that ∇ is **integrable** if $\nabla^2 = 0$.

Note that non-integrability is the obstruction to having local solutions of the equation $\nabla f = 0$.

If ∇ is integrable, then automatically the higher iterations of ∇ also behave as differentials, so we get a complex of sheaves of $\mathcal{O}_Y$-modules on Y:

$$\Omega^\bullet_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{F}\colon \quad \mathcal{F} \xrightarrow{\nabla} \Omega^1_{Y/k} \otimes \mathcal{F} \xrightarrow{\nabla} \Omega^2_{Y/k} \otimes \mathcal{F} \xrightarrow{\nabla} \Omega^3_{Y/k} \otimes \mathcal{F} \to \cdots$$

and the **deRham cohomology with coefficients** defined to be its hypercohomology:

$$H^i_{\text{dR}}(Y, (\mathcal{F}, \nabla)) \stackrel{\text{def}}{=} \mathbb{H}^i\left(\Omega^\bullet_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{F}\right)$$

14 The Gauss-Manin Connection

We let k be a field, and Y/k a smooth curve of finite type. Let $f\colon X \to Y$ a smooth morphism. The goal is to define a connection on the sheaf $\mathcal{F} = H^i_{\mathrm{dR}}(X/Y)$, following quite closely [KO68].

If all the involved schemes were affine, say $X = \mathrm{Spec}(B), Y = \mathrm{Spec}(A)$, then we would get one of the fundamental exact sequences of B-modules (see [Har77], chapter II):

$$\Omega^1_{A/k} \otimes_A B \longrightarrow \Omega^1_{B/k} \longrightarrow \Omega^1_{B/A} \longrightarrow 0$$

In the global situation, we get as well the fundamental exact sequence, where we recall that $f^*\Omega^1_{Y/k} \stackrel{\text{def}}{=} f^{-1}\Omega^1_{Y/k} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$:

$$f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

Thanks to $X \to Y$ being smooth, the first map is actually injective (so we get an extra 0 on the left). The terms in this exact sequence are locally-free, and so we get a canonical filtration of the complex $\Omega^\bullet_{X/k}$:

$$\Omega^\bullet_{X/k} = F^0(\Omega^\bullet_{X/k}) \supseteq F^1(\Omega^\bullet_{X/k}) \supseteq F^2(\Omega^\bullet_{X/k}) \supseteq \cdots$$

with

$$F^i = F^i(\Omega^\bullet_{X/k}) = \mathrm{image}[\Omega^{\bullet - i}_{X/k} \otimes_{\mathcal{O}_X} f^*\Omega^i_{Y/k} \to \Omega^\bullet_{X/k}]$$

and such that:

$$\mathrm{gr}^i = \mathrm{gr}^i(\Omega^\bullet_{X/k}) \stackrel{\text{def}}{=} F^i/F^{i+1} = f^*\Omega^i_{Y/k} \otimes_{\mathcal{O}_X} \Omega^{\bullet - i}_{X/Y}$$

(see [Har77], Exercise $II.5.16d$).

Note that in our case $F^2 = 0$ because Y has dimension 1 over k, so the filtration is equivalent to the following exact sequence of complexes:

$$0 \to f^{-1}\Omega^1_{Y/k} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^{\bullet - 1}_{X/Y} \to \Omega^\bullet_{X/k} \to \Omega^\bullet_{X/Y} \to 0$$

and in turn this yields a long exact sequence in hypercohomology (that is, one takes the hyperderived functors of f_*). In particular, there exist boundary maps:

$$H^i_{\mathrm{dR}}(X/Y) = \mathbb{H}^i(\Omega^\bullet_{X/Y}) \xrightarrow{\delta} \mathbb{H}^{i+1}\left(f^{-1}\Omega^1_{Y/k} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^{\bullet - 1}_{X/Y}\right)$$

As $f^{-1}\Omega^1_{Y/k}$ is locally free and the differential of the complex is $f^{-1}\mathcal{O}_Y$-linear, the term on the right is isomorphic to:

$$\Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} \mathbb{H}^{i+1}(\Omega^{\bullet - 1}_{X/Y}) = \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} \mathbb{H}^i(\Omega^\bullet_{X/Y}) = \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} H^i_{\mathrm{dR}}(X/Y)$$

and so the connecting homomorphism can be seen as a morphism:

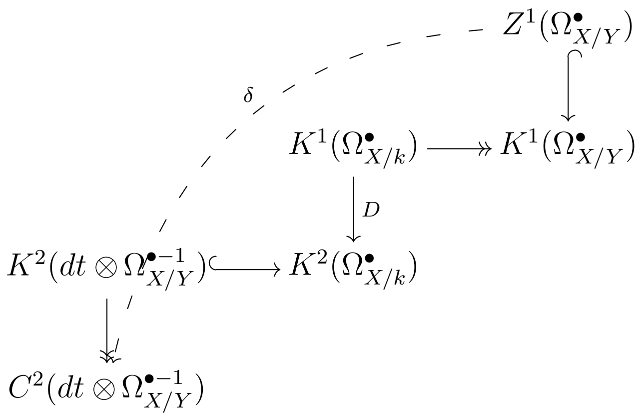
$$\nabla_i\colon H^i_{\mathrm{dR}}(X/Y) \to \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} H^i_{\mathrm{dR}}(X/Y)$$

Definition 14.1. The **Gauss-Manin connection** is ∇_i.

Example. *We compute* $\nabla_1 \colon H^1_{dR}(X/Y) \to \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} H^1_{dR}(X/Y)$*. Let's assume that* Y *is affine,* $Y = \operatorname{Spec} A$*. After localizing, we can assume that* $\Omega^1_{Y/k} \simeq A dt$*, for some* $t \in A$*.*

Consider an affine cover of X*,* $X = \cup U_i$*, with* $U_i = \operatorname{Spec} B_i$*.*

Let $x \in H^1_{dR}(X/Y)$*, represented by* $((\omega_i)_{i\in I}, (f_{ij})_{i<j}) \in K^1(\Omega^\bullet_{X/Y})$*, with* $\omega_i \in \Omega^1_{B_i/A}$ *and* $f_{ij} \in B_{ij}$ *satisfying the conditions we have worked out before. To compute* $\nabla_1(x)$*, we just need to follow the definition of the connecting homomorphism, induced by the morphism on the terms in the total complex (we denote by* Z *and* C *the kernel and cokernel of the differentials on each complex):*



That is, choose lifts $\overline{\omega_i}$ *of* ω_i*, apply* D *to the element* $((\overline{\omega_i})_i, (f_{ij})_{i<j})$*, and we will be able to write:*

$$D\left((\overline{\omega}_i), (f_{ij})\right) = dt \otimes ((\eta_i), (g_{ij})_{i<j})$$

and then

$$\nabla_1(x) = dt \otimes ((\eta_i), (g_{ij})_{i<j})$$

15 The Poincaré Pairing (lite)

Let C/k be a smooth proper curve. We assume $\operatorname{char} k = 0$ and, for simplicity, we will also assume that k is algebraically closed (otherwise, just minor modifications need to be done). We will see a way to compute a perfect, alternating pairing:

$$\langle \cdot, \cdot \rangle_{\text{Poinc}} \colon H^1_{\text{dR}}(C/k) \times H^1_{\text{dR}}(C/k) \to k$$

We just show to compute it, without giving the proper definition. Let $K = k(C)$ be the function field of C, and let $\Omega_K \overset{\text{def}}{=} \Omega^1_{K/k}$ be the K-vectorspace of *global meromorphic* differentials on C. Consider also the sheaves $\mathcal{O}_C$ and $\Omega^1_{C/k}$.

For each closed point $P \in C(k)$, we have a discrete valuation:

$$\operatorname{ord}_P \colon K^\times \to \mathbb{Z}$$

Let K_P be the completion of K at ord_P, and $\mathcal{O}_P$ be the completion of $\mathcal{O}_{C,P}$ (the stalk at P, which is seen as a subring of K). Define also $\Omega_{K_P} \overset{\text{def}}{=} \Omega^1_{K_P/k}$, and Ω_P the completion of the stalk $\Omega_{C,P}$.

Example. *If t is a uniformizer at P (that is, $\operatorname{ord}_P(t) = 1$), then:*

$$\mathcal{O}_P \simeq k[\![t]\!] \qquad K_P \simeq k((t))$$
$$\Omega_P \simeq \mathcal{O}_P \cdot dt \qquad \Omega_{K_P} \simeq K_P \cdot dt$$

We have also, for each $P \in C(k)$, the *residue map*:

$$\operatorname{res}_P \colon \Omega_K \to k$$

defined as follows: if $\omega \in \Omega_K$, let ω_P be its image in $\Omega_{K_P}(= k((t)) \cdot dt)$. Then $\operatorname{res}_P(\omega) \stackrel{\text{def}}{=} \operatorname{res}_P(\omega_P) \stackrel{\text{def}}{=} a_{-1}$ (if $\omega = (\frac{a_{-n}}{t^n} + \cdots + \frac{a_{-1}}{t} + a_0 + a_1 + \cdots)dt$). One needs to check that this is actually well defined...

Denote by Ω_K^{II} the k-vectorspace of meromorphic differential forms of the second kind:

$$\Omega_K^{\mathrm{II}} \stackrel{\text{def}}{=} \{\omega \in \Omega_K \mid \operatorname{res}_P \omega = 0 \, \forall P \in C(k)\}$$

Remark. If $\omega \in \Omega_K^{\mathrm{II}}$, then we can locally integrate it: for each $P \in C(k)$, there exists some $\gamma_P \in K_P(= k((t)))$ such that $d\gamma_P = \omega_P$.

We have seen that, for elliptic curves, the first deRham cohomology coincided with the differential forms of the second kind. In general, a similar result is true:

Proposition 15.1. *The deRham cohomology of C can be computed as:*

$$H^1_{dR}(C/k) \simeq \frac{\Omega_K^{II}}{dK}$$

(for elliptic curves, $dK = 0$).

Now, if we believe the previous proposition, then given $[\alpha], [\beta] \in H^1_{\mathrm{dR}}(C/k)$, with $\alpha, \beta \in \Omega_K^{\mathrm{II}}$, we proceed as follows: for each $P \in C(k)$, let $\alpha_P = d\gamma_P$ for some $\gamma_P \in K_P$. Then:

$$\langle [\alpha], [\beta] \rangle_{\mathrm{Poinc}} \stackrel{\text{def}}{=} \sum_{P \in C(k)} \operatorname{res}_P(\gamma_P \beta_P) \quad \in k$$

(this is actually well-defined, alternating, and non-degenerate, but we won't prove any of these facts here).

Example. *Let E/k be an elliptic curve over a field k, $\operatorname{char} k = 0$. Consider the equation for E that we have found:*

$$E\colon \quad y^2 = 4x^3 - g_2 x - g_3$$

and recall that x, y were chosen in such a way so that, if T is a parameter at ∞, then:

$$x_\infty = \frac{1}{T^2} + a_0 + a_1 T + \cdots$$
$$y_\infty = \frac{2}{T^3} + b_0 + b_1 T + \cdots$$

(note that the 2 in the expression for y_∞ is so that we get the 4 in the equation for E).

Then $\omega = dx/y, \eta = xdx/y$, and we will compute the Poincaré pairing. As it is alternating, $\langle \omega, \omega \rangle_{Poinc} = \langle \eta, \eta \rangle_{Poinc} = 0$, and so it is enough to calculate $\langle \eta, \omega \rangle_{Poinc}$:

$$\langle \eta, \omega \rangle_{Poinc} = \sum_{P \in E(k)} \mathrm{res}_P(\gamma_P \omega_P) = \mathrm{res}_\infty(\gamma_\infty \omega_\infty)$$

So we compute the expansions of ω and η at ∞, and we find that $\langle \eta, \omega \rangle_{Poinc} = -1$ (and hence $\langle \omega, \eta \rangle_{Poinc} = 1$).

16 The Gauss-Manin Connection on an family of Elliptic Curves

Recall the set-up that we had some sections ago:

$$\begin{array}{ccc} (\tau, v) & & \mathbb{E} = (\mathfrak{h} \times \mathbb{C})/\mathbb{Z}^2 \\ \big\downarrow & & \big\downarrow \big\uparrow {\scriptstyle e\colon \tau \mapsto [(\tau,0)]} \\ \tau & & \mathfrak{h} \end{array}$$

(where $\mathbb{Z}^2$ acts on $\mathfrak{h} \times \mathbb{C}$ on the right, by $(\tau, v) \cdot (a, b) \overset{\text{def}}{=} (\tau, v + a\tau + b)$).

Then the fibers above $\tau \in \mathfrak{h}$ are precisely:

$$f^{-1}(\tau) = \tau \times (\mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z})) = \tau \times E_\tau \simeq E_\tau$$

Let $R \overset{\text{def}}{=} \{f : \mathfrak{h} \to \mathbb{C} \text{ holomorphic }\}$, and we get in this way an analytic family $\mathbb{E}/R$. This can be made to be the analyfication of an algebraic family (by GAGA?, because $\mathfrak{h}$ doesn't have sheaf cohomology?). We can thus write:

$$\mathbb{E}/R\colon \quad y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) \qquad g_i(\tau) \in R$$

Recall also that, in this setting, $x = \wp = \wp(\tau, z)$, and $y = \wp'(\tau, z)$ (the derivative taken with respect to z). Then, as we have seen:

$$H^1_{\mathrm{dR}}(\mathbb{E}/R) = R\omega \oplus R\eta$$

with $\omega = dx/y, \eta = xdx/y$.

Remark. Note that $d\tau = 0$ because $\tau \in R$ (we treat it as a constant). So d will take derivatives with respect to z only.

The Gauss-Manin connection, becomes, by noting that $\Omega^1_{\mathfrak{h}/\mathbb{C}} \simeq R \cdot d\tau$:

$$\nabla\colon H^1_{\mathrm{dR}}(\mathbb{E}/R) \to d\tau \otimes_R H^1_{\mathrm{dR}}(\mathbb{E}/R)$$

Write $\partial \overset{\text{def}}{=} \frac{d}{d\tau}$ for the dual of $d\tau$. We can then define $\nabla_\tau \overset{\text{def}}{=} \partial \circ \nabla$ by contracting:

$$H^1_{\mathrm{dR}}(\mathbb{E}/R) \xrightarrow{\nabla} \Omega^1_{\mathfrak{h}/\mathbb{C}} \otimes_R H^1_{\mathrm{dR}}(\mathbb{E}/R) \xrightarrow{\partial} H^1_{\mathrm{dR}}(\mathbb{E}/R)$$

and this carries the same information (to recover ∇, we just set $\nabla(\alpha) \stackrel{\text{def}}{=} d\tau \otimes \nabla_\tau(\alpha)$. Note that ∇_τ satisfies:

$$\nabla_\tau(a\alpha) = \frac{\partial a}{\partial \tau}\alpha + a\nabla_\tau(\alpha) \quad \forall a \in R, \alpha \in H^1_{\text{dR}}(\mathbb{E}/R)$$

We will now compute ∇_τ. For this, we will exploit the fact that we have a lot more structure (coming from $\mathbb{E}$ and $\mathfrak{h}$ being analytic varieties). In particular, we can consider homology ($H_1(\mathbb{E}/R)$).

Let γ_1 be the image in $\mathbb{E}$ of the family of closed paths $\{\tau \times [0,\tau] \mid \tau \in \mathfrak{h}\}$. Similarly, let γ_2 be the image in $\mathbb{E}$ of the family $\mathfrak{h} \times [0,1]$. Then, inside $H_1(\mathbb{E}/\mathfrak{h}, R) = \gamma_1 R \oplus \gamma_2 R$ there is a lattice $H_1(\mathbb{E}/\mathfrak{h}, \mathbb{Z}) = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$.

Consider the pairing that gives the duality between homology and deRham cohomology (deRham's theorem):

$$(\cdot,\cdot)\colon H_1(\mathbb{E}/\mathfrak{h}, R) \times H^1_{\text{dR}}(\mathbb{E}/R) \to R$$

We get a chain of isomorphisms:

$$H_1(\mathbb{E}/\mathfrak{h}, R) \simeq H^1_{\text{dR}}(\mathbb{E}/R)^\vee \simeq H^1_{\text{dR}}(\mathbb{E}/R)$$

(the first one thanks to this previous pairing, the second one thanks to the Poincaré pairing)

As ∇_τ acts on $H^1_{\text{dR}}(\mathbb{E}/R)$, it induces an action on $H_1(\mathbb{E}/\mathfrak{h}, R)$. The "horizontal paths" give us the complex homology:

$$H_1(\mathbb{E}/\mathfrak{h}, R)^{\nabla_\tau = 0} = H_1(\mathbb{E}/\mathfrak{h}, \mathbb{C})$$

(and note that $\nabla_\tau(\gamma_i) = 0$ for $i = 1, 2$). So ∇_τ is the dual of the connection on $H_1(\mathbb{E}/R)$ for which γ_i are the horizontal sections.

We identify as well, through this isomorphisms, the paths γ_i as elements in $H^1_{\text{dR}}(\mathbb{E}/R)$. They satisfy, for all $\xi \in H^1_{\text{dR}}(\mathbb{E}/R)$,

$$\langle \xi, \gamma_i \rangle_{\text{Poinc}} = \int_{\gamma_i} \xi$$

It follows that:

$$\langle \gamma_2, \gamma_1 \rangle_{\text{Poinc}} = -\langle \gamma_1, \gamma_2 \rangle_{\text{Poinc}} = \int_{\gamma_1} \gamma_2 = 1$$

(note that in the integral, γ_1 is seen as a path, and γ_2 is seen as a 1-form!). Also, of course $\langle \gamma_i, \gamma_i \rangle_{\text{Poinc}} = 0$ by alternancy.

In this way, we have two "natural" basis for $H^1_{\text{dR}}(\mathbb{E}/R)$: one given by $\{\omega, \eta\}$, the other given by $\{\gamma_1, \gamma_2\}$. We want to relate them, so write:

$$\begin{pmatrix} \omega \\ \eta \end{pmatrix} = A \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix}, \quad A \in M_{2\times 2}(\mathbb{C})$$

Define:

$$\omega_1 \stackrel{\text{def}}{=} \int_{\gamma_1} \omega = \langle \omega, \gamma_1 \rangle_{\text{Poinc}} \qquad \omega_2 \stackrel{\text{def}}{=} \int_{\gamma_2} \omega = \langle \omega, \gamma_2 \rangle_{\text{Poinc}}$$

$$\eta_1 \stackrel{\text{def}}{=} \int_{\gamma_1} \eta = \langle \eta, \gamma_1 \rangle_{\text{Poinc}} \qquad \eta_2 \stackrel{\text{def}}{=} \int_{\gamma_2} \eta = \langle \eta, \gamma_2 \rangle_{\text{Poinc}}$$

(the **periods** of ω and η, respectively). It is easy to show (it is logically equivalent to the fact that $\langle \omega, \eta \rangle_{\text{Poinc}} = 1$) that they satisfy the *Legendre relation*:

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$$

With these, the matrix A becomes:

$$A = \begin{pmatrix} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{pmatrix}$$

and, inverting it (the Legendre relation gives its determinant) we get:

$$2\pi i \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

Now, we apply ∇_τ, which kills γ_i, as we have said. Denote by a prime $'$ the derivative with respect to τ. Then:

$$0 = \begin{pmatrix} -\eta_2' & \omega_2' \\ -\eta_1' & \omega_1' \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} + \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \nabla_\tau \omega \\ \nabla_\tau \eta \end{pmatrix}$$

Solving for the last term, we get:

$$\begin{pmatrix} \nabla_\tau \omega \\ \nabla_\tau \eta \end{pmatrix} = \frac{-1}{2\pi i} \begin{pmatrix} \eta_1' \omega_2 - \eta_2' \omega_1 & \omega_1 \omega_2' - \omega_2 \omega_1' \\ \eta_2 \eta_1' - \eta_1 \eta_2' & \eta_1 \omega_2' - \eta_2 \omega_1' \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

Now, note that $\omega_1 = \tau$ and $\omega_2 = 1$, so the Legendre relation becomes $\eta_1 - \tau\eta_2 = 2\pi i$. We conclude:

$$\begin{pmatrix} \nabla_\tau \omega \\ \nabla_\tau \eta \end{pmatrix} = \frac{-1}{2\pi i} \begin{pmatrix} \eta_2 & -1 \\ \eta_2^2 - 2\pi i \eta_2' & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

We just have to compute η_2, and this is not formal.

Lemma 16.1. *We have:*

$$\eta_2 = -\frac{-\pi 2}{3} P$$

where $P = E_2$ is the weight-2 Eisenstein series normalized so that, if $q = e^{2\pi i \tau}$,

$$P(q) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$$

Proof. The function P can be expressed in terms of τ (in fact this is how it is defined) as:

$$P(\tau) = \frac{3}{\pi^2} \sum_{m,n}{}' \frac{1}{m\tau + n}$$

so it is enough to prove that: $\eta_2 = \sum'_{m,n} \frac{1}{m\tau+n}$. For this, note that $\eta = x dx/y = \wp(z) dz$. Define the Weierstrass ζ-function by:

$$\zeta(z) \stackrel{\text{def}}{=} \frac{1}{z} + \sum_{m,n}{}' \left(\frac{1}{z - m\tau - n} + \frac{1}{m\tau + n} + \frac{z}{(m\tau + n)^2} \right)$$

and remark that $\eta = -d\zeta(z)$, so that:

$$\eta_2 = \int_{\gamma_2} \eta = \int_0^1 -d\zeta(z) = \int_z^{z+1} -d\zeta(z) = \zeta(z) - \zeta(z+1)$$

and a simplifying this expression we get the result. □

Remark. By changing the order of summation in the expression of $\zeta(z)$ we can also see that

$$\eta_1 = \zeta(z) - \zeta(z+\tau) = -\sum_{n,m}{}' \frac{\tau}{(m\tau+n)^2}$$

This, combined with the Legendre relation $\eta_1(\tau) - \tau\eta_2(\tau) = 2\pi i$ yields the transformation function for $P(\tau)$:

$$P\left(\frac{-1}{\tau}\right) = \tau^2 P(\tau) - \frac{6i\tau}{\pi}$$

From the previous lemma we obtain the explicit matrix for the Gauss-Manin connection:

$$\begin{pmatrix} \nabla_\tau \omega \\ \nabla_\tau \eta \end{pmatrix} = \frac{1}{2\pi i} \begin{pmatrix} \frac{\pi^2 P}{3} & 1 \\ \frac{\pi^4}{9}P^2 - \frac{12}{2\pi i}P' & -\frac{\pi^2}{3}P \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

17 The Kodaira-Spencer Map

Given an elliptic curve E/R, we have defined so far (assuming $\operatorname{Spec} R$ to be smooth and one-dimensional):

- The hodge filtration: $0 \to \underline{\omega}_{E/R} \to H^1_{\mathrm{dR}}(E/R) \xrightarrow{\beta} \underline{\omega}_{E/R}^{-1} \to 0$.
- The Poincaré pairing $\langle \cdot, \cdot \rangle_P \colon H^1_{\mathrm{dR}}(E/R) \times H^1_{\mathrm{dR}}(E/R) \to R$, compatible with the Serre pairing.
- The Gauss-Manin connection $\nabla \colon H^1_{\mathrm{dR}}(E/R) \to \Omega^1_{R/k} \otimes_R H^1_{\mathrm{dR}}(E/R)$.

Definition 17.1. The **Kodaira-Spencer map** is the R-module homomorphism:

$$\begin{aligned} \varphi_{\mathrm{KS}} \colon \quad & \underline{\omega}_{E/R}^{\otimes 2} \longrightarrow \Omega^1_{R/k} \\ & \omega_1 \otimes \omega_2 \longmapsto \langle \omega_1, \nabla\omega_2 \rangle_{\mathrm{Poinc}} \end{aligned}$$

(we pair ω_1 with the deRham part of $\nabla\omega_2$, and we get an element in $\Omega^1_{Y/k}$).

It turns out that the Kodaira-Spencer map is an isomorphism if C is the Tate elliptic curve over $\mathbb{Z}((q))$. For that, we just need to compute:

$$\begin{aligned} \varphi_{\mathrm{KS}}(\omega \otimes \omega) &= \langle \omega, \nabla\omega \rangle_{\mathrm{Poinc}} = \langle \omega, \frac{\pi}{6i} P d\tau \otimes \omega + \frac{1}{2\pi i} d\tau \otimes \eta \rangle_{\mathrm{Poinc}} = \\ &= \frac{\pi}{6i} P d\tau \otimes \langle \omega, \omega \rangle_{\mathrm{Poinc}} + \frac{1}{2\pi i} d\tau \otimes \langle \omega, \eta \rangle_{\mathrm{Poinc}} = \frac{1}{2\pi i} d\tau \end{aligned}$$

(the last equality follows because $\langle \omega, \omega \rangle_{\mathrm{Poinc}} = 0$ and $\langle \omega, \eta \rangle_{\mathrm{Poinc}} = 1$). As $\frac{1}{2\pi i} \in \mathbb{C}^\times$, the map φ_{KS} is an isomorphism.

18 Relationship between Katz's and Classical Modular Forms

18.1 The Moduli Problem $\Gamma(N)$

Fix an integer $N \geq 3$, and let $\mathbf{Sch}_{\mathbb{Z}[1/N]}$ denote the category of schemes over $\operatorname{Spec}(\mathbb{Z}[1/N])$. Define a (contravariant) functor:

$$\begin{aligned} P_N \colon \mathbf{Sch}_{\mathbb{Z}[1/N]} &\to \mathbf{Sets} \\ S &\mapsto \left\{(E,\alpha) \mid E/S \text{ is an elliptic curve }, (\mathbb{Z}/N\mathbb{Z})^2_S \overset{\alpha}{\simeq} E[N]_S\right\}/_{\simeq} \end{aligned}$$

(α gives a level-N structure to E). Note that $P_N(S)$ might be void (for certain schemes S there might not be such a level-N structure).

The problem is to prove whether or not P_N is representable: that is, whether there exists a scheme $Y(N)$ such that:

$$P_N \simeq \hom_{\mathbf{Sch}_{\mathbb{Z}[1/N]}}(-, Y(N))$$

Theorem 18.1. *Let $N \geq 3$. Then P_N is represented by a* fine moduli scheme*, called $Y(N)$, defined over $\mathbb{Z}[1/N]$.*

Proof. This is quite difficult and way beyond the scope of these notes. You can look at the original paper [KM85]. □

Remark. This means that there exists a universal triple $(Y(N), E(N), \alpha_N)$ where $Y(N)$ is a scheme over $\mathbb{Z}[1/N]$, $E(N)$ is an elliptic curve over $Y(N)$, and $\alpha_N \colon (\mathbb{Z}/N\mathbb{Z})^2 \simeq E(N)[N]_{Y(N)}$ is a level-N structure, such that for every scheme S over $\mathbb{Z}[1/N]$ and every pair $(E,\alpha) \in P_N(S)$, there exists a unique morphism $\varphi \colon S \to Y(N)$ such that (E,α) is obtained by taking the fiber product:

$$\begin{array}{ccc} (E,\alpha) & \longrightarrow & (E(N),\alpha_N) \\ \downarrow & & \downarrow \\ S & \xrightarrow{\ \varphi\ } & Y(N) \end{array}$$

Remark. This is a very strong statement. In particular, if k is a field such that $\operatorname{char} k \nmid N$, and (E,α) is a pair over k, then the theorem ensures the existence of a unique point $s \in Y(N)(k)$ (which is the image of $\operatorname{Spec} k$ under φ) such that $[(E,\alpha)]$ comes from it.

There exists a natural morphism

$$Y(N) \to \operatorname{Spec}(\mathbb{Z}[1/N][j])$$

which, on points, sends the pair $[(E,\alpha)]$ to the j-invariant $j(E)$ of E. This turns out (not obvious) to be a finite, flat morphism, and so it follows that $Y(N)$ is affine of relative dimension 1 over $\operatorname{Spec}(\mathbb{Z}[1/N])$. This natural map is also smooth, so that $Y(N)$ is actually a curve over the j-line.

If ζ' is a fixed primitive N^{th} root of 1, then:

$$Y(N) \otimes_{\mathbb{Z}[1/N]} \mathbb{Z}[1/N, \zeta'] = \coprod_{\zeta} Y(N)_\zeta$$

(where ζ runs over the primitive N^{th} roots of 1). Also, $Y(N)_\zeta$ is smooth over $\operatorname{Spec}(\mathbb{Z}[1/N,\zeta])$.

Over $\mathbb{C}$, we can take $\zeta_N \overset{\text{def}}{=} e^{\frac{2\pi i}{N}}$, and then

$$(Y(N)_{\mathbb{C}})_{\zeta_N} \simeq \Gamma(N)\backslash\mathfrak{h}$$

Moreover, we have a commutative diagram:

$$\begin{array}{ccc} E(N)_{\mathbb{C},\zeta_N} & \xrightarrow{\simeq} & \Gamma(N)\backslash\mathbb{E} \\ \downarrow & & \downarrow \\ Y(N)_{\mathbb{C},\zeta_N} & \xrightarrow{\simeq} & \Gamma(N)\backslash\mathfrak{h} \end{array}$$

18.2 From Katz's to Classical

Let $k \in 2\mathbb{Z}_{\geq 2}$ and let $R_0 \overset{\text{def}}{=} \mathbb{C}$. Consider $Y(N)_{\zeta_N,\mathbb{C}}$, which is the ζ_N component of $Y(N)$ base-extended to $\mathbb{C}$. We have explained that this will be affine, so:

$$Y(N)_{\zeta_N,\mathbb{C}} = \operatorname{Spec}(R) \quad \text{for some } \mathbb{C}\text{-algebra } R$$

Given a Katz's modular form of weight k and level N over $R_0 = \mathbb{C}$, we can evaluate it at the pair:

$$(E(N)_{\zeta_N,\mathbb{C}}/R, \alpha_N)$$

and we will obtain a section of

$$\underline{\omega}^{\otimes k}_{E(N)_{\zeta_N,\mathbb{C}}} = \left(\underline{\omega}^{\otimes 2}_{E(N)_{\zeta_N,\mathbb{C}}}\right)^{\otimes k/2}$$

We can apply then the Kodaira-Spencer map to get a global section of

$$\left(\Omega^1_{R/\mathbb{C}}\right)^{\otimes k/2}$$

This will give us a classical modular form, provided that it is holomorphic at infinity. We will deal with this issue in the next section.

18.3 From Classical to Katz's

Let now g be a classical modular form of weight $k \in 2\mathbb{Z}_{\geq 2}$ and level N. We have seen that, if we define $\omega_g \overset{\text{def}}{=} g(\tau)(d\tau)^{k/2}$, then we have

$$\omega_g \in H^0\left(Y(N)_{\zeta_N,\mathbb{C}}, (\Omega^1_{Y(N)_{\zeta_N,\mathbb{C}}})^{\otimes k/2}\right)$$

As before, we can apply the inverse of the Kodaira-Spencer map and get an element:

$$\omega_g \in \underline{\omega}^{\otimes k}_{E(N)_{\zeta_N,\mathbb{C}}/Y(N)_{\zeta_N,\mathbb{C}}}$$

which, after applying the morphism $\varphi(N)$ yields f^u (u for *universal*):

$$f^u \overset{\text{def}}{=} \omega_g^{\varphi(N)} \in \underline{\omega}^{\otimes k}_{E(N)/\mathbb{C}}$$

Let now R be any $\mathbb{C}$-algebra, and let $(E/R, \alpha)$ be an elliptic curve together with a level-N structure (over R). Then, by representability, there exists a unique morphism $\varphi\colon S \to Y(N)$, defined over $\mathbb{C}$, such that the following diagram is cartesian:

$$\begin{array}{ccc} (E, \alpha) & \longrightarrow & (E(N), \alpha_N) \\ \big\downarrow & & \big\downarrow \\ S = \operatorname{Spec} R & \xrightarrow{\ \varphi\ } & Y(N) \end{array}$$

In particular, we can pull f^u back to $\underline{\omega}_{E/R}^{\otimes k}$ via the map:

$$\varphi^*\colon \left(\underline{\omega}_{E(N)/Y(N)}\right)^{\otimes k} \to \left(\underline{\omega}_{E/R}\right)^{\otimes k}$$

to get an element:

$$f(E/R, \alpha) \stackrel{\text{def}}{=} \varphi^*(f^u)$$

The assignment $(E/R, \alpha) \mapsto f(E/R, \alpha) \in \underline{\omega}_{E/R}^{\otimes k}$ is a modular form in the sense of Katz, of weight k and level N, and it is associated (via the previous section) to the classical modular form g.

Remark. Note that the level-N structure has only intervened for the existence and uniqueness of the morphism φ.

19 q-expansions and Holomorphicity at ∞

19.1 Tate Curves

Let $\tau \in \mathfrak{h}$, and consider its associated elliptic curve $E_\tau \stackrel{\text{def}}{=} \mathbb{C}/(\tau\mathbb{Z} \oplus \mathbb{Z})$. We have an (analytic isomorphism):

$$\begin{array}{ccc} E_\tau & \xrightarrow{\ \simeq\ } & \mathbb{C}^\times / q_\tau^{\mathbb{Z}} \\ z & \longmapsto & e^{2\pi i z} \end{array}$$

where $q_\tau \stackrel{\text{def}}{=} e^{2\pi i \tau}$, and $q_\tau^{\mathbb{Z}}$ is defined to be the multiplicative subgroup of $\mathbb{C}^\times$ generated by q_τ.

We can consider the formal spectrums:

$$\operatorname{Spf}(\mathbb{C}[\![q]\!]) \hookrightarrow \mathrm{SL}_2(\mathbb{Z})\backslash\overline{\mathfrak{h}}, \text{ and } \operatorname{Spf}(\mathbb{C}((q))) \hookrightarrow \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$$

as formal neighborhoods of ∞ (the second one with ∞ itself removed). By pull-back we get a cartesian diagram:

$$\begin{array}{ccc} \operatorname{Tate}(q) & \longrightarrow & \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{E} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spf}(\mathbb{C}((q))) & \longrightarrow & \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h} \end{array}$$

We want to derive equations for $\operatorname{Tate}(q)$. For this, let

$$L \stackrel{\text{def}}{=} L_\tau \stackrel{\text{def}}{=} 2\pi i(\tau\mathbb{Z} \oplus \mathbb{Z})$$

and let X and Y be:

$$X \stackrel{\text{def}}{=} \wp(2\pi i z, L), \quad Y \stackrel{\text{def}}{=} \wp'(2\pi i z, L)$$

We get the equations:

$$Y^2 = 4X^3 - g_2(L)X - g_3(L) = 4X^3 - \frac{E_4(q)}{12}X - \frac{E_6(q)}{216}$$

where:

$$E_4(q) = 12(2\pi i)^4 g_4(\tau) = 12 g_2(L) = 1 + 240 \sum \sigma_3(n) q^n$$
$$E_6(q) = 216(2\pi i)^6 g_6(\tau) = 216 g_3(L) = 1 - 504 \sum \sigma_5(n) q^n$$

(note that the q-expansions are in $\mathbb{Z}[\![q]\!]$, and hence the equation for $\mathrm{Tate}(q)$ is defined over $\mathbb{Z}[1/6]((q))$. We want to remove also these denominators. So replace $X = x + 12$, and $Y = x + 2y$, and we obtain:

$$y^2 + xy = x^3 + B(q)x + C(q)$$

where:

$$B(q) = -5\frac{E_4(q)-1}{240} = -5\sum \sigma_3(n)q^n C(q) \quad = \frac{1}{12}\left(-5\frac{E_4(q)-1}{240} - 7\frac{E_6(q)-1}{504}\right) = -\sum \frac{5\sigma_3(n) + 7\sigma_5}{12}$$

It's an elementary number theoretic calculation to show that $C(q)$ has coefficients in $\mathbb{Z}$ as well (won't do the embarrassingly easy high-school number theory exercise).

So we arrive at the definition of the Tate curve:

Definition 19.1. The **Tate curve** over $\mathbb{Z}((q))$ is the elliptic curve given by the equation $y^2 + xy = x^3 + B(q)x + C(q)$, together with the canonical differential $\omega_{\text{can}} \stackrel{\text{def}}{=} \frac{dx}{x+2y}$.

Similarly define, for $N \geq 3$:

Definition 19.2. The **level-N Tate curve** over $\mathbb{Z}((q))$ is the elliptic curve given by the equation $y^2+xy = x^3+B(q^N)x+C(q^N)$, together with the canonical differential $\omega_{\text{can}} \stackrel{\text{def}}{=} \frac{dx}{x+2y}$.

Note that the level-N Tate curve fits in the cartesian diagram (the lower map induced by the mapping $q \mapsto q^N$):

$$\begin{array}{ccc} (\mathrm{Tate}(q), \omega_{\text{can}}) & \longleftarrow & \left(\mathrm{Tate}(q^N), \omega_{\text{can}}\right) \\ \downarrow & & \downarrow \\ \mathrm{Spf}\,(\mathbb{Z}((q))) & \longleftarrow & \mathrm{Spf}\,\left(\mathbb{Z}((q^N))\right) \end{array}$$

Note also that the map $\mathbb{Z}((q)) \to \mathbb{C}$ given by mapping $q \mapsto e^{2\pi i \tau}$ induces a cartesian diagram:

$$\begin{array}{ccc} \left(\mathbb{C}^\times / q_\tau^{\mathbb{Z}}, dt/t\right) & \longrightarrow & (\mathrm{Tate}(q), \omega_{\text{can}}) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(\mathbb{C}) & \longrightarrow & \mathrm{Spf}(\mathbb{Z}((q))) \end{array}$$

19.2 Tate Curves (formal-schemes-free version)

In this subsection we partly redo the previous section without mentioning formal schemes. This wasn't assumed initially as required background, so the previous subsection is not necessarily understandable.

consider $q \overset{\text{def}}{=} e^{2\pi i\tau}$, and $q^N = e^{2\pi i N\tau}$, and $E_\tau = \mathbb{C}^\times/(q^{N\mathbb{Z}}$. Then its N-torsion group is:

$$E_\tau[N] = \{\zeta^q q^b \mid 0 \leq a, b \leq N-1\}, \quad \zeta \text{ a primitive } N^{\text{th}} \text{ root of } 1$$

A level-N structure is then an ordered basis $\alpha = (e_1, e_2)$ of $E_\tau[N]$, as a $\mathbb{Z}/N\mathbb{Z}$-module. Note that if $e_1 = \zeta^a q^b$ and $e_2 = \zeta^c q^d$, then α is a basis if, and only if,

$$\delta \overset{\text{def}}{=} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^\times$$

Moreover, the Weil pairing is easy to compute:

$$\langle e_1, e_2 \rangle_{\text{Weil}} = e^{\frac{2\pi i\delta}{N}}$$

so that the ζ_N component corresponds to those bases $\alpha = (e_1, e_2)$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

From all this we can deduce that there is a one-to-one correspondence between level-N structures on $\mathrm{Tate}(q^N)$ and cusps of $X(N)$. More on this can be found in [Sil86].

Fix now $\mathrm{Tate}(q^N)/\mathbb{C}((q))$, together with ω_{can} and a level-N structure α. We have just seen that α corresponds to a cusp $x_\alpha \in X(N)_\zeta$ (if α corresponds to the primitive N^{th} root of unity ζ).

We have then an embedding of $Y(N)_\zeta = \mathrm{Spec}(R_\zeta)$ into $X(N)_\zeta$, because R_ζ is a $\mathbb{C}$-algebra of finite type, and an integral domain. So we can take its fraction field $K_\zeta \overset{\text{def}}{=} Q(R_\zeta) = \mathbb{C}(Y(N)_\zeta) = \mathbb{C}(X(N)_\zeta)$.

As $R_\zeta \subseteq K_\zeta$, then x_α will correspond to a discrete valuation v_α on K_ζ. Let $\hat{K_\zeta}$ be the completion of K_ζ with respect to the valuation v_α, and let $q_\alpha \in K_\zeta$ be a uniformizer.

As $X(N)_\zeta$ is smooth, we have:

$$\hat{K_\zeta} \simeq \mathbb{C}((q_\alpha)) \simeq \mathbb{C}((q))$$

Hence we have a $\mathbb{C}$-algebra homomorphism $R_\zeta \to \mathbb{C}((q))$, which is given by inclusions depending on the level-N structure. This yields a map:

$$\mathrm{Spec}\left(\mathbb{C}((q))\right) \to Y(N)_\zeta$$

as we had in the previous subsection.

19.3 The q-expansion of a Modular Form

Let $N \geq 3$, and $k \in \mathbb{Z}_{\geq 0}$. Let R_0 be a $\mathbb{Z}[1/N]$-algebra, and assume that R_0 contains a primitive N^{th} root of 1. Let f be a Katz's modular form of weight k and level N defined over R_0. Fix α a level-N structure of $\mathrm{Tate}(q^N)$ over $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$.

Definition 19.3. The **q-expansion of f at** α (we think of it as at the cusp in $X(N)_{R_0}$ corresponding to α) is:

$$f\left(\mathrm{Tate}(q^N)/\mathbb{Z}((q)) \otimes R_0, \omega_{\mathrm{can}}, \alpha\right) \in \mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$$

Remark. Using this definition, every Katz's modular form is automatically meromorphic at ∞ (i.e. at all cusps).

Definition 19.4. We say that f is **holomorphic at** ∞ if, for every level-N structure α of $\mathrm{Tate}(q^N)$, its q-expansion at α actually belongs to $\mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} R_0$. We say that f is a **cusp form** if it belongs to $q\mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} R_0$ (also for all α).

19.4 The q-expansion Principle

Theorem 19.5 (The q-expansion principle). *Let R_0 be a $\mathbb{Z}[1/N, \zeta]$-algebra, and let f be a Katz's modular form of weight k and level $N \geq 3$ defined over R_0, and holomorphic at ∞. Suppose that, for every ζ primitive N^{th} root of 1, there exists a level-N structure $\alpha_\zeta = (e_1, e_2)$ of $\mathrm{Tate}(q^N)$, with $\langle e_1, e_2\rangle_{Weil} = \zeta$ such that the q-expansion of f at α is 0. Then f is zero as a modular form.*

Corollary 19.6. *If $K \subseteq R_0$ is a $\mathbb{Z}[1/N, \zeta]$-subalgebra such that all q-expansions of f are in $\mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} K$, then f is in fact defined over K.*

Remark. One can also define modular forms over modules, and then K in the previous corollary needs only to be a sub-module for the result to hold.

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