p-adic Hodge theory

Recall: $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) := \varprojlim_{K/\mathbb{Q} \text{ finite Galois}} \mathrm{Gal}(K/\mathbb{Q}) \subseteq \overbrace{\prod_K \underbrace{\mathrm{Gal}(K/\mathbb{Q})}_{\text{w/ discrete topology}}}^{\text{w/ product topology}}$

w/ subspace topology

Facts: 1) $G_{\mathbb{Q}}$ is compact and totally disconnected.

2) Let $\psi_K : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q})$ be the natural restriction maps.

Then ψ_K is continuous, and $\{\ker(\psi_K)\}_{K/\mathbb{Q} \text{ fin Gal.}}$ is a system of open (sbgps) neighborhoods of $1 \in G_{\mathbb{Q}}$.

3) A <u>subgroup</u> of $G_{\mathbb{Q}}$ is open iff it's closed and of finite index.

For each ℓ, there is a canonical inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$, which can be lifted (need to choose an extension of ℓ) to an inclusion $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

This gives an inclusion $G_{\mathbb{Q}_\ell} := \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \hookrightarrow G_{\mathbb{Q}}$.

(if the extension is changed, then $G_{\mathbb{Q}_\ell}'$ is a conjugate of the starting $G_{\mathbb{Q}_\ell}$)

Theorem (Čebotarev): $G_{\mathbb{Q}}$ is topologically generated by $G_{\mathbb{Q}_\ell}$'s for all ℓ's

(in fact, only need a set of density one).

For each ℓ, $G_{\mathbb{Q}_\ell}$ acts on the ring of integers of $\overline{\mathbb{Q}}_\ell$, and on the maximal ideal as well, so we get

$$0 \to \underset{\text{inertia}}{I_\ell} \to G_{\mathbb{Q}_\ell} \to G_{\mathbb{F}_\ell} \to 0$$

$$G_{\mathbb{F}_\ell} = \mathrm{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell) = \overline{\langle \underset{x \mapsto x^\ell}{\mathrm{Frob}_\ell} \rangle} \cong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

As for the inertia group, we define $W_\ell \subseteq I_\ell$ to be ~~the~~ its maximal pro-ℓ group.

The quotient $I_\ell / W_\ell \overset{\text{non-canonically}}{\simeq} \prod_{p \neq \ell} \mathbb{Z}_p$

it's the Tame inertia

So at the end it "boils down" to study the (wild ramification) W_ℓ, which is quite hard.

Fun fact/conjecture: every finite group is a quotient of $G_{\mathbb{Q}}$.

So $G_{\mathbb{Q}}$ is (conjecturally) pretty complicated. To understand it, we look at its representations:

$$\rho : G_{\mathbb{Q}} \to GL_n(K) \simeq Aut(V)$$

(where K is some field).

Natural choices for K:
- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- $\mathbb{Q}_p$, $\mathbb{C}_p$ / ...
- $\mathbb{F}_p$, $K/\mathbb{F}_p$ finite

We will give each K above its usual topology, and study only the continuous representations. ($GL_n(K) \subseteq K^{n^2}$ ← vectorspace topology)

Note: 1) if $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$ then the image(ρ) is finite. (uses that all compact totally-disconnected of $GL_n(\mathbb{C})$ are finite).

(the source is compact and totally-disconnected)

2) this is also true if $K/\mathbb{F}_p$ is finite, of course!

3) Not true in general, if $K = \mathbb{Q}_p$ or $K/\mathbb{Q}_p$ is finite.

Moreover, if $K = \mathbb{Q}_p, \mathbb{F}_p, \ldots$ then ρ need not be semisimple.

(the usual proof of averaging doesn't carry over, because no good notion of integral).

We restrict ourselves to cont. representations $\rho : G_{\mathbb{Q}_\ell} \to GL_n(\mathbb{Q}_p)$.

Rmk: if $\ell \neq p$, then $\rho(W_\ell) =$ finite, so it makes the problem much easier.
if $\ell = p$, then $\rho(W_\ell)$ can be infinite, as in the following example.

Example: Let $\mu_{p^\infty} := \varinjlim \mu_{p^n}(\overline{\mathbb{Q}_p})$ ← gp of p^n-th roots of 1 in $\overline{\mathbb{Q}_p}$.

We get an action of $G_{\mathbb{Q}} \ni g$ on μ_{p^∞}, by sending $\zeta \mapsto \zeta^{\chi(g)}$

Then $\chi(g) \in \mathbb{Z}_p^\times$, and so we get a character, which is continuous

$$\chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times = GL_1(\mathbb{Z}_p) \qquad \text{(called the cyclotomical character)}$$

Fact: $\mathbb{Q}_p(\mu_{p^n})$ is totally ramified, of degree $\varphi(p^n) = (p-1)p^{n-1}$ over $\mathbb{Q}_p$.

By a HW exercise, $\chi(I_p)$ is infinite $\Rightarrow$ $\chi(W_p)$ infinite.
↑ Why?

Some examples of why the p-adic reps of $G_{\mathbb{Q}_p}$ are useful.

Ex: (Tate modules) Let E/F be an elliptic curve, $F = \#$ field.

Let $\bar{F}$ be a fixed alg. closure of F.

We have isomorphisms $E[p^n](\bar{F}) \simeq \left(\mathbb{Z}/p^n\mathbb{Z}\right)^2$

$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ G_F & \dashrightarrow & G_p \end{array}$$

So we get a two-dimensional representation $\rho_E : G_F \to GL_2(\mathbb{Z}/p^n\mathbb{Z})$

Making compatible choices of the isomorphisms, it gives rise to a cont. rep:

$$\rho_E : G_F \to GL_2(\mathbb{Z}_p)$$

For each prime $\mathfrak{p}$ of F, fix an embedding $G_{F_\mathfrak{p}} \hookrightarrow G_F$. Again, we have an exact sequence:

$$0 \to I_\mathfrak{p} \to G_{F_\mathfrak{p}} \to \underset{\overline{\langle \mathrm{Frob}_\mathfrak{p} \rangle} \ \ x \mapsto x^{q(\mathfrak{p}) = \# k(\mathfrak{p})}}{\overset{}{G_{k(\mathfrak{p})}}} \to 0$$

Def: we say that ρ is unramified at $\mathfrak{p}$ if $\rho|_{I_\mathfrak{p}}$ is trivial $\left(\rho(I_\mathfrak{p}) = id_{GL_n}\right)$.

(then ρ factors through $G_{k(\mathfrak{p})}$).

In this case, ρ is determined by the image of $\mathrm{Frob}_\mathfrak{p}$.

<u>Theorem</u>: if $\underline{\mathfrak{p} \nmid p}$, then E has good reduction at $\mathfrak{p}$ if, and only if, $\rho_E : G_F \to GL_2(\mathbb{Z}_p)$ is unramified at $\mathfrak{p}$.

this is the Tate module!

In this case, $G_{F_\mathfrak{p}}$ acts on $\varprojlim_n E[p^n](\overline{F})$ $\left(\simeq (\mathbb{Z}_p)^2\right)$ through $G_{k(\mathfrak{p})} \simeq \overline{\langle \mathrm{Frob}_\mathfrak{p} \rangle}$, and the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ is $X^2 - a_{E,\mathfrak{p}} X + \overset{= \# k(\mathfrak{p})}{q_\mathfrak{p}} \in \mathbb{Z}[X] \subseteq \mathbb{Z}_p[X]$

(where $q_\mathfrak{p} + 1 - \# \tilde{E}(k(\mathfrak{p})) = a_{E,\mathfrak{p}}$).

(note that this is independent of p, as long as $\mathfrak{p} \nmid p$!).

Def: Let $$L(E,s) := \prod_{\mathfrak{p}\text{ bad}} \Big(1 - \underbrace{a_\mathfrak{p}}_{\substack{\shortparallel \\ -1,0,1}} q_\mathfrak{p}^{-s}\Big)^{-1} \cdot \prod_{\mathfrak{p}\text{ good}} \Big(1 - \overset{q_\mathfrak{p}+1-\#\tilde{E}(k(\mathfrak{p}))}{a_\mathfrak{p}} q_\mathfrak{p}^{-s} + q_\mathfrak{p}^{1-2s}\Big)^{-1}$$

$$= \prod_\mathfrak{p} L_\mathfrak{p}(E,s)$$

where $L_\mathfrak{p}(E,s) = \left(q_\mathfrak{p}^{-2s} \cdot P_\mathfrak{p}(q_\mathfrak{p}^s) \right)^{-1}$

$I_\mathfrak{p}$-invariants.

(where $P(X)$ is the char. poly of $\mathrm{Frob}_\mathfrak{p}$ acting on $T_p(E)^{I_\mathfrak{p}}$) for each $\mathfrak{p}$, choose p s.t. $\mathfrak{p} \nmid p$!

The point of the previous example is that the representation ρ determines $L(E,s)$, and $L(E,s)$ ~~determines~~ determines E (by Faltings' theorem) (up to isogeny)

Conj (BSD):

1) $L(E,s)$ admits a ~~meromorphic~~ analytic continuation to all of $s\in\mathbb{C}$.

(Known when E is modular).

2) $\mathrm{ord}_{s=1} L(E,s) = \mathrm{rk}_{\mathbb{Z}} E(F)$

3) ...

Q: is there an analogue of the theorem about good reduction when $\mathfrak{p}|p$?

That is, can the padic rep. tell us about good reduction of the curve?

N.B. The determinant of ρ_E, $\det(\rho_E) = \chi$ (the cyclotomic character)

$\Rightarrow$ unramified is not the right condition for ρ... "infinitely-ramified"

However: if we replace "unramified at $\mathfrak{p}$" is replaced by "crystalline at $\mathfrak{p}$", then it gives a true answer to our question.

Example (generalizes tate modules). Let X be an alg. variety $/F$, $\mathrm{char}(F)\neq p$. (any field)

Assume X to be smooth and projective.

For $i\geq 0$, consider $H^i_{et}(X_{\bar{F}}, \mathbb{Z}_p)$. They are finitely-generated $\mathbb{Z}_p$-modules, with an action of G_F. So they give representations

$$\rho : G_F \longrightarrow \mathrm{Aut}\left(H^i_{et}(X_{\bar{F}}, \mathbb{Z}_p)\right)$$

continuous according to the definition that we will see

They can have lots of torsion, so not the kind that we look at.

Def: Let Γ be a profinite group, $\left(= \varprojlim \overset{\text{finite}}{\Gamma_i} \right)$.

A <u>continuous</u> representation of Γ on a finitely-gen $\mathbb{Z}_p$-module Λ is a $\mathbb{Z}_p[\Lambda]$-module structure on Λ such that the action map

$$\begin{aligned} \Gamma \times \Lambda &\to \Lambda \\ (g, \lambda) &\mapsto g \cdot \lambda \end{aligned} \quad \text{is continuous.}$$

(in the homework, we will see an equivalent definition).

Let $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma)$ be the category of such representation (morphisms are Γ-equivariant morphisms).

<u>Fact</u>: If X is smooth & projective over F then the action of G_F on $H^i_{\acute{e}t}(X_{\bar{F}}, \mathbb{Z}_p)$ is unramified for almost all $\mathfrak{p}$ of F, but it is almost never unramified at those $\mathfrak{p} \mid p$.

Q: Does $\rho : G_F \to H^i_{\acute{e}t}(X_{\bar{F}}, \mathbb{Z}_p)$ satisfy some "nice" property when restricted to $G_{F_\mathfrak{p}}$ for $\mathfrak{p} \mid p$?

A: Yes! That's what p-adic Hodge theory is for.

Fix a prime p.

__Def__: Let Γ be any profinite group (eg $\Gamma = G_{\mathbb{Q}}, \dots$).

A __p-adic representation__ of Γ V is a continuous hom. $\rho : \Gamma \to \mathrm{Aut}(V)$, where V is a fin.dim. $\mathbb{Q}_p$-vectorspace.

(continuous is taken to mean: choose a basis for V, so $\mathrm{Aut}(V) \simeq GL_n(\mathbb{Q}_p)$, then $GL_n(\mathbb{Q}_p) \subseteq \mathbb{Q}_p^{n^2}$ gets the subspace topology, and Γ has the profinite topology).

__Def__: A __p-adic field__ is any field of characteristic zero, complete wrt a discrete valuation, and with perfect residue field of char p.

(__recall__: an $\mathbb{F}_p$-algebra A is __perfect__ if the endomorphism $a \mapsto a^p$ is __bijective__).

__Examples__ (of p-adic fields): $\mathbb{Q}_p$, $K/\mathbb{Q}_p$ finite, $\widehat{\mathbb{Q}_p^{nr}}$, $\widehat{K^{nr}}$ ($K/\mathbb{Q}_p$ finite).

The goal of p-adic Hodge theory is to classify p-adic representations of the Galois groups G_K, with K a p-adic field.

__Def__: Let $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$ be the category of p-adic (continuous) representations of Γ.

(the maps are Γ-equivariant and $\mathbb{Q}_p$-linear).

$$\begin{array}{ccc}
\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma) & & \text{Vectorspaces}/F \leftarrow \text{sometimes it will be } K \text{ itself} \\
\cup\!\!\!| & \text{faithful, but not full at the beginning} & \cup\!\!\!| \\
& (\text{exact}, \dots) & \\
\left\{\begin{array}{c}\text{"good" subcategories} \\ \text{defined in terms of} \\ \text{"simple" rep-theoretic} \\ \text{criteria}\end{array}\right\} & \xrightarrow[\text{functors def. by "period rings"}]{} & \left\{\begin{array}{c}\text{"nice" subcategories} \\ \text{of Vectorspaces}/F\end{array}\right\}
\end{array}$$

Fix now K a p-adic field, and fix an alg. closure $\overline{K}$.
Set $\mathbb{C}_K := \widehat{\overline{K}}$ (it is algebraically-closed). Write $|\cdot|$ (resp ord_K) to denote the abs. value (resp valuation) on $\mathbb{C}_K$ (on $\mathbb{C}_K^\times$) that continuously extend that of K. (and normalize them to $|p| = \frac{1}{p}$, $|x| = p^{-\mathrm{ord}_K(x)}$).

• The cyclotomic character.

Set $\mu_{p^n}(\overline{K}) := \{ \xi \in \overline{K} \mid \xi^{p^n} = 1 \}$

Have natural maps
$$\begin{aligned} \mu_{p^{n+1}}(\overline{K}) &\to \mu_{p^n}(\overline{K}) \\ \xi &\longmapsto \xi^p \end{aligned}$$

In this way, we get an inverse system of abelian groups.

Define: $\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}(\overline{K})$

Concretely, $\mathbb{Z}_p(1) = \{ (\xi_{p^n})_n : \xi_{p^n}^p = \xi_{p^{n-1}} \ \forall n, \text{ and } \xi_1 = 1 \}$

We have (non-canonical) group isomorphisms $\mu_{p^n}(\overline{K}) \xleftarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$.
We can choose compatible isomorphisms $\forall n$ (that is,

$$\begin{array}{ccc} \mu_{p^{n+1}}(\overline{K}) & \xleftarrow{\sim} & \mathbb{Z}/p^{n+1}\mathbb{Z} \\ \downarrow (\ )^p & & \downarrow \bmod p^n \\ \mu_{p^n}(\overline{K}) & \xleftarrow{\sim} & \mathbb{Z}/p^n\mathbb{Z} \end{array}$$

to get a (non-canonical) isomorphism $\mathbb{Z}_p(1) \simeq \mathbb{Z}_p$.

Moreover G_K acts on $\mathbb{Z}_p(1)$ as follows:

For each n, have:

$$\begin{tikzcd} \chi_n : G_K \arrow[r] \arrow[dr] & \mathrm{Aut}\left(\mu_{p^n}(\bar{K})\right) \\ & \mathrm{Gal}\left(K(\mu_{p^n})/K\right) \arrow[u, dashed] \end{tikzcd}$$

, and χ_n factors through $\mathrm{Gal}\left(K(\mu_{p^n})/K\right)$

Set $\chi := \varprojlim \chi_n : G_K \to \mathbb{Z}_p^{\times}$, which is a <u>continuous</u> homomorphism.

So $\mathbb{Z}_p(1)$ is a G_K-module, and if we choose a basis for $\mathbb{Z}_p(1)$ as a $\mathbb{Z}_p$-module, then the G_K-action is given by:

$$g(e) = \chi(g)\cdot e$$

<u>Def</u>: $\mathbb{Z}_p(r) := \begin{cases} \mathbb{Z}_p(1)^{\otimes r} & \text{if } r>0 \quad (\text{as } G_K\text{-module, with diagonal } G_K\text{-action}) \\ \mathbb{Z}_p & \text{if } r=0 \quad (\text{with trivial } G_K\text{-action}) \\ \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p(-r), \mathbb{Z}_p) & \text{if } r<0 \\ \quad (\text{with } G_K\text{-action } (g\cdot\varphi)(x) := \varphi(g^{-1}\cdot x)). & \end{cases}$

$(r \in \mathbb{Z})$

If we choose a basis e_1 of $\mathbb{Z}_p(1)$, we get a basis e_r of $\mathbb{Z}_p(r)$ $\forall r \in \mathbb{Z}$,

and $\mathbb{Z}_p(r) \cong \mathbb{Z}_p \cdot e_r$, with G_K-action $g(e_r) = \chi^r(g)\cdot e_r$.

<u>Def</u>: If M is any $\mathbb{Z}_p[G_K]$-module, define then

$$M(r) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r) \quad (\text{as } \mathbb{Z}_p[G_K]\text{-module!})$$

As before, if e_1 is a basis for $\mathbb{Z}_p(1)$, then $M(r) \simeq M$ as a $\mathbb{Z}_p$-module, but the G_K action is

$$\underset{\substack{\text{action} \\ \text{on } M(r)}}{g\cdot m} := \chi(g)^r \cdot \underset{\text{given action on } M}{g(m)}$$

Note: G_K acts on $\mathbb{C}_K = \widehat{\bar{K}}$, by continuity.

Def: A $\underline{\mathbb{C}_K\text{-representation of } G_K}$ is a finite dimensional $\mathbb{C}_K$-vectorspace with a continuous $\underline{\mathbb{C}_K\text{-semilinear}}$ G_K-action.

i.e. $g(c\cdot v) = g(c).g(v)$

The category of such representations is $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

We want to study the functor:

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{Rep}_{\mathbb{C}_K}(G_K)$$
$$V \longmapsto \mathbb{C}_K \otimes_{\mathbb{Q}_p} V$$

with G_K acting on $\underline{\text{both}}$ factors.

$\underline{\text{Motivation}}$: There is an amazing theorem of Faltings, which says that, if X is a smooth projective variety $/K$, then:

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} \underbrace{H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)}_{\text{a VERY misterious object}} \underset{\substack{\text{canonical iso} \\ \text{in } \mathrm{Rep}_{\mathbb{C}_K}(G_K)}}{\cong} \bigoplus_q \mathbb{C}_K(-q) \otimes_K \underbrace{H^{n-q}(X, \Omega^q_{X/K})}_{\text{with trivial } G_K\text{-action}}$$

So if we define $h^{p,q} := \dim_K H^p(X, \Omega^q_{X/K})$, then:

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p) \underset{\substack{\text{non can} \\ \text{in } \mathrm{Rep}_{\mathbb{C}_K}(G_K)}}{\cong} \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_K(-q)^{h^{n-q,q}}$$

Theorem (Tate - Sen, simplified by Ax):

1) $\mathbb{C}_K^{G_K} = K$ (there are no transcendental invariants).

More generally, $\mathbb{C}_K(r)^{G_K} = 0$ if $r \neq 0$.

2) $H^1_{cont}(G_K, \mathbb{C}_K(r)) = \begin{cases} 1\text{-dim'l}/K & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$

More generally, if $\eta : G_K \to \mathbb{Z}_p^\times$ is any continuous character, and we define $\mathbb{C}_K(\eta) := \mathbb{C}_K$ with the G_K-action twisted by η, then:

a) $\mathbb{C}_K(\eta)^{G_K} = \begin{cases} 0 & \text{if } \eta(I_K) \text{ is infinite} \\ 1\text{-dim'l}/K & \text{if } \eta(I_K) \text{ is finite.} \end{cases}$

b) $H^1_{cont}(G_K, \mathbb{C}_K(\eta)) = 0$ if $\eta(I_K)$ is infinite.

Def: If $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and $q \in \mathbb{Z}$, set $W\{q\} := (W(q))^{G_K} = (\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W)^{G_K}$

Note:

1) $W\{q\}$ is a K-subspace of $W(q)$

2) $W\{q\}$ contains no $\mathbb{C}_K$-lines. (i.e., if $w \in W\{q\}$ and $c \cdot w \in W\{q\}$ for all $c \in \mathbb{C}_K$, then $w = 0$).

We have natural G_K-equivariant K-linear maps (coming from $W\{q\} \hookrightarrow W(q)$)

$$K(-q) \otimes_K W\{q\} \longrightarrow K(-q) \otimes_K W(q) \simeq W$$

So by extending scalars, we get maps in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$:

$$\mathbb{C}_K(-q) \otimes_K W\{q\} \longrightarrow W \qquad \forall q \in \mathbb{Z}.$$

Hence we get a map:

$$\xi_W : \bigoplus_{q \in \mathbb{Z}} C_K(-q) \otimes_K W\{q\} \longrightarrow W$$

<u>Lemma</u> (Serre-Tate): ξ_W is injective ($W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ arbitrary).

~~Pf~~ Later

<u>Consequences</u>:

1) $W\{q\} = 0$ for all but finitely-many q.

2) $\dim_K W\{q\} < \infty$ $\forall q \in \mathbb{Z}$.

3) $\sum_q \dim_K W\{q\} \leq \dim_{C_K} W$, and equality holds iff ξ_W is an iso.

<u>Pf of lemma</u>: For any $(x_q) \in \bigoplus C_K(q) \otimes_K W\{q\}$, define:

$$\ell((x_q)) := \sum_q \ell(x_q) \quad \text{where } \ell(x_q) = \text{least } n_q \geq 0 \text{ s.t. } x_q \in \mathbb{C}_K(-q) \otimes W\{q\}$$

has a rep. as a sum of n_q elements in the $\otimes$.

(<u>N.B.</u> ℓ is $\mathbb{C}_K^\times$-invariant).

Choose $(v_q) \in \operatorname{Ker} \xi_W$ of minimal length, and assume $(v_q) \neq 0$ (to derive a contradiction)

There exists q_0 with $v_{q_0} \neq 0$. By renaming $W(q_0)$ as W, we can assume $q_0 = 0$.

$\downarrow$

(cont pf of lemma).
We will construct another element in the kernel, of smaller length. ($\Rightarrow$!!).

By $\mathbb{C}_K^\times$-scaling, we can assume that v_0 has a minimal-length expression:

$$v_0 = \sum c_{j,0} \otimes y_{j,0} \quad {}^{\in W\{0\}} \qquad \text{with some } c_{j_0,0} = 1$$

with $c_{j,0} \in \mathbb{C}_K^\times$.

For $\underset{\text{min-length}}{v_q} = \sum_j \underset{\in \mathbb{C}_K^\times}{c_{j,q}} \otimes y_{j,q}^{\ \in W\{q\}}$, then:

$$g\cdot v_q = \sum_j \chi(g)^{-q} g(c_{j,q}) \otimes y_{j,q}, \quad \text{and so:}$$

$$g v_q - v_q = \sum \left(\chi(g)^{-q} g(c_{j,q}) - c_{j,q}\right) \otimes y_{j,q}$$

So $\ell(gv_q - v_q) \le \ell(v_q)$, and $\ell(gv_0 - v_0) < \ell(v_0)$ b/c $c_{j_0,0}=1$.

So $\ell\left(g(v_q)_q - (v_q)_q\right) < \ell\left((v_q)_q\right)$, and $g(v_q)_q - (v_q)_q$ is still in the kernel of ξ_W, as it is G_K-invariant. ~~[illegible]~~

$\Rightarrow g(v_q)_q = (v_q)_q \Rightarrow g v_q = v_q$. ~~So $v_q \in (\mathbb{C}_K(q) \otimes W\{q\})^{G_K} = \mathbb{C}_K(-q)^{G_K} \otimes W\{q\}$~~

So $v_q \in \left(\mathbb{C}_K(-q) \otimes W\{q\}\right)^{G_K} = \mathbb{C}_K(-q)^{G_K} \otimes W\{q\} = \begin{cases} 0 & q \neq 0 \\ W\{0\} & q = 0 \end{cases}$

So $\underset{\substack{\uparrow \\ \text{it is a} \\ \mathbb{C}_K\text{-subspace}}}{\operatorname{Ker} \xi_W} \subseteq \underset{\substack{\uparrow \\ \text{contains no } \mathbb{C}_K\text{-lines}}}{W\{0\} = W^{G_K}}$ $\Rightarrow \operatorname{Ker} \xi_W = 0$

//

Def: $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is <u>Hodge-Tate</u> if ξ_W is an isomorphism.

(equivalently, if $\sum \dim_K W\{q\} = \dim_{\mathbb{C}_K}(W)$)

. $W \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is <u>Hodge Tate</u> iff $W \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ is Hodge-Tate.

<u>Concretely</u>: $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is H.T. iff: $W \underset{\substack{\uparrow \\ \text{in } \mathrm{Rep}_{\mathbb{C}_K}(G_K)}}{\simeq} \bigoplus \mathbb{C}_K(-q)^{h_q}$ $(h_q \geq 0)$

(clearly, HT reps look like this, because the action of G_K on $W\{q\}$ is trivial, so $\mathbb{C}_K(-q) \otimes_K W\{q\} \simeq \mathbb{C}_K(-q)^{h_q}$, $h_q = \dim_K W\{q\}$.

Conversely, if $W \simeq \bigoplus_q \mathbb{C}_K(-q)^{h_q}$, then:

$$W\{q_0\} = \left(\mathbb{C}_K(q_0) \otimes_{\mathbb{C}_K} W\right)^{G_K} = \bigoplus_q \left(\mathbb{C}_K(q_0 - q)^{G_K}\right)^{h_q} \underset{\substack{\uparrow \\ \text{Ax Sen-Tate}}}{=} K^{h_{q_0}}$$

So that $\sum_q \dim_K W\{q\} = \sum_K h_q = \dim_{\mathbb{C}_K} W$ //)

Def: Let $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ be Hodge-Tate. The <u>HT-weights</u> of W are those $q \in \mathbb{Z}$ such that $W\{q\} \neq 0$. The <u>multiplicity</u> of a HT-weight q is $\dim_K W\{q\}$.

Concretely, $q \in \mathbb{Z}$ is a HT-weight iff $\exists$ injection $\mathbb{C}_K(-q) \overset{\text{in } \mathrm{Rep}_{\mathbb{C}_K}(G_K)}{\hookrightarrow} W$.

<u>Remark!</u>: Some people define HT-weights those q s.t. $\exists$ injection $\mathbb{C}_K(q) \hookrightarrow W$ (that is, the negatives of those defined here).

Def: Let F be a field. Write

Gr_F := category of graded F-vectorspaces.

$ob(Gr_F)$: F-vectorspaces D ~~together~~ with a decomposition:

$$D \simeq \bigoplus_q D_q \quad , \text{ with } D_q \ F\text{-subspaces } \forall q.$$

$$\mathrm{mor}(D' \to D) = \{ \varphi \ F\text{-linear maps s.t. } \varphi_q := \varphi|_{D'_q} \text{ maps } D'_q \to D_q \}.$$

Ex: $D = \mathbb{C}_K[t, \frac{1}{t}]$, we can define $D_q := \mathbb{C}_K \cdot t^q$.

Fact: Gr_F is an abelian category. For example:

$$\mathrm{Ker}(\varphi : D' \to D) = \bigoplus_q (\mathrm{Ker}\, \varphi_q)$$

There is also a good notion of duality: $(D^\vee)_q := (D_{-q})^\vee$

and of $\otimes$: $(D \otimes D')_q = \bigoplus_{i+j=q} (D_i \otimes D'_j)$

NB: exact sequences correspond to exact sequence in every degree.

Def: Let $Gr_{F,f}$:= full subcategory of Gr_F of finite dimensional objects.

Def: There is a family of object, that we are interested in:

$$F\langle r\rangle := \text{graded } F\text{-vectorspace}, \quad (F\langle r\rangle)_q := \begin{cases} 0 & q \neq r \\ F & q = r \end{cases}$$

We define a covariant functor $\underline{D} = \underline{D}_K : \mathrm{Rep}_{\mathbb{C}_K}(G_K) \longrightarrow \mathrm{Gr}_K$

by $\underline{D}(W) := \bigoplus_q \underbrace{W\{q\}}_{\text{the } q^{\text{th}} \text{ graded piece}}$

$$W\{q\} = \left(\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W\right)^{G_K}$$

<u>Rk</u>: $\underline{D}$ is left exact ($\otimes$ is left exact, G_K is left exact too).

With this, we can restate Serre–Tate: $\underline{D}$ takes values in $\mathrm{Gr}_{K,f}$,

and $\dim_K \underline{D}(W) \leq \dim_{\mathbb{C}_K} W$, with equality iff W is H-T.

<u>Example</u>: $\underline{D}(\mathbb{C}_K(r)) = \bigoplus_q \left(\mathbb{C}_K(q) \otimes \mathbb{C}_K(r)\right)^{G_K} = \bigoplus_q \mathbb{C}_K(q+r)^{G_K} \underset{\substack{\uparrow \\ \text{Ax Sen-Tate}}}{=} K\langle -r \rangle$

<u>Prop</u>: If $0 \to W' \to W \to W'' \to 0$ is exact in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and W is HT, then W' and W'' are also HT, and

$$0 \to \underline{D}(W') \to \underline{D}(W) \to \underline{D}(W'') \to 0 \quad \text{is exact.}$$

Pf/ We know that $0 \to \underline{D}(W') \to \underline{D}(W) \to \underline{D}(W'') \to 0$ is <u>left</u>-exact.

So $\dim_K \underline{D}(W) \leq \dim_K \underline{D}(W') + \dim_K \underline{D}(W'') \leq \dim_{\mathbb{C}_K} W' + \dim_{\mathbb{C}_K} W''$

$= \dim_{\mathbb{C}_K} W$

But $\dim_K \underline{D}(W) \overset{W \text{ HT}}{=} \dim_{\mathbb{C}_K} W \Rightarrow$ equality everywhere. //

But the converse of this false: an extension of HT is not necessarily HT.

↓

<u>Example</u>: By the thm of Ax-Sen-Tate,

$$H^1_{cont}(G_K, \mathbb{C}_K) \neq 0$$

By a HW exercise, this means that there exists an exact sequence in $Rep_{\mathbb{C}_K}(G_K)$: $(*)$ $0 \to \mathbb{C}_K \to W \to \mathbb{C}_K \to 0$ which is non-split.

<u>Claim</u>: W is <u>not</u> H-T.

Pf Assume it was. Then by the previous theorem,

$$0 \to \underline{D}(\mathbb{C}_K) \to \underline{D}(W) \to \underline{D}(\mathbb{C}_K) \to 0$$

is exact. By what we computed before, this is:

$$0 \to K\langle 0 \rangle \to \underline{D}(W) \to K\langle 0 \rangle \to 0$$

The 0^{th} degree says:

$0 \to K \to W\{0\} \to K \to 0$ is exact.

Any choice of K-linear splitting gives (because $W\{0\} = W^{G_K}$) a G_K-equivariant $\mathbb{C}_K$-splitting of $(*)$ in $Rep_{\mathbb{C}_K}(G_K)$ which is a contradiction. //

<u>Def</u>: The <u>HT ring</u> is $B_{HT} := \bigoplus_q \mathbb{C}_K(q)$, with multiplication

$$\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} \mathbb{C}_K(q') \simeq \mathbb{C}_K(q+q')$$

Note that it is a graded $\mathbb{C}_K$-~~vectorspace~~ algebra, with G_K-action, which respects the ring structure and the grading.

If we choose a basis of $\mathbb{C}_K(1)$ (thus giving a basis for $\mathbb{C}_K(r)$), then we can identify:

$$B_{HT} = \mathbb{C}_K[t, \tfrac{1}{t}] = \bigoplus_q \mathbb{C}_K t^q$$

with the G_K-action $g.t^q = \chi(g)^q t^q$.

Note that by Ax-Sen-Tate, $(B_{HT})^{G_K} = K$.

We have, for any $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$:

$$\underline{D}(W) = \bigoplus_q W\{q\} = \bigoplus_q \left(\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W\right)^{G_K} = \left(B_{HT} \otimes_{\mathbb{C}_K} W\right)^{G_K}$$

Def: Let $D \in \mathrm{Gr}_K$. Define

the 0^{th} graded piece

$$\underline{V}(D) = \underline{V}_K(D) := gr^0\left(B_{HT} \otimes_K D\right) = \bigoplus_q \mathbb{C}_K(-q) \otimes D_q$$

We will consider $\underline{V}$ as a functor $\underline{V}: \mathrm{Gr}_{K,f} \to \mathrm{Rep}_{\mathbb{C}_K}(G_K)$

Note that it actually has image in HT-reps $\subseteq \mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

NB: $\underline{V}$ is covariant and exact.

Example: $\underline{V}(K\langle r\rangle) = \mathbb{C}_K(-r)$.

For any $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$, we get a B_{HT}-linear, G_K-equivariant graded map:

$$\gamma_W : B_{HT} \otimes_K \underline{D}(W) = B_{HT} \otimes_K \left(B_{HT} \otimes_{\mathbb{C}_K} W\right)^{G_K} \hookrightarrow B_{HT} \otimes_K B_{HT} \otimes_{\mathbb{C}_K} W \xrightarrow{\text{mult in } B_{HT}} B_{HT} \otimes_{\mathbb{C}_K} W$$

Lemma: The map γ_W is injective, and it is an isomorphism iff W is HT.

Pf

$$gr^n(\gamma_W): \bigoplus_q \mathbb{C}_K(n-q) \otimes_K W\{q\} \longrightarrow \mathbb{C}_K(n) \otimes_{\mathbb{C}_K} W$$

Then it is easy to realize that $gr^n(\gamma_W) = \mathbb{C}_K(n) \otimes \left(\xi_W : \bigoplus_q \mathbb{C}_K(-q) \otimes W\{q\} \to W\right)$

Theorem: The covariant functors:

$$\text{HT-reps} \underset{\underline{V}}{\overset{\underline{D}}{\rightleftarrows}} Gr_{K,f}$$

are quasi-inverse (exact) equivalences of categories.

Pf

For any $D \in Gr_{K,f}$, we have isomorphisms:

$$\gamma_{V(\underline{D})} : B_{HT} \otimes_K \underline{D}(\underline{V}(D)) \xrightarrow{\sim} B_{HT} \otimes_{\mathbb{C}_K} \underline{V}(D).$$

Taking G_K-invariants, we get: $\gamma_{V(\underline{D})}$ is G_K-equivariant.

$$B_{HT}^{G_K} \otimes_K \underline{D}(\underline{V}(D)) \xrightarrow{\sim} \left(\left(\bigoplus_n \mathbb{C}_K(n)\right) \otimes_{\mathbb{C}_K} \left(\bigoplus_r \mathbb{C}_K(-r) \otimes_K D_r\right)\right)^{G_K}$$

$$\underline{D}(\underline{V}(D)) \qquad \left(\bigoplus_{n,r} \mathbb{C}_K(n-r) \otimes_{\mathbb{C}_K} D_r\right)^{G_K} \underset{A\text{-}S\text{-}T}{=} \bigoplus_n D_n = D$$

As for the other order of composition, if $V \in Rep_{\mathbb{C}_K}(G_K)$ is HT,

$$\gamma_V : B_{HT} \otimes \underline{D}(V) \xrightarrow{\sim} B_{HT} \otimes_{\mathbb{C}_K} V$$

Passing to the 0^{th}-graded pieces:

$$\underline{V}(\underline{D}(V)) \simeq V$$

Def: Let $\mathrm{Rep}_{HT}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ be the full subcategory of HT-reps.

(recall that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is HT $\Leftrightarrow$ $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ is HT!)

We get a functor:

$$\underline{D}_{HT} : \mathrm{Rep}_{HT}(G_K) \to \mathrm{Gr}_{K,f}$$

$$V \longmapsto \underline{D}_{HT}(V) := \underline{D}_K(\mathbb{C}_K \otimes_{\mathbb{Q}_p} V)$$

Claim: $\underline{D}_{HT}$ is faithful:

Let $\varphi : V' \to V$ be in $\mathrm{Rep}_{HT}(G_K)$.

$\underline{D}_{HT}(\varphi) : D_{HT}(V') \to D_{HT}(V)$

We have a comm. diagram:

$$\begin{tikzcd} B_{HT} \otimes D_{HT}(V') \arrow[r, "\gamma_{\mathbb{Q}_p \otimes V}", "\sim"'] \arrow[d, "1 \otimes D_{HT}(\varphi)"] \arrow[dr, phantom, "\circlearrowleft"] & B_{HT} \otimes_{\mathbb{Q}_p} V \arrow[d, "1 \otimes \varphi"] \\ B_{HT} \otimes D_{HT}(V) \arrow[r, "\gamma_{\mathbb{C}_K \otimes V}"'] & B_{HT} \otimes_{\mathbb{Q}_p} V \end{tikzcd}$$

//

However, $\underline{D}_{HT}$ is not full!!

Pf Let $\eta : G_K \to \mathbb{Z}_p^\times$ be any finite-order character.

$$D_{HT}(\mathbb{Q}_p(\eta)) = \left(B_{HT} \otimes_{\mathbb{C}_K} \mathbb{C}_K(\eta)\right)^{G_K} = \bigoplus_q \mathbb{C}_K(\chi^q \eta)^{G_K} = K\langle 0 \rangle.$$

On the other hand, $D_{HT}(\mathbb{Q}_p) = K\langle 0 \rangle$ as well.

However there are no nonzero homs. $\mathbb{Q}_p \to \mathbb{Q}_p(\eta)$ $\left(\begin{array}{l} 1 \mapsto x \\ 1 \mapsto g(x) = \eta(g)x \end{array} \Leftrightarrow x = 0 \right)$ //

Let K be a p-adic field. Let $K_\infty = K(\mu_{p^\infty})$ (adjoin all p^n-th roots of 1).

Then $\Gamma = Gal(K_\infty/K) \overset{\chi}{\hookrightarrow} \mathbb{Z}_p^\times$ has finite-index image. Also, im χ is closed (by a HW exercise). So Γ is an open subgroup of $\mathbb{Z}_p^\times$.

$$\left\{ \begin{array}{l} \text{(separable) algebraic extensions} \\ \text{of } K_\infty \end{array} \right\} \overset{\text{they are equivalent}}{\underset{\text{Fontaine-Wintenberger}}{\longleftrightarrow}} \left\{ \begin{array}{l} \text{separable alg. extensions} \\ \text{of } \underset{\text{char } p}{E} := E(K_\infty/K) \simeq k((u)) \end{array} \right\}$$

(this equivalence is due to the theory of Norm fields)

In particular, $G_{K_\infty} \simeq G_E = Gal(E^s/E)$, where E^s is the <u>separable</u> algebraic closure of E.

Note that we have an exact sequence:

$$0 \to G_{K_\infty} \to G_K \to \Gamma \to 0$$

So that $Rep_{\mathbb{Z}_p}(G_K) \longleftrightarrow \left\{ \begin{array}{l} \text{continuous reps of } G_{K_\infty} \simeq G_E \\ + \text{ some "}\Gamma\text{-descent data"} \end{array} \right\}$

So the question becomes: How to describe $Rep_{\mathbb{Z}_p}(G_E)$ for $E = k((u))$?

This involves étale φ-module.

(in fact, to describe $Rep_{\mathbb{Z}_p}(G_K)$, one will need what's called (φ, Γ)-modules. We will not do this in this course, though)

To start, we will describe $Rep_{\mathbb{F}_p}(G_E)$.

Fix E a field of characteristic $p > 0$, and let E_s be a fixed separable closure of E. Write $G_E = \mathrm{Gal}(E_s/E)$.

Denote by φ_E and φ_{E_s} the Frobenius $x \mapsto x^p$, acting on E and E_s, resp.

(note that they commute with the G_E-action)

Then $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ = cts reps of G_E on f.d. $\mathbb{F}_p$-vectorspaces.

(and continuous means that they factor through $\mathrm{Gal}(E'/E)$ for some finite separable E').

There is an analogy:

$$\begin{array}{ll|ll}
B_{HT} & B_{HT}^{G_K} = K & E_s & E_s^{G_E} = E \\
G_K \circlearrowright \quad \circlearrowleft \text{grading} & \mathrm{gr}^0(B_{HT}) = \mathbb{C}_K & G_E \circlearrowright \quad \circlearrowleft \varphi_{E_s} & E_s^{\varphi=1} = \mathbb{F}_p \\
\text{(compatible)} & & \text{(compatible)} &
\end{array}$$

Def: An <u>étale φ-module over E</u> is a pair (M, φ_M) consisting of a finite-dimensional E-vs M, with an endomorphism $\varphi_M : M \to M$, (as abelian groups) satisfying:

1) φ_M is φ_E-semilinear: $\varphi_M(e \cdot m) = \varphi_E(e) \cdot \varphi_M(m)$

2) The <u>E-linearization</u> of φ_M is an isomorphism (étaleness).

(Given $\varphi_M : M \to M$, we construct $M \otimes_{E, \varphi_E} E \to M$, $m \otimes e \longmapsto \varphi_M(m) \cdot e$, and $M \otimes_E E$ is an E-vectorspace via the right factor. Then this becomes linear.

We write $\varphi_E^*(M) := M \otimes_{E, \varphi_E} E$).

<u>Exercise</u>: Condition (2) is equivalent to saying that the matrix of $\varphi_M : M \to M$ in $GL_d(E)$ is invertible.

Denote by $\Phi M_E^{\text{ét}}$ the category of étale φ-modules $/E$. The morphisms are E-linear and φ_M-compatible maps.

<u>Example</u>: $(E, \varphi_E) \in \Phi M_E^{\text{ét}}$.

- The category $\Phi M_E^{\text{ét}}$ has $\otimes$: $(M, \varphi_M) \otimes (M', \varphi_{M'}) := (M \otimes M', \varphi_M \otimes \varphi_{M'})$.
- Duality: $(M, \varphi_M)^\vee = (M^\vee, \varphi_{M^\vee})$, where $M^\vee$ is the E-linear dual,

and $\varphi_{M^\vee}$ is the map:

$$\begin{aligned} M^\vee &\longrightarrow (\varphi_E^*(M))^\vee \xrightarrow{\ \simeq\ } M^\vee \\ \ell &\longmapsto \big(m \otimes e \mapsto \varphi_E(\ell(m)) \cdot e\big) \end{aligned}$$

(check that this map is φ_E semilinear)

(the isomorphism is the dual of the inverse of $\varphi_E^*(M) \xrightarrow[\varphi_M^{\text{lin.}}]{\sim} M$).

<u>Exercise</u>: check that $\varphi_{M^\vee}$ linearizes to an isomorphism.

<u>Lemma</u>: $\Phi M_E^{\text{ét}}$ is an abelian category. If $h : M' \to M$ is a morphism, then $\operatorname{im} h$, $\ker h$, $\operatorname{Coker} h$ are étale φ-modules.

Pf Consider:

$$\begin{tikzcd} 0 \arrow[r] & \underbrace{\ker(\varphi_E^*(h))}_{\substack{\varphi_E^*(\ker h) \\ \text{b/c ext. of scalars} \\ \text{commutes with kernel}}} \arrow[r] \arrow[d, dashed, "\simeq"'] & \varphi_E^*(M') \arrow[r, "\varphi_E^*(h) = h \otimes 1"] \arrow[d, "\varphi_E^*(\varphi_M') \ \simeq"] \arrow[dr, phantom, "\circlearrowleft"] & \varphi_E^*(M) \arrow[r] \arrow[d, "\simeq\ \varphi_E^*(\varphi_M')"] & \underbrace{\operatorname{Coker}(\varphi_E^*(h))}_{\varphi_E^*(\operatorname{Coker}(h))} \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\ 0 \arrow[r] & \ker h \arrow[r] & M' \arrow[r, "h"] & M \arrow[r] & \operatorname{Coker} h \arrow[r] & 0 \end{tikzcd}$$

//

For $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$, define $\mathbb{D}_E(V) := \left(E_s \otimes_{\mathbb{F}_p} V\right)^{G_E}$

This is an E-vectorspace, with an endomorphism $\varphi_{\mathbb{D}_E(V)} := (\varphi_{E_s} \otimes 1)|_{G_E\text{-invariants}}$

which is E-semilinear.

We will see that $(\mathbb{D}_E(V), \varphi_{\mathbb{D}_E(V)})$ is an étale φ-module.

But first, we go the other way: for $M \in \Phi M_E^{ét}$, define:

$$\mathbb{V}_E(M) := \left(E_s \otimes_E M\right)^{\varphi = 1} \qquad \left(\text{where } \varphi = \varphi_{E_s} \otimes \varphi_M \text{ acting on } E_s \otimes_E M\right).$$

Since $E_s^{\varphi_{E_s}=1} = \mathbb{F}_p$, this is an $\mathbb{F}_p$-vectorspace (maybe not fin-dim?), together with a G_E-action (only acting nontrivially on the E_s-factor)

Example:

$$\mathbb{D}_E(\mathbb{F}_p) = \left(E_s \otimes_{\mathbb{F}_p} \mathbb{F}_p\right)^{G_E} = \underset{\circlearrowleft \varphi_{E_s}|_{G_E\text{-inv}}}{E_s^{G_E}} = \underset{\circlearrowleft \varphi_E}{E}$$

$$\mathbb{V}_E(E) = \left(E_s \otimes_E E\right)^{\varphi=1} = E_s^{\varphi_{E_s}=1} = \mathbb{F}_p + \text{trivial } G_E\text{-action.}$$

We also define a pair of contravariant functors:

$$\mathbb{D}_E^*(\cancel{M}) := \mathbb{D}_E(V^\vee) \quad \text{the dual } \mathbb{F}_p\text{-}\cancel{\text{vector}}\text{space.}$$

$$\mathbb{V}_E^*(M) := \mathbb{V}_E(M^\vee) \quad \text{the dual étale } \varphi\text{-module.}$$

Note that $\mathbb{D}_E(V^\vee) = \left(E_s \otimes \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)\right)^{G_E} \cong \left(\mathrm{Hom}_{\mathbb{F}_p}(\cancel{V}, E_s)\right)^{G_E} =$

$$= \mathrm{Hom}_{\mathbb{F}_p[G_E]}(V, E_s) \qquad \text{(exercise).}$$

and similarly for $\mathbb{V}_E$: $\mathbb{V}_E^*(M) = \mathrm{Hom}_{E,\varphi}(M, E_s)$ $\quad E,\varphi \simeq E$-linear, φ equivariant

<u>Lemma</u>: For any $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$, $\mathbb{D}_E(V)$ is finite-dimensional$/E$, and $\dim_E \mathbb{D}_E(V) = \dim_{\mathbb{F}_p}(V)$,

Moreover, $\varphi_{\mathbb{D}_E(V)}$ linearizes to an isomorphism.

(this lemma is saying that $\mathbb{D}_E$ takes values in ΦM_E^{et}, and it is rank-preserving).

Pf/ First, $E_s \otimes_{\mathbb{F}_p} V$ is finite-dimensional E_s-vectorspace, with a diagonal G_E action which is continuous wrt the discrete topology. (i.e. any element of $E_s \otimes_{\mathbb{F}_p} V$ has open stabilizer in G_E).

<u>Claim</u>:

$$\begin{tikzcd} E_s \otimes_E \mathbb{D}_E(V) = E_s \otimes_E (E_s \otimes_{\mathbb{F}_p} V)^{G_E} \arrow[rr, bend right, "(*)"'] & E_s \otimes_E E_s \otimes_{\mathbb{F}_p} V \arrow[r, "\text{mult on } E_s"] & E_s \otimes_{\mathbb{F}_p} V \end{tikzcd}$$

The claim is that $(*)$ is an iso.

"pf" (of claim)

Galois descent, using continuity of the action $G_E \circlearrowright E_s \otimes_{\mathbb{F}_p} V$ as above.

loose step in Serre's Galois Cohomology

(i.e. $E_s \otimes_{\mathbb{F}_p} V$ has a G_E-invariant basis). //

By the claim, $\dim_{E_s}(E_s \otimes \mathbb{D}_E(V)) = \dim_{E_s}(E_s \otimes_{\mathbb{F}_p} V) = \dim_{\mathbb{F}_p} V$

$$\dim_{E_s}(E_s \otimes \mathbb{D}_E(V)) = \dim_E \mathbb{D}_E(V).$$

Remains to show that $\varphi_{\mathbb{D}_E(V)}$ linearizes to an iso. But this is an iso $\Leftrightarrow$ the E_s-linearization of $\varphi_{E_s} \otimes \varphi_{\mathbb{D}_E(V)}$ is an iso (because linearization commutes with ext. of scalars $E \to E_s$).

(cont'd pf of lemma)

But by the claim, this is an iso iff the E_s-linearization of $\varphi_{E_s} \otimes 1$ on $E_s \otimes_{\mathbb{F}_p} V$ is an iso, which is equivalent to saying that the map $E_s \underset{E_s, \varphi_{E_s}}{\otimes} E_s \longrightarrow E_s$ is an iso, and this is obvious //.

Lemma: For any ~~...~~ $M \in \Phi M_E^{et}$, $\mathbb{V}_E(M)$ is finite-dimensional $/\mathbb{F}_p$, with $\dim_{\mathbb{F}_p} \mathbb{V}_E(M) = \dim_E M$.

(So $\mathbb{V}_E$ takes values in $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$, and it is rank-preserving).

Pf

$$E_s \otimes_{\mathbb{F}_p} \mathbb{V}_E(M) = E_s \otimes_{\mathbb{F}_p} (E_s \otimes_E M)^{\varphi} \to E_s \otimes_{\mathbb{F}_p} E_s \otimes_E M \xrightarrow{\text{mult on } E_s} E_s \otimes_E M \quad (**)$$

The claim, again, is that this composition is an isomorphism, which is G_E and φ equivariant.

Pf (claim):

Injectivity: It is enough to show that, if $v_1, \dots, v_r$ are ~~linearly-independent~~ in $\mathbb{V}_E(M)$ are $\mathbb{F}_p$-linearly-indep, then when considered in $E_s \otimes M$ (under $**$), they are E_s-linearly-indep.

Suppose the contrary, and let $r \geq 1$ be a minimal counterexample.

So $v_1, \dots, v_r$ $\mathbb{F}_p$-lin. indep, $\sum a_i v_i = 0$, $a_i \in E_s$.

By minimality, each $a_i \neq 0$. Scale by $E_s^\times$, to assume $a_1 = 1$.

~~Note~~ then $v_1 = -\sum_{i>1} a_i v_i \Rightarrow \underset{v_1}{\varphi(v_1)} = -\sum_{i>1} \varphi(a_i) v_i$ (v_i's are φ-invariant).

Hence $0 = \sum_{i>1} (a_i - \varphi(a_i)) v_i \Rightarrow a_i = \varphi(a_i) \ \forall i > 1$.

So $a_i \in E_s^{\varphi_{E_s}} = \mathbb{F}_p \Rightarrow !!$ because they were $\mathbb{F}_p$-independent //

(cont pf. of lemma)

<u>Surjectivity</u>: Since $(**)$ is injective, it's enough to show that

$$\dim_{E_s} E_s \otimes_{\mathbb{F}_p} \mathbb{V}_E(M) = \dim_{E_s}\left(E_s \otimes_E M\right).$$

So if $d = \dim_E M$, we want to show that

$$\dim_{\mathbb{F}_p} \mathbb{V}_E(M) \leq d \quad \text{is an equality.}$$

But $\mathbb{V}_E(M) = \mathbb{V}_E(M^{\vee\vee}) = \mathrm{Hom}_{E,\varphi}(M^\vee, E_s)$.

Choose an E-basis $m_1, \ldots, m_d$ of M, and give $M^\vee$ the dual basis, say $m_1^*, \ldots, m_d^*$.

Then $\varphi_{M^\vee}(m_j^*) = \sum_i c_{ij} m_i^*$ with $(c_{ij}) \in GL_d(E)$ <u>invertible</u> (M is an étale φ-module!)

So $\mathrm{Hom}_{E,\varphi}(M^\vee, E_s) = \left\{ \begin{array}{l} e_i = \theta(m_i^*) \\ \text{s.t. } e_j^p = \sum_j c_{ij} e_i \end{array} \right\} = \cancel{\mathrm{Hom}_{E\text{-alg}}}$

$$= \mathrm{Hom}_{E\text{-alg}}\left(\underbrace{\frac{E[X_1, \ldots, X_d]}{\left(X_j^p - \sum_i c_{ij} X_i\right)_{1 \leq j \leq d}}}_{=A}, E_s\right).$$

<u>Claim</u>: As an E-algebra, A is a product of finite separable field extensions.

Note that, from the claim, we get $\# \mathbb{V}_E(M) = \# \mathrm{Hom}_{E\text{-alg}}\left(\prod E_i, E_s\right) =$

$$= \sum_i \dim E_i = \dim_E A = p^d \quad \text{as we wanted.}$$

For the proof of the claim, note that it is equivalent to having $\Omega^1_{A/E} = 0$.

But $\Omega^1_{A/E} = \dfrac{\bigoplus A\, dX_i}{\left(-\sum c_{ij} dX_i\right)_j} = 0$ because (c_{ij}) is <u>invertible</u>

// (lemma)

What we've done: (E a field of char p).

$$\begin{array}{ccc} \text{Ab Category} & & \mathrm{Gal}(E_s/E) \\ & & \| \\ \Phi M_E^{et} & \underset{\mathbb{D}_E}{\overset{\mathbb{V}_E}{\rightleftarrows}} & \mathrm{Rep}_{\mathbb{F}_p}(G_E) \\ \| & & \end{array}$$

$$\left\{\begin{array}{l} \text{fin. dim } E\text{-vs} \\ M \ + \varphi_M : M \to M \\ \text{semilinear over } \varphi : E \to E \\ + \ \varphi_M \text{ linearizes to } \cong \end{array}\right\}$$

$$\mathbb{V}_E(M) = (E_s \otimes_E M)^{\varphi=1}$$

$$\mathbb{D}_E(V) = (E_s \otimes_{\mathbb{F}_p} V)^{G_E}$$

Both categories have $\otimes$, $\vee$.

We proved that the natural maps:

$$(*) \quad E_s \otimes_E \mathbb{D}_E(V) = E_s \otimes_E (E_s \otimes_{\mathbb{F}_p} V)^{G_E} \longrightarrow E_s \otimes_E E_s \otimes_{\mathbb{F}_p} V \xrightarrow{\text{mult}} E_s \otimes_{\mathbb{F}_p} V$$

$$(**) \quad E_s \otimes_{\mathbb{F}_p} \mathbb{V}_E(M) = E_s \otimes_{\mathbb{F}_p} (E_s \otimes_E E)^{\varphi=1} \hookrightarrow E_s \otimes_{\mathbb{F}_p} E_s \otimes_E M \xrightarrow{\text{mult}} E_s \otimes_E M$$

are E_s-linear, φ-equivariant isomorphisms.

As a corollary, we got that $\mathbb{V}_E$, $\mathbb{D}_E$ are rank-preserving, exact and take values in the expected categories.

<u>Theorem</u>: $\mathbb{V}_E$, $\mathbb{D}_E$ are quasi-inverse equivalences of categories, $\otimes$-compatible and duality-compatible.

Pf Pass to φ-invariants on $(*)$, to get

$$\mathbb{V}_E(\mathbb{D}_E(V)) \simeq (E_s \otimes_{\mathbb{F}_p} V)^{\varphi} \simeq V$$ because the action of φ on V is trivial.

Take G_E-invariants on $(**)$:

$$\mathbb{D}_E(\mathbb{V}_E(M)) \simeq (E_s \otimes_E M)^{G_E} \simeq M.$$

(cont pf of Thm):

Need to check compatibility with $\otimes$. (for V, HW!).

Let $V, V' \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$. There is a natural map:

$$\begin{tikzcd}
\mathbb{D}_E(V)\otimes \mathbb{D}_E(V') = \left(E_s \otimes_{\mathbb{F}_p} V\right)^{G_E} \underset{E}{\otimes} \left(E_s \otimes_{\mathbb{F}_p} V'\right)^{G_E} \arrow[r, hook] \arrow[drr, "F"] & \left(E_s \otimes_{\mathbb{F}_p} V\right) \underset{E}{\otimes} \left(E_s \otimes_{\mathbb{F}_p} V'\right) \arrow[r, "\mathrm{mult}"] & E_s \otimes_{\mathbb{F}_p} \left(V \otimes_{\mathbb{F}_p} V'\right) \\
 & & \mathbb{D}_E(V\otimes V') \arrow[u]
\end{tikzcd}$$

We want to see that F is an iso.

Enough to check it after extending scalars to E_s.

But using $(*)$, this becomes:

$$\left(E_s \otimes_{\mathbb{F}_p} V\right) \otimes \left(E_s \otimes_{\mathbb{F}_p} V'\right) \xrightarrow{\mathrm{mult}} E_s \otimes_{\mathbb{F}_p}\left(V \otimes V'\right)$$

and this is obviously an isomorphism. $\boxtimes$

We want a similar description of $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ (instead of $\mathbb{F}_p$-reps)

(finitely-generated $\mathbb{Z}_p$-modules with continuous G_E-action).

Since we will be lifting coefficients, we will to lift also the "period ring" E_s.

<u>Assumption</u> (although it will always be satisfied, as we will see later):

There exists a complete DVR $\mathcal{O}_{\mathcal{E}}$, of char. 0, with uniformizer p and residue field E, together with a map $\varphi : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ lifting φ_E on E.

Let $\mathcal{E} = \mathrm{Frac}(\mathcal{O}_{\mathcal{E}}) = \mathcal{O}_{\mathcal{E}}[\frac{1}{p}]$.

<u>Fact</u>: $(\mathcal{O}_{\mathcal{E}}, \varphi)$ always exists.

Example: $E = k((u))$ (k = residue field of some p-adic field. So k is perfect).

Then we can consider $W(k)[[u]]$ ($W(k)$ the ring of witt vectors)

Then $W(k)[[u]]$ is regular local, of dim 2, $\mathfrak{m} = (p, u)$, and

its residue field is ~~[illegible]~~ k.

Now, ~~[illegible]~~ localize at (p), and complete wrt the $\mathfrak{m}$-adic topology:

$$\mathcal{O}_{\mathcal{E}} := \left[\left(W(k)[[u]] \right)_{(p)} \right]^{\wedge}$$ is a complete, local, 1-dim'l, and its residue field is E.

Note that $$\mathcal{O}_{\mathcal{E}} = \left\{ \sum_{n \in \mathbb{Z}} a_n u^n \;\middle|\; \begin{array}{l} a_n \in W(k) \\ a_n \to 0 \text{ as } n \to -\infty \end{array} \right\}$$

The Frobenius also lifts:

$\varphi : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ by $\sum a_n u^n \mapsto \sum \sigma(a_n) u^{pn}$

(where $\sigma(a_n)$ is the unique lift of Frob on k to $W(k)$).

$$\longleftarrow\!\!\!\longrightarrow$$

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{E}} & \hookrightarrow & \mathcal{O}_{\mathcal{E}}^{un} \\ \downarrow & & \downarrow \\ E & \hookrightarrow & E_s \end{array}$$

$\mathcal{O}_{\mathcal{E}}^{un}$ = strict henselization of $\mathcal{O}_{\mathcal{E}}$ (it is a DVR, with uniformizer p, residue field E_s but maybe not complete).

Set also $\mathcal{E}^{un} := \mathrm{Frac}(\mathcal{O}_{\mathcal{E}}^{un})$.

Remark: $\mathcal{O}_{\mathcal{E}}^{un}$ is functorial in the following sense:

Let $f: \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$, $\bar{f}$ its restriction to $E \to E$, and let $\bar{f}'$ be a lift of $\bar{f}$ to $E_s \xrightarrow{\bar{f}'} E_s$. Then $\exists!\, f'$ making the following commute:

$$\begin{tikzcd}
& \mathcal{O}_{\mathcal{E}} \arrow[rr, hook] \arrow[dd] & & \mathcal{O}_{\mathcal{E}}^{ur} \arrow[dd] \\
\mathcal{O}_{\mathcal{E}} \arrow[ur, "f"] \arrow[rr, hook] \arrow[dd] & & \mathcal{O}_{\mathcal{E}}^{ur} \arrow[ur, "\exists!\, f"'] \arrow[dd] & \\
& E \arrow[rr, hook] & & E_s \\
E \arrow[ur, "\bar{f}"] \arrow[rr, hook] & & E_s \arrow[ur, "\bar{f}'"'] &
\end{tikzcd}$$

<u>Example</u>: $(f, \bar{f}, \bar{f}') = (id, id, g \in G_E)$.

Then we get that G_E acts <u>continuously</u> on $\mathcal{O}_{\mathcal{E}}^{ur}$.

<u>Example</u>. $(f, \bar{f}, \bar{f}') = (\varphi, \varphi_E, \varphi_{E_s})$

Then there is φ acting on $\mathcal{O}_{\mathcal{E}}^{ur}$.

Note that φ and G_E commute, by uniqueness of the liftings again.

Finally, we complete this ring: $\widehat{\mathcal{O}_{\mathcal{E}}^{ur}}$, which is also equipped with an action of G_E and a Frobenius φ. We also complete $\mathcal{E}^{ur}$, to get $\widehat{\mathcal{E}^{ur}}$.

<u>Def</u>: An étale φ-module over $\mathcal{O}_{\mathcal{E}}$ is a pair (M, φ_M), where M is a finitely-generated $\mathcal{O}_{\mathcal{E}}$-module, and φ_M is a φ-semilinear map $\varphi_M : M \to M$, such that it linearizes to an isomorphism.

We denote by $\Phi M_{\mathcal{O}_\mathcal{E}}^{ét}$ the category of such objects, where the morphisms are $\mathcal{O}_\mathcal{E}$-module hom., and φ-compatible.

Rk: Now, we can't reduce the étaleness condition to a matrix being invertible, because we can have torsion...

Def: Denote by $\Phi M_{\mathcal{E}}^{ét}$ the full subcategory of $\Phi M_{\mathcal{O}_\mathcal{E}}^{ét}$ consisting of p-torsion elements.

Lemma: $\Phi M_{\mathcal{O}_\mathcal{E}}^{ét}$ is an abelian category.

Pf: We need to check that the induced φ-maps on im, ker, coker on ~~a~~ map $f: M' \to M$ are étale.

Exercise: If (M, φ_M) is a f.gen $\mathcal{O}_\mathcal{E}$ module with ~~an~~ $\varphi_{\mathcal{O}_\mathcal{E}}$-semilinear endomorphism, then it linearizes to an iso $\Leftrightarrow$ φ_M mod p does.
(i.e. étaleness can be checked mod p).

Idea:

$$\varphi^*(M) : M \otimes_{\mathcal{O}_\mathcal{E}, \varphi} \mathcal{O}_\mathcal{E} \longrightarrow M, \quad m \otimes x \longmapsto \varphi_M(m) \cdot x$$

Both have the same rank + invariant factors. So they are abstractly isomorphic, so it's enough to check surjectivity, and this can be done mod p.

Using this, we can show that $\operatorname{coker}(f)$ is an étale φ-module.

$\downarrow$

$$\begin{tikzcd}
M' \arrow[r, "\cdot p"] \arrow[d, "f"'] & M' \arrow[r] \arrow[d, "f"] & M'/pM' \arrow[r] \arrow[d, "f \bmod p"] & 0 \\
M \arrow[r, "\cdot p"] \arrow[d] & M \arrow[r] \arrow[d] & M/pM \arrow[r] \arrow[d] & 0 \\
(\operatorname{coker} f) \arrow[r, "\cdot p"] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d] & \operatorname{coker}(f \bmod p) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

So $(\operatorname{coker} f) \pmod p) = \operatorname{coker}(f \bmod p)$. $\varphi^* \operatorname{coker}(f \bmod p) \overset{\text{121}}{}$

$$\begin{tikzcd}
\varphi^*(M'/pM') \arrow[r, "\varphi^*(f \bmod p)"] \arrow[d, "\cong"', "\varphi_{M'} \bmod p"] & \varphi^*(M/pM) \arrow[r] \arrow[d, "\cong"', "\varphi_M^{lin}"] & \operatorname{coker}(\varphi^*(f \bmod p)) \arrow[d, "\varphi^{lin}_{\operatorname{coker}(f \bmod p)}"] \\
M'/pM' \arrow[r, "f"'] & M/pM \arrow[r] & \operatorname{coker}(f \bmod p) \arrow[r] & 0
\end{tikzcd}$$

~~We~~ get iso on the third column, so the linearization of $\varphi_{\operatorname{coker}(f)}$ is an iso (mod p), by the exercise, we are done.

For the image and kernel note that the formation of ker, im commutes with the flat scalar extension $\mathcal{O}_\mathcal{E} \xrightarrow{\varphi} \mathcal{O}_\mathcal{E}$.

So $\varphi^*(\operatorname{im}(f)) = \operatorname{im}(\varphi^*(f))$, and the same for ker.

Now, use:

$$\begin{tikzcd}
0 \arrow[r] & \varphi^*(\operatorname{im} f) \arrow[r] \arrow[d] & \varphi^* M \arrow[r] \arrow[d, "\cong"] & \varphi^* \operatorname{coker} f \arrow[r] \arrow[d, "\cong \leftarrow \text{by what we did.}"] & 0 \\
0 \arrow[r] & \operatorname{im} f \arrow[r] & M \arrow[r] & \operatorname{coker} f \arrow[r] & 0
\end{tikzcd}$$

So we get it for im. The same argument works for ker. $\square$

We also have $\otimes$ and $\vee$ for $\Phi M^{et}_{\mathcal{O}_\mathcal{E}}$, and for $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$, at least in some cases:

	$\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$	$\Phi M^{et}_{\mathcal{O}_\mathcal{E}}$
fin. length	$\mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Q}_p/\mathbb{Z}_p)$	$\mathrm{Hom}_{\mathcal{O}_\mathcal{E}}(-,\mathcal{E}/\mathcal{O}_\mathcal{E})$
fin. free	$\mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Z}_p)$	$\mathrm{Hom}_{\mathcal{O}_\mathcal{E}}(-,\mathcal{O}_\mathcal{E})$

Lemma: The natural maps $\mathcal{O}_\mathcal{E} \to \left(\widehat{\mathcal{O}_\mathcal{E}^{un}}\right)^{G_E}$, $\mathbb{Z}_p \to \left(\widehat{\mathcal{O}_\mathcal{E}^{un}}\right)^{\varphi=1}$

and $\mathcal{E} \to \left(\widehat{\mathcal{E}^{un}}\right)^{G_E}$, $\mathbb{Q}_p \to \left(\widehat{\mathcal{E}^{un}}\right)^{\varphi=1}$ are all isomorphisms.

Pf Enough to show it for the integral versions. Note that all these maps are injections.

Claim If $\theta: R' \hookrightarrow R$ is a local injective map of p-adically separated and complete rings, and θ is surjective mod p, then θ is an isomorphism.

Pf/ Successive approximation: Let $r \in R$. Let $r_0' \in R'$ be s.t. $\theta(r_0') \equiv r \bmod p$, so $\theta(r_0') = r + pr_1^*$, some $r_1^* \in R$). Then let $r_1' \in R$ st $\theta(r_1') = r_1 + pr_2$, $r_2 \in R$.

Then $\underbrace{\theta(r_0' - pr_1' + p^2 r_2' - \cdots)}_{\text{since source is p-adically complete}} \underset{\text{by p-adic separatedness of the target.}}{=} r$

∎(claim)

Now, we have an exact sequence:

$$0 \to \widehat{\mathcal{O}_\mathcal{E}^{un}} \xrightarrow{\cdot p} \widehat{\mathcal{O}_\mathcal{E}^{un}} \to E_s \to 0$$

~~Taking G_E-invariants, we get:~~

~~$0 \to$~~

(cont pf of lemma)

From the exact sequence $0 \to \widehat{\mathcal{O}_{\mathcal{E}}^{un}} \xrightarrow{\cdot p} \widehat{\mathcal{O}_{\mathcal{E}}^{un}} \to E_s \to 0$

we can take G_E-invariants (resp φ-inv) to get injections:

$$\left(\widehat{\mathcal{O}_{\mathcal{E}}^{un}}\right)^{G_E} \big/ (p) \hookrightarrow E_s^{G_E} = E \qquad \left(\text{resp. } \left(\widehat{\mathcal{O}_{\mathcal{E}}^{un}}\right)^{\varphi=1} \hookrightarrow E_s^{\varphi=1} = E\right)$$

and these are isomorphisms for dimension reasons.

<u>Theorem</u> (Fontaine): There are covariant (quasi-inverse) equivalences of abelian categories:

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_E) \underset{\mathbb{V}_{\mathcal{E}}}{\overset{\mathbb{D}_{\mathcal{E}}}{\rightleftarrows}} \Phi M_{\mathcal{O}_{\mathcal{E}}}^{et}$$

where:
$$\mathbb{D}_{\mathcal{E}}(V) := \left(\widehat{\mathcal{O}_{\mathcal{E}}^{un}} \otimes_{\mathbb{Z}_p} V\right)^{G_E}$$

$$\mathbb{V}_{\mathcal{E}}(D) = \left(\widehat{\mathcal{O}_{\mathcal{E}}^{un}} \otimes_{\mathcal{O}_{\mathcal{E}}} D\right)^{\varphi=1}$$

These functors are exact, preserve ranks, are compatible with $\otimes$, $\vee$ (for duality $\vee$, note that we only have it for torsion and for torsion free objects)

<u>Lemma</u>: Let R be a complete DVR, with residue field k, and uniformizer p. Let k_s = sep closure of k, $G_k = \mathrm{Gal}(k_s/k)$.

Let $R' := \widehat{R^{un}}$, and G_k acts on R' in the canonical way.

Then for any finitely-generated R'-module M with continuous semilinear (with the natural topology) G_k-action,

1) The R-module M^{G_k} is finitely-generated (over R).

2) The natural map $R' \otimes_R M^{G_k} \xrightarrow{\alpha_M} M$ is an iso.

Proof (of the lemma):

First suppose that M is p^r-torsion, for some $r \geq 1$. We induct on r.

The $r=1$ case is Galois descent, using that M has open stabilizers for G_K.

For $r \geq 2$, let $M' := p^{r-1}M \subseteq M$, $M'' = M/M'$.

$$(*) \qquad 0 \to \underset{\substack{\uparrow \\ p\text{-torsion}}}{M'} \to M \to \underset{\substack{\uparrow \\ p^{r-1}\text{-torsion}}}{M''} \to 0$$

Apply G_K-invariants on $(*)$:

$$0 \to (M')^{G_K} \to M^{G_K} \to (M'')^{G_K} \to H^1(G_K, M') \to \cdots$$

By the case $r=1$, $M' \simeq k_s \otimes_K (M')^{G_K}$ as G_K-modules, additive H90

So $M' \simeq k_s^{\oplus d}$ (as G_K-modules) so $H^1(G_K, M') = H^1(G_K, k_s)^{\oplus d} = 0$.

Since $R \to R'$ is flat, we can extend scalars to R':

$$\begin{tikzcd}
0 \arrow[r] & R' \otimes_R (M')^{G_K} \arrow[r] \arrow[d, "\simeq\ \alpha_{M'}"] & R' \otimes_R M^{G_K} \arrow[r] \arrow[d, "\alpha_M"] & R' \otimes_R (M'')^{G_K} \arrow[r] \arrow[d, "\simeq\ \alpha_{M''}"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow$ α_M is an iso as well, and M is fin-gen as well! $\blacksquare$ (torsion case)

Now, in the general case: for any $m \geq n$:

$$M/p^m \xrightarrow{p^n} M/p^m \to M/p^n \to 0$$

hence get an exact sequence:

$$(M/p^m)^{G_K} \xrightarrow{p^n} (M/p^m)^{G_K} \to (M/p^n)^{G_K} \to 0$$

(cont pf of Lemma)
Next, pass to $\varprojlim_m$, to get:

$$\begin{array}{ccccccc}
\underbrace{\varprojlim \left(M/p^m\right)^{G_K}} & \xrightarrow{p^n} & \varprojlim \left(M/p^m\right)^{G_K} & \longrightarrow & \left(M/p^n\right)^{G_K} & \to & 0 \\
\wr\| \Leftarrow HW & & \wr\| & & \| & & \\
M^{G_K} & \longrightarrow & M^{G_K} & \longrightarrow & \left(M/p^n\right)^{G_K} & \to & 0
\end{array}$$

So we conclude that $M^{G_K}/(p^n) \cong \left(\frac{M}{p^n}\right)^{G_K}$.

$\Rightarrow$ For $n=1$, $M^{G_K}/(p)$ is fin-dim as a k-vectorspace. So by successive approximations, any lift of a k-basis to M^{G_K} spans M^{G_K} over R. Hence M^{G_K} is fin-gen over R.

Now, $\alpha_M : R' \otimes_R M^{G_K} \to M$ is R'-linear of fin.gen R'-modules, so we can check that it is an isomorphism modulo p^n $\forall n$.

But by using $\frac{M^{G_K}}{p^n} \cong \left(M/p^n\right)^{G_K}$, we have $\alpha_M \bmod p^n = \alpha_{(M/p^n)}$ and the RHS is an iso by the torsion case. $\boxtimes$

Consequence: apply the lemma with $R = \mathcal{O}_{\mathcal{E}}$, $R' = \widehat{\mathcal{O}}_{\mathcal{E}}^{un}$, $M = \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} V$, for $V \in \mathrm{Rep}_{\mathbb{Z}_p}(G_E)$:

$$(\dagger) \quad \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}_E(V) = \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} \left(\widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} V\right)^{G_E} \cong \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} V \xrightarrow{\text{mult}} \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} V$$

is an isomorphism, which is G_E-equivariant and φ-eq.

As a consequence, $\varphi_{\mathbb{D}_\mathcal{E}(V)}$ linearizes to an isomorphism, as we can check after tensoring $- \otimes_{\mathcal{O}_\mathcal{E}} \widehat{\mathcal{O}_\mathcal{E}^{un}}$, and then it becomes:

$$\left(\widehat{\mathcal{O}_\mathcal{E}^{un}} \underset{\substack{\mathcal{O}_\mathcal{E}^{un} \\ \uparrow \\ \varphi}}{\otimes} \widehat{\mathcal{O}_\mathcal{E}^{un}}\right) \otimes_{\mathbb{Z}_p} V \longrightarrow \widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes V \quad \text{is an iso}$$

(but this is obvious, because what's in parenthesis is just the multiplication map)

Similarly, we get that $\mathbb{D}_\mathcal{E}$ is exact, rank preserving, compatible with tensor products: these properties can all be checked after a <u>faithfully-flat</u> extension $\mathcal{O}_\mathcal{E} \to \widehat{\mathcal{O}_\mathcal{E}^{un}}$. Then, using $(\dagger)$, it is the same as the functor $V \rightsquigarrow V \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_\mathcal{E}^{un}}$, for which they are obvious.

Now, we want the other half (the other comparison isomorphism).

<u>Lemma</u>: For $D \in \Phi M^{\acute{e}t}_{\mathcal{O}_\mathcal{E}}$, $\mathbb{V}_\mathcal{E}(D)$ is fin. gen over $\mathbb{Z}_p$, and the natural $\widehat{\mathcal{O}_\mathcal{E}^{un}}$-linear G_E, φ-equivariant map:

$$\widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes_{\mathbb{Z}_p} \mathbb{V}_\mathcal{E}(D) = \widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes_{\mathbb{Z}_p} \left(\widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes_{\mathcal{O}_\mathcal{E}} D\right)^{\varphi=1} \hookrightarrow \widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes_{\mathcal{O}_\mathcal{E}} D \xrightarrow{\text{mult}} \widehat{\mathcal{O}_\mathcal{E}^{un}} \otimes_{\mathcal{O}_\mathcal{E}} D$$

Pf: we only deal with the p^r-torsion objects. The rest is mimicking the inverse limit argument that we did in the other lemma.

So let D be p^r-torsion. We induct on r.

For $r=1$, it is already known, as $\Phi M^{\acute{e}t}_E$ is the full subcategory of p-torsion objects in $\Phi M^{\acute{e}t}_\mathcal{E}$.

(cont pf of lemma)

For $r \geq 1$, let again $D' = p^{r-1} D \subseteq D$, $D'' = D/D'$.

$$0 \to D' \to D \to D'' \to 0$$

Extend (flatly) scalars:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & \mathbb{V}_E(D') \arrow[d] \arrow[r] & \mathbb{V}_E(D) \arrow[d] \arrow[r] & \mathbb{V}_E(D'') \arrow[d] \arrow[dddll, bend left] & \\
 0 \arrow[r] & & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D' \arrow[r] \arrow[d, "\varphi-1"] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D \arrow[r] \arrow[d, "\varphi-1"] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D'' \arrow[r] \arrow[d, "\varphi-1"] & 0 \\
 0 \arrow[r] & & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D' \arrow[r] \arrow[d] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D \arrow[r] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D'' \arrow[r] & 0 \\
 & & \mathrm{Coker}(\varphi-1) \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

Claim: $\mathrm{Coker}(\varphi-1) = 0$:

pf/ by the already known comparison iso for ΦM_E^{et}

$$\widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D' \simeq E_s \otimes_E D' \simeq E_s \otimes_{\mathbb{F}_p} \mathbb{V}_E(D')$$

φ, G_E-equivariant

To show $\mathrm{Coker}(\varphi-1) = 0$, it is enough to show that:

$$E_s \otimes_{\mathbb{F}_p} \mathbb{V}_E(D') \xrightarrow{\varphi-1} E_s \otimes_{\mathbb{F}_p} \mathbb{V}_E(D') \quad \text{is surjective.}$$

But this is cos $\varphi - 1 : E_s \to E_s$ is surjective.

And note that $x \mapsto x^p - x$ is surj. because E_s is separably-closed.

$\boxtimes$ (claim)

From the diagram we obtain an exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} \mathbb{V}_{\mathcal{E}}(D') \arrow[r] \arrow[d, "\cong"] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} \mathbb{V}_{\mathcal{E}}(D) \arrow[r] \arrow[d] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathbb{Z}_p} \mathbb{V}_{\mathcal{E}}(D'') \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D' \arrow[r] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D \arrow[r] & \widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D'' \arrow[r] & 0
\end{tikzcd}$$

So the middle map is an iso. ∎

<u>Consequences</u>: $\mathbb{V}_{\mathcal{E}}$ is exact, rank-preserving, $\otimes$-compatible, ...
(Pf: use the faithfully-flat extension $- \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}}^{un}$).

Also, $\mathbb{V}_{\mathcal{E}}(D)$ is continuous as a G_E-rep. By the previous HW, it is equivalent to show that

$$\mathbb{V}_{\mathcal{E}}(D)/p^n \quad \text{has open stabilizers for all } n.$$

By exactness of $\mathbb{V}_{\mathcal{E}}$, we have:

$$\mathbb{V}_{\mathcal{E}}(D)/p^n \cong \mathbb{V}_{\mathcal{E}}(D/p^n)$$

Since $\mathbb{V}_{\mathcal{E}}(D/p^n) = \left(\widehat{\mathcal{O}}_{\mathcal{E}}^{un} \otimes_{\mathcal{O}_{\mathcal{E}}} D/p^n\right)^{\varphi=1} = \left(\widehat{\mathcal{O}}_{\mathcal{E}}^{un}/p^n \otimes_{\mathcal{O}_{\mathcal{E}}/p^n} D/p^n\right)^{\varphi=1} =$

$$\overset{\text{without the hat!}}{=} \left(\mathcal{O}_{\mathcal{E}}^{un}/p^n \otimes_{\mathcal{O}_{\mathcal{E}}/p^n} D/p^n\right)^{\varphi=1}$$

So it's enough to show that $\mathcal{O}_{\mathcal{E}}^{un}/p^n$ has open stabilizers, which is to say that $\mathcal{O}_{\mathcal{E}}^{un}$ has open stabilizers.

But $\mathcal{O}_{\mathcal{E}}^{un} = \varinjlim \mathcal{O}_{\mathcal{E}'}$ where $\mathcal{E}'/\mathcal{E}$ is finite unramified. ∎

Proof (of the Thm of Fontaine)

We pass to φ-invariants on:

$$\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathcal{O}_\mathcal{E}} \mathbb{D}_\mathcal{E}(V) \xrightarrow{\sim} \widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathbb{Z}_p} V \quad \rightsquigarrow \mathbb{V}_\mathcal{E}(\mathbb{D}_\mathcal{E}(V)) \simeq \left(\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes V\right)^{\varphi=1}$$

and $G_\mathcal{E}$-invariants on:

$$\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathbb{Z}_p} \mathbb{V}_\mathcal{E}(D) \xrightarrow{\sim} \widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathcal{O}_\mathcal{E}} D \quad \rightsquigarrow \mathbb{D}_\mathcal{E}(\mathbb{V}_\mathcal{E}(D)) \simeq \left(\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes D\right)^{G_\mathcal{E}}$$

Then, we must show that

$$h : D \to \left(\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathcal{O}_\mathcal{E}} D\right)^{G_\mathcal{E}} \qquad \text{and} \qquad h' : V \to \left(\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathbb{Z}_p} V\right)^{\varphi=1}$$

$$d \mapsto 1 \otimes d \qquad\qquad v \mapsto 1 \otimes v$$

are isomorphisms. It is enough to show that $h \otimes \widehat{\mathcal{O}}_\mathcal{E}^{ur}$ is an iso (resp. $h' \otimes \widehat{\mathcal{O}}_\mathcal{E}^{ur}$), but this becomes an ~~isomorphism~~ the identity map.

for h', this requires another little argument.

[This is how $h : D \to \left(D \otimes_{\mathcal{O}_\mathcal{E}} \widehat{\mathcal{O}}_\mathcal{E}^{ur}\right)^{G_\mathcal{E}}$, $d \mapsto 1 \otimes d$ is an iso, and why also $h' : V \to \left(V \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_\mathcal{E}^{ur}\right)^{\varphi=1}$ is also:

Recall that from a previous lemma, if R = complete DVR, with residue field k, and uniformizer p, and $R' := \widehat{R^{ur}}$ (with action of $G_k = Gal(k^s/k)$), then $R' \otimes_R M^{G_k} \to M$ is an iso.

Apply it for $R = \mathcal{O}_\mathcal{E}$, $R' = \widehat{\mathcal{O}}_\mathcal{E}^{ur}$, $M = D \otimes_{\mathcal{O}_\mathcal{E}} \widehat{\mathcal{O}}_\mathcal{E}^{ur}$, to get:

$$\widehat{\mathcal{O}}_\mathcal{E}^{ur} \otimes_{\mathcal{O}_\mathcal{E}} \left(D \otimes_{\mathcal{O}_\mathcal{E}} \widehat{\mathcal{O}}_\mathcal{E}^{ur}\right)^{G_\mathcal{E}} \xrightarrow{\simeq} D \otimes_{\mathcal{O}_\mathcal{E}} \widehat{\mathcal{O}}_\mathcal{E}^{ur}.$$

This is the scalar extension of h to $\widehat{\mathcal{O}}_\mathcal{E}^{ur}$, which is faithfully flat$/\mathcal{O}_\mathcal{E}$.

(for h)

As for h', first note that the following seq. is exact:

$$(*) \qquad 0 \to \mathbb{Z}_p \to \widehat{\mathcal{O}_{\mathcal{E}}^{un}} \xrightarrow{\varphi-1} \widehat{\mathcal{O}_{\mathcal{E}}^{un}} \to 0$$

(we know that $\mathbb{Z}_p$ is the ker of $\varphi-1$. So just need to see surjectivity. But this can be checked mod p (by successive approximation). So we need $\begin{array}{ccc} E_s & \xrightarrow{\varphi-1} & E_s \\ x & \longmapsto & x^p - x \end{array}$ surjective. But E_s is separably-closed, so done).

Apply $- \otimes_{\mathbb{Z}_p} V$ to $(*)$, to get:

$$\mathrm{Tor}^1_{\mathbb{Z}_p}(\widehat{\mathcal{O}_{\mathcal{E}}^{un}}, V) \to V \xrightarrow{h'} V \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathcal{E}}^{un}} \xrightarrow{\varphi-1} V \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathcal{E}}^{un}} \to 0$$

Since $\widehat{\mathcal{O}_{\mathcal{E}}^{un}}$ is $\mathbb{Z}_p$-flat, $\mathrm{Tor}^1_{\mathbb{Z}_p} = 0$, so h' maps isomorphically

$V \to \ker \varphi - 1$. ✓.

• <u>From $\mathrm{Rep}_{\mathbb{Z}_p}$ to $\mathrm{Rep}_{\mathbb{Q}_p}$</u>:

<u>Notation</u>: We will now be more careful for how we write φ:

So we have (E, φ_E), $(\mathcal{E}, \varphi_{\mathcal{E}})$, $(\mathcal{O}_{\mathcal{E}}, \varphi_{\mathcal{O}_{\mathcal{E}}})$, ...

<u>Def</u>: For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_E)$, let:

$$D_{\mathcal{E}}(V) := \left(\widehat{\mathcal{E}^{un}} \otimes_{\mathbb{Q}_p} V\right)^{G_E} \quad (\text{: an } \mathcal{E}\text{-vectorspace, not fin-dim'l a priori!}).$$

This comes equipped with a $\varphi_{\mathcal{E}}$-semilinear endomorphism, induced from $\varphi_{\widehat{\mathcal{E}^{un}}} \otimes 1$.

Prop: The $\mathcal{E}$-vectorspace $\mathbb{D}_{\mathcal{E}}(V)$ is fin. dim'l $/\mathcal{E}$, with $\dim_{\mathcal{E}} \mathbb{D}_{\mathcal{E}}(V) = \dim_{\mathbb{Q}_p} V$, and the linearization of $\varphi_{\mathbb{D}_{\mathcal{E}}(V)}$ is an isomorphism.

Moreover, there exists a $\varphi_{\mathbb{D}_{\mathcal{E}}(V)}$-stable lattice (over $\mathcal{O}_{\mathcal{E}}$), $L \subseteq \mathbb{D}_{\mathcal{E}}(V)$, such that $(L, \varphi_{\mathbb{D}_{\mathcal{E}}(V)}|_L) \in \Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$

Pf: By a HW exercise, there exists $\Lambda \in Rep_{\mathbb{Z}_p}(G_E)$ free $/\mathbb{Z}_p$, such that $V = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (as G_E-modules).

So, by definition of $\mathbb{D}_{\mathcal{E}}$, we have:

$$\mathbb{D}_{\mathcal{E}}(V) = \mathbb{D}_{\mathcal{E}}(\Lambda) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{E} = \mathbb{D}_{\mathcal{E}}(\Lambda)[\tfrac{1}{p}]$$

Set $L := \mathbb{D}_{\mathcal{E}}(\Lambda)$, and note that $L \in \Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$.

All the proposition follows then by what we've done for $Rep_{\mathbb{Z}_p}(G_E)$.

We now want to define $\Phi M^{ét}_{\mathcal{E}}$.

"Naive guess": An étale φ-module over $\mathcal{E}$ is a fin-dim'l $\mathcal{E}$-vectorspace with $\varphi_{\mathcal{E}}$-semilinear ~~linearizes to~~ endo. $\varphi_D : D \to D$ that linearizes to $\cong$.

Example (this is not a good definition): A fin-dim $\mathcal{E}$-vectorspace D, together with φ_D, with no stable lattice:

$D := \mathcal{E}$, $\varphi_D := \frac{1}{p}\varphi_{\mathcal{E}}$. This is $\varphi_{\mathcal{E}}$-semilinear and it linearizes to iso.

The $\mathcal{O}_{\mathcal{E}}$-lattices in $\mathcal{E}$ are of the form $\mathcal{O}_{\mathcal{E}} \cdot x$, $x \neq 0$.

But $\varphi_D(x) = \frac{1}{p}\varphi_{\mathcal{E}}(x) = \frac{1}{p} \frac{\varphi_{\mathcal{E}}(x)}{x} \cdot x$!

Claim: $\varphi_{\mathcal{E}}(x)/x \in \mathcal{O}_{\mathcal{E}}^{\times}$:

Claim: $\frac{\varphi_\mathcal{E}(x)}{x} \in \mathcal{O}_\mathcal{E}^\times$

pf/ enough to show that ~~for~~ $x \in \mathcal{O}_\mathcal{E}^\times$

(because $\frac{\varphi_\mathcal{E}(p^r x)}{p^r x} = \frac{\varphi_\mathcal{E}(x)}{x}$).

But $\varphi_\mathcal{E}(u) = u^p + py$ for some $y \in \mathcal{O}_\mathcal{E}$.

Hence $|\varphi_\mathcal{E}(u)| = 1 = |u|$ ✓. ▣ (claim).

// (Example)

Def: An étale φ-module over $\mathcal{E}$ is a finite-dimensional $\mathcal{E}$-vectorspace D, with a $\varphi_\mathcal{E}$-semilinear endomorphism φ_D, that linearizes to iso, together with the existence of a φ_D-stable $\mathcal{O}_\mathcal{E}$-lattice $L \subseteq D$ s.t. $(L, \varphi_D|_L) \in \Phi M^{\text{ét}}_{\mathcal{O}_\mathcal{E}}$

Rk: We don't need to ask for φ_D to linearize to an iso, as it does automatically given the lattice condition.

Denote the category of these objects by $\Phi M^{\text{ét}}_\mathcal{E}$.

We have a functor:

$$\Phi M^{\text{ét}}_{\mathcal{O}_\mathcal{E}} \longrightarrow \Phi M^{\text{ét}}_\mathcal{E}$$

$$D \longmapsto D[\tfrac{1}{p}] = D \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{E} \qquad L \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{E}$$

Clearly, $\mathrm{Hom}_{\Phi M^{\text{ét}}_{\mathcal{O}_\mathcal{E}}}(L, L')[\tfrac{1}{p}] \xrightarrow{\cong} \mathrm{Hom}_{\Phi M^{\text{ét}}_\mathcal{E}}(\overline{L[\tfrac{1}{p}]}, L'[\tfrac{1}{p}])$

So $\Phi M^{\text{ét}}_\mathcal{E}$ = isogeny category of $\Phi M^{\text{ét}}_{\mathcal{O}_\mathcal{E}}$.

Consequence: $\Phi M^{\text{ét}}_\mathcal{E}$ is an abelian category.

Theorem: The functors

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_E) \underset{\mathbb{V}_\mathcal{E}}{\overset{\mathbb{D}_\mathcal{E}}{\rightleftarrows}} \Phi M_\mathcal{E}^{ét}$$

given by:

$$\mathbb{D}_\mathcal{E}(V) = \left(\hat{\mathcal{E}}^{un} \otimes_{\mathbb{Q}_p} V\right)^{G_E}$$

$$\mathbb{V}_\mathcal{E}(D) = \left(\hat{\mathcal{E}}^{un} \otimes_{\mathcal{E}} D\right)^{\varphi=1}$$

are quasi-inverse exact equivalences of categories, which are $\otimes$-compatible, dimension-preserving, and compatible with duality.

pf/ By the HW, pick $\Lambda \subseteq V$ that is G_E-stable. By definition, then,

$$\mathbb{D}_\mathcal{E}(V) = \overbrace{\mathbb{D}_{\mathcal{O}_\mathcal{E}}(\Lambda)}^{L} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{E}$$ and, on the other hand, for $D \in \Phi M_\mathcal{E}^{ét}$,

we have a φ_D-stable L, and $\mathbb{V}_\mathcal{E}(D) = \mathbb{V}_\mathcal{E}(L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Everything follows from the integral case (3.2.4), applied to the full subcategories of free objects. ∎

We won't do more on this, but it should be regarded as a nice "toy example" of what follows.

Turn the page!

Fix a p-adic field K, and let $G_K = \mathrm{Gal}(\bar{K}/K)$.

Recall: $B_{HT} = \bigoplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)$, which is a graded $\mathbb{C}_K$-algebra, with a semilinear G_K-action, respecting the grading and ring structure.

We had proven that

$$\left\{ \begin{matrix} \text{Hodge-Tate} \\ \mathbb{C}_K\text{-reps} \end{matrix} \right\} \overset{D}{\underset{V}{\rightleftarrows}} \mathrm{Gr}_{K,f}$$ was an equivalence of categories.

However, we then proved that, if we look at $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, then the functor:

$$\underset{\mathrm{Rep}_{\mathbb{Q}_p}(G_K)}{\overset{\cup\!\shortmid}{}}\ \mathrm{Rep}_{HT}(G_K) \longrightarrow \mathrm{Gr}_{K,f}$$ is not full.

Goal: Refine $\mathrm{Rep}_{HT}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ to a category that includes all reps coming from geometry (which, by Faltings' thm, are HT) and which admits a fully-faithful $\otimes$-compatible functor to a nice category of semi-linear algebra data.

Actually, we want some period ring B, with G_K action + ... s.t the functor looks like $V \rightsquigarrow (B\otimes V)^{G_K}$.
Moreover, we want also to recover B_{HT} from B.

Motivation:
Faltings' Thm: X/K smooth proper, then $H^n_{Hodge}(X) := \bigoplus_q H^{n-q}(X, \Omega^q_{X/K})$ is isomorphic to $\left(B_{HT} \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)\right)^{G_K}$

We should refine $H^n_{Hodge}(X)$.

Let $H^n(X) := H^n_{dR}(X/K)$. This is a finitely-dimensional K-vs, equipped with a collection of subspaces $\{Fil^i H^n(X)\}$,

s.t ~~$Fil^i(H^n(X))$~~

$$H^n(X) = Fil^0(H^n(X)) \supseteq Fil^1 H^n(X) \supseteq \cdots \supseteq Fil^{n+1} H^n(X) = 0$$

and $$Fil^q H^n(X) \big/ Fil^{q+1} H^n(X) = H^{n-q}(X, \Omega^q_{X/K})$$

(i.e. $gr^\bullet\left(H^n_{dR}(X/K)\right) = H^n_{Hodge}(X)$).

So we will find B_{dR}, with $D_{dR}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$,

s.t $D_{dR}\left(H^n_{ét}(X/\bar{K}, \mathbb{Q}_p)\right) = H^n_{dR}(X/K)$

Def: A <u>filtered module</u> over a (commutative) ring R is an R-module M, equipped with a collection $\{Fil^i M\}_{i\in\mathbb{Z}}$ of R-submodules, that is decreasing (i.e $Fil^{i+1}M \subseteq Fil^i M$)

We say that the filtration is <u>exhaustive</u> if $\bigcup_{i\in\mathbb{Z}} Fil^i M = M$,

and we say that it is <u>separated</u> if $\bigcap_i Fil^i M = \{0\}$

Def: If K is (a field, or a ring) a <u>filtered K-algebra</u> is a K-algebra A, equipped with a filtration $\{Fil^i A\}_{i\in\mathbb{Z}}$, $Fil^i A$ a ~~K-subalgebra~~ K-vectorspaces $\forall i$, such that $(Fil^i A)\cdot(Fil^j A) \subseteq Fil^{i+j} A$, and such that $1 \in Fil^0 A$.

Given a filtered module (resp k-algebra), we associate to it the associated ~~modul~~ graded module (resp k-vectorspace) as:

$$gr^\bullet(M) := \bigoplus_{i\in\mathbb{Z}} \frac{Fil^i M}{Fil^{i+1} M} \qquad \text{(or resp. replace } M \text{ by } A\text{)}.$$

A filtered ring is a filtered $\mathbb{Z}$-algebra.

We want to define B_{dR} so to have a functor:

$$D_{dR} : \mathrm{Rep}_{dR}(G_K) \longrightarrow Fil_{K,f}$$
$$V \rightsquigarrow \left(B_{dR} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$$

So B_{dR} should (at least!) be a filtered K-algebra, with a G_K-action.
We also want to recover B_{HT} from B_{dR}, via $B_{HT} = gr^\bullet(B_{dR})$.

Q: How to construct filtered K-algebras?

<u>Example</u>: Let R be a dvr, with a uniformizer π, and residue field k.
Set $A := Q(R) = \mathrm{Frac}(R)$. Set then $A^i := \pi^i R \subseteq A$.
This makes A into a filtered object. This filtration is both exhaustive and separated, with associated grading:

$$gr^\bullet(A) = \bigoplus_{i\in\mathbb{Z}} \frac{\pi^i}{\pi^{i+1}} \simeq k\left[t, \tfrac{1}{t}\right]$$

the iso depends on the choice of π.

Based on the example, we should look for a dvr B_{dR}^+ with G_K-action, residue field $\mathbb{C}_K$ ~~and~~ ~~$\mathbb{C}_K$~~ and in a G_K-equivariant way, such that $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathbb{C}_K(1)$ ($\mathfrak{m}$ = max. ideal of B_{dR}). Then $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq \mathbb{C}_K(i)$, so if $B_{dR} = \mathrm{Frac}(B_{dR}^+)$, then $gr^\bullet(B_{dR}) = B_{HT}$.

<u>A first guess:</u> $B_{dR}^{+,\text{naive}} := \mathbb{C}_K[\![t]\!]$, with G_K-action via

$$g\cdot t := \chi(g)\cdot t$$

(so that $(t)/(t^2) \simeq \mathbb{C}_K(1)$ as G_K-modules).

We'll see that we don't get anything new with that: a rep. is HT $\Leftrightarrow$ it is dR^{naive}.

The reason is that this ring is too close to be graded.

<u>Better</u>: use Witt vectors:

$$\left\{\begin{array}{c}\text{perfect field}\\ k \text{ of}\\ \text{characteristic } p\end{array}\right\} \xrightarrow[\text{functorial}]{\text{Witt vectors}} \left\{\begin{array}{c}\text{complete dvr } W(k) \text{ of char } 0\\ \text{with uniformizer } p,\\ \text{and residue field } k\end{array}\right\}$$

$$\mathbb{F}_p \rightsquigarrow \mathbb{Z}_p = W(\mathbb{F}_p).$$

$$\mathbb{F}_q \rightsquigarrow W(\mathbb{F}_q) = \text{ring of integers in "}\mathbb{Q}_q\text{"}$$

<u>Rk</u>: We can't immediately do it with $k = \mathbb{C}_K$ (b/c $\mathbb{C}_K$ has char. 0!).

But $\mathcal{O}_{\mathbb{C}_K} = \varprojlim_n \mathcal{O}_{\mathbb{C}_K}/(p^n)$ ← p-power torsion...

However, $\mathcal{O}_{\mathbb{C}_K}/(p)$ is <u>not</u> perfect (b/c $x \mapsto x^p$ is not an automorphism — it is not injective —)

The Witt-vectors of non-perfect rings are pretty bad. So what we will do is to form a ring R_K, which will be a "perfect" replacement for $\mathcal{O}_{\mathbb{C}_K}/(p)$.

R_K is a height-1 valuation ring of equicharacteristic p, with algebraically-closed fraction field ($\Rightarrow$ perfect), equipped with a G_K-action.

Once we have R_K, we will consider $W(R_K)[\frac{1}{p}]$, together with a surjection:

$$\theta_K : W(R_K)[\tfrac{1}{p}] \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}[\tfrac{1}{p}] = \mathbb{C}_K$$

Fact: $\ker(\theta_K)$ = principal, nonzero. So we will set $B_{dR}^+ :=$ $\ker(\theta_K)$-adic completion of $W(R_K)[\frac{1}{p}]$

Review of Witt vectors (Serre, "Local Fields", II §6)

Motivation: $k = \mathbb{F}_q$, A = valuation ring of the unique finite unramified ~~ff~~ extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ ~~(A = A)~~

We have the Teichmüller map:

$$[\cdot] : k \to A \quad \text{sending } 0 \mapsto 0, \text{ and } a \in k^\times \text{ to the}$$

unique $\tilde{a} \in \mu_{q-1}(A)$ s.t. $\tilde{a} \bmod p = a$.

$[\cdot]$ is multiplicative, and it's the unique set-theoretic section to the reduction map which is compatible with pth power.

Note that $\alpha \in A$ can be uniquely written

$$\alpha = \sum_{i \geq 0} [a_i] p^i, \quad a_i \in k.$$

Q: If $\alpha' = \sum_{i \geq 0} [a_i'] p^i$, how to write $\alpha + \alpha'$ and $\alpha\alpha'$ in this form?

Ans: Fix $X_0, X_1, \ldots$ an infinite # of variables. The Witt polynomials are:

$$W_0 = X_0, \quad W_1 = X_0^p + pX_1, \quad \ldots \quad W_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}}.$$

(note that can solve for X_i from the W's).

Theorem: For any polynomial $\Phi \in \mathbb{Z}[X, Y]$, there are unique $\Phi_0, \Phi_1, \ldots$ $\Phi_i \in \mathbb{Z}[X_0, X_1, \ldots, Y_0, Y_1, \ldots]$ s.t.:

$$\Phi(W_n(X_0, \ldots), W_n(Y_0, \ldots)) = W_n(\Phi_0, \ldots, \Phi_n)$$

(cont. motivation)

In particular, for $\bullet\ \Phi := X+Y$, let $(S_i) := (\Phi_i)$

$\bullet\ \Phi := X \cdot Y$, let $(P_i) := (\Phi_i)$

Def: For any ring A, define the Witt vectors are:

$W(A) := A^{\mathbb{N}}$ as a set, with operations:

if $\alpha = (a_0, \ldots)$, $\beta = (b_0, \ldots) \in W(A)$, then:

$$\alpha + \beta := \left(S_0(\underline{a}, \underline{b}), S_1(\underline{a}, \underline{b}), \ldots \right)$$
$$\alpha \cdot \beta := \left(M_0(\underline{a}, \underline{b}), M_1(\underline{a}, \underline{b}), \ldots \right)$$

Remark: S_i and M_i only involve the variables $\{X_0, \ldots, X_i, Y_0, \ldots, Y_i\}$!

Theorem: $W(A)$ is a commutative ring with unit $(1, 0, 0, \ldots)$

Moreover, there is a natural map (of sets):

$$(W_n) : W(A) \to A^{\mathbb{N}}$$

which is a ring homomorphism ($A^{\mathbb{N}}$ with the natural ring structure), which is an isomorphism if p is invertible in A.

Def: The Witt vectors of length n is the ring-quotient $W_n(A)$ of $W(A)$, given by projection to the first n components (makes sense because of the above remark).

N.B: The operation W is a functor on rings, because P_i, S_i have coefficients in $\mathbb{Z}$, so that they are "universal".

The multiplication-by-p map:

$$p : W(A) \longrightarrow W(A)$$

is computed to be $(a_0, a_1, \dots) \mapsto (0, a_0^p, a_1^p, \dots)$

So if A is a perfect $\mathbb{F}_p$-algebra, then:

$$p^n : W(A) \to W(A) \text{ is } \underline{\text{injective}}, \text{ with image} = (\underbrace{0, 0, \dots 0}_{n}, *, *, *, \dots)$$

Hence, again if A is a perfect $\mathbb{F}_p$-algebra,

$$W(A)/p^n W(A) \cong W_n(A)$$

So $\varprojlim W(A)/p^n W(A) = \varprojlim W_n(A) = W(A)$, so $W(A)$ is p-adically separated and complete, and $W(A)/pW(A) \cong A$.

<u>Def</u>: A <u>p-ring</u> is a ring B, equipped with a filtration by ideals $b_1 \supseteq b_2 \supseteq \dots$ such that $b_n b_m \subseteq b_{n+m}$, and such that B/b_1 is a perfect $\mathbb{F}_p$-algebra (so $p \in b_1$), and such that B is separated and complete for the topology defined by the $\{b_i\}_{i \geq 1}$

We say that B is a <u>strict p-ring</u> if $b_i = p^i B$, and $p : B \to B$ is <u>injective</u>.

<u>Example</u>: $W(A)$ is a strict p-ring, if A is a perfect $\mathbb{F}_p$-algebra.

Prop: If $\mathcal{B}$ is any strict p-ring, then the natural reduction map

require homs to be continuous there

$(*)$: $\mathrm{Hom}(\mathcal{B}, B') \longrightarrow \mathrm{Hom}(\mathcal{B}/p, B'/\mathfrak{b}_1)$ is

bijective for all p-rings B'.

Cor: $\mathcal{B} \simeq W(\mathcal{B}/p)$ for any strict p-ring $\mathcal{B}$.

Pf (of cor): By the prop, $W(\mathcal{B}/p)$ and $\mathcal{B}$ satisfy the same universal property.

In fact, the map
$$W(\mathcal{B}/p) \longrightarrow \mathcal{B}$$
$$(\beta_0, \beta_1, \dots) \longmapsto \sum_{n \geq 0} [\beta_n^{p^{-n}}] \cdot p^n$$
is an isomorphism of rings.

($[\cdot] : \mathcal{B}/p \to \mathcal{B}$ is the Teichmüller map that we will see in the proof of prop).

Lemma: If B is a p-ring, then there is a unique section of sets:
$$r_B : B/\mathfrak{b}_1 \to B$$
to the reduction map, such that $r_B(x^p) = r_B(x)^p$.
Moreover, r_B is multiplicative, and $r_B(1) = 1$.

Pf: Uses the claim: if $x \equiv y \bmod \mathfrak{b}_i$, then $x^p \equiv y^p \bmod \mathfrak{b}_{i+1}$.
($x = y + z$, with $z \in \mathfrak{b}_i$. Then $x^p = y^p + \underset{\mathfrak{b}_1}{p}\,\underset{\mathfrak{b}_i}{z(\dots)} + \underset{\mathfrak{b}_{pi}}{z^p}$ ✓)

Define then $r_B(x) := \lim_{n \to \infty} \left(\widehat{x^{p^{-n}}}\right)^{p^n}$ (where $\wedge$ is any lift to B)

The limit exists because $\left(\widehat{x^{p^{-n'}}}\right)^{p^{n'-n}} \equiv \widehat{x^{p^{-n}}} \bmod \mathfrak{b}_1$.

By the claim, we get (taking p^n-powers) $\left(\widehat{x^{p^{-n'}}}\right)^{p^{n'}} \equiv \left(\widehat{x^{p^{-n}}}\right)^{p^n} \bmod \mathfrak{b}_{n+1}$.

So the limit exists because the sequence is Cauchy.

(cont pf of lemma)

Independence of choice $\widehat{x^{p^{-n}}}$:

$$\left(\widehat{x^{p^{-n}}}\right)^{p^n} \equiv \left(\widetilde{x^{p^{-n}}}\right)^{p^n} \mod b_{n+1} \quad \text{(for two lifts)} \quad ✓.$$

Clearly, r_B is p-power compatible.

<u>Uniqueness</u>: If r_B' is any p-power compatible section, then we could twice for defining r_B: $\widehat{x^{-p^n}} := r_B'(x^{-p^n})$.

Then $r_B(x) = \lim_{n\to\infty}\left(r_B'(x^{-p^n})\right)^{p^n} = \lim_{n\to\infty} r_B'(x) = r_B'(x)$ ✓.

It is clear that $r_B(1)=1$. For multiplicability, lift a product ~~to the~~ as:

$$\widehat{(xy)^{p^{-n}}} := r_B(x^{-p^n})\cdot r_B(y^{-p^n}) \quad ✓.$$

<u>Consequence (of the lemma)</u>

If B is a strict p-ring, then any $\beta\in B$ can be uniquely represented by $\beta = \sum_{i\geq 0} r_B(b_i)p^i$, $b_i \in b_1$.

<u>Proof (of the Prop)</u>: First, injectivity:

If $h\in \mathrm{Hom}(B, B')$, then $h\left(\sum_i r_B(b_i)p^i\right) = \sum_i h(r_B(b_i))p^i =$

$= \sum_i r_{B'}(\bar h(b_i))\, p^i$. Hence $\bar h$ determines h $\Rightarrow$ (*) is injective.

<u>Surjectivity</u>: Given $\bar h$, want to produce h. Define

$$h\left(\sum r_B(b_i)p^i\right) := \sum r_{B'}(\bar h(b_i))\, p^i$$

Must check that this is a ring map.

(cont pf of Prop).

To check: $h\left(\sum r_B(b_i)p^i\right) := \sum r_{B'}(\bar{h}(b_i))\,p^i$ is a ring map.

It's easy to check that it preserves 0 and 1.

<u>Claim</u>. For any p-ring C, and $c = \sum r_C(c_i)p^i$, $c' = \sum r_C(c_i')p^i$,

then $$c + c' = \sum r_C\left(S_i(c_0, c_1^p, \dots, c_i^{p^i}, c_0', c_1'^p, \dots, c_i'^{p^i})^{p^{-i}}\right)p^i$$

$$c \cdot c' = \sum r_C\left(P_i(\qquad)^{p^{-i}}\right)p^i$$

where S_i, P_i are the witt sum / product polynomials (coeffs in $\mathbb{Z}$!).

Pf/ Look it up in Serre's Local Fields, II §5 Prop 9. ■

■ (Prop)

<u>Application</u>: Let K be a p-adic field, with residue field k.

Consider $\mathcal{O}_K$, with $b_i := m^i = (\pi^i)$, which is a p-ring.

So we get a map $W(k) \hookrightarrow \mathcal{O}_K$ which is injective.

(this lifts, by the universal property, the map $k \xrightarrow{\sim} \mathcal{O}_K/m$).

Df: Let $K_0 := \mathrm{Frac}(W(k))$. ~~It~~ is the maximal unramified subextension of K. (by definition ??) ← no! it's a "lemma": K/K_0 is finite and totally ramified (prove it!).

Let A be any $\mathbb{F}_p$-algebra. We can associate to it:

$$R(A) := \varprojlim_{x \mapsto x^p} A = \left\{(x_0, x_1, \dots) \in \prod_{\mathbb{Z}_{\geq 0}} A : x_{i+1}^p = x_i \ \forall i\right\}.$$

(with ring-structure induced by the ring structure in $\prod_{\mathbb{Z}_{\geq 0}} A$). ↓

Claim: $R(A)$ is perfect (i.e $x \mapsto x^p$ is bijective)

Pf/ Inj: $(x_0, x_1, \ldots)^p = (x_0^p, x_0, x_1, x_2, \ldots)$ ✓.

Surj: $(x_1, x_2, \ldots)^p = (x_1^p, x_1, x_2, \ldots)$ ∎

Moreover, $R(A)$ is final among the perfect $\mathbb{F}_p$-algebras with maps to A,

via
$$R(A) \longrightarrow A$$
$$(x_i) \longmapsto x_0$$

The given map $R(A) \to A$ is an iso whenever A is perfect.

N.B: $R(A)$ is covariant (functorial) in A.

Def: If K is a p-adic field, define then:

$$R := R_K := R\left(\mathcal{O}_{\mathbb{C}_K}/(p)\right) \left(= R\left(\mathcal{O}_{\bar{K}}/(p)\right)\right).$$

equipped with a G_K-action by the functoriality of R.

Note: R is canonically a $\bar{k}$-algebra.

Pf/ Let K_0 be the maximal unramified subextension of K. Since

$[K:K_0] < \infty$, $\mathbb{C}_{K_0} \cong \mathbb{C}_K$, and $W(\bar{k})$ = valuation ring of $\widehat{K_0^{un}}$

But $\widehat{K_0^{un}} \hookrightarrow \widehat{\bar{K}} = \mathbb{C}_K$, so $\mathbb{C}_K$ is a $W(\bar{k})$-algebra, and

hence $\mathcal{O}_{\mathbb{C}_K}/(p)$ is a $W(\bar{k})/(p) = \bar{k}$-algebra.

Now apply R, and note that $R(\bar{k}) = \bar{k}$. ∎

Rk: There is no quotient of $\mathcal{O}_{\mathbb{C}_K}$ which is perfect, so that this is not a p-ring!

There is a G_K-equivariant surjection:

$$R \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/(p)$$
$$(x_i) \longmapsto x_0$$

If $\mathcal{O}_{\mathbb{C}_K}/(p)$ was perfect, then we would get $W(R) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$ which, after inverting p, would $W(R)[\frac{1}{p}] \twoheadrightarrow \mathbb{C}_K$.

But: $\mathcal{O}_{\mathbb{C}_K}/(p)$ is not perfect. So we need to do something else.

We would like to show that R is a valuation ring. For that, we need first:

Prop: Let A be a p-adically separated and complete ring. Then the map of sets (which is multiplicative):

$$\varprojlim_{x \mapsto x^p} A \longrightarrow R(A/pA)$$
$$(x^{(n)})_{n\geq 0} \longmapsto (x^{(n)} \bmod p)_{n\geq 0}$$

is bijective. (the LHS is not a priori a ring, as $x \mapsto x^p$ is not a ring map).

The inverse is given by:

$$(x_n) \longmapsto (x^{(n)}) \quad , \quad x^{(n)} := \lim_{m\to\infty} \left(\widehat{x_{n+m}}\right)^{p^m}$$

Pf. The limit exists by the same argument as before: $\left(\widehat{x_{n+m'}}\right)^{p^{m'-m}} \equiv \left(\widehat{x_{n+m}}\right) \pmod p$

So, taking p^m-th powers, get congruence $\pmod{p^{m+1}}$.

The same argument as before shows the independence of liftings.

This gives an inverse by uniqueness of liftings. ∎

Consequence:

1) $R(A/pA)$ is a domain if A is.

2) $\varprojlim_{x \mapsto x^p} A$ has a ring structure. Concretely, for $(x^{(n)}), (y^{(n)}) \in \varprojlim A$

then
$$(x^{(n)})(y^{(n)}) = \left(x^{(n)} y^{(n)} \right)_{n \geq 0}$$
$$(x^{(n)}) + (y^{(n)}) = \left(\lim_{m \to \infty} \left(x^{(n+m)} + y^{(n+m)} \right)^{p^m} \right)_{n \geq 0}$$

Back to our p-adic field K, $R = R_K = R(\mathcal{O}_{\mathbb{C}_K}/(p))$, then we have

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K} \overset{\cong}{\hookrightarrow} R$$

The LHS ~~can~~ be given a valuation as follows:

Let $|\cdot|_p : \mathbb{C}_K \to p^{\mathbb{Q}} \cup \{\infty\}$ be the usual abs. value, normalized to $|p|_p = \frac{1}{p}$.

Then define: $|\cdot|_R : R \to p^{\mathbb{Q}} \cup \{\infty\}$ by:

$$|(x^{(n)})|_R := |x^{(0)}|_p$$

Lemma: This is an absolute value in R. If v_R is the associated valuation on $\mathrm{Frac}(R)$ (extending $-\log|\cdot|_R$), then R is the valuation ring of $\mathrm{Frac}(R)$. Also, R is v_R-adically separated and complete.

Moreover, the map $\bar{k} \to R$ maps isomorphically ~~onto~~ the residue field.

Proof (of the lemma) :

1) $x \in \varprojlim \mathcal{O}_{\mathbb{C}K}$ is 0 iff $x^{(0)} = 0$. So $|x|_R = 0 \Leftrightarrow x = 0$.

2) Clear that $|\cdot|_R$ is multiplicative.

3) Want $|x+y|_R \leq \max\{|x|_R, |y|_R\}$

For this, it suffices to treat the case $|y^{(0)}|_p \geq |x^{(0)}|_p$ and $x, y \neq 0$.

In that case, $|x^{(n)}|_p = |x^{(0)}|_p^{p^{-n}} \leq |y^{(0)}|_p^{p^{-n}} = |y^{(n)}|_p$.

So $\frac{x^{(n)}}{y^{(n)}} \in \mathcal{O}_{\mathbb{C}K}$ $\forall n$, hence if $z = \left(\frac{x^{(n)}}{y^{(n)}}\right)$, we have:

$x = yz$, so $|x+y|_R = |y(1+z)|_R = |y|_R |1+z|_R \leq |y|_R$ ✓.

4) For v_R-adic completedness + separatedness, note:

$v_p(x^{(0)}) = v_p(x^{(n)\, p^n}) = p^n v_p(x^{(n)})$,

So ~~$x \in R$~~ $v_R(x) \geq p^n \Leftrightarrow x^{(n)} \equiv 0 \pmod p$.

Hence $\{x \in R : v_R(x) \geq p^n\} = \ker \theta_n$, where $\theta_n : R \to \mathcal{O}_{\mathbb{C}K}/p$
$(x^{(n)}) \mapsto x^{(n)} \bmod p$

But $\operatorname{Ker} \theta_n = \left\{ (x_m) \in \prod \mathcal{O}_{\mathbb{C}K}/(p) \;:\; \begin{array}{l} x_{m+1}^p = x_m \\ x_\ell = 0 \text{ for } \ell \leq n \end{array} \right\}$

which is a basis of open neighborhoods of 0 in the product topology. (top. on $\mathcal{O}_{\mathbb{C}K}/(p)$ is given by the discrete topology)

This gives the proof, as the product topology is separated + complete, and R is closed inside the product.

∎

Thm: Frac(R) is algebraically closed.

Pf: Since R is perfect, it's enough to show that Frac(R) is ~~separably~~ separably closed.

If $P(x) \in R[x]$ monic, separable, of positive degree, then P has a root in R (by using v_R to pass from Frac(R) to R). ← Gauss lemma.

Since P is separable, $(P, P') = (1)$ so $\exists\, U, V \in R[x]$ s.t.

$UP + VP' = r \in R \setminus \{0\}, \quad v_R(r) \in \mathbb{Z}_{\geq 0}$ (? R is not discrete?)

Let $m := v_R(r)$. We will construct a root of P by successive approximation.

Lemma (Hensel's lemma): If $n \geq 2m+1$ and $\xi \in R$ is s.t.

$$v_R(P(\xi)) \geq n, \text{ then } \exists\, y \in R,\ v_R(y) \geq n-m, \text{ s.t.}$$

$$v_R(P(\xi + y)) \geq n+1$$

Granting the lemma, we just need to find $\rho_1 \in R$ with $v_R(P(\rho_1)) \geq 2m+1$; applying the lemma successively, with $n = 2m+1$, $\xi = \rho_1$, get y_1 with $v_R(y_1) \geq m+1$, and $v_R(P(\rho_1 + y_1)) \geq 2m+2$.

Setting $\rho_2 := \rho_1 + y_1$, apply lemma again with $n = 2m+2$, ... and use that R is complete and separated to get $\rho = \lim_i \rho_i$, $v_R(P(\rho_i)) \to \infty$, so $P(\rho) = 0$.

To find ρ_1: consider $\theta_j : R[x] \to \mathcal{O}_{\mathbb{C}_K/(p)}[x]$ be the proj. onto the j^{th} factor $\left(R \subseteq \prod_{i \geq 0} \mathcal{O}_{\mathbb{C}_K}/(p)\right)$.

Then θ_j is a ring map so it carries P to a monic polynomial $Q_j \in \mathcal{O}_{\mathbb{C}_K}/(p)[x]$ of positive degree. Let $z \in \mathcal{O}_{\mathbb{C}_K}/(p)[x]$ be a root of Q_j (use that $\mathcal{O}_{\mathbb{C}_K}/(p)$ is algebraically closed), and pick $\rho_{1j} \in \theta_j^{-1}(z)$ (use that $\mathcal{O}_{\mathbb{C}_K}/(p)$ is alg. closed ~~because~~ to see that $R \to \mathbb{C}_K/(p)$ is surjective).

(cont pf of Thm)

Then $\theta_j(P(\rho_{1,j})) = Q_j(z) = 0$, so $P(\rho_{1,j}) \in \operatorname{Ker}\theta_j$.

Recall that $\operatorname{Ker}\theta_j = \{x \in R : v_R(x) \geq p^j\}$.

So choose $\rho_1 := \rho_{1,j}$ for any j such that $p^j \geq 2m+1$. $\blacksquare$ (Thm)

<u>Pf of the lemma (Hensel's lemma):</u>

Given n, and ξ, we want to find y.

$$P(X+Y) = P(X) + Y P'(X) + \sum_{j \geq 2} Y^j P_j(X) \quad , \; P_j \in R[X].$$

For any $y \in R$ with $v_R(y) \geq n-m$,

$$v_R(P(\xi+y)) \geq \min\{ v_R(P(\xi)+yP'(\xi)) , \; v_R(y^j P_j(\xi))_{j \geq 2} \}$$

But $v_R(y^j P_j(\xi)) \geq j\, v_R(y) \geq 2(n-m) \geq n+1$ (b/c $m \leq \frac{n-1}{2}$).

So just need to find y, $v_R(y) \geq n-m$, such that $v_R(P(\xi)+yP'(\xi)) \geq n+1$.

Let $y := -\frac{P(\xi)}{P'(\xi)}$. It's enough to show that $v_R(P'(\xi)) \leq m$.

We have $U(\xi)P(\xi) + V(\xi)P'(\xi) = r$. But $v_R(U(\xi)P(\xi)) \geq n > m$,

so that $v_R(V(\xi)P'(\xi)) = v_R(r) = m \Rightarrow v_R(P'(\xi)) \leq m$. $\blacksquare$ (lemma)

We want now to construct a G_K-equivariant surjection

$$\theta : W(R) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$$

lifting the map $\theta_0 : R = W(R)/(p) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/(p)$

↓

Def: Let θ be defined (as a map of sets) as the extension by continuity of $[r] \mapsto r^{\circ}$. That is:

$$\theta\left(\sum_{n\geq 0}[c_n]p^n\right) := \sum_{n\geq 0} c_n^{(0)} p^n \qquad \theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$$

Also, we can use the representation $(r_0, r_1, r_2, \dots) \leftrightarrow \sum_{n\geq 0}[r_n^{p^{-n}}]p^n$

and let $\theta((r_0, r_1, r_2, \dots)) = \sum_{n\geq 0} r_n^{(n)} p^n$

Lemma: θ is a ring map.

Pf: one could try to do the computation explicitly, but it's rather messy.

Recall that $\mathcal{O}_{\mathbb{C}_K} = \varprojlim \mathcal{O}_{\mathbb{C}_K}/(p^n)$. We will identify $W(R)$ as an inverse limit, and then construct individual maps for each n.

Let θ_n be, as before, $\theta_n : R \to \mathcal{O}_{\mathbb{C}_K}/(p)$ be proj. onto the n^{th} component.

Note that $(\theta_{n+1}(x))^p = x_{n+1}^p = x_n$. So $\mathrm{Frob} \circ \theta_{n+1} = \theta_n$.

By functoriality of W, we get:

$$\begin{tikzcd}
W(R) \arrow[d, "W(\theta_{n+1})"'] \arrow[dr, "W(\theta_n)"] & \\
W(\mathcal{O}_{\mathbb{C}_K}/(p)) \arrow[r, "\varphi = W(\mathrm{Frob})"'] & W(\mathcal{O}_{\mathbb{C}_K}/(p))
\end{tikzcd}$$

Calculate that $\varphi(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$ (exercise)

Let $\Theta_n : W(R) \xrightarrow{W(\theta_n)} W(\mathcal{O}_{\mathbb{C}_K}/(p)) \to W_n(\mathcal{O}_{\mathbb{C}_K}/(p))$ be the composition.

$$(a_0, \dots) \mapsto (a_0, a_1, \dots, a_{n-1})$$

We get: $f_n \circ \Theta_{n+1} = \Theta_n$, where $f_n : W_{n+1}(\mathcal{O}_{\mathbb{C}_K}/(p)) \to W_n(\mathcal{O}_{\mathbb{C}_K}/p)$

$$(a_0, \dots, a_n) \longmapsto (a_0^p, \dots, a_{n-1}^p)$$

(cont pf of lemma)

Claim: The natural ring map $\alpha : W(R) \longrightarrow \varprojlim_{f_n} W_n\left(\mathcal{O}_{\mathbb{C}_K}/(p)\right)$

$$x \longmapsto \{ \Theta_n(x) \}_{n \geq 0}$$

is a topological isomorphism and it is G_K-equivariant.

Pf (claim)

G_K-equivariance is easy, just unwinding the definitions.

$$\Theta_n(w_0, w_1, \ldots) = \left(w_0^{(n)} \bmod p, \ldots, w_{n-1}^{(n)} \bmod p\right).$$

Clearly, α is injective: if $w_i^{(n)} = 0$ $\forall n$, then $w_i = 0$

Also, as $\mathcal{O}_{\mathbb{C}_K}$ is alg. closed, can extract p^{th} powers, so Θ_n is surjective, hence α is.

It remains to see that it is a homeomorphism:

Note that $\Theta_n(w_0, \ldots) = (0, 0, \ldots, 0) \iff w_i^{(n)} \equiv 0 \pmod p$ $\forall i \leq n-1$.

$\iff v_R(w_i) \geq p^n$ $\forall i \leq n-1$.

So α (Base of opens in $W(R)$) = Base of opens in $W_n\left(\mathcal{O}_{\mathbb{C}_K}/(p)\right)$, and this readily implies the result. $\blacksquare$ (claim)

To construct θ, we just need now to define $\psi_n : W_n\left(\mathcal{O}_{\mathbb{C}_K}/(p)\right) \to \mathcal{O}_{\mathbb{C}_K}/p^n$ inducing θ on the inverse limit.

By definition,

$$\psi_n(c_0, \ldots, c_{n-1}) := \mathbb{W}_n(\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n-1}, 0) := \sum_{j=0}^{n} p^j (\hat{c}_j)^{p^{n-j}} \pmod{p^n}$$

lifts of c_j to $\mathcal{O}_{\mathbb{C}_K}/(p^n)$

$\left(\mathbb{W}_n : W_n(A) \to A\right.$ sends $(a_0, \ldots, a_n) \mapsto \sum_{j=0}^{n} p^j a_j^{p^{n-j}}$, which is a ring map$\left.\right)$

n^{th} Witt polynomial.

Note that Ψ_n is well-defined, because if $\tilde{c}_j \equiv \hat{c}_j \mod p$, then
$\tilde{c}_j^{\,p^{n-j}} \equiv \hat{c}_j^{\,p^{n-j}} \mod p^{n-j+1} \Rightarrow p^j \tilde{c}_j^{\,p^{n-j}} \equiv p^j \hat{c}_j^{\,p^{n-j}} \mod p^{n+1}$ ✓.

The fact that Ψ_n is a ring hom follows from:

$$\begin{tikzcd}
W_{n+1}(\mathcal{O}_{C_K}) \arrow[r, "W_n"] \arrow[d, "\mathrm{proj}"'] \arrow[dr, "\rho"] & \mathcal{O}_{C_K} \arrow[r, two heads] & \mathcal{O}_{C_K}/(p^n) \\
W_n(\mathcal{O}_{C_K}) \arrow[r, "\mathrm{red}"] & W_n(\mathcal{O}_{C_K}/(p)) \arrow[ur, dashed, "\Psi_n"'] &
\end{tikzcd}$$

Note that $\ker \rho = \{(pb_0, pb_1, \dots, pb_{n-1}, b_n)\}$.

and that $W_n\big((pb_0, \dots, pb_{n-1}, b_n)\big) = \sum_{j=0}^{n-1} p^j (pb_j)^{p^{n-j}} + p^n b_n \equiv 0 \mod p^n$.

Hence we get a factorization Ψ_n.

Now, TFC:

$$\begin{tikzcd}
W_{n+1}(\mathcal{O}_{C_K}/(p)) \arrow[r, "\Psi_{n+1}"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_{C_K}/p^{n+1} \arrow[d, "\mathrm{red}"] \\
W_n(\mathcal{O}_{C_K}/(p)) \arrow[r, "\Psi_n"] & \mathcal{O}_{C_K}/p^n
\end{tikzcd}$$

So the $\{\Psi_n\}$ induce a map on the inverse limit, which is continuous wrt the p-adic topology on $\mathcal{O}_{C_K}$.
We get a continuous, equivariant map G_K-equivariant map:

$$\begin{tikzcd}
W(R) \arrow[r, "\sim", "\alpha"'] \arrow[rr, bend right, "\theta"'] & \varprojlim_{f_n} W_n(\mathcal{O}_{C_K}/(p)) \arrow[r, "\{\Psi_n\}"] & \mathcal{O}_{C_K}
\end{tikzcd}$$

Finally, just compute:

$$\theta([r]) = \varprojlim \Psi_n\left(\Theta_n([r])\right) = \varprojlim \Psi_n\left(\Theta_n(r,0,0,\ldots)\right) =$$
$$= \varprojlim \Psi_n\left((r^{(n)} \bmod p), 0, 0, \ldots, 0\right) = \varprojlim (r^{(n)})^{p^n} = \varprojlim r^{(0)} = r^{(0)}$$

Recall that we have:

$$\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$$
$$\sum_{n\geq 0} [r_n] p^n \mapsto \sum_{n \geq 0} r_n^{(0)} p^n$$

this is surjective because the map $r \mapsto r^{(0)}$ is surjective.

By inverting p, we get:

$$\theta_K : W(R)[\tfrac{1}{p}] \twoheadrightarrow \mathbb{C}_K.$$

<u>Prop (4.4.4)</u>: Let $\pi = (p, p^{\frac{1}{p}}, p^{\frac{1}{p^2}}, \ldots) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K} = R$.

(or, more generally, let $\pi \in R$ any element s.t. $\pi^{(0)} = p$).

Let $\xi \in W(R)$ be $\xi := [\pi] - p$. Then:

1) ξ generates $\operatorname{Ker}\theta$.

2) If $w = (r_0, r_1, \ldots) \in W(R)$, then w generates $\operatorname{Ker}\theta \iff r_1 \in R^\times$

Pf/ First note that $\theta(\xi) = \theta([\pi]) - p = \pi^{(0)} - p = p - p = 0$. So $\xi \in \operatorname{Ker}\theta$, at least.

We want to show that $\operatorname{Ker}\theta = (\xi)$.

First, we show that $\operatorname{Ker}\theta \subseteq (\xi, p) = ([\pi], p)$:

↓

(cont pf of Prop).

If $w=(r_0,r_1,\dots)\in\ker\theta$, then $\theta(w)=\sum r_n^{(0)}p^n=0$, so that $r_0^{(0)}\equiv 0 \pmod p$.

$\therefore$ $v_R(r_0)\geq 1 = v_R(\pi)$. So $r_0\in\pi R$.

As the Teichmüller lift is multiplicative,

$$([r_0],p)\subseteq([\pi],p)\subseteq W(R).$$

But $w=\sum[r_n]p^n\in([r_0],p)$. ✓

Now, we know that $\ker\theta\subseteq(\xi,p)$.

So if $w\in\ker\theta$, $w=a\xi+bp$, $a,b\in W(R)$.

Applying θ, we get $0=\theta(w)=\theta(a)\overbrace{\theta(\xi)}^{0}+\theta(b)\theta(p)$.

So $\theta(b)=0$, because $W(R)$ is a strict p-ring. (mult by p is injective).

Repeating the argument, and using that $W(R)$ is p-adically complete and separated, we get $w=\xi\cdot x$, $x\in W(R)$. $\square$(a)

As for part (b), if $w\in\ker\theta$, we have:

$$w=(r_0,r_1,\dots)=\xi\cdot w'=(\pi,-1,\dots)\cdot(r_0',r_1',\dots)\underset{\substack{\uparrow\\ \text{use product polynomials}}}{=}(\pi r_0',\pi^p r_1'-r_0'^p,\dots)$$

So $r_1=\pi^p r_1'-r_0'^p$ is in $R^\times \Leftrightarrow r_0'\in R^\times \Leftrightarrow w'\in W(R)^\times \Leftrightarrow w$ generates $\ker\theta$.

$\square$

Lemma: $\bigcap_{j\geq 0} (\ker\theta)^j = \bigcap_{j\geq 0} (\ker\theta_K)^j = 0$.

Pf: By HW3, $\bigcap_j (\ker\theta)^j = W(R) \cap \left(\bigcap_j (\ker\theta_K)^j\right)$.

So $\left(\bigcap (\ker\theta)^j\right)[\tfrac{1}{p}] = \bigcap (\ker\theta_K)^j$, and hence it's enough to show that $\bigcap_j (\ker\theta)^j = 0$.

Now, let $w \in \bigcap (\ker\theta)^j$. So $w = \xi^j w_j \ \forall j \ (w_j \in W(R))$.

If $w = (r_0, \dots)$, then $r_0 = \pi^j s_j \ \forall j \Rightarrow r_0 = 0$ by the v_R-adic separatedness of R (using $v_R(\pi) = 1$).

But this means that:

$$w = (0, r_1, r_2, \dots) = \underbrace{p(r_1', r_2', \dots)}_{=pw'} \text{ in } W(R).$$

So then $w' \in \bigcap (\ker\theta)^j [\tfrac{1}{p}] = \bigcap (\ker\theta_K)^j$. As also $w' \in W(R)$, it is in $\bigcap (\ker\theta)^j$.

This says that $I := \bigcap (\ker\theta)^j$ is p-divisible. As $W(R)$ is a strict p-ring ($\Rightarrow$ separated + complete), $I = 0$.

$\blacksquare$

Corollary: We get a G_K-map:

$$W(R)[\tfrac{1}{p}] \hookrightarrow \varprojlim_j \frac{W(R)[\tfrac{1}{p}]}{(\ker\theta_K)^j} = \underbrace{W(R)[\tfrac{1}{p}]}_{}\!\!\widehat{\ }$$

$\ker\theta_K$-adic completion.

Def: $B_{dR}^+ := \varprojlim_j \left(\dfrac{W(R)[\tfrac{1}{p}]}{(\ker\theta_K)^j} \right)$

The map θ_K induces a G_K-equivariant surjective map

$$\theta_{dR}^+ : B_{dR}^+ \twoheadrightarrow \mathbb{C}_K$$

we conclude that B_{dR}^+ is a complete dvr, with uniformizer ξ, residue field $\mathbb{C}_K$.

<u>Def</u>: The <u>field of p-adic periods</u> is:

$$B_{dR} := \mathrm{Frac}(B_{dR}^+) \left(= B_{dR}^+[\tfrac{1}{\xi}]\right)$$

equipped with a G_K-action and the filtration (which is G_K-stable) given by powers of the maximal ideal of B_{dR}^+

We want to topologize B_{dR}^+ in a way that reflects <u>both</u> its d.v.-topology and the v_R-adic topology on R.

<u>Def</u>: For $\mathfrak{a} \subseteq R$ any open ideal, define for each $N \geq 0$:

$$U_{N,\mathfrak{a}} := \bigcup_{j \geq N} \left(p^{-j} W(\mathfrak{a}^{p^j}) + p^N W(R) \right)$$

where $W(J) := \{ w \in W(R)$ with all of its components in $J \}$

A HW3-exercise shows that $U_{N,\mathfrak{a}}$ are G_K-stable $W(R)$-submodules of $W(R)[\frac{1}{p}]$, defining a topology for which the $U_{N,\mathfrak{a}}$ are a basis of open nbhds of the identity.

This makes $W(R)[\frac{1}{p}]$ into a topological ring with continuous G_K-action.

1) By HW3 again, θ_K is continuous and open wrt the valuation on $\mathbb{C}_K$.

2) Also, $(\ker \theta_K)^j$ is closed $\forall j$.

3) Giving each $W(R)[\frac{1}{p}]/(\ker\theta_K)^j$ its quotient topology, and B_{dR}^+ the corresponding limit topology, we get a Hausdorff topological ring, with continuous G_K-action.

Lemma: The map

$$\log([\cdot]) : \{x \in 1 + m_R : x^{(0)} = 1\} \longrightarrow B_{dR}^+$$
$$x \longmapsto \sum_{n\geq 1} (-1)^{n+1} \frac{([x]-1)^n}{n}.$$

is continuous wrt to the v_R-top on the LHS, and the $U_{N,\mathfrak{a}}$ top on RHS.

Pf: The sum makes sense b/c $\theta([x]-1) = x^{(0)} - 1 = 0$.

For continuity, it is enough to show it at the identity.

For any ideal $\mathfrak{a} \subseteq R$, if $x \in 1 + \mathfrak{a}$, then $[x]-1 \in W(\mathfrak{a})$, and therefore $\frac{([x]-1)^n}{n} \in p^{-j} W(\mathfrak{a}^{p^j})$ where $j = \mathrm{ord}_p(n)$

This implies the claimed continuity. $\boxtimes$

Def: Fix $\varepsilon = (\varepsilon^{(n)})_{n\geq 0} = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K} = R$

where $\zeta_p \neq 1$ is a p^{th} root of 1.

(Note that any two choices of ε differ by a $\mathbb{Z}_p^\times$-power.)

Now,

$$v_R(\varepsilon-1) = v_p\left((\varepsilon-1)^{(0)}\right) = v_p\left(\lim_{n\to\infty}\left(\varepsilon^{(n)}+(-1)^{(n)}\right)^{p^n}\right) =$$

for $p>2$. Different argument for $p=2$

$$= v_p\left(\lim_{n}\left(\zeta_{p^n}-1\right)^{p^n}\right) = \lim_{n\to\infty} p^n\, v_p\left(\zeta_{p^n}-1\right) = \lim_{n\to\infty}\frac{p^n}{p^{n-1}(p-1)} = \frac{p}{p-1}$$

In particular, the following def. makes sense:

Def: Define $t := \log([\varepsilon]) \in B_{dR}^+$ $\quad\left(\text{N.B. } \varepsilon^{(0)}=1 \ \& \ v_R(\varepsilon-1)\geq 1\right)$

By the continuity of log,

$$\log([\varepsilon^a]) = \log([\varepsilon]^a) = a\log([\varepsilon]) \qquad (\text{for } a\in\mathbb{Z}_p^\times).$$

Since ε is unique up to a $\mathbb{Z}_p^\times$-power, the element t is unique up to a $\mathbb{Z}_p^\times$-multiple.

In particular, the line $t\cdot\mathbb{Z}_p$ in B_{dR}^+ is intrinsic. Also, it is G_K-stable:

$$g\cdot t = \log([g\varepsilon]) = \log[\varepsilon^{\chi(g)}] = \chi(g)\cdot t$$

So B_{dR}^+ contains an intrinsic copy of $\mathbb{Z}_p(1)$.

The analogy with $\mathbb{C}$ is that $\mathbb{C}$ contains a copy of $\mathbb{Z}(1)$, the kernel of $\exp: \mathbb{C}\to\mathbb{C}$ (i.e. $2\pi i\,\mathbb{Z}$).

So we think of $t\in B_{dR}^+$ as $2\pi i\in\mathbb{C}$.

<u>Prop</u>: $t \in B_{dR}^+$ is a uniformizer.

Pf/ Write $t = \log([\varepsilon]) = \sum (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n}$.

So to show that $t \notin (\mathrm{Ker}\, \theta_{dR}^+)^2$, it's enough to show that

$[\varepsilon] - 1 \notin (\mathrm{Ker}\, \theta_{dR}^+)^2$, i.e. $[\varepsilon]-1 \notin (\mathrm{Ker}\, \theta)^2$ (i.e. $\neq \xi^2 \cdot x$, $x \in W(R)$)

We just need to look at the 0^{th}-component,

$([\varepsilon]-1) = (\varepsilon - 1, \ldots)$ and $\xi^2 = (\pi^2, \ldots)$. $\pi \in R$ $\text{``}(p, p^{1/p}, \ldots)\text{''}$.

So just need that $\varepsilon - 1 \notin \pi^2 R$.

But $v_R(\varepsilon - 1) = \frac{p}{p-1}$ and $v_R(\pi^2) = 2 v_R(\pi) = 2 v_p(p) = 2$.

Hence at least for $p \geq 3$, we are done.

For $p = 2$, we need to look at the next component. (HW3). ∎

What we've done so far: Let K be a p-adic field. k = residue field.

$R = R_K = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/(p)$ = valuation ring w/ perfect fraction field (of equichar p)

$\theta_K : W(R)[\tfrac{1}{p}] \twoheadrightarrow \mathbb{C}_K$

G_K-equivariant continuous surjection, as $W(\bar{k})$-algebras.

We then defined:

$B_{dR}^+ := \varprojlim_n \frac{W(R)[\frac{1}{p}]}{(\mathrm{Ker}\, \theta_K)^n}$, a complete dvr w/ uniformizer $t = \log([\varepsilon])$

with continuous G_K-action, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$

and a G_K-equivariant continuous surjection

$\theta_{dR}^+ : B_{dR}^+ \to \mathbb{C}_K$. and we saw that, for $g \in G_K$, $gt = \chi(g) \cdot t$

We then defined $B_{dR} := \mathrm{Frac}(B_{dR}^+)$, which is a filtered $W(k)$-algebra with a (cont) G_K-action.

As a corollary, we obtain:

<u>Cor</u>: $gr^\bullet(B_{dR}) = B_{HT} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_K(n)$.

Pf/ Need:

1) The residue field of B_{dR}^+ is G_K-equiv. isomorphism to $\mathbb{C}_K$. ✓

2) $m/m^2 \simeq \mathbb{C}_K(1)$, which is the same as to say $\mathbb{Z}_p \cdot t \cong \mathbb{Z}(1)$ ✓.

☒

Now, the map $k \hookrightarrow R$ induces:

$$\begin{tikzcd}
K_0 = W(k)[\tfrac{1}{p}] \arrow[r, hook] \arrow[drr, bend right] & W(R)[\tfrac{1}{p}] \arrow[r, hook] & B_{dR}^+ \arrow[r, two heads] & \mathbb{C}_K \\
 & & \overline{K} \arrow[u, dashed, "\exists! \text{ by Hensel}"] \arrow[ur, hook] &
\end{tikzcd}$$

So B_{dR}^+ is a $\overline{K}$-algebra, in a canonical way.

<u>Theorem</u>: The map $K \hookrightarrow B_{dR}^{G_K}$ is an isomorphism.

Pf/ As the G_K-action respects the filtration, we get an embedding:

$$gr^\bullet(B_{dR}^{G_K}) \hookrightarrow (gr^\bullet(B_{dR}))^{G_K} = B_{HT}^{G_K} = K$$

So that $gr^\bullet(B_{dR}^{G_K})$ is 1-dimensional over $K \Rightarrow \dim_K(B_{dR}^{G_K}) = 1$

☒

• Formalism of classifying representations by "period" rings

Let F be a field (eg $F = \mathbb{Q}_p$) and let G be a group (e.g. G_K).
Let B be an F-algebra, which is a domain, together ~~with~~ a G-action via F-algebra endomorphisms.

Let $E := B^G$, and assume that E is a field.

Let $C := \mathrm{Frac}(B)$.

Def: B is called (F, G)-regular if:

1) $C^G = B^G$; and

2) Every nonzero $b \in B$ that spans a G-stable F-line is in $B^\times$.

Note that, if B is a field, then B is (F,G)-regular (in a "trivial" way).

Example: B_{dR} is $(\mathbb{Q}_p, G_K)$-regular, with $E = B_{dR}^{G_K} = K$.

but B_{dR}^+ is not, because t contradicts the second condition.

Example: $B_{HT} \overset{n.c}{\cong} \mathbb{C}_K[T, \frac{1}{T}]$ is $(\mathbb{Q}_p, G_K)$-regular:

Pf/ Assume $\frac{f_1}{f_2} \in \mathrm{Frac}(B_{HT}) \overset{n.c}{\cong} \mathbb{C}_K(T)$ is G_K-invariant, where we force $f_1, f_2 \in \mathbb{C}_K[T]$, $(f_1, f_2) = 1$, and $f_2 = T^{d_2} + \dots$.
(makes f_1/f_2 unique).

Then $g \cdot \frac{f_1}{f_2} = \frac{g f_1}{g f_2} = \frac{g f_1}{\chi(g)^{d_2} \cdot T^{d_2} + \dots} = \frac{\chi(g)^{-d_2} \cdot g \cdot f_1}{\underbrace{\chi(g)^{-d_2} g \cdot f_2}_{\text{monic}}} = \frac{f_1}{f_2}$

So, by uniqueness, $g \cdot f_2 = \chi(g)^{d_2} \cdot f_2$ and $g \cdot f_1 = \chi(g)^{d_2} \cdot f_1$

Writing $f_1 = \sum c_j T^j$, this says $g \cdot c_j \, \chi(g)^j = \chi(g)^{d_2}$ $\forall g \in G$, $\forall j$.
So that $c_j \in \mathbb{C}_K(d_2 - j)^{G_K}$

(cont ~~pf~~ of example).

From $c_j \in \mathbb{C}_K(d_2 - j)^{G_K} = \begin{cases} 0 & d_2 \neq j \\ K & \text{o.w.} \end{cases}$

Hence $f_1 = c \cdot T^{d_2}$. For the same reason, $f_2 = T^{d_2}$, so $\frac{f_1}{f_2} = c \in K$.

2) We want to show that, if $b \in B_{HT}$ is nonzero and G_K acts on b through a $\mathbb{Q}_p^\times$-valued character, then $b \in B_{HT}^\times$.

As the G_K-action is continuous, we must have:

$$g \cdot b = \Psi(g) \cdot b \quad \text{for} \quad \Psi : G_K \to \mathbb{Z}_p^\times.$$

If $b = \sum c_j T^j$ (some j's negative), then,

$$g \cdot c_j \chi(g)^j = \Psi(g) \cdot c_j. \quad \text{So that} \quad c_j \in \mathbb{C}_K(\Psi^{-1}\chi^j)^{G_K}$$

$$\text{"} \begin{cases} 0 & \text{if } \eta|_{I_K} \text{ not of finite order} \\ \text{1-dim'l o.w.} \end{cases}$$

$(\eta = \Psi^{-1}\chi^j)$

Then, if $c_j, c_{j'} \neq 0$, then we would get that

$\Psi^{-1}\chi^j$, $\Psi^{-1}\chi^{j'}$ have both finite order when restricted to I_K,

so that $\chi^{j-j'}|_{I_K}$ has finite order, which is not true.

Hence $b = c \cdot T^j$ for some $j \in \mathbb{Z}$, so $b \in B_{HT}^\times$. $\blacksquare$

Def: Let B be (F,G)-regular. We define:

$$D_B : \mathrm{Rep}_F(G) \longrightarrow \text{vectorspaces over } E = B^G$$

$$V \longmapsto (B \otimes_F V)^G$$

We have a B-linear, G-equivariant comparison ~~isomorphism~~ homomorphism:

$$\alpha_V : B \otimes_E D_B(V) = B \otimes_E (B \otimes_F V)^G \hookrightarrow B \otimes_E B \otimes_F V \xrightarrow{mult} B \otimes_F V$$

Def: We say that V is B-admissible if ~~α_V is an isomorphism.~~

$$\dim_E D_B(V) = \dim_F(V)$$

Example: B_{HT}-admissible is the same as being Hodge-Tate.
B_{dR}-admissible is called deRham.

(5.2.1 in the notes)
Thm: Let $V \in \mathrm{Rep}_F(G)$. Then:

1) α_V is injective, and $\dim_E D_B(V) \leq \dim_F(V)$, with equality if, and only if, α_V is an isomorphism.

2) The functor $D_B : \mathrm{Rep}_F^B(G) \longrightarrow \mathrm{Vec}_E$ = fin. dim vectorspaces $/E$

full subcategory of B-admissible reps

is exact and faithful.

2') Any subquotient (i.e subobject or quotient) of a B-admissible rep is B-admissible.

3) If $V_1, V_2 \in \mathrm{Rep}_F^B(G)$, then $V_1 \otimes V_2 \in \mathrm{Rep}_F^B(G)$, and

$$D_B(V_1) \otimes_E D_B(V_2) \simeq D_B(V_1 \otimes_F V_2)$$

Also, if $V \in \mathrm{Rep}_F^B(G)$, then $V^\vee = \mathrm{Hom}_F(V, F)$ with the usual G-action (by the inverse) is B-admissible,

and $D_B(V^\vee) = D_B(V)^\vee$ (through $D_B(V) \otimes D_B(V^\vee) \xrightarrow{\sim} D_B(V \otimes_F V^\vee) \to D_B(F) = E$)

Furthermore, D_B commutes with exterior and symmetric powers of B-admissible reps.

Pf:

1)
$$\begin{tikzcd} B \otimes_E D_B(V) \arrow[r, "\alpha_V"] \arrow[d, hook, "\text{b/c } B \text{ is a domain}"'] & B \otimes_F V \arrow[d, hook] \\ C \otimes D_C(V) \arrow[r, "\alpha_V"] & C \otimes_F V \end{tikzcd}$$

So it is enough to prove (1) with C replacing B. So WLOG we can assume that B is a field. (note: $C^G = E$ is a field, so it is (F,G)-regular)

Now, if $x_1, .., x_\ell \in B \otimes_F V$ are E-linearly-independent and G-invariant, then we want to see that they are B-linearly independent.

If we have a B-relation on the x's, we can assume that it has the form: $x_r = \sum_{i<r} b_i x_i$, $b_i \neq 0\ \forall i$, of minimal length

Now, $\overset{x_r}{\overset{\shortparallel}{g x_r}} = \sum_{i<r} g b_i\, g x_i = \sum_{i<r} (g b_i) x_i$, so subtracting and using minimality, we will get $g b_i = b_i\ \forall i$, ie. $b_i \in B^G = E$, so that x's are not E-independent (!!).

It is clear that if α_V is an iso, then $\dim_E D_B(V) = \dim_F(V)$

Conversely, given an equality of dimensions, we know that:

$$1 \otimes \alpha_V : C \otimes_E D_B(V) \longrightarrow C \otimes_F V \text{ is an iso.}$$

we need to pass from C to B.

Let $d = \dim_E D_B(V) = \dim_F(V)$, and pick an E-basis $\{e_j\}$ for $D_B(V)$, and an F-basis $\{v_i\}$ for V.

(cont pf of long Thm).

So α_V has a matrix $\alpha_V = (b_{ij})_{i,j}$ $b_{ij} \in B$, with $\det(b_{ij}) \neq 0$.

We want to see that the determinant is a unit in B.

We have that $\alpha_V(e_j) = \sum_i b_{ij} \otimes v_i$, so:

$$(\Lambda^d(\alpha_V))(e_1 \wedge \dots \wedge e_d) = \det(b_{ij})\, v_1 \wedge \dots \wedge v_d.$$

The vector $e_1 \wedge \dots \wedge e_d$ is G-invariant, and G acts through a character on $v_1 \wedge \dots \wedge v_d$.

Hence $b = \det(b_{ij})$ spans a G-stable F-line. As $b \neq 0$, then $b \in B^\times$ by (F,G)-regularity. $\square$(of 1)

(Proof of (2)):
By (1), we know that α_V's induce a natural isomorphism of functors:

$$B \otimes_E D_B|_{\mathrm{Rep}^B_F(G)}(-) \xrightarrow[\alpha_{(-)}]{\sim} B \otimes_F (-)$$

So it is exact and faithful, because the identity functor is. $\square$(of 2)

(of 2'):
If $0 \to V' \to V \to V'' \to 0$ is exact in $\mathrm{Rep}_F(G)$ and V is B-admissible, then $0 \to D_B(V') \to D_B(V) \to D_B(V'')$ is exact, because $\otimes$ and $(\ )^G$ are left-exact.

So: $\dim_E D_B(V) \leq \dim_E D_B(V') + \dim_E D_B(V'') \leq \dim_F V' + \dim_F V'' = \dim_F V$. So all inequalities are in fact equalities. $\square$

(cont Pf of long Thm)

(Part 3):

Let $V_1, V_2 \in \mathrm{Rep}_F^B(G)$. Then we have a B-linear G-equivariant map:

$$t: D_B(V_1)\otimes_E D_B(V_2) \longrightarrow (B\otimes_F V_1)\otimes_E (B\otimes_F V_2) \simeq (B\otimes_E B)\otimes_F (V_1\otimes_F V_2) \longrightarrow B\otimes_F (V_1\otimes_F V_2)$$

Notice that, by G-equivariance, t factors through $D_B(V_1\otimes_F V_2)$.

<u>Claim</u>: t is injective.

~~Pf~~ Consider the following commutative diagram:

$$\begin{tikzcd}
D_B(V_1)\otimes_E D_B(V_2) \arrow[rr, "t"] \arrow[d, hook] & & D_B(V_1\otimes_F V_2) \arrow[d, hook] \\
B\otimes_E\left(D_B(V_1)\otimes_E D_B(V_2)\right) \arrow[d, equal] \arrow[rr, phantom, "\circlearrowleft \text{ (by construction)}"] & & B\otimes_F(V_1\otimes_F V_2) \arrow[d, phantom, "\simeq"] \\
\left(B\otimes_E D_B(V_1)\right)\otimes_B\left(B\otimes_E D_B(V_2)\right) \arrow[rr, "\cong"', "\alpha_{V_1}\otimes\alpha_{V_2}"] & & (B\otimes_F V_1)\otimes_B(B\otimes_F V_2)
\end{tikzcd}$$

b/c V_i's are B-adm.

built from multiplication on B.

So t is injective. (~~first part~~ claim)

Now, note that the source of t has E-dimension $\dim_F(V_1)\cdot\dim_F(V_2)$, and the dimension of the target is $\leq \dim_F(V_1\otimes F_2)$. Hence equality, so that t is an isomorphism, and so $V_1\otimes_F V_2$ is B-admissible and D_B commutes with $\otimes$.

}

(cont pf)

We have seen now that $V^{\otimes r}$ is B-admissible for any $V \in \mathrm{Rep}_F^B(G)$.

As $\Lambda^r V$ and $\mathrm{Sym}^r V$ are quotients of this, they are B-admissible as well.

We work now with $\Lambda^r V$, but for Sym^r it's done in the same way. We have:

$$\begin{tikzcd} D_B(V)^{\otimes r} \arrow[r, "\simeq"] \arrow[d, two heads] & D_B(V^{\otimes r}) \arrow[d, two heads] \\ \Lambda^r D_B(V) \arrow[r, dotted] & D_B(\Lambda^r V) \end{tikzcd}$$

b/c the map is $D_B(V^{\otimes r} \twoheadrightarrow \Lambda^r V)$ and D_B is exact on B-admissibles.

b/c of the universal mapping property of Λ^r.

The dotted map has to be surjective, then.

Since $\dim_E(\Lambda^r D_B(V)) = \dim_F(\Lambda^r V) = \dim_E D_B(\Lambda^r V)$, we are done.

Now, we need to prove the duality statement which uses a nice trick.

For any $V \in \mathrm{Rep}_F(G)$, we have a natural G-equivariant F-linear iso:

$$\det(V^\vee) \underset{F}{\otimes} \Lambda^{d-1}(V) \xrightarrow{\simeq} V^\vee$$

by, if $d = \dim_F V$, sending:

$$(\ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_d) \otimes (v_2 \wedge v_3 \wedge \cdots \wedge v_d) \longmapsto \left(v_1 \mapsto \det\left(\ell_i(v_j)\right)\right).$$

So, if V is B-admissible and $\det(V^\vee) = (\det V)^\vee$ is B-admissible, then $V^\vee$ is also B-admissible.

We already know that $\det V$ is B-admissible, so we just ~~need~~ to show that, for any 1-dimensional admissible, duality works.

(cont pf)

Pick now an F-basis v_0 of V (V 1-dim'l, B-admissible).

Then we have $D_B(V) = E\cdot(b\otimes v_0)$ for $b\in B\setminus 0$.

(any element of V is a single tensor!).

~~Σ for b~~

The map

$$\begin{array}{ccc} \alpha_V : B\otimes_E D_B(V) & \xrightarrow{\simeq} & B\otimes_F V \\ \cong & & \cup \\ B(1\otimes v_0) \ni b\,v_0 & \mapsto & b\cdot(1\otimes v_0) \end{array}$$

As α_V is an iso, $b(1\otimes v_0)$ is an ~~[illegible]~~ F-basis in $B\otimes_F V$,

so $b\in B^\times$. (we could use an argument using (F,G)-regularity, also).

We just need to find a nonzero element in $D_B(V^\vee)$

As $b\otimes v_0$ is G-invariant, $g(b)\otimes g(v_0) = b\otimes v_0$, so that

$g(b)\eta(g) = b$, where $\eta : G\to F^\times$ gives the G-action on v_0.

Let v_0^* be the dual basis for $V^\vee$. We compute: (note $b^{-1}\otimes v_0^*\in B\otimes V^\vee$)

$$g(b^{-1}\otimes v_0^*) = g(b^{-1})\otimes g(v_0^*) = g(b^{-1})\otimes(\eta(g)^{-1}v_0^*) = (g(b)\eta(g))^{-1}\otimes v_0^* = b^{-1}\otimes v_0^*.$$

So $b^{-1}\otimes v_0^* \in (B\otimes V^\vee)^G = D_B(V^\vee)$, thus $D_B(V^\vee)\neq 0$, and

hence $\dim_E D_B(V^\vee) = 1$ ✓.

Now, we want to show that the pairing $(\dagger) : D_B(V)\otimes_E D_B(V^\vee) \to D_B(F) = E$

is perfect. If $d = \dim_F V$ and we take Λ^d, we get a pairing:

$(\dagger\dagger) : \Lambda^d D_B(V)\otimes \Lambda^d D_B(V^\vee) \to E$, ~~which is perfect because each of the factors is 1-dimensional.~~

(cont pf)

Fact from linear algebra: $(\dagger\dagger)$ is perfect $\Leftrightarrow$ $(\dagger)$ is perfect.

We know that $\Lambda^d D_B(V) \simeq D_B(\Lambda^d V)$ and $\Lambda^d D_B(V^\vee) \simeq D_B(\Lambda^d V^\vee)$ by B-adm. of V and $V^\vee$. As $\Lambda^d(V^\vee) = (\Lambda^d V)^\vee$, we get $(\dagger\dagger)$ is equivalent to:

$$D_B(\Lambda^d V) \otimes_E D_B((\Lambda^d V)^\vee) \to E$$

which is the pairing induced by $\Lambda^d V \otimes (\Lambda^d V)^\vee \to F$ by applying $D_B(-)$.

Hence, again we can reduce to 1-dimensional V. But in that case, it follows easily from our calculation:

$$E(b\otimes v_0) \otimes_E E\cdot(b^{-1}\otimes v_0^*) \longrightarrow E$$

$$\alpha(b\otimes v_0)\otimes \beta(b^{-1}\otimes v_0^*) \longmapsto \alpha\beta$$

which is clearly perfect. $\blacksquare$ (of Thm)

<u>Note</u>: $V \in Rep_F(G)$ is B-admissible iff

$$B \otimes_F V \underset{\substack{\uparrow \\ \text{B-linearly} \\ \text{G-equ.}}}{\simeq} B^{\oplus d} \qquad \text{where } d = \dim_F(V).$$

• de Rham representations

We saw that B_{dR} is $(\mathbb{Q}_p, K)$-regular, with $B_{dR}^{G_K} = K$.

Def: $D_{dR} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{Vec}_K$

$$V \longmapsto (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

We know that $\dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p}(V)$.

Def: We say V is <u>de Rham</u> if $\dim_K D_{dR}(V) = \dim_{\mathbb{Q}_p}(V)$.

Let $\mathrm{Rep}_{\mathbb{Q}_p}^{dR}(G_K)$ = full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of de Rham reps.

We saw that this is stable under $\otimes$, duality, passing to subquotients, and that $D_{dR}|_{\mathrm{Rep}^{dR}(G_K)}$ is compatible with all these operations, and is moreover <u>exact</u> and <u>faithful</u>.

<u>Example</u>: $V = \mathbb{Q}_p(n)$ is dR for all n, with $D_{dR}(V) = K \cdot t^{-n} \subseteq B_{dR} \otimes \mathbb{Q}_p(n)$

But note that D_{dR} takes values in $\mathrm{Fil}_K \to \mathrm{Vec}_K$.

faithful functor, it's not a subcategory!

<u>Q</u>: Is it the case that $D_{dR} : \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(G_K) \longrightarrow \mathrm{Fil}_K$ enjoys the same properties as $D_{dR} : \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(G_K) \longrightarrow \mathrm{Vec}_K$?

(this makes sense, because the "inclusion" $\mathrm{Fil}_K \subseteq \mathrm{Vec}_K$ is not full!)

We will now review the filtered vectorspaces.

· Review of filtered vectorspaces

Let F be a field, Vec_F := cat. of finite-dimensional v.s /F.

Recall: A filtered v.s /F is a pair $(D, \{Fil^i_D\})$ where

D = fin. dim vectorspace, and $\{Fil^i D\}$ is a collection of F subspaces of D, with $Fil^{i+1} D \subseteq Fil^i D$ $(i \in \mathbb{Z})$.

We say that the filtration is:

a) exhaustive if $D = \bigcup Fil^i D$

b) separated if $\bigcap Fil^i = \{0\}$.

Def: Let Fil_F be the category whose objects are filtered ($\Rightarrow$ fin. dim) v.s. with exhaustive & separated filtration, and whose morphisms are required to be Fil-compatible:

$$T : D' \to D \quad F\text{-linear},$$

$$T(Fil^i D') \subseteq Fil^i D$$

Let $T : D' \to D$ in Fil_F.

Def: The kernel of T is:

$Ker(T)$ = F-linear kernel of T with the subspace filtration:

$$Fil^i(Ker T) = Ker(T) \cap Fil^i D.$$

The cokernel of T is:

$coker(T)$ = F-linear cokernel of T with quotient filtration:

$$Fil^i(coker(T)) = \frac{Fil^i(D) + T(D')}{T(D')}$$

(Cont defs):

3) $\mathrm{Im}(T)$ = F-linear image of T, with subspace filtration:

$$\mathrm{Fil}^i(\mathrm{Im}\,T) = \mathrm{im}(T) \cap \mathrm{Fil}^i(D)$$

4) $\mathrm{Coim}(T)$ = F-linear image of T = $\dfrac{D'}{\mathrm{Ker}\,T}$, with quotient filtration:

$$\mathrm{Fil}^i(\mathrm{coim}\,T) = \frac{\mathrm{Fil}^i(D') + \mathrm{Ker}\,T}{\mathrm{Ker}\,T}$$

In general, we have a map in Fil_F, $\mathrm{Coim}(T) \to \mathrm{im}(T)$, induced by:

$$\begin{tikzcd}
\mathrm{Fil}^i(D') + \mathrm{Ker}\,T \arrow[r, "T"] \arrow[dr, dashed] & D \\
 & \mathrm{Fil}^i(D) \arrow[u]
\end{tikzcd}$$

which is an isomorphism in Vec_F, but not (generally) in Fil_F.

<u>Def</u>: We say that $T: D' \to D$ is <u>strict</u> if $\mathrm{coim}(T) \to \mathrm{im}(T)$ is an isomorphism in Fil_F.

<u>Example</u>: $D = D'$ as V.S./F, and define:

$$\mathrm{Fil}^i(D) = \begin{cases} D & \text{if } i \leq 1 \\ 0 & \text{else} \end{cases} \qquad \mathrm{Fil}^i(D') = \begin{cases} D' & i \leq 0 \\ 0 & \text{else} \end{cases}$$

Then the identity map $\mathrm{id}: D' \to D$ is a map in Fil_F, but Im and Coim do <u>not</u> coincide.

<u>NB</u>: Fil_F is not abelian!

Def: 1) Let $D, D' \in Fil_F$. The tensor product $D' \otimes D$ is:

- $D' \otimes_F D$ as F-vectorspaces.
- $Fil^n(D' \otimes D) := \sum_{i+j=n} Fil^i(D') \otimes Fil^j(D) \subseteq D' \otimes_F D$

(check that this is exhaustive & separated).

2) The unit object is $F[0]$: F as a vectorspace, with $Fil^i(F[0]) = \begin{cases} F & i \leq 0 \\ 0 & i > 0 \end{cases}$

3) For D in Fil_F, the dual of D is:

$D^* = Hom(D, F)$ as F-vectorspace, with:

$$Fil^i(D^*) := (Fil^{1-i}(D))^\perp = \{ \ell \in D^* : Fil^{1-i}(D) \subseteq Ker(\ell) \}$$

4) A short exact sequence in Fil_F is a diagram in Fil_F:

$$0 \to D' \to D \to D'' \to 0$$

Such that $D' \simeq Ker(D \to D'')$ and $D'' \simeq coker(D' \to D)$ in Fil_F.

This is equivalent to:

$$0 \to Fil^i D' \to Fil^i D \to Fil^i D'' \to 0$$

being exact in Vec_F, $\forall i$.

5) The shift functor $[n] : Fil_F \to Fil_F$ is given by:

$Fil^i(D[n]) := Fil^{i+n}(D)$ $\quad$ ($D[n] = D$ as F-vs).

6) $Hom(D', D) = Hom_{Vec_F}(D', D)$, with the filtration

internal hom

$$Fil^i(Hom(D', D)) = Hom_{Fil_F}(D', D[i]) = \{ T \in Hom_{Vec}(D', D) : T(Fil^j(D')) \subseteq Fil^{i+j}(D) \ \forall j \}$$

We have a functor $gr: Fil_F \to Gr_{F,f}$ (finite), given by:

$$gr(D) := \bigoplus_{i\in\mathbb{Z}} \frac{Fil^i(D)}{Fil^{i+1}(D)}$$

that preserves F-dimension (thanks to exhaustive + separated).

<u>Rmk</u>: It is easy to check that $gr(D'\otimes D) \simeq gr(D')\otimes gr(D)$

From now on, we will view D_{dR} as taking values on Fil_K, via $Fil^i(D_{dR}(V)) = \left(Fil^i(B_{dR})\otimes_{\mathbb{Q}_p} V\right)^{G_K}$, where the filtration on B_{dR} is $Fil^i B_{dR} = t^i B_{dR}^+$.

The resulting filtration on D_{dR} is exhaustive & separated.

<u>Example</u>: We computed before $D_{dR}(\mathbb{Q}_p(n)) = K\cdot t^{-n}$, so that

$$Fil^i(D_{dR}(\mathbb{Q}_p(n))) = \begin{cases} 0 & \text{if } i > -n \\ D_{dR}(\mathbb{Q}_p(n)) & \text{if } i \leq -n \end{cases}$$

i.e. $gr(D_{dR}(\mathbb{Q}_p(n)))$ = concentrated in degree $-n$.

<u>Prop 6.3.2</u>: For $V \in Rep_{\mathbb{Q}_p}(G_K)$, there is an injection in $Gr_{K,f}$

$$gr(D_{dR}(V)) \hookrightarrow D_{HT}(V)$$

which is an equality if V is deRham.

Pf We have an exact sequence:

$$0 \to Fil^{i+1}B_{dR}\otimes V \to Fil^i B_{dR}\otimes V \to gr^i B_{dR}\otimes V \to 0.$$

Taking G_K-invariants, gives:

$$0 \to Fil^{i+1}D_{dR}(V) \to Fil^i D_{dR}(V) \to gr^i(B_{HT}(V))$$

When V is deRham, we get:

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{dR}(V) = \dim_K gr(D_{dR}(V)) \leq \dim_K D_{HT}(V) \leq \dim_{\mathbb{Q}_p} V$$

so all ~~equalities~~ are equalities. ∎

<u>Corollary</u>: If V is deRham, then it is Hodge-Tate.

<u>Prop (6.3.3)</u>: $D_{dR}: Rep^{dR}_{\mathbb{Q}_p}(G_K) \longrightarrow Fil_K$ is exact, faithful and compatible with $\otimes$ and duality.

<u>Pf</u>. • <u>Faithful</u>: it follows from the fact that $Fil_K \xrightarrow{\text{forget}} Vec_K$ is faithfull.

• <u>Exactness</u>: Let $0 \to V' \to V \to V'' \to 0$ be exact in $Rep_{\mathbb{Q}_p}(G_K)$.

Then, $\forall i$, $Fil^i B_{dR} \otimes_{\mathbb{Q}_p} V$

$0 \to Fil^i V' \to Fil^i V \to Fil^i V'' \to 0$ is exact as well.

By taking G_K-invariants, get:

$$0 \to Fil^i(D_{dR}(V')) \to Fil^i(D_{dR}(V)) \to Fil^i(D_{dR}(V'')) \xrightarrow{?} H^1 \ldots$$

If V is deRham, then so are V' and V''. So all three are also Hodge-Tate.

So, for any j, we have an exact sequence:

$$0 \to gr^j(D_{HT}(V')) \to gr^j(D_{HT}(V)) \to gr^j(D_{HT}(V'')) \to 0$$

$$\qquad \| \qquad\qquad\qquad \| \qquad\qquad\qquad \|$$

$$0 \to gr^j(D_{dR}(V')) \to gr^j(D_{dR}(V)) \to gr^j(D_{dR}(V'')) \to 0$$

That is, $\dim_K gr^j(D_{dR}(V)) = \dim_K gr^j(D_{dR}(V')) + \dim_K gr^j(D_{dR}(V''))$.

By summing for $j \geq i$, we get $\dim_K Fil^i D_{dR}(V) = \dim_K Fil^i D_{dR}(V') + \dim_K Fil^i D_{dR}(V'')$

Hence the left-exact sequence becomes also right-exact.

- Compatibility w/ $\otimes$ & duality:

We ~~know~~ that, if $V_1, V_2 \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$, then the map:

$$\begin{tikzcd}
D_{dR}(V_1) \otimes D_{dR}(V_2) \arrow[r, hook] \arrow[drr] & (B_{dR} \otimes_{\mathbb{Q}_p} V_1) \otimes_K (B_{dR} \otimes_{\mathbb{Q}_p} V_2) \arrow[r, "\mathrm{mult}"] & B_{dR} \otimes_{\mathbb{Q}_p} (V_1 \otimes_{\mathbb{Q}_p} V_2) \\
(\dagger) & & D_{dR}(V_1 \otimes_{\mathbb{Q}_p} V_2) \arrow[u, hook]
\end{tikzcd}$$

is a map in Fil_K, and it is known to be an iso in Vec_K.

<u>General pple</u>: a map in Fil_K which is an iso in Vec_K, is an iso in Fil_K iff the associated map in $\mathrm{Gr}_{K,f}$ is an isomorphism.

But when we apply $gr : \mathrm{Fil}_K \to \mathrm{Gr}_{K,f}$, the diagram $(\dagger)$ becomes the corresponding diagram for HT-$\otimes$-comparison map, because:

1) $gr(B_{dR}) = B_{HT}$

2) $gr(D_{dR}(V)) = D_{HT}(V)$ for any $V \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$.

For duality, it's done in the same way. $\boxtimes$

<u>Cor 1</u>: $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is de Rham iff $V \otimes_{\mathbb{Q}_p} W$ is de Rham for all $W \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$.

Pf/ $\Rightarrow$) $\otimes$-compat

$\Leftarrow$) take $W := \mathbb{Q}_p$.

<u>Corollary 2</u>: $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is deRham iff $V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ is deRham <u>for some n</u>.

<u>Prop (6.3.6)</u>: If $V \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$, then the B_{dR}-linear G_K-equivariant map:

$$\alpha : B_{dR} \otimes_K D_{dR}(V) \longrightarrow B_{dR} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism in Fil_K.

Pf/ Apply gr to get the HT-comparison map (etc) ☒

<u>Claim</u>: $D_{dR} : \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K) \to \mathrm{Fil}_K$ is <u>not</u> full.

Pf/ To see this, we will write $D_{dR,K}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$ depends on K as well, but not too much!

<u>Prop (6.3.7)</u>: For any complete, discretely-valued extension K'/K in $\mathbb{C}_K$, and for any $V \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$, the natural map:

pf in the notes

$$K' \otimes_K D_{dR,K}(V) \longrightarrow D_{dR,K'}(V)$$

is an isomorphism (in $\mathrm{Fil}_{K'}$).

<u>Consequence</u>: If $\rho : G_K \to \mathrm{Aut}(V)$ is in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $\rho(I_K)$ is finite, then ρ is deRham, and

$$D_{dR}(V) = (K[0])^{\oplus \dim(V)}$$

Pf (of the consequence):

Choose L/K finite with $\rho(I_L)=1$. Set $K' := \widehat{L^{un}}$. So $G_{K'} = I_L$.

Then $D_{dR,K'}(V) = K' \otimes_{\mathbb{Q}_p} V \cong (K'[0])^{\oplus \dim V}$

Since $K' \otimes D_{dR,K}(V) = (K'[0])^{\oplus \dim V}$, we get it. ☒.

Example: There exist HT-reps that are not deRham.

Pf (sketch):

Consider an extension in $Rep_{\mathbb{Q}_p}(G_K)$:

$$(\dagger) \quad 0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p(1) \to 0$$

Such extensions are classified by $H^1_{cont}(G_K, \mathbb{Q}_p(-1))$.

By Tate duality (look-up Euler systems in Rubin's book):

$$\dim_{\mathbb{Q}_p} H^0(G_K, \mathbb{Q}_p(-1)) + \dim_{\mathbb{Q}_p} H^2(G_K, \mathbb{Q}_p(-1)) - \dim_{\mathbb{Q}_p} H^1(G_K, \mathbb{Q}_p(-1)) = -1$$

So $\dim_{\mathbb{Q}_p} H^1_{cont}(G_K, \mathbb{Q}_p(-1)) \geq 1$ (in fact $=1$).

So there exists a non split $(\dagger)$, Fix such a V.

Claim: V is HT:

pf $H^1_{cont}(G_K, \mathbb{C}_K(-1)) = \{0\}$, so $V_{\mathbb{C}_K} \simeq \mathbb{C}_K \oplus \mathbb{C}_K(1)$, which is HT. ☒

Claim: V is not dR:

"pf" Since dR ⟺ pot. sst, we can find K'/K with V = sst $G_{K'}$-rep, with K' finite over K.

Now, we have a fully faithful exact functor $D_{st,K'}: Rep^{sst}_{\mathbb{Q}_p}(G_K) \to MF^{\phi,N}_{K'}$

We'll see later that $D_{st}((\dagger))$ is split

To improve on D_{dR}: search for a "finer" period rings $B \subseteq B_{dR}$ with more structure (eg Frob).

Correspondingly, need to enrich the target Fil_K.

The motivation is geometry, of course:

Let $\mathfrak{X}$ = smooth, proper $\mathcal{O}_K$-scheme, $\mathcal{O}_K \subseteq K$ the valuation ring.
Write $X := \mathfrak{X}_K$, and $\mathfrak{X}_0 := \mathfrak{X}_k$ ← smooth, proper over k.

Then we have $H^i_{cris}(-/W(k)) : \left\{ \begin{matrix} \text{smooth + proper} \\ k\text{-schemes} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{finite } W(k)\text{-modules} \\ \text{(maybe not free)} \end{matrix} \right\}$.

So if $D := H^i_{cris}(\mathfrak{X}_0/W(k))[\frac{1}{p}]$, then D is a finite-dim $K_0 = W(k)[\frac{1}{p}]$-vect-or space equipped with a σ-semilinear Frobenius $\phi : D \to D$,

where σ = Frob on K_0.

Moreover, we have a comparison isomorphism:

$$H^i_{cris}(\mathfrak{X}_0/W(k))[\tfrac{1}{p}] \otimes_{K_0} K \simeq H^i_{dR}(X/K)$$

So $D_K := D \otimes_{K_0} K$ is an object in Fil_K (coming from the Hodge filtration).

However, Frob. doesn't extend from D to D_K!

Def: Let K be a p-adic field. A <u>filtered ϕ-module over K</u> is a triple $(D, \phi, \mathrm{Fil}^\bullet)$, where D = fin. dim K_0-vectorspace, ϕ is a σ-semilinear bijective endomorphism of D, and $\mathrm{Fil}^\bullet$ is a separated + exhaustive filtration on $D_K := D \otimes_{K_0} K$.

<u>Rk</u>: There's so far no compatibility btw Frob and $\mathrm{Fil}^\bullet$

Def: A morphism $(D_1, \phi_1, Fil_1^\bullet) \to (D_2, \phi_2, Fil_2^\bullet)$ is a K_0-linear map

$D_1 \to D_2$ that intertwines ϕ_1, ϕ_2 and has a scalar extension $D_{1K} \to D_{2K}$

which is a morphism in Fil_K.

We call the resulting category MF_K^ϕ.

HW: If $\phi: D \to D$ w/ D = fin-dim K_0-vs and ϕ is σ-semilinear + injective,

then ϕ is bijective.

The category MF_K^ϕ has notions of Ker, coker, Im, Coim, $\otimes$, duality, Hom,

exact sequences, ...

Examples: Given $\alpha: D' \to D$

$$\begin{tikzcd}
0 \arrow[r] & \ker(\alpha) \arrow[r] \arrow[d] & D' \arrow[r, "\alpha"] \arrow[d, "\phi'_{(bij)}"] & D \arrow[r] \arrow[d, "\phi_{(bij)}"] & \mathrm{Coker}(\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(\alpha) \arrow[r] & D' \arrow[r] & D \arrow[r] & \mathrm{Coker}(\alpha) \arrow[r] & 0
\end{tikzcd}$$

$Fil^\bullet(\ker \alpha)$ is the subspace filtration, $Fil^\bullet(\mathrm{coker}(\alpha))$ the quotient filt.

- Short exact sequences:

$$0 \to D' \to D \to D'' \to 0$$

where the morphisms are in MF_K^ϕ

+ exact in Fil_K after $\otimes_{K_0} K$.

- Internal Hom: for $D, D' \in MF_K^\phi$, define $\mathrm{Hom}(D,D') \in \mathrm{Ob}\, MF_K^\phi$, as:

$\mathrm{Hom}_{K_0}(D,D')$ as vectorspace, with Frob $\phi = (\alpha \mapsto \phi_{D'} \circ \alpha \circ \phi_D^{-1})$.

Note $\mathrm{Hom}_{K_0}(D,D')_K = \mathrm{Hom}_K(D_K, D'_K)$, which can be given the Hom-Fil in Fil_K.

Later, we'll construct a $K_0[G_K]$-subalgebra $B_{cris} \subseteq B_{dR}$, containing $W(R)[\frac{1}{p}]$, and equipped with an injective, σ-semilinear end. $\varphi: B_{cris} \to B_{cris}$.

Also, the natural map $K \otimes_{K_0} B_{cris} \to B_{dR}$ will be injective.

We'll then have $K \otimes_{K_0} B_{cris}^{G_K} \hookrightarrow B_{dR}^{G_K} = K$, so $B_{cris}^{G_K} = K_0$, and we will see that $B_{cris} = (\mathbb{Q}_p, G_K)$ - regular.

Our formalism will yield a faithful functor:

$$D_{cris} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vec}_{K_0}$$

given by $D_{cris}(V) := (V \otimes_{\mathbb{Q}_p} B_{cris})^{G_K}$

In fact, $D_{cris}(V) \in \mathrm{Ob}\, MF_K^{\phi}$, with ϕ induced by the Frob. on B_{cris}, with filtration on $D_{cris}(V) \otimes_{K_0} K \hookrightarrow D_{dR}(V)$ (given by the inclusion $K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR}$) given by the subspace filtration.

We'll get a faithful functor:

$$D_{cris} : \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(G_K) \longrightarrow MF_K^{\phi}$$

An annoying fact is that MF_K^{ϕ} is not abelian.

However, $D_{cris}|_{\mathrm{Rep}_{\mathbb{Q}_p}^{cris}(G_K)}$ factors through a subcategory $MF_K^{\phi, w.a.}$ (weakly-admissible), which is an abelian full subcategory of MF_K^{ϕ}.

· To define $MF_K^{\phi,wa}$, we need:

1) Hodge polygon : a convex polygon encoding information about $Fil^\bullet(D_K)$.

2) Newton Polygon : " " " " " " ϕ.

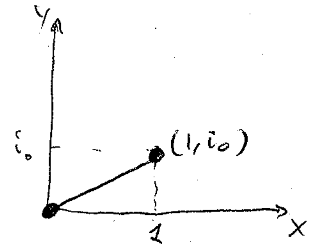
Def: Let F be a field, $D \in Fil_F$. Let $i_0 < i_1 < \dots < i_r$ be the distinct "HT-weights" (i.e. those i for which $gr^i(D) \neq 0$).

The Hodge polygon of D is $P_H(D)$ is the polygon in the $(x-y)$ plane with leftmost endpoint $(0,0)$ and with $\dim_F gr^{i_j}(D)$ segments of horizontal length 1 and slope i_j $(j=0,1,\dots,r)$.

If $D=0$, $P_H(D) := (0,0)$.

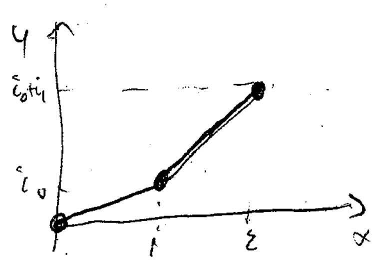
NB: the rightmost endpoint of $P_H(D)$ is $\sum_{i\in\mathbb{Z}} i \dim_F gr^i(D) =: t_H(D)$ ← y-coordinate, with x-coordinate $\dim D$.

Ex: if $\dim D = 1$, $\exists!\, i_0$ st $gr^{i_0}(D) \neq 0$.

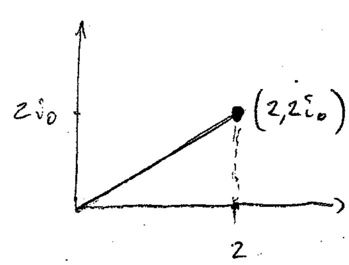


if $\dim_F D = 2$,

a) $\exists\, i_0 < i_1$ w/ $\dim gr^{i_j}(D) \neq 0$



b) $\exists!\, i_0$ w/ $\dim gr^{i_0} D \neq 0$



Lemma: If $D \in \mathrm{Fil}_F$ has dim $d > 0$, let $\det D := \Lambda^d D$, w/ filtration the quotient filtration of $D^{\otimes d}$, then:

$$t_H(D) = t_H(\det D).$$

Pf/ Write:

$$0 \subsetneq \mathrm{Fil}^{i_0} D \subsetneq \mathrm{Fil}^{i_1} D \subsetneq \cdots \subsetneq \mathrm{Fil}^{i_r} D = D$$

$i_0 \ldots i_r$ HT-weights. but now we write it as an increasing filtration!

Choose a bases:

$$\underbrace{e_1, \ldots, e_{j_0}}_{\substack{\text{basis for} \\ \mathrm{Fil}^{i_0} D}};\ e_{j_0+1}, \ldots, e_{j_0+j_1};\ \cdots;\ \cdots\ e_{j_0+j_1+\cdots+j_r}$$

We have:

$$\mathrm{Fil}^i D^{\otimes d} = \sum_{\alpha_1+\cdots+\alpha_d=i} \mathrm{Fil}^{\alpha_1} D \otimes \cdots \otimes \mathrm{Fil}^{\alpha_d} D.$$

Since $\mathrm{Fil}^i(\Lambda^d D) = \dfrac{\mathrm{Fil}^i(D^{\otimes d}) + A}{A}$ (write $\Lambda^d D = D^{\otimes d}/A$)

then the terms in $\mathrm{Fil}^i(D^{\otimes d})$ have 0 in $\Lambda^d D$ unless

$$\underbrace{\mathrm{Fil}^{i_0} D \otimes \cdots \otimes \mathrm{Fil}^{i_0} D}_{j_0} \otimes \cdots \otimes \underbrace{\mathrm{Fil}^{i_r} D \otimes \cdots \otimes \mathrm{Fil}^{i_r} D}_{j_r}$$

Thus there is a unique jump in the filtration of $\Lambda^d D$ is at:

$$i_0 j_0 + \cdots + i_r j_r$$

But $j_s = \dim_F gr^s(D)$, so $t_H(\Lambda^d D) = \sum_{i \in \mathbb{Z}} i \dim_F gr^i(D)$ ∎

For D', D in Fil_F with dimensions d' and d, we have isomorphisms in Fil_F: $\det(D^*) \simeq (\det(D))^*$ and $\det(D)^{d'} \otimes \det(D')^{d} \simeq \det(D \otimes D')$

Also, for $0 \to D' \to D \to D'' \to 0$ in Fil_F, we have:

$$\det(D) \simeq \det(D') \otimes \det(D'') \quad \text{in } \mathrm{Fil}_F.$$

<u>Prop</u>: For $D \in \mathrm{Fil}_F$, $t_H(D^*) = -t_H(D)$

and, for $D, D' \in \mathrm{Fil}_F$,

$$t_H(D \otimes D') = (\dim D)\, t_H(D') + (\dim D')\, t_H(D).$$

and, for $0 \to D' \to D \to D'' \to 0$,

$$t_H(D) = t_H(D') + t_H(D'')$$

Pf/ exercise.

Next, we want to define $P_N(D)$, the Newton polygon.

<u>Def</u>: An <u>isocrystal</u> over K_0 is a finite-dimensional K_0-vectorspace with a σ-semilinear bijective endomorphism $\phi: D \to D$.

<u>Preliminary definition</u>: We assume first that k (= residue field) is finite, $\#k = q = p^r$.

<u>Def</u>: The slopes of $\phi: D \to D$ are the ratios (counted with multiplicity):

$$\frac{\mathrm{ord}_p(\lambda)}{\mathrm{ord}_p(q)} \quad \text{for } \lambda = \text{eigenvalues of } \phi^r \text{ (in } \overline{K_0}\text{)}.$$

(This is independent of the choice of k, up to finite extensions).

In general, we'll use the Dieudonné-Manin classification of isocrystals.

Def: Given $\alpha \in \mathbb{Q}$, $\exists!\, r>0$, $s \in \mathbb{Z}$ with $(r,s)=1$ and $\alpha = r/s$.

$$D_\alpha := K_0[\phi] / (\phi^r - p^s) \qquad \text{where } K_0[\phi] = \text{"twisted polynomial ring" with } \phi \cdot c = \sigma(c)\cdot\phi \ \forall c \in K_0.$$

Clearly, $D_\alpha = D_{r,s}$ is a fin. dim'l K_0-vectorspace of dim r, and $\phi : D_\alpha \to D_\alpha$ given by left multiplication by ϕ is σ-semilinear + bijective.

Hence D_α is an isocrystal$/K_0$. Morally, the slopes on $D_{r,s}$ are $\frac{s}{r}$, with multiplicity $r = \dim_{K_0} D_{r,s}$

Theorem (Dieudonné-Manin): If $k = \bar{k}$, then the category of isocrystals over K_0 is semisimple, with simple objects D_α, for $\alpha \in \mathbb{Q}$ (up to iso).

For general k, we have a functor

$$\text{Isocrystals}/K_0 \longrightarrow \text{Isocrystals}/\widehat{K_0^{un}} = W(\bar{k})[\tfrac{1}{p}]$$

$$(D,\phi) \longmapsto \left(\widehat{K_0^{un}} \otimes_{K_0} D,\ \sigma \otimes \phi\right) =: (D', \phi')$$

Thus, for an isocrystal D/K_0, we get its <u>isotypic decomposition</u>:

$$D' = \bigoplus_{\alpha \in \mathbb{Q}} D'(\alpha) \quad , \text{ where } D'(\alpha) = D_\alpha^{\oplus e_\alpha} \leftarrow \text{have pure slope } \alpha.$$

Def: The <u>slopes</u> of D are those $\alpha \in \mathbb{Q}$ for which $D'_\alpha \neq 0$, and the <u>multiplicity</u> of α is $\dim_{\widehat{K_0^{un}}} D'(\alpha) = r \cdot e_\alpha \quad (\alpha = s/r)$.

Def: The Newton polygon $P_N(D)$ of an isocrystal D over K_0 is defined as follows;

Let $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ be the slopes of D, with multiplicity $\mu_0, \mu_1, \ldots, \mu_n$.

Then $P_N(D)$ is the convex polygon in the (x,y)-plane, with left endpoint $(0,0)$, followed by μ_i segments of horizontal length 1 and slope α_i, for $i = 0, 1, \ldots, n$.

If $D = \{0\}$, $P_N(D) = \{(0,0)\}$.

N.B. All corners of $P_N(D)$ are in $\mathbb{Z}^2$. $\left(\mu_i \cdot \alpha_i = r \cdot e_{\alpha_i} \cdot \frac{s}{r} = s \cdot e_{\alpha_i} \in \mathbb{Z}\right)$.

Def: The rightmost endpoint of $P_N(D)$ has height:

$$t_N(D) = \sum_{i=0}^{n} \alpha_i \underbrace{\dim_{K_0^{un}} D(\alpha_i)}_{\mu_i}$$

Prop 7.2.5: For any isocrystal D/K_0, $t_N(D) = t_N(\det D)$.

Also, $t_N(D^*) = -t_N(D)$, $t_N(D \otimes D') = (\dim D) t_N(D') + (\dim D') t_N(D)$, for D' another isocrys$/K_0$, and also t_N is additive on short-exact seq.

Pf HW.

Example: How do $P_N(D)$ and $P_H(D)$ relate for $D \in MF_K^{\phi}$?

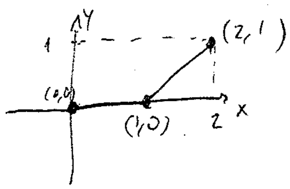
Let E/K be an elliptic curve with good reduction. Let $\mathcal{E}/\mathcal{O}_K$ be its integral model.

Then $\mathcal{E}_0 = \mathcal{E}_k$ = closed fiber, is an elliptic curve as well.

$$D := H^1_{cris}(\mathcal{E}_0/W(k))\left[\tfrac{1}{p}\right] \in \text{iso}/K_0 \quad \text{Then } D \otimes_{K_0} K \simeq H^1_{dR}(E/K) \in \mathrm{Fil}_K$$

(cont example)

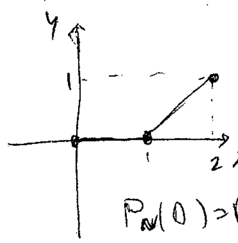
In this case (recall the Hodge filtration $0 \to H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/K}) \to H^1_{dR} \to H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) \to 0$) has gr^0, gr^1 of dim 1. So:



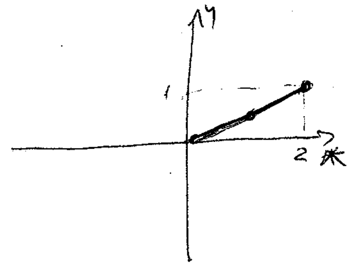
$\leftarrow P_H(D)$

For $P_N(D)$, we use that $H^1_{cris}(\mathcal{E}_0/W(k)) \simeq \mathbb{D}(\mathcal{E}_0[p^\infty])$ (Dieudonné module). Using Dieudonné theory, we get that D has slopes:

$$\begin{cases} 0,\ 1 & \text{with } \cancel{\text{mult}}\ 1 \quad \text{if } \mathcal{E}_0 \text{ is ordinary.} \\ \frac{1}{2} & \text{with mult } 2 \quad \text{if } \mathcal{E}_0 \text{ is supersingular.} \end{cases}$$



$P_N(D) = P_H(D)$



Note that $P_N(D)$ lies on or above $P_H(D)$, and $t_N(D) = t_H(D)$.

<u>Lemma (7.2.7)</u> If $D/K_0 \neq 0$ is an isocrystal with slopes $\alpha_0 < \alpha_1 < \dots < \alpha_n$, then there is a unique decomposition <u>over K_0</u>:

$$D = \bigoplus D(\alpha_i), \quad D(\alpha_i) \text{ subobjects of } D$$

s.t. $D(\alpha_i) \otimes_{K_0} \widehat{K_0^{un}}$ has pure slope α_i (one says that $D(\alpha_i)$ is isoclinic of slope α_i).

Pf We have $D' = \oplus D'(\alpha_i)$ with $D'(\alpha_i)$ isoclinic of slope α_i. We also have an action of $G_k = G_K/I_K = G_{\widehat{K_0^{un}}}$ on $D' = \widehat{K_0^{un}} \otimes_{K_0} D$, with $(D')^{G_k} = D$.

(cont of pf of lemma):

Claim: $D'(\alpha_i)$ is G_K-stable

Pf Indeed, $D'(\alpha_i)$ is spanned by the images of all maps $D_{\alpha_i} \to D'$ as isocrystals (i.e by all $v \in D'$ s.t $\phi^{r_i} v = p^{s_i} v$, $\alpha_i = s_i/r_i$).

Since G_K commutes with ϕ, we get the claim. ∎

By "completed unramified descent" (look it up in Chap. I in Conrad's notes), we get $\cancel{(\widehat{K_0^{un}} \otimes_{K_0} D'(\alpha_i))^{G_K} \simeq D(\alpha_i)}$ $\widehat{K_0^{un}} \otimes D'(\alpha_i)^{G_K} \simeq D'(\alpha_i)$

So we define $D(\alpha_i) := D'(\alpha_i)^{G_K} \subseteq D'^{G_K} = D$.

Now, $\bigoplus D(\alpha_i) \to D$ is an isomorphism after $- \otimes \widehat{K_0^{un}}$, hence it's an iso over K_0 as well. ∎

Theorem (Berthelot-Ogus): Let D be the filtered ϕ-module attached to $\mathfrak{X}/\mathcal{O}_K$, a smooth, proper scheme over $\mathcal{O}_K$. Then:

$$P_N(D) \text{ lies on or above } P_H(D), \text{ and } t_N(D) = t_H(D).$$

Lemma (Fontaine): For $D \in MF_K^{\phi}$, TFAE:

i) For all subobjects $D' \subseteq D$, $P_N(D')$ lies on or above $P_H(D')$.

ii) For all subobjects $D' \subseteq D$, $t_N(D') \geq t_H(D')$.

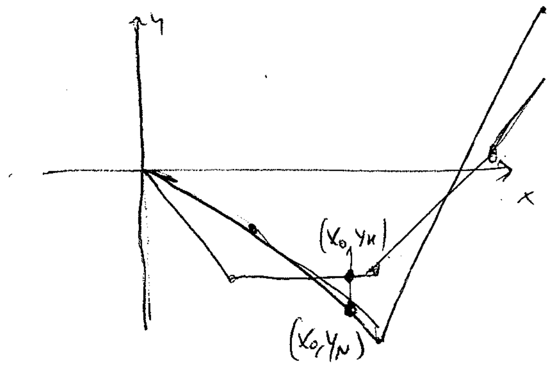
Moreover, these hold iff they do after extending scalars to $\widehat{K_0^{un}}$.

Pf (i ⟹ ii) Obvious.

Now, given $D' \subseteq D$ with $P_N(D')$ below $P_H(D')$ at some $x = x_0$, we want $D'' \subseteq D$ with $t_N(D'') < t_H(D'')$.

(cont pf of lemma)

We may assume now that $t_N(D') \geq t_H(D')$ (otherwise, we are done).



By continuity, we may assume first that x_0 isn't a corner of P_N or P_H.
So P_N and P_H have well-defined slopes at x_0.
By convexity and moving to the left or to the right depending on which slope is bigger, we may assume that (x_0, y_N) is the final point of D' with slope α_r ($y_N < y_H$).

Consider now $D' = \bigoplus_{\alpha \in \mathbb{Q}} D(\alpha)$, and set $D'' := \bigoplus_{\alpha \leq \alpha_r} D(\alpha)$, giving $D''_K \subseteq D'_K$ the subspace filtration.

Then, by construction, $t_N(D'') = y_N$.
On the other hand, the k^{th} jump on the filtration of D''_K is $\geq$ k^{th} jump on Fil for D'_K (bc D''_K is a subobject of D'_K).
That is, $P_H(D'')$ is on or above $P_H(D')$ for $0 \leq x \leq \dim D''$.
So $t_H(D'') \geq t_H(D') = y_H$.
Hence $t_N(D'') = y_N < y_H \leq t_H(D'')$, as we wanted.
For the second part, just note that P_N and P_H do not change when we extend scalars to $\widehat{K_0^{un}}$.

Def: We say that $D \in MF_K^\phi$ is weakly admissible (w.a.) if $t_N(D') \geq t_H(D')$ for all subobjects $D' \subseteq D$ (in MF_K^ϕ) with equality for $D' = D$.

Note: $\{\text{subobjects } D' \subseteq D\} \xleftrightarrow{1:1} \{\text{quotient objects } D \twoheadrightarrow D''\}$

$$D' \longmapsto D'' = D/D'$$

Since t_* is additive in s.e.s, w.a. is equivalent to asking that $t_N(D'') \leq t_H(D'')$ for all quotients D'' of D, with equality if $D'' = D$.

HW: Show that $D \in MF_K^\phi$ is w.a. iff D^* is w.a.

We want to enlarge MF_K^ϕ.

Motivation: Consider $\rho: G_K \to GL(W)$, with $W = \mathbb{Q}_\ell$-vectorspace, and focus on $\rho|_{I_K}$. Since I_K^{wild} = pro-p, $\rho(I_K^{wild})$ = finite, so $\exists\, K'/K$ finite s.t. $\rho(I_{K'}^{wild}) = 1$.

Hence $\rho|_{I_{K'}}$ factors through $I_{K'}^t = I_{K'}/I_{K'}^{wild}$

Thm (Grothendieck): $\rho|_{I_{K'}^t}: I_{K'}^t \longrightarrow GL(W)$ is given by:

$\rho(g) = \exp\left(t_\ell(g) \cdot N_{K'}\right)$, where
- $t_\ell: I_{K'}^t \twoheadrightarrow \mathbb{Z}_\ell(1)$ is the ℓ-comp. of the iso. $I_{K'}^t \overset{\pi}{\simeq} \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ (choose a unif.)
- $N_{K'} \in \mathrm{Hom}(W, W(-1))$ is nilpotent.

By setting $N := \frac{1}{e(K'/K)} N_{K'}$, we get a nilpotent $N \in \mathrm{Hom}(W, W(-1))$ that is independent of K'.

"Moral": N encodes $\rho|_{I_K}$. (unramified $\Leftrightarrow N=0$).

This motivates the following:

Def: A (ϕ, N)-module over K_0 is an isocrystal (D, ϕ) over K_0, with a K_0-linear map $N : D \to D$ (called the monodromy operator) such that $N\phi = p\,\phi N$.

A morphism of (ϕ,N)-modules is a morphism of isocrystals that respects N.

The category of (ϕ,N)-modules is written $\mathrm{Mod}_{K_0}^{(\phi,N)}$ $\left(\text{Isocrystals} = \mathrm{Mod}_{K_0}^{(\phi,0)}\right)$.

Def: A filtered (ϕ,N)-module over K is a (ϕ,N)-module D over K_0, such that D_K has the structure of an object in Fil_K.

The morphisms are those maps of (ϕ,N)-modules whose extension $\otimes_{K_0} K$ is a morphism in Fil_K.

The category of filtered (ϕ,N)-modules over K is written $MF_K^{\phi,N}$.

N.B. MF_K^{ϕ} is the full subcategory of $M_F^{\phi,N}$ with $N=0$.

We have: Ker, coker, im, coim, s.e.s, in $\mathrm{Mod}_{K_0}^{\phi,N}$ & $\mathrm{Mod}_K^{\phi,N}$.

Also, we have duals, $\otimes$, internal Homs, with monodromy.

e.g. $N_{D\otimes D'} = 1_D \otimes N_{D'} + N_D \otimes 1_{D'}$; $N_{D^*} = -N_D^*$; $N_{\mathrm{Hom}(D,D')}(L) = N_{D'} \circ L - L \circ N_D$

The unit object of $MF_K^{\phi,N}$ is $(K_0, \overset{\phi}{\overset{\shortparallel}{\sigma}}, \overset{N}{\overset{\shortparallel}{0}})$ with trivial filtration on $K = K_0 \otimes_{K_0} K$ ($Fil^0 = K$, $Fil^1 = 0$).

These definitions behave nicely as we would expect.

Example: We can consider $D \in MF_K^{\phi,N}$, and consider the isoclinic decomposition:

$$D = \bigoplus_{\alpha \in \mathbb{Q}} D(\alpha)$$

where $D(\alpha)$ is spanned by all $v \in D$ s.t. $\phi^r v = p^s v$ for $\alpha = \frac{s}{r}$, $r \geq 0$, $(s,r)=1$

Since $\phi \circ N = \frac{1}{p} N \circ \phi$, it is an easy computation to see

$$N(D(\alpha)) \subseteq D(\alpha - 1)$$

Hence N is nilpotent as only finitely-many $D(\alpha) \neq 0$.

Note: if $\dim D = 1$, $N_D = 0$!

Def: $D \in MF_K^{\phi,N}$ is weakly-admissible if for all subobjects $D' \subseteq D$ (in $MF_K^{\phi,N}$) we have $t_N(D') \geq t_H(D')$, with equality when $D = D'$.

Warning: There is a functor $MF_K^{\phi,N} \to MF_K^{\phi,N}$

$$(D, \phi, N, Fil) \to (D, \phi, 0, Fil)$$

but this doesn't take w.a. to w.a ! There are more subobjects (in general) when $N=0$ than when $N \neq 0$.

We define then:

Def: $MF_K^{\phi,N,w.a}$ = full subcategory of w.a. objects in $MF_K^{\phi,N}$
$MF_K^{\phi,wa}$ = " " — " MF_K^{ϕ}.

Prop: If $0 \to D' \to D \to D'' \to 0$ is short-exact in $MF_K^{\phi,N}$ and any two terms are w.a, so is the third.

To prove this, we need:

Lemma: If $h: M' \to M$ is a bijective morphism in Fil_K, then $t_H(M') \leq t_H(M)$, and equality holds iff it is an iso in Fil_K.

Pf First, h is iso $\Leftrightarrow$ $\Lambda^d h$ is iso, for $d = \dim M = \dim M'$.
Using that $t_H(\det(-)) = t_H(-)$, we are reduced to $d=1$.
In this case, $t_H(M)$ is the unique i such that $gr^i(M) \neq 0$.
In fact, $gr^i(M) = M$, then.
Since h is iso $\Leftrightarrow$ it is on $gr^\bullet$ and the induced maps on $gr^\bullet$ is iso $\Leftrightarrow$ $t_H(M) = t_H(M')$, we are done. ∎

Pf (of prop): $0 \to D' \to D \to D'' \to 0$

1) If D, D'' are w.a: if $D_1 \subseteq D'$ in $MF_K^{\phi,N}$, we view $D_1 \subseteq D$ in $MF_K^{\phi,N}$, so $t_N(D_1) \geq t_H(D_1)$ as D is w.a. For $D_1 = D'$, by additivity of t_*, we get $t_N(D') = t_N(D) - t_N(D'') = t_H(D) - t_H(D'') = t_H(D')$ ✓.

2) If D', D are w.a, dualize ~~to get~~ and use D'' wa $\Leftrightarrow (D'')^*$ wa, and (1).

3) D', D'' w.a. Let $D_1 \subseteq D$ be a subobject ($D_1 = D$ is done). Set $D_1' := D_1 \cap D'$ w/ subspace filtration. $D_1' \subseteq D'$ is a subobject. Also, let $D_1'' := D_1/D_1'$ with the quotient filtration from $(D_1)_K$.

(cont of pf of prop)

We have a bijective morphism in $MF_K^{\phi,N}$ $j: D_1'' \hookrightarrow D''$, and we let $j(D_1'') = \operatorname{im} j$ (with subspace filtration from D''_K).

We have:

$t_H(D_1'') \leq t_H(j(D_1''))$ by the lemma.

Also, $t_N(D_1'') = t_N(j(D_1''))$ as j is an iso of isocrystals

We calculate:

$$t_H(D_1) = t_H(D_1') + t_H(D_1'') \overset{D_1' \subseteq D_1^{wa} \text{ is a subobject}}{\leq} t_N(D_1') + t_H(D_1'') \overset{\text{lemma}}{\leq} t_N(D_1') + t_H(j(D_1'')) \overset{D'' \text{ wa}}{\leq}$$

$$\leq t_N(D_1') + t_N(j(D_1'')) = t_N(D_1') + t_N(D_1'') = t_N(D_1) \checkmark$$

$\boxtimes$.

<u>Theorem</u>: Let $h: D' \to D$ be a morphism in $MF_K^{\phi,N\text{-}wa}$. Then h is <u>strict</u> (the induced map coim $\to$ im is an iso), and $\ker(h)$, $\operatorname{coker}(h)$, $\operatorname{im}(h) \simeq \operatorname{coim}(h)$ are w.a.

(i.e. $MF_K^{\phi,N\,wa}$ is an abelian category).

Pf: We have a diagram in $MF_K^{\phi,N}$ (don't know wa yet!)

$$\underset{\text{sub fil}}{\ker(h)} \hookrightarrow D' \twoheadrightarrow \underset{\text{quo fil}}{\operatorname{coim}(h)} \to \underset{\text{sub fil}}{\operatorname{im}(h)} \hookrightarrow D \twoheadrightarrow \underset{\text{quo fil}}{\operatorname{coker}}$$

Since coim $\to$ im is a bijective morphism in $MF_K^{\phi,N}$, the lemma implies $t_H(\operatorname{coim}) \leq t_H(\operatorname{im})$ (with $= \Leftrightarrow \cong$).

Hence $t_N(\operatorname{coim}) \underset{\substack{\text{coim is a quo} \\ \text{of w.a } D'}}{\leq} t_H(\operatorname{coim}) \leq t_H(\operatorname{im}) \underset{\substack{\text{im is sub} \\ \text{of w.a } D}}{\leq} t_N(\operatorname{im})$

But coim $\to$ im is an iso of isocrystals, so $t_N(\operatorname{coim}) = t_N(\operatorname{im})$, and all ineq are $=$.

(finish pf of Thm)

So $t_N(\mathrm{im}) = t_H(\mathrm{im}) = t_H(\mathrm{coim}) = t_N(\mathrm{coim})$, so the previous lemma implies that $\mathrm{im} \simeq \mathrm{coim}$.

Also, $\mathrm{im} \simeq \mathrm{coim}$ is weakly-admissible.

Finally, ~~the~~ ~~maps~~ have $0 \to \ker h \to D' \to \mathrm{coim}\, h \to 0$ and $0 \to \mathrm{im}\, h \to D \to \cdots$

so by the prop we are done. ∎

Example: Calculation of $D_{cris}(\mathbb{Q}_p(n)) \in MF_K^{\varphi, N, w.a.}$

Recall: $B_{cris} \subseteq B_{dR}$ is a $K_0[G_K]$-subalgebra, containing $W(R)[\frac{1}{p}]$, (rec: $K_0 = W(k)[\frac{1}{p}]$)

So (since $\bar{k} \hookrightarrow R$) B_{cris} contains $W(\bar{k})[\frac{1}{p}] = \widehat{K_0^{ur}}$.

We'll also see later that $t^n \in B_{cris}$, $\forall n$. ($t = \log([\varepsilon])$, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$)

Observe that, since $\varphi : B_{cris} \to B_{cris}$ is compatible with $\sigma : \widehat{K_0^{ur}} \to \widehat{K_0^{ur}}$, then we have:

$$\varphi(t^n) = p^n \cdot t^n$$

Thus:

$$D_{cris}(\mathbb{Q}_p(n)) \overset{def}{=} \left(B_{cris} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)\right)^{G_K}$$

has basis: $t^{-n} \otimes 1$

just need to know $D_{HT}(\mathbb{Q}_p(n))$

So $\mathbb{Q}_p(n)$ is crystalline, and $D_{cris}(\mathbb{Q}_p(n)) = \left(K_0, p^{-n}\sigma, N=0, Fil^j K = \begin{cases} K & j \leq -n \\ 0 & \text{else} \end{cases}\right)$

In contravariant terms, let $D^*_{cris}(-) := \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(-, B_{cris})$, then:

$$D^*_{cris}(\mathbb{Q}_p(n)) = \left(K_0, p^{-n}\sigma, N=0, Fil^j K = \begin{cases} K & j \leq n \\ 0 & j > n \end{cases}\right).$$

The previous example motivates the following

<u>Def</u>: Let $(D, \phi, N, Fil^\bullet) \in MF_K^{\phi,N}$. The <u>$i$-fold Tate twist</u> of D is:

$$D\langle i\rangle := \left(D,\ p^{-i}\phi,\ N,\ Fil^r(D\langle i\rangle_K) = Fil^{r+i}(D_K)\right)$$

Observe that: Newton or Hodge polygons

$P_*(D\langle i\rangle)$ is obtained from $P_*(D)$ by decreasing all slopes by i,

so that: $t_*(D\langle i\rangle) = t_*(D) - i\cdot\dim D$.

<u>Note</u>: There is a bijection:

$$\left\{\begin{matrix}\text{subobjects}\\ D' \subseteq D\langle i\rangle\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{subobjects}\\ D'' \subseteq D\end{matrix}\right\}$$

(ie $D\langle i\rangle$ is w.a $\Leftrightarrow$ D is).

By construction,

$$D^*_{cris}\left(V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(i)\right) = D^*_{cris}(V)\langle i\rangle$$

- <u>Classification of 1-dimensional filtered (ϕ, N)-modules $/K$ which are w.a.</u>

First, since N is nilpotent, $N=0$.

Write $D = K_0\cdot e$, and $\phi(e) = \lambda e$, $\lambda \in K_0^\times$.

If we replace e by $c\cdot e$, $c \in K_0^\times$, then $\phi(c\cdot e) = \left(\frac{\sigma(c)}{c}\lambda\right)(ce)$, so at least $ord_p(\lambda)$ is independent of e (and $t_N(D) = ord_p(\lambda)$!).

By Tate-twisting, we can assume that $gr^0(D_K) \neq 0$ $\left(\text{i.e. } Fil^0(D_K) = D_K,\ Fil^1(D_K) = 0\right)$.

In this case, w.a holds $\Leftrightarrow$ $\overset{0}{\overset{\shortparallel}{t_H(D)}} = t_N(D) = ord_p(\lambda)$. So $\lambda \in W(k)^\times$.

So we get:

$$\left\{\begin{array}{l}1\text{-dimensional}\\ D\in MF_K^{\phi,N,w.a}\\ \text{up to twist}\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\lambda\in W(k)^\times\\ \text{with } \lambda \text{ up to}\\ \text{mult by } \frac{\sigma(c)}{c},\\ \text{for } c\in W(k)^\times\end{array}\right\}$$

Consider now:

$$0\to \underset{\left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times}{\underset{\shortparallel}{W_n(\mathbb{F}_p)^\times}} \longrightarrow W_n^\times \longrightarrow W_n^\times \longrightarrow 1 \qquad (\text{of alg. gps }/k)$$

$$x \longmapsto \wp(x) := \frac{\sigma(x)}{x}$$

By passing to k-points, get:

$$\frac{W_n(k)^\times}{\operatorname{im}(\wp)} \xrightarrow{\sim} H^1_{cont}\left(G_K, \left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times\right)$$

Taking inverse limits on n, get:

$$\frac{W(k)^\times}{\operatorname{im}\wp} \simeq H^1_{cont}\left(G_K, \mathbb{Z}_p^\times\right) = Hom_{cont}\left(G_K, \mathbb{Z}_p^\times\right) = Hom^{un}_{cont}\left(G_K, \mathbb{Z}_p^\times\right)$$

That is, unramified continuous characters $\eta: G_K\to \mathbb{Q}_p^\times$.

On the LHS, we have $\lambda\in W(k)^\times$, up to mult by $\sigma(c)/c$.

So:

All crystalline reps which are 1-dim'l are unramified continuous characters.

<u>Lemma</u>: The correspondence $\eta \longmapsto D_\eta \left(\begin{array}{l}= \text{w.a. } D\in MF_K^{\phi,N,w.a.} \text{ parametrized}\\ \text{with } \lambda\in W(k)^\times \text{ assoc. to } \eta\end{array}\right)$.

<u>is</u> the functor D^*_{cris}.

$\downarrow$

Pf (lemma): Let $\eta : G_K \to \mathbb{Z}_p^\times$ be an unramified character.
Via the previous construction

$$W(k)^\times / \mathrm{Im}(\wp) \xrightarrow{\sim} \mathrm{Hom}^{un}_{cts}(G_K, \mathbb{Z}_p^\times)$$

we have $\eta \mapsto \lambda$ where, for any $w \in W(\bar{k})^\times$ s.t. $\frac{\sigma(w)}{w} = \lambda$,

$$\eta(g) = \frac{gw}{w}$$

(and any two such w differ by the image of $\mathbb{Z}_p^\times \to W(\bar{k})^\times$).

So the line $D := K_0 \cdot w \subseteq W(\bar{k})$ is indep of w.

As B_{cris} contains $W(R)[\frac{1}{p}]$, hence it contains also $W(\bar{k})[\frac{1}{p}]$.

So $w \in B_{cris}$, and

$$D^*_{cris}(\mathbb{Q}_p(\eta)) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(\eta), B_{cris})$$

contains $e = (1 \mapsto w)$, so $D^*_{cris}(\mathbb{Q}_p(\eta)) = K_0 \cdot e$

and $gr^i(D_{cris}(\mathbb{Q}_p(\eta))_K) = \begin{cases} 0 & \text{if } i \neq 0 \\ D_{cris}(\mathbb{Q}_p(\eta))_K & \text{else} \end{cases}$

Finally, $\varphi(e) = \lambda \cdot e$, since $\frac{\sigma(w)}{w} = \lambda$.

Thus $D_\eta = D^*_{cris}(\mathbb{Q}_p(\eta))$. ∎

This calculation shows that every $D \in MF_K^{\phi,N,w.a}$ of dim 1 is D^*_{cris} applied to ~~the~~ a Tate twist of an unramified character.
Granting that D^*_{cris} is full, we get an equivalence of categories

$$\left\{ \begin{matrix} \text{1-dim crystalline} \\ \text{reps of } \mathbb{Q}_p \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} MF_K^{\phi,N,w.a.,\text{1-dim'l}}$$

- Classification of all 2-dimensional $D \in MF_K^{\phi,N,w.a.}$, when $K = \mathbb{Q}_p$.

As $K = \mathbb{Q}_p$, then $K_0 = \mathbb{Q}_p$ as well, and ϕ is linear.

Fix $(D, \phi, N, Fil) \in MF_{\mathbb{Q}_p}^{\phi,N,w.a.}$.

By twisting, we can assume $gr^0(D) \neq 0$ (i.e. 0 is the smallest HT-weight).

First, assume $N=0$

1) Only one HT-weight.

Then $Fil^1(D)=0$. So $P_H(D)$ is a horizontal segment on the x-axis.

By weak-admissibility, $P_N(D)$ is the same polygon.

Hence there is a unique slope of ϕ, namely 0. ($\Rightarrow N=0$ in any case!)

and $\phi : D \xrightarrow{\sim} D$ is a $\mathbb{Q}_p$-linear iso, whose char-poly is:

$$f_\phi(X) = X^2 + aX + b \in \mathbb{Z}_p[X]$$

with both roots in $\overline{\mathbb{Z}_p}^\times$ (so the poly f_ϕ has coeffs in $\mathbb{Z}_p$ and not just in $\mathbb{Q}_p$)

Notice: Any subobject $D' \subseteq D$ is ϕ-stable, so $t_N(D')=0$ and $t_H(D')=0$ because D' has the subspace filtration. (in any case, we knew that by weak-admissibility).

(i.e. $P_H(D') = P_N(D')$).

Hence any subobject of D is also weakly-admissible!

Claim: $\exists$ a ϕ-~~stable~~ $\mathbb{Z}_p$-lattice $\Lambda \subseteq D$ on which ϕ acts by an automorphism T.

Pf: If ϕ is a scalar, any $\mathbb{Z}_p$-lattice works. Otherwise, pick $e_1 \in D$ with $\phi(e_1) \notin \mathbb{Q}_p \cdot e_1$. Let $e_2 = \phi(e_1)$ and let $\Lambda := \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2$.

(ϕ has then matrix $\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix} \in M_2(\mathbb{Z}_p)$, and $b \in \overline{\mathbb{Z}_p}^\times \cap \mathbb{Z}_p = \mathbb{Z}_p^\times$, so ϕ acts as an aut.).

$\square$ (Claim)

Thus we have a bijection: $D \rightsquigarrow (\Lambda, T)/\sim$

where $(\Lambda_1, T_1) \sim (\Lambda_2, T_2) \iff \exists\ \mathbb{Q}_p$-linear iso $\theta : \underset{D}{\overset{\shortparallel}{\Lambda_1 \otimes \mathbb{Q}_p}} \xrightarrow{\sim} \underset{D}{\overset{\shortparallel}{\Lambda_2 \otimes \mathbb{Q}_p}}$, carrying T_1 to T_2.

i.e.:

$$\left\{ \begin{array}{l} \text{2-dim'l w.a filtered } (\phi, N)\text{-mod} \\ \text{over } \mathbb{Q}_p \text{ with single HT-weight } 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} GL_2(\mathbb{Q}_p)\text{-conjugacy classes} \\ \text{of } GL_2(\mathbb{Z}_p) \end{array} \right\}$$

2) Two HT-weights: $0, r$ (with $r > 0$).

Set $L := \mathrm{Fil}^1 D$ (1-dim'l), and note $\mathrm{Fil}^i D = \begin{cases} D & i \leq 0 \\ L & 1 \leq i \leq r \\ 0 & i > r \end{cases}$

Let $f_\phi := X^2 + aX + b \in \mathbb{Q}_p[X]$ be the char. poly of ϕ on D.

Note: $t_N(D) = \mathrm{ord}_p(b) = t_H(D) = r$, i.e. $b \in p^r \mathbb{Z}_p^\times$.

Case A: f_ϕ ~~has distinct roots, dist~~ is irreducible $/\mathbb{Q}_p$.

In this case, $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ switches the two roots in $\bar{\mathbb{Q}}_p$, so that they have the same ord_p, i.e $a \in p^{\lceil r/2 \rceil} \cdot \mathbb{Z}_p$

Note: there are no nontriv ϕ-stable subobjects:

b/c f_ϕ is irred $/\mathbb{Q}_p$.

So, in particular, $\phi(L) \neq L$.

Let $e_1 :=$ $\mathbb{Q}_p$-basis of L, $e_2 := \phi(e_1)$. Then D has basis e_1, e_2, (and $[\phi] = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$) and $\mathrm{Fil}^j D = \begin{cases} D & j \leq 0 \\ L & 1 \leq j \leq r \\ 0 & j > r \end{cases}$

This gives a bijection

$$\left\{ \begin{array}{l} \text{w.a 2-dim'l filtered } (\phi, N)\text{-mod}, \\ \text{with } N = 0, /\mathbb{Q}_p, \text{ least HT-weight } 0 \text{ and greatest } r \\ \text{with irreducible } f_\phi \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{pairs } (a, b) \in p^{\lceil r/2 \rceil} \mathbb{Z}_p \times p^r \mathbb{Z}_p^\times \\ \text{s.t } b^2 - 4a \notin \mathbb{Q}_p^2 \end{array} \right\}$$

(continues classification)

→ Two HT-weights $0, r$, $r>0$.

Case B: f_ϕ is reducible over $\mathbb{Q}_p$: $f_\phi(X) = (X-\lambda_1)(X-\lambda_2)$, $\mathrm{ord}(\lambda_1) \leq \mathrm{ord}(\lambda_2)$.

In this case,

$$t_H(D) = r \quad , t_N(D) = \mathrm{ord}(\lambda_1) + \mathrm{ord}(\lambda_2) = r.$$

Case B.1: $\lambda_1 \neq \lambda_2$. We can choose eigenvectors e_1, e_2 for ϕ, with eigenv's λ_1, λ_2.

Then $\mathbb{Q}_p e_1$, $\mathbb{Q}_p e_2$ are the only nontrivial subobjects s.t. $D \simeq \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$

$\mathbb{Q}_p$-vector sp.

Observe then that $t_N(\mathbb{Q}_p e_i) = \mathrm{ord}(\lambda_i)$

Let $L := \mathrm{Fil}^1 D$ (a line in D)

Then $t_H(\mathbb{Q}_p e_i) = \begin{cases} r & \text{if } \mathbb{Q}_p e_i \subseteq L \\ 0 & \text{ow.} \end{cases}$

So by w.a., $t_N(\mathbb{Q}_p e_i) \geq t_H(\mathbb{Q}_p e_i) \geq 0$

Hence $\lambda_i \in \mathbb{Z}_p$, and $\mathrm{ord}(\lambda_1) < r$

$\Rightarrow t_H(\mathbb{Q}_p e_1) = 0$ (b/c w.a forces $t_H(\mathbb{Q}_p e_1) \leq t_N(\mathbb{Q}_p e_1) < r$)

So $\mathbb{Q}_p e_1 \neq L$

Case B.1.1: If $\mathbb{Q}_p e_2 = L$

This can only happen if $\mathrm{ord}_p(\lambda_1) = 0$ (and $\mathrm{ord}_p(\lambda_2) = t_N(\mathbb{Q}_p e_2) = r$)

Then we have $D \simeq \mathbb{Q}_p e_2 \oplus \mathbb{Q}_p e_1$ as w.a filtered (ϕ, N)-modules

(they correspond to $\psi \oplus \psi'(r)$ where ψ, ψ' are unramified chars of $G_{\mathbb{Q}_p}$, which are classified by $\lambda_1 \in \mathbb{Z}_p^\times$, $\lambda_2/p^r \in \mathbb{Z}_p^\times$.

Case B.1.2. If $\mathbb{Q}_p e_2 \neq L$.

We can then scale e_1, e_2 to assume $L = \mathbb{Q}_p(e_1+e_2)$.

Then: $\left\{ \begin{array}{l} \lambda_1, \lambda_2 \in \mathbb{Z}_p \\ \mathrm{ord}_p(\lambda_1) \leq \mathrm{ord}_p(\lambda_2) \\ \text{distinct} \\ \mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = r \end{array} \right\} \overset{1:1}{\rightsquigarrow} \left\{ \begin{array}{l} D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 \\ \varphi e_i = \lambda_i e_i \\ \mathrm{Fil} = \begin{cases} D & j \leq 0 \\ \mathbb{Q}_p(e_1+e_2) & 1 \leq j \leq r \\ 0 & j > r \end{cases} \end{array} \right\}$

(note: $t_N(\mathbb{Q}_p e_2) = \mathrm{ord}(\lambda_2) > 0 = t_H(\mathbb{Q}_p e_2)$

So $\mathbb{Q}_p e_2$ is never a w.a. subobject, and $\mathbb{Q}_p e_1$ is w.a subobject $\Leftrightarrow \lambda_1 \in \mathbb{Z}_p^\times$)

In terms of representations,

$\lambda_1 \notin \mathbb{Z}_p^\times$ corresponds to irred. crystalline 2-dim'l reps, with HT-weights $0, r$.

$\lambda_1 \in \mathbb{Z}_p^\times$ " " extensions (non-split)

$$0 \to \psi'(r) \to V \to \psi \to 0$$

V crystalline

where ψ, ψ' are unramified chars classified by $\lambda_1, \lambda_2/p^r \in \mathbb{Z}_p^\times$.

<u>Case B.2</u> $f_\phi(X) = (X-\lambda)^2$.

Then $t_N(D) = 2\,\mathrm{ord}(\lambda) = t_H(D) = r \implies r$ is even.

Again, let $L = \mathrm{Fil}^1 D$.

<u>Claim</u>: $\phi: D \to D$ is <u>not</u> a scalar.

pf/ If it were, then L would be ϕ-stable, so a subobject. But then, by w.a,

$t_N(L) = \mathrm{ord}(\lambda)$ & $t_H(L) = r = 2\,\mathrm{ord}(\lambda) > 0 \implies !!$ ☒

So the λ-eigenspace of ϕ is 1-dim'l, Let e_1 be a basis.

We can choose e_2 s.t $\{e_1, e_2\}$ is a basis of D, and s.t

$$[\phi] = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Then $L = \mathbb{Q}_p(e_1 + c e_2)$, for some $c \in \mathbb{Q}_p^\times$. So:

$$\left\{ \begin{array}{l} \text{nonzero } \lambda \in p\mathbb{Z}_p \\ c \in \mathbb{Q}_p^\times \\ \mathrm{ord}(\lambda) = r/2 \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 \\ [\phi] = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \\ \mathrm{Fil}^i D = \begin{cases} D & i \le 0 \\ \mathbb{Q}_p(e_1 + c e_2) & 1 \le i \le r \\ 0 & i > r \end{cases} \end{array} \right. \longleftrightarrow \left\{ \begin{array}{l} \text{irreducible crystalline reps} \\ \text{of HT-weights } 0, r, \ r \text{ even} \\ f_\phi(X) = (X-\lambda)^2. \end{array} \right\}$$

and there are <u>no</u> w.a. subobjects, as the only ϕ-stable subspace is $\mathbb{Q}_p e_1$,

but $t_N(\mathbb{Q}_p e_1) = \mathrm{ord}(\lambda) = r/2$, $t_H(\mathbb{Q}_p e_1) = 0$.

Lastly, assume $N \neq 0$.

Let $D \in MF_{\mathbb{Q}_p}^{\phi, N, w.a}$, 2-dim'l, and let α, β be the slopes of ϕ.

By Dieudonné-Manin classification, $D \simeq \begin{cases} D(\alpha) & \text{if } \alpha = \beta \\ D(\alpha) \oplus D(\beta) & \text{else.} \end{cases}$

But $N(D(\alpha)) \subseteq D(\alpha - 1)$, so if $N \neq 0$, we have the second case, ie $\alpha \neq \beta$, and actually $\beta = \alpha - 1$.

So $\lambda_1 \neq \lambda_2$ (the eigenvalues of ϕ) and $\underset{m}{\underset{\shortparallel}{\mathrm{ord}(\lambda_1)}} = \mathrm{ord}(\lambda_2) - 1$

Then $\lambda_1, \lambda_2 \in \mathbb{Q}_p^\times$: they can't be conjugate under $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$!

Write then $L := Fil^1(D)$

Claim: We must have $Fil^1(D) \neq 0$

Pf/ If $Fil^1 D$ were 0, then $\exists!$ HT-weight !! (check two pages ago) ⊠

So L is a line, and there is $r > 0$ s.t. $Fil^j D = L$ for $1 \leq j \leq r$.

Note: $Ker(N) \subseteq D$ is (ϕ, N)-stable. Hence it is a subobject.

As $\phi(Nv) = \frac{\lambda}{p}(Nv)$ for v an eigenvector of ϕ, with eigenvalue λ.

So the eigenvalue of ϕ on the line $Ker\, N$ is necessarily λ_1:
(because an eigenvector for λ_1 is in the $Ker\, N$ and $Ker\, N \neq D$).

Recall that $m := \mathrm{ord}(\lambda_1) = \mathrm{ord}(\lambda_2) - 1$, and $r = t_H(D) = 2m + 1 > 0$.

So $m \geq 0$, and $t_N(Ker\, N) = m$. By w.a, $t_H(Ker\, N) \leq m$.

As $N(\lambda_2\text{-eigenspace}) \subseteq (\lambda_1\text{-eigenspace})$, let e_2 be a basis for λ_2-eig.,
and let $e_1 := N(e_2)$. So $\{e_1, e_2\}$ give a basis of D

We have then:

$$\lambda_2 e_1 = N(\lambda_2 e_2) = N\varphi(e_2) = p\varphi N(e_2) = p\,\varphi(e_1) = p\lambda_1 e_1 \implies \boxed{\lambda_2 = p\lambda_1}$$

So far, we have $\lambda = \lambda_1 \in p^m \mathbb{Z}_p^\times$, with $2m+1 = r \geq 1$, and

$D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ as a $\mathbb{Q}_p$-vectorspace, and $[N] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $[\phi] = \begin{bmatrix} \lambda & 0 \\ 0 & p\lambda \end{bmatrix}$

How about Fil?

Note that the only nontrivial subobject of D is $\mathbb{Q}_p \cdot e_1 = \ker N$.

(b/c if v spans an N-stable line, then $Nv = \alpha v$, so $\underset{0}{N^k v} = \alpha^k v \Rightarrow \alpha = 0$) N nilpotent

Now, as $t_N(\mathbb{Q}_p e_1) = m$, $t_H(\mathbb{Q}_p e_1) \in m$.

On the other hand, $t_H(L) = 2m+1$. So ~~$\mathbb{Q}_p e_1$~~ $\mathbb{Q}_p e_1 \neq L$.

$\therefore L = (\alpha e_1 + \beta e_2)$ with $\beta \neq 0$. By scaling, $L = \mathbb{Q}_p(ce_1 + e_2)$ for a unique c.

So then $t_H(\mathbb{Q}_p e_1) = 0 \leq m = t_N(\mathbb{Q}_p e_1)$.

Hence $\mathbb{Q}_p e_1$ is not w.a unless $m = 0$

We get:

$$\left\{ (\lambda, c) \in p^m \mathbb{Z}_p^\times \times \mathbb{Q}_p^\times \right\} \longleftrightarrow \left\{ \begin{array}{l} D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 \\ N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \phi = \begin{bmatrix} \lambda & 0 \\ 0 & p\lambda \end{bmatrix} \\ \mathrm{Fil}^j D = \begin{cases} D & j \leq 0 \\ \mathbb{Q}_p(ce_1 + e_2) & 1 \leq j \leq r \\ 0 & j > r \end{cases} \end{array} \right\}$$

- if $m \neq 0$, this corresponds to irreducible 2-dim'l semistable reps.
- if $m = 0$, this corresponds to reducible, non-semisimple 2-dimensional semistable (and non-crystalline) reps (i.e. semistable extensions):

e.g.

$$0 \to \psi(1) \to V \to \psi \to 0$$

We will now construct B_{cris}. So let K be a p-adic field.

We constructed $R = \varprojlim_{x \mapsto x^p} \left(\mathcal{O}_{\mathbb{C}_K}/(p) \right) \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}$

$(x_n) \qquad (x^{(n)})$

and a map: (continuous, G_K-equivariant surjection)

$$\Theta : W(R) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}, \quad (r_0, r_1, \dots) \longmapsto \sum_{n \geq 0} r_n^{(n)} p^n$$

Recall $\operatorname{Ker} \Theta = (\xi_\pi)$, $\xi_\pi := [\pi] - p$ and $\pi = (p, p^{\frac{1}{p}}, \dots)$

We then defined $B_{dR}^+ := \varprojlim_n \dfrac{W(R)[\frac{1}{p}]}{(\operatorname{Ker} \Theta_K)^n} = \Big\{ \underbrace{\sum_{n \geq 0} \alpha_n \xi_\pi^n}_{\text{non-unique!}} : \alpha_n \in W(R)[\tfrac{1}{p}] \Big\}$

But there is no Frobenius acting on B_{dR}^+!

Def: Let $A^{\circ}_{cris} := W(R)\left[\frac{\alpha^m}{m!} : \begin{smallmatrix} m \geq 1 \\ \alpha \in \operatorname{Ker} \Theta \end{smallmatrix} \right] = W(R)\left[\frac{\xi^m}{m!} : m \geq 1 \right] \subsetneq W(R)[\tfrac{1}{p}]$

for any ξ s.t. $\operatorname{Ker} \Theta = (\xi)$

(this is called the divided power envelope of $\operatorname{Ker} \Theta$).

Let $A_{cris} := \varprojlim_n A^{\circ}_{cris}/(p^n) = \left\{ \sum_{n \geq 0} \alpha_n \frac{\xi^n}{n!} : \begin{smallmatrix} \alpha_n \in W(R) \\ \alpha_n \xrightarrow[n]{} 0 \\ p\text{-adically} \end{smallmatrix} \right\}$

non-unique!

Claim:
There's a unique, continuous, (G_K-equivariant) map of $W(R)$-algebras:

$$A_{cris} \longrightarrow B_{dR}^+$$

such that T.F.C.:

$$\begin{tikzcd} A_{cris} \arrow[r, dashed] & B_{dR}^+ \\ A^{\circ}_{cris} \arrow[u] \arrow[r, hook] & W(R)[\frac{1}{p}] \arrow[u] \end{tikzcd}$$

$\leftarrow$ be careful! the completions are not wrt the same ideal, so one needs some argument here!

Pf/ See it in the notes

Lemma: The map $A_{cris} \to B_{dR}^{+}$ is injective.

Pf/ Let $x = \sum_{n \geq 0} \alpha_n \frac{\xi^n}{n!}$, $\alpha_n \in W(R)$ be in A_{cris}.

We can assume wlog that $\alpha_n \neq 0 \Rightarrow \alpha_n \notin \operatorname{Ker}\theta$ (by adjusting powers of ξ).

So the image of x in B_{dR}^{+} is:

$\xi^{n_0}(a_{n_0} + \xi y)$, for some $y \in B_{dR}^{+}$.

This is $0 \Leftrightarrow a_{n_0} + \xi y = 0$. But $\theta(a_{n_0} + \xi y) = \theta(a_{n_0}) \neq 0$ by hypothesis.

Thus $A_{cris} \hookrightarrow B_{dR}^{+}$ is a $\mathbb{Z}_p$-flat domain, and the composition

$A_{cris} \hookrightarrow B_{dR}^{+} \twoheadrightarrow \mathbb{C}_K$ has image in $\mathcal{O}_{\mathbb{C}_K}$.

Def: Let $B_{cris}^{+} := A_{cris}[\frac{1}{p}] \subseteq B_{dR}^{+}$.

Rk: This is a G_K-stable $W(R)[\frac{1}{p}]$-subalgebra of B_{dR}^{+}.

Let $t = \log([\varepsilon]) = \sum_{n \geq 1} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in B_{dR}^{+}$.

Prop: The element t is in A_{cris}. Moreover, for any $a \in \operatorname{Ker}(A_{cris} \to B_{dR}^{+} \twoheadrightarrow \mathbb{C}_K)$ we have $a^m/m! \in A_{cris}$ for all $m \geq 1$.

Pf/ Write $t = \sum (-1)^{n+1} \underbrace{w^n (n-1)!}_{\in W(R)} \frac{\xi^n}{n!}$ where $[\varepsilon] - 1 = w \cdot \xi$, $w \in W(R)$.

and $w^n (n-1)! \to 0$ p-adically (thanks to $(n-1)!$).

If $a \in \operatorname{Ker}\theta|_{A_{cris}}$, have $a = \sum_{n \geq 0} \alpha_n \frac{\xi^n}{n!}$, $\alpha_n \to 0$

$\downarrow$

(cont pf of lemma)

So from $a = \sum_{n \geq 0} \alpha_n \frac{\xi^n}{n!}$, $\alpha_n \to 0$, by the binomial thm:

$$\frac{a^m}{m!} = \sum \text{ terms like } \beta_I \frac{1}{i_1!}\left(\frac{\xi^{n_1}}{n_1!}\right)^{i_1} \frac{1}{i_2!}\left(\frac{\xi^{n_2}}{n_2!}\right)^{i_2} \cdots \frac{1}{i_r!}\left(\frac{\xi^{n_r}}{n_r!}\right)^{i_r}$$

with the β_I "$\to 0$".

So we can reduce to the case of $a = \frac{\xi^n}{n!}$.

But: $\frac{1}{m!}\left(\frac{\xi^n}{n!}\right)^m = \frac{(mn)!}{m!(n!)^m} \frac{\xi^{mn}}{(mn)!}$ and $\frac{(mn)!}{m!(n!)^m} \in \mathbb{Z}$ ← theory of divided powers. ☒

Def: $B_{cris} := B_{cris}^+ \left[\frac{1}{t}\right] \subseteq B_{dR}^+\left[\frac{1}{t}\right] = B_{dR}$, which is a G_K-stable, $W(R)\left[\frac{1}{p}\right]$ subalgebra.

Rk: The topology on B_{cris} is the p-adic one, not the subspace topology!

Rk: $\frac{t^p}{p!} \in A_{cris}$, so $t^p \in pA_{cris}$. So $B_{cris} = A_{cris}\left[\frac{1}{p}\right]\left[\frac{1}{t}\right] = A_{cris}\left[\frac{1}{t}\right]$

Rk: B_{cris}, A_{cris}, .. only use K via $\mathbb{C}_K$. Hence can replace K with finite extensions of $\widehat{K^{unram}}$ without changing them.

Note that $W(\bar{k}) \hookrightarrow W(R) \subseteq A_{cris}$, so we get $\widehat{K_0^{un}} \subseteq B_{cris}$,

so that $K_0 \subseteq B_{cris}^{G_K} \subseteq B_{dR}^{G_K} = K$

The following theorem establishes what $B_{cris}^{G_K}$ is:

↓

__Theorem__: The natural G_K-map $K \otimes_{K_0} B_{cris} \xrightarrow{(*)} B_{dR}$ is __injective__.

Pf: Fontaine's 2^{nd} exposé ☒

__Cor__: The map $(*)$ induces an isomorphism on the associated graded ~~algebra~~ vectorspaces, where the filtration on $K \otimes_{K_0} B_{cris} \subseteq B_{dR}$ is the __subspace__ filtration.

Pf/ Just use that $t \in B_{cris}$ ☒

__Cor__: $B_{cris}^{G_K} = K_0$.

__Prop__: The domain B_{cris} (it's not a field) is $(\mathbb{Q}_p, G_K)$-regular.

Pf/ B_{dR} is a field, so since $K \otimes_{K_0} B_{cris} \to B_{dR}$ is injective, then so is $\mathrm{Frac}(K \otimes_{K_0} B_{cris}) \hookrightarrow B_{dR}$.

$$\mathrm{Frac}(K \otimes_{K_0} B_{cris}) = K \otimes_{K_0} \mathrm{Frac}(B_{cris})$$

Hence:

$$(\mathrm{Frac}(B_{cris}))^{G_K} = K_0$$

As for the second condition for regularity, let $b \in B_{cris} \setminus \{0\}$ be any nonzero elt, and suppose that $\mathbb{Q}_p \cdot b$ is G_K-stable, via some character $\eta: G_K \to \mathbb{Q}_p^\times$.

Since $t \in B_{dR}^+$ is a uniformizer, and $t \in B_{cris}^\times$, we can replace b with $t^i b$, to assume $b \in B_{dR}^+ \setminus \mathrm{Fil}^1$. i.e. $\bar{b} := \theta_K(b) \in \mathbb{C}_K^\times$.

By hypothesis, $\mathbb{Q}_p \cdot \bar{b} \subseteq \mathbb{C}_K$ is G_K-stable, with G_K action given by η.

So $\eta: G_K \to \mathbb{Z}_p^\times$, is continuous, and $\bar{b} \neq 0$ is in $\mathbb{C}_K(\eta^{-1})^{G_K}$.

By Sen-Tate, $\eta(I_K)$ is finite.

↓

(cont of pf of Prop)

By choosing $K'/\widehat{K^{un}}$ finite that trivializes η and using the fact that

$$\mathbb{C}_{K'}^{G_{K'}} = \mathbb{C}_K^{G_{K'}} = K', \text{ we get } \bar{b} \in K'.$$

Let now $\beta \in B_{dR}^+$ be the unique lift of $\bar{b} \in \mathbb{C}_K$ (Hensel's lemma).

Then, $b-\beta \in \mathrm{Fil}^1 B_{dR}^+$. If $b-\beta \neq 0$, then it spans a G_K-stable $\mathbb{Q}_p$-line by uniqueness of β.

But $\dfrac{\mathrm{Fil}^i B_{dR}^+}{\mathrm{Fil}^{i+1} B_{dR}^+} \simeq \mathbb{C}_K(i)$ which has trivial G_K-invariants and separatedness of $\mathrm{Fil}^\bullet \Rightarrow b-\beta = 0$.

Now, $b=\beta$ is algebraic over $\widehat{K^{un}}$, or even over $\widehat{K_0^{un}}$.

<u>But</u>: B_{cris} is insensitive to replacing K with $\widehat{K^{un}}$, so

$$K' \otimes_{\widehat{K_0^{un}}} B_{cris} \hookrightarrow B_{dR} \quad \forall K'/\widehat{K_0^{un}} \text{ finite}$$

Now, $b \in B_{cris}$ has $1 \otimes b \in K' \subseteq B_{dR}$, but this makes $b \in \widehat{K_0^{un}} \subseteq B_{cris}$

So $b \in B_{cris}^{\times}$ ∎

<u>Lemma</u>: The $W(R)$-subalgebra $A_0^{cris} \hookrightarrow W(R)[\frac{1}{p}]$ is ϕ_R-stable (ϕ_R = Frob on $W(R)$ induced from $x \mapsto x^p$ on R).

Pf: $\phi_R(\xi) = \phi_R([\pi]-p) = [\pi^p]-p = [\pi]^p - p = (\xi+p)^p - p = \xi^p + p\cdot w$, for some $w \in W(R)$.

So $$\phi_R\left(\frac{\xi^m}{m!}\right) = \frac{1}{m!}\phi_R(\xi)^m = \frac{1}{m!}\left(p\left(w+(p-1)!\frac{\xi^p}{p!}\right)\right)^m = \overset{\in \mathbb{Z}_p}{\frac{p^m}{m!}}\underbrace{\left(w+(p-1)!\frac{\xi^p}{p!}\right)^m}_{\in A_0^{cris}}$$ ∎

Finally, by p-adic completion we get a unique continuous extension of ϕ_R to:

$$\phi : A_{cris} \to A_{cris} \quad \text{and} \quad \phi : B_{cris} \to B_{cris}$$

<u>Claim</u>: $\phi(t) = p.t$

<u>Pf</u> $t = \log([\varepsilon]) = \sum_{n\geq 1} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in A_{cris}$

Since ϕ is defined on A_{cris} by p-adic continuity,

$$\phi(t) = \sum_{n\geq 1} (-1)^{n+1} \frac{([\varepsilon^p]-1)^n}{n} = \log([\varepsilon^p]) = \log([\varepsilon]^p) = p.t \qquad \blacksquare$$

<u>Rk</u>: The series for log was defined in one topology, and now we use it in another topology! So something is going on...

<u>Fact</u>: $\phi : A_{cris} \to A_{cris}$ is <u>injective</u>.

So also $\phi : B_{cris} \to B_{cris}$ is, as well.

Thus, we get $D_{cris} : Rep_{\mathbb{Q}_p}(G_K) \longrightarrow MF_K^{\phi}$

<u>Def</u>: $V \in Rep_{\mathbb{Q}_p}(G_K)$ is called <u>crystalline</u> if it is B_{cris}-admissible.

We write $Rep_{\mathbb{Q}_p}^{cris}(G_K)$ for the full subcategory of $Rep_{\mathbb{Q}_p}(G_K)$.

The formalism we introduced, + filtration arguments (as in the dR-case), show that $D_{cris} : Rep_{\mathbb{Q}_p}^{cris}(G_K) \to MF_K^{\phi}$ is exact and faithful, commuting with $\otimes$ and duality.

Prop: For $V \in \mathrm{Rep}^{cris}_{\mathbb{Q}_p}(G_K)$, the natural map:

$$K \otimes_{K_0} D_{cris}(V) \longrightarrow D_{dR}(V)$$

is an isomorphism in Fil_K

In particular, any crystalline rep is deRham.

Moreover, the B_{cris}-linear, ϕ, G_K-compatible comparison isomorphism

$$\alpha : B_{cris} \otimes_{K_0} D_{cris}(V) \longrightarrow B_{cris} \otimes_{\mathbb{Q}_p} V$$

has $\alpha_K = \alpha \otimes_{K_0} K$ is an isomorphism in Fil_K.

Pf First, note that the filtration on $K \otimes_{K_0} D_{cris}(V)$ is the subspace filtration from $D_{dR}(V)$.

So it's enough to check dimensions.

$$\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(D_{cris}(V)) = \dim_K(K \otimes_{K_0} D_{cris}(V)) \leq \dim_K(D_{dR}(V)) \leq \dim_{\mathbb{Q}_p}(V)$$

So all inequalities are $=$, and hence V is dR and map is $\cong$.

As for the second part, note that α_K is already a map in Fil_K, so enough to check iso on $\mathrm{gr}^\bullet$.

But we saw last time that

$$\mathrm{gr}^\bullet(K \otimes_{K_0} B_{cris}) = \mathrm{gr}(B_{dR}) = B_{HT}.$$

Chasing definitions, we can see that $\mathrm{gr}(\alpha_K)$ is the comparison map for HT. (which is an iso, as $dR \Rightarrow HT$). $\blacksquare$

Theorem: Give B_{cris} the subspace filtration from $B_{cris} \hookrightarrow K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR}$.

Then: $\mathrm{Fil}^0(B_{cris})^{\phi=1} := \{ b \in \mathrm{Fil}^0 B_{cris} : \phi(b) = b \} = \mathbb{Q}_p$.

Pf/ Ommitted. (can be found in Fontaine, "exposé II Prop 5-3.6 / Thm 5.37", Asterisque 223)

Corollary: the functor $D_{cris} : \mathrm{Rep}^{cris}_{\mathbb{Q}_p}(G_K) \longrightarrow MF_K^{\phi}$ is fully faithful, with inverse on the essential image given by:

$$V_{cris}(-) = \mathrm{Fil}^0 (B_{cris} \otimes_{K_0} -)^{\phi=1}$$

Pf/ Since $\alpha : B_{cris} \otimes_{K_0} D_{cris}(V) \longrightarrow B_{cris} \otimes_{\mathbb{Q}_p} V$ is B_{cris}-linear, ϕ, G_K-compatible, with α_K a filtered iso, then by passing to Fil^0 we get a G_K-equivariant K_0-iso $\mathrm{Fil}^0(B_{cris} \otimes_{K_0} D_{cris}(V)) \to \mathrm{Fil}^0(B_{cris} \otimes_{\mathbb{Q}_p} V)$ which is ϕ-compatible.

So, by passing to $(\phi = 1)$, get:

$$V_{cris}(D(V)) = \mathrm{Fil}^0(B_{cris} \otimes_{\mathbb{Q}_p} V)^{\phi=1} \overset{\text{Thm}}{=} V$$

and this identification is G_K-equivariant. ∎

(So we can recover V from $D_{cris}(V)$!)

On Morphisms:

Given $T : D' \to D$ in MF_K^{ϕ} where $D' = D_{cris}(V')$, $D = D_{cris}(V)$, with $V, V' \in \mathrm{Rep}^{cris}_{\mathbb{Q}_p}(G_K)$, then we get a commutative diagram of G_K-equiv' maps:

$$\begin{tikzcd} B_{cris} \otimes D' \arrow[r, "1 \otimes T"] \arrow[d, "\alpha'", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & B_{cris} \otimes D \arrow[d, "\wr", "\alpha"'] \\ B_{cris} \otimes V' \arrow[r, "\tilde{T}"] & B_{cris} \otimes V \end{tikzcd}$$

$\tilde{T}$ is ϕ, G_K compat, + Fil-compat, So $\tilde{T}$ carries $V' = \mathrm{Fil}^0(B_{cris} \otimes V')^{\phi=1}$ into $V = \mathrm{Fil}^0(B_{cris} \otimes V)^{\phi=1}$ via a G_K-equiv. map T''. By functoriality of D_{cris}, $D_{cris}(T'') = T$. ∎

<u>Note</u>: There are contravariant versions of V_{cris}, D_{cris}:

$$V^*_{cris}(-) = \mathrm{Hom}_{Fil,\phi}(-, B_{cris}) \quad \text{etc.}$$

<u>Next</u>: we introduce B_{st}. The idea is that there aren't that many crystalline reps (as opposed to deRham). So we want to find something in between, without loosing fully-faithfulness.

<u>Example</u>: Recall, for $q \in K^\times$, $|q|<1$, we have $E_q = \overline{K}^\times / q^{\mathbb{Z}}$, and

V_q = Tate module of E_q.

It has basis $\underline{\varepsilon} = (\zeta_{p^n}) \in R$, $\underline{q} = (q, q^{1/p}, \dots) \in R$.

Relative to that basis, the G_K-action on V_q is given by:

$$\begin{pmatrix} \chi & \eta_{\underline{q}} \\ 0 & 1 \end{pmatrix}$$

with $\dfrac{g\underline{q}}{\underline{q}} = \varepsilon^{\eta_{\underline{q}}(g)}$, with $\eta_{\underline{q}} : G_K \to \mathbb{Z}_p(1)$ a 1-cocycle.

Using $t \in B_{cris}$, and "$\log([\underline{q}])$" $\in B_{dR}$, we have:

$$g(t) = \chi(g)\cdot t, \quad g(\log[\underline{q}]) = \eta_{\underline{q}}(g)\cdot t + \log[\underline{q}].$$

That is, we have a G_K-equivariant map $V_q \hookrightarrow B_{dR}$.

<u>Idea</u>: B_{st} should be a B_{cris}-algebra ~~generated by~~ containing V_q, and still as small as possible.

So we will "universally" adjoin log's to B_{cris}.

Lemma: For $x \in 1 + \mathfrak{m}_R \subseteq R$, and $n >> 0$, the elements

$$\frac{([x]-1)^n}{n} \in W(R)\left[\tfrac{1}{p}\right]$$ is in A_{cris}, and tend to 0 p-adically (as $n \to \infty$).

So $$\log_{cris}([x]) := \sum_{n \geq 1} (-1)^{n+1} \frac{([x]-1)^n}{n} \in B^+_{cris}$$ for all $x \in 1 + \mathfrak{m}_R$.

Pf/ postponed.

We get in this way a G_K-equivariant logarithm:

$$\lambda : R^\times \cong \bar{k}^\times \times (1 + \mathfrak{m}_R) \longrightarrow B^+_{cris}$$

by $\lambda|_{\bar{k}^\times} = 0$ and $\lambda|_{1+\mathfrak{m}_R} = \log_{cris}([-])$.

Consider the category of pairs (S, λ_S), where S is a B^+_{cris}-algebra, and $\lambda_S : \mathrm{Frac}(R^\times) \longrightarrow S$ is an extension of $\lambda : R^\times \to B^+_{cris} \to S$.

(morphisms are mor. of B^+_{cris}-algebras, compatible with λ_S's).

Then (B^+_{st}, λ) is the initial object in this category.

Construction: As B^+_{cris} is a $\mathbb{Q}$-algebra, we get a G_K-eq. map of $\mathbb{Q}$-algebras

$$\lambda : \mathrm{Sym}_{\mathbb{Q}}(R^\times) \longrightarrow B^\times_{cris}$$

the symmetric algebra on the $\mathbb{Q}$-vectorspace $\mathbb{Q} \otimes_{\mathbb{Z}} R^\times$ mult !!

We have an G_K-equivariant sequence of abelian groups

$$1 \to R^\times \to (\mathrm{Frac}\, R)^\times \xrightarrow{v_R} \mathbb{Q} \to 1$$

So $$(\mathrm{Frac}(R)^\times) / R^\times \cong \mathbb{Q}.$$

So $\mathrm{Sym}_{\mathbb{Q}}((\mathrm{Frac}\, R)^\times)$ is a 1-variable poly. algebra over $\mathrm{Sym}_{\mathbb{Q}}(R^\times)$

We define then:

$$B_{st}^{+} := \mathrm{Sym}_{\mathbb{Q}}\left((\mathrm{Frac}(R))^{\times}\right) \underset{\mathrm{Sym}_{\mathbb{Q}}(R^{\times})}{\otimes} B_{cris}^{+}$$

Concretely, $B_{st}^{+} \simeq B_{cris}^{+}[X]$ where X "is" "$\log(\tilde{p})$".

<u>Proof of lemma in the prev page.</u>

Let $x \in 1 + m_R \subseteq R$, $n >> 0$. Want to see that

$$\frac{([x]-1)^n}{n} \in W(R)[\tfrac{1}{p}] \text{ is in } A_{cris}, \text{ and } \xrightarrow[n\to\infty]{} 0 \text{ p-adically.}$$

Note that:

$$\theta([x]-1) = x^{(0)} - 1 \in m_{\mathcal{O}_{\mathbb{C}_K}}.$$

Hence $\exists N > 0$ s.t. $\theta\left(([x]-1)^N\right) \in p\,\mathcal{O}_{\mathbb{C}_K} \Rightarrow ([x]-1)^N = p\cdot w_1 + \xi w_2$, with $w_i \in W(R)$.

Because $\frac{p^j}{j!} \in \mathbb{Z}_p$, and $\frac{\xi^j}{j!} \in A_{cris}$, we conclude:

$$\frac{([x]-1)^{Nj}}{j!} \in A_{cris}$$

Now, consider $\frac{([x]-1)^n}{n}$ for $n \geq 0$. Write $n = Nq_n + r_n$, $0 \leq r_n < N$.

Then: $\frac{q_n!}{n}([x]-1)^{r_n} \underbrace{\frac{([x]-1)^{Nq_n}}{q_n!}}_{\in A_{cris}}$, so we just need to show that

$\frac{q_n!}{n} \in A_{cris}$ for suff. large n:

usual inequalities for ord_p of a factorial

$$\frac{q_n}{p-1} \geq \mathrm{ord}_p(q_n!) \geq \frac{q_n - 1}{p-1} - \log_p(q_n)$$

Since $q_n = \lfloor \frac{n}{N} \rfloor$, it grows linearly with n. On the other hand,

$\mathrm{ord}_p(n) \leq \log_p(n)$, so $\mathrm{ord}_p(n)$ grows at most like $\log_p(n)$

Topological remark

Write, as usual, $\varepsilon = (1, \zeta_p, \dots) \in R$, and set $\varepsilon_1 := (\zeta_p, \dots) \in R$

So $\varepsilon_1^p = \varepsilon$. Set $\omega := \dfrac{[\varepsilon]-1}{[\varepsilon_1]-1} \in W(R)$.

Looking at the second component of ω, we can show $(\omega) = \mathrm{Ker}\,\theta$.

Doing as in the previous lemma, we can show that the sequence

$$x_n := \frac{\omega^{p^n-1}}{(p^n-1)!} \in A_{cris}$$

does not tend to 0 p-adically in A_{cris}.

However, $\omega \cdot x_n = \dfrac{\omega^{p^n}}{(p^n-1)!} = p^n \dfrac{\omega^{p^n}}{p^n!} \in A_{cris}$ does tend to 0 p-adically.

Since $\mathrm{Ker}\,\theta = (\omega) = (t)$, we conclude that $t \cdot x_n \to 0$ in A_{cris},

hence $t x_n \to 0$ in B_{cris}^+ and so in B_{cris}.

So, in B_{cris}, $x_n \to 0$!

This means that the resulting topology on A_{cris} induced from B_{cris} is not the p-adic.

A much better ring (topologically, at least) is:

$$B_{max} := \left\{ \sum a_n \frac{\xi^n}{p^n} : a_n \in W(R),\ a_n \to 0 \text{ p-adically} \right\}.$$

Back to business, recall that we had just defined

$$B_{st}^+ := \mathrm{Sym}_{\mathbb{Q}}\left((\mathrm{Frac}\,R)^\times\right) \otimes_{\mathrm{Sym}_{\mathbb{Q}}(R^\times)} B_{cris}^+$$

By construction, the pair (B_{st}, λ) is initial in the category of pairs (S, λ_S), S a B^+_{cris}-algebra, λ_S an extension of $\lambda: R^\times \to B^+_{cris}$ to $(\mathrm{Frac}\, R)^\times \to S$.

A choice of $y \in (\mathrm{Frac}\, R)^\times$ not in $R^\times$ yields, by setting $X = \lambda(y)$, an isomorphism:

$$B^+_{st} \simeq B^+_{cris}[X]$$

Def: $B_{st} := B^+_{st}[\frac{1}{t}]$.

- We equip B_{st} and B^+_{st} with a continuous G_K-action via the G_K-action on B^+_{cris} and on $R^\times$ and $(\mathrm{Frac}\, R)^\times$. (it makes λ G_K-equivariant, too)
- Frob: extends from ϕ on B^+_{cris} by acting on X $\left(B^+_{st} = B^+_{cris}[X] \right)$, and

$$\phi(X) = p \cdot X.$$

- The monodromy operator: choose $y_0 \in \mathfrak{m}_R - \{0\}$ and put $v_0 := v_R(y_0) \in \mathbb{Q}_{>0}$.

  Set $X := \lambda(y_0)$, and identify $B^+_{st} = B^+_{cris}[X]$.

  Consider the action on $B^+_{cris}[X]$ of $\frac{d}{dX}$:

  Observe that, if y_0' has the same valuation v_0, then $y_0' = u \cdot y_0$, with $u \in R^\times$. So $X' = \lambda(y_0') = \lambda(u) + X$, and $\lambda(u) \in B^+_{cris}$,

  so $\frac{d}{dX'} = \frac{d}{dX}$. Hence $\frac{d}{dX}$ depends only on v_0, not on y_0.

  Let then $N := v_0 \frac{d}{dX}$

  It is then independent of all choices. Also, it is nilpotent. Need to check that $(N \circ \phi)(X) = N(p \cdot X) = pN(X) = p \cdot v_0$ and $(\phi \circ N)(X) = \phi(v_0) = v_0$ ✓.

Also, N is defined on B_{st}, and clearly $B_{cris} = (B_{st})^{N=0}$.

Now, we want an injective, G_K-equivariant B_{cris}-algebra map

$$B_{st} \longrightarrow B_{dR}, \quad \text{carrying } B_{st}^+ \text{ into } B_{dR}^+$$

By universality, it is enough to construct a logarithm on B_{dR}^+.

To do this, we will need to make a choice of a G_K-homomorphism

$$\log_{\overline{K}} : \overline{K}^\times \longrightarrow \overline{K}$$

extending the usual log on 1-units, and trivial on Teichmüller lifts.

Since $\overline{K}^\times = \overline{k}^\times \times (1 + m_{\mathcal{O}_{\overline{K}}}) \times p^{\mathbb{Q}}$, such a map is uniquely determined by choosing $\log_{\overline{K}}(p)$.

We will take $\log_{\overline{K}}(p) := 0$

Now, it is enough to define

$$\lambda : (\operatorname{Frac} R)^\times \longrightarrow B_{dR}^{*} \quad \text{extending } \lambda : R^\times \to B_{cris}^+ \hookrightarrow B_{dR}^+$$

To do this, note that we already have a map

$$\begin{tikzcd} R \setminus \{0\} \arrow[r, "{[-]}"] \arrow[d, hook] & (B_{dR}^+)^\times \\ (\operatorname{Frac} R)^\times \arrow[ur, dashed, "{[\cdot]}"'] & \end{tikzcd}$$

, which uniquely extends to

Similarly, we have $(\operatorname{Frac} R)^\times \longrightarrow \mathbb{C}_K^\times$

$$y \longmapsto (y^{(0)})$$

Every coset $(\operatorname{Frac} R)^\times / R^\times$ is represented by some $y \in (\operatorname{Frac} R)^\times$, with $y^{(0)} \in \overline{\mathcal{O}}_p^\times$.

So via the embedding $\overline{\mathcal{O}}_p^\times \hookrightarrow B_{dR}^+$, the ratio $\frac{[y]}{y^{(0)}}$ makes sense, and it projects to 1 in $\mathbb{C}_K$. So $\frac{[y]}{y^{(0)}} \in 1 + \ker \theta$

We finally define:

$$\lambda(y) := \log_{dR}\left(\frac{[y]}{y^{(0)}}\right) + \log_{\bar{K}}(y^{(0)}) \in B_{dR}^{+}$$

$$\log_{dR} \;\text{``=''}\; \sum_{n\geq 1} (-1)^{n+1} \frac{\left(\frac{[y]}{y^{(0)}} - 1\right)^n}{n}$$

Clearly, λ is G_K-equivariant.

<u>Lemma</u>: If $y \in R^\times$, then $\log_{cris}([y]) = \lambda(y) = \log_{dR}\left(\frac{[y]}{y^{(0)}}\right) + \log_{\bar{K}}(y^{(0)})$

<u>Remark</u>: the convergence on LHS and RHS is <u>different</u>! So there's something to prove.

Pf ommited (see notes).

By universality of (B_{st}^+, λ) we get a B_{cris}-algebra G_K-equivariant map

$$B_{st}^+ \longrightarrow B_{dR}^+ \qquad (\text{and also } B_{st} \to B_{dR}).$$

Concretely, if $\underline{q} \in \mathfrak{m}_R \setminus \{0\}$ and $\underline{q}^{(0)} =: q \in \mathcal{O}_{\mathbb{C}_K}$ we have

just notation!

$$B_{st}^+ \simeq B_{cris}^+[X], \quad X \mapsto \log_{dR}\left(\frac{[\underline{q}]}{q}\right) + \log_{\bar{K}}(q) =: \log([\underline{q}])$$

gives the map $B_{st}^+ \longrightarrow B_{dR}^+$.

If we change the choice of $\log_{\bar{K}}(p)$, what changes is that the image of X is changed by an additive constant (which will belong to K_0 if we require $\log_{\bar{K}}(p) \in K_0$).

Thus the image of $B_{st} \longrightarrow B_{dR}$ is <u>independent</u> of the choice of $\log_{\bar{K}}(p)$ as long as this is in K_0^{un} ($\subseteq B_{dR}^+$).

Theorem: The natural map

$$K \otimes_{K_0} B_{st} \longrightarrow B_{dR} \quad \text{is injective.}$$

Pf/ Can find it in Asterisque 223, Exp II ~~4.3.2~~ & 4.3.3.

Corollary: $B_{st}^{G_K} = K_0$, $\mathrm{Frac}(B_{st})^{G_K} = K_0$.

* Henceforth, we choose $\log_{\bar{K}}(p) = 0$.

Prop: B_{st} is $(\mathbb{Q}_p, G_K)$-regular.

Pf/ The previous Corollary is part of that definition. It remains to show that, if $b \overset{\neq 0}{\in} B_{st}$ spans a G_K-stable $\mathbb{Q}_p$-line, we have $b \in B_{st}^{\times}$.

By replacing K with $\widehat{K^{un}}$, we may assume that $k = \bar{k}$.

We fix an iso $B_{st} \simeq B_{cris}[X]$ by

$$g.X = X + \eta(g)t \qquad \text{with} \quad \begin{cases} t = \log([\varepsilon]) \\ \eta: G_K \to \mathbb{Q}_p^{\times} \text{ def. by} \\ \quad g(\underline{q}) = \underline{q}\cdot\varepsilon^{\eta(g)} \end{cases}$$

(we could even specify $\underline{q}$ by $q^{(0)} := p$).

The G_K-action on $\mathbb{Q}_p b$ is via $\psi: G_K \to \mathbb{Q}_p^{\times}$.

We write $b = b_0 + b_1 X + \cdots + b_r X^r$. ~~for~~ $(b_i \in B_{cris})$. So:

$$\psi(g)b = g(b) = gb_0 + gb_1(X + \eta(g)\cdot t) + \cdots + gb_r(X + \eta(g)t)^r$$

Need to show that $r = 0$ (i.e. $b \in B_{cris}$, as B_{cris} is $(\mathbb{Q}_p, G_K)$-regular, and $B_{st}^{\times} = B_{cris}^{\times}$!).

↓

(cont pf of prop).

Comparing X^r and X^{r-1} coeffs give:

(*) $\psi(g) b_r = g(b_r)$

(**) $\psi(g)\, b_{r-1} = g(b_{r-1}) + g(b_r)\cdot r \cdot \eta(g) \cdot t$

By (*), b_r spans a G_K-stable $\mathbb{Q}_p$-line in B_{cris}, with action via ψ.

We saw in the proof of B_{cris} being $(\mathbb{Q}_p, G_K)$-regular that ψ must be continuous, therefore crystalline.

So $\psi = \chi^n \cdot (\text{unramified char}) \underset{\substack{\uparrow \\ G_K = I_K \text{ b/c } K = \widehat{K^{un}}}}{=} \chi^n$

Replacing b with $t^{-n} b$ ($t \in B_{cris}^\times$), we can assume ψ is the trivial char.

Hence $g(b_r) = b_r \ \forall g$. Hence $b_r \in (B_{cris}^\times)^{G_K} = K_0^\times$

By (**), $g(b_{r-1}) - b_{r-1} = -r b_r \eta(g) t$.

If $r > 0$, then $\dfrac{g(b_{r-1})}{-r b_r} - \dfrac{b_{r-1}}{-r b_r} = \eta(g)\cdot t = g(x) - x$

$g\left(\dfrac{b_{r-1}}{-r b_r}\right) - \dfrac{b_{r-1}}{-r b_r}$

So we get $g\left(x + \dfrac{b_{r-1}}{r b_r}\right) = x + \dfrac{b_{r-1}}{r b_r}$ $(\forall g)$.

Get then that $x + \dfrac{b_{r-1}}{r b_r} \in K_0 \Rightarrow x \in B_{cris} \Rightarrow !!$.

So $r = 0$, as we wanted to show.

By our formalism, we conclude:

$$D_{st} : Rep_{\mathbb{Q}_p}(G_K) \longrightarrow Vec_{K_0}$$
$$V \longmapsto (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

satisfies:

$\dim_{K_0} D_{st}(V) \leq \dim_{\mathbb{Q}_p}(V)$, with $=$ iff V is semistable (ie B_{st}-admissible).

As usual, write $Rep^{st}_{\mathbb{Q}_p}(G_K)$ to be the full subcategory of semistable representations.

Since B_{st} has ϕ and N, and $K \otimes_{K_0} B_{st} \hookrightarrow B_{dR}$ we also have a filtration on $K \otimes_{K_0} B_{st}$, actually D_{st} takes values on the more refined category $MF_K^{\phi,N}$.

The restriction of D_{st} to semistable representations $Rep^{st}_{\mathbb{Q}_p}(G_K)$ is faithful, exact and commutes with $\otimes$, duals, ... (Look back at the corresp facts for D_{dR}).
Moreover, for $V \in Rep^{st}_{\mathbb{Q}_p}(G_K)$, we have a B_{st}-linear, G_K-equivariant, N, ϕ-compat. isomorphism $\alpha : B_{st} \otimes_{K_0} D_{st}(V) \xrightarrow{\sim} B_{st} \otimes_{\mathbb{Q}_p} V$,

and α_K is an iso in Fil_K (pf: same as before).

As $B_{st}^{N=0} = B_{cris}$,

$$D_{st}(V)^{N=0} \underset{\text{in } MF_K^{\phi}}{\cong} D_{cris}(V) \qquad \text{for any } V \in Rep_{\mathbb{Q}_p}(G_K)$$

so usually one doesn't use D_{cris}, just D_{st} and check if $N=0$ or not.

Remark: As $D_{cris}(V) = D_{st}(V)^{N=0} \hookrightarrow D_{st}(V)$,

if V is crystalline then $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0} D_{cris}(V) \leq \dim_{K_0} D_{st}(V) \leq \dim_{\mathbb{Q}_p} V$

$\Rightarrow$ V is semistable.

Likewise, V sst $\Rightarrow$ V dR.

Thm: $D_{st} : \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K) \longrightarrow MF_K^{\phi,N}$ is full, with inverse functor on the essential image given by

$V_{st} : MF_K^{\phi,N} \longrightarrow \mathbb{Q}_p[G_K]$-modules. ~~given by~~

$$V_{st}(D) = \mathrm{Fil}^0 \left(B_{st} \otimes_{K_0} D \right)^{N=0, \phi=1}$$

where Fil on B_{st} is via $B_{st} \hookrightarrow K \otimes_{K_0} B_{st} \hookrightarrow B_{dR}$

Proof:

$$\mathrm{Fil}^0 (B_{st})^{N=0, \phi=1} = \mathrm{Fil}^0 (B_{cris})^{\phi=1} = \mathbb{Q}_p .$$

Follow now the other proof we did, it carries through. ∎

Recall: For any complete, discretely-valued K'/K, we have:

$$K' \otimes_K D_{dR,K}(V) \simeq D_{dR,K'}(V) \qquad \forall\, V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K).$$

That is, V is dR as G_K-rep $\Leftrightarrow$ V is dR as $G_{K'}$-rep.

(even if K'/K is (finitely) ramified).

On the contrary, this is false for crystalline representations: crystalline characters are Tate twists of unramified characters.

Prop: For $K' = \widehat{K^{ur}}$, we have, $\forall V \in Rep_{\mathbb{Q}_p}(G_K)$,

$$K_0' \otimes_{K_0} D_{st,K}(V) \simeq D_{st,K'}(V)$$

$\nwarrow$ in $MF_K^{\phi,N}$

Thus V is $\left\{\begin{matrix} s.st \\ crys \end{matrix}\right\}$ as a G_K-rep $\Leftrightarrow$ V is $\left\{\begin{matrix} s.st \\ crys \end{matrix}\right\}$ as a $G_{K'}$-rep $\overset{\shortparallel}{I_K}$

Pf/ Using $D_{st}(V)^{N=0} = D_{cris}(V)$, it is enough to concentrate on D_{st}.

It is clear that $(K')^{\text{o there!}} \otimes_{K_0} D_{st,K}(V) \to D_{st,K'}(V)$ is a map in $MF_K^{\phi,N}$.

To see that it is an iso, we use "completed unramified descent" (see the analogous proof for D_{dR}). $\blacksquare$

Corollary: Let $\rho: G_K \to GL(V)$ be a p-adic rep. with open kernel.

(i.e. ρ factors through $Gal(L/K)$, L/K finite galois).

Then TFAE:

1) ρ is unramified.

2) ρ is crystalline

3) ρ is semistable.

Pf
(1)$\Rightarrow$(2) by the prop. ($I_K = \{1\}$).

(2)$\Rightarrow$(3) always.

(3)$\Rightarrow$(1): may assume $k = \bar{k}$ by replacing K with $\widehat{K^{ur}}$.

Let L/K be a finite ext. corresp to $\ker\rho$. We have $L_0 = K_0$ ($\nwarrow_{k=\bar{k}}$).

Thus $B_{st}^{G_L} = L_0 = K_0 = B_{st}^{G_K} \Rightarrow D_{st,K}(V) = D_{st,L}(V)^{Gal(L/K)} = (B_{st}^{G_L} \otimes_{\mathbb{Q}_p} V)^{Gal(L/K)}$

$= (K_0 \otimes_{\mathbb{Q}_p} V)^{Gal(L/K)}$. As V is s.st, $\dim_{\mathbb{Q}_p} V = \dim_{\mathbb{Q}_p} V^{Gal(L/K)} \Rightarrow \rho$ is trivial. $\blacksquare$

Corollary: If $\eta : G_K \to \mathbb{Q}_p^\times$ is a continuous character, then TFAE:

1) η is a Tate twist of an unramified char.
2) η is crystalline.
3) η is semistable.

Pf: (1) ⇒ (2) ⇒ (3) clear from what we've already done.

(3) ⇒ (1) If η has HT-weight n, we can twist by $\chi(-n)$ to get to the case $n = 0$.

Also, we may suppose that $k = \bar{k}$, as in the previous proof.

Since η is sst; $\mathbb{Q}_p(\eta)^{G_K} \neq 0$. So by Tate-Sen, $\eta(I_K)$ is finite, so since $I_K = G_K$ we conclude that η has open kernel.

Now we can apply the previous Corollary. ∎

Theorem: Let $V \in \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K)$. Then $D_{st}(V)$ is weakly-admissible.

In particular, if V is crystalline, then $D_{cris}(V)$ is weakly-admissible.

Pf: We know that passing from K_0 to $\widehat{K_0^{un}}$ doesn't affect w.a.-ility.

The same is true for semistability. So we will assume $k = \bar{k}$.

Let $D := D_{st}(V)$, and let $D' \subseteq D$ be a subobject. We want $t_H(D') \leq t_N(D')$, with equality with $D = D'$. WLOG, assume $D' \neq 0$ (ie $\dim_{K_0} D' = d' > 0$).

$\Lambda^{d'} D = D_{st}(\Lambda^{d'} V)$ and $\Lambda^{d'} V$ is semistable (it's a quotient of $V^{\otimes d'}$).

Also, t_* $(* = H, N)$ satisfy $t_*(\Lambda^{d'} D') = t_*(D')$. So we may assume that $d' = 1$.

If $D' = D$, then V is 1-dimensional, so $V \simeq \mathbb{Q}_p(n)$ (by previous cor). Then $1 \otimes t^{-n}$ spans $D_{st}(V)$, so $t_H(D) = -n = t_N(D)$ ✓

↓

(cont pf of thm):

In general, let e' be a K_0-basis for D'. Then $\phi(e') = \lambda \cdot e'$, $\lambda \in K_0^\times$,

and $t_N(D') = \mathrm{ord}_p(\lambda)$.

Put $s := t_H(D')$, so:

$$(*) \quad e' \in \mathrm{Fil}^s(B_{dR} \otimes V) = (\mathrm{Fil}^s B_{dR}) \otimes V$$
$$e' \notin (\mathrm{Fil}^{s+1} B_{dR}) \otimes V$$

Let $v_1, \dots, v_n$ be a $\mathbb{Q}_p$-basis for V. So via $D' \hookrightarrow D$ we can

write $e' = \sum b_i \otimes v_i$, $b_i \in B_{st}$.

So: $\phi(b_i) = \lambda \cdot b_i$ and, as D' is 1-dim'l, so $N_{D'} = 0$, have $N(b_i) = 0$.

Hence $b_i \in B_{cris}$ ($\forall i$)

Thus, by $(*)$, $b_i \in \mathrm{Fil}^s(B_{cris})$ $\forall i$, and $b_{i_0} \notin \mathrm{Fil}^{s+1}(B_{cris})$ for some i_0.

Now we have $b := b_{i_0} \neq 0$ in $\mathrm{Fil}^s(B_{cris}) / \mathrm{Fil}^{s+1}$, s.t. $\phi(b) = \lambda b$.

Claim: $s \leq \mathrm{ord}_p(\lambda)$.

Pf/ Let $m := \mathrm{ord}_p(\lambda)$, and assume $s \geq m+1$.

Then $b \in \mathrm{Fil}^s(B_{cris}) \subseteq \mathrm{Fil}^{m+1}(B_{cris})$

But the only $b \in \mathrm{Fil}^{m+1}(B_{cris})$ with $\phi(b) = \lambda b$ and $\mathrm{ord}_p(\lambda) = m$ is 0:

By multiplying by t^{-m}, we are reduced to the case $m = 0$.

Then ~~$\mathrm{Fil}^1(B_{cris})$~~ $\phi(b) = u \cdot b$, $u \in W(k)^\times$. Since $k = \bar{k}$, we

can find $u' \in W(k)^\times$ with $\frac{\sigma u'}{u'} = u$.

Replacing b with $b' := \frac{b}{u'}$, we have $b' \in \mathrm{Fil}^1(B_{cris})^{\phi=1} = (\mathrm{Fil}^1 B_{cris}) \cap \underbrace{(\mathrm{Fil}^0 B_{cris})^{\phi=1}}_{= \mathbb{Q}_p} =$

$= \{0\}$.

$\blacksquare$

<u>Theorem</u> (Fontaine–Colmez): The fully-faithful exact and $\otimes$-compatible functor:

$$D_{st} : \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K) \longrightarrow MF_K^{\phi,N,w.a}$$

is an equivalence, with an inverse functor

$$V_{st} : MF_K^{\phi,N,w.a} \longrightarrow \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K)$$

$$D \rightsquigarrow \mathrm{Fil}^0(B_{st} \otimes D)^{N=0,\phi=1}$$

Pf/ VERY hard, no way we will do this. We'll see ~~the~~ at least some partial results, though. $\boxtimes$

<u>Thm</u> (Berger, Kedlaya, Mebkhout, André).
$V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is de Rham iff it is potentially-semistable (i.e. $\exists K'/K$ finite such that $V \in \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_{K'})$.)

Pf/ Hard, also. Uses p-adic DE's, Robba ring, ... $\boxtimes$

<u>Lemma</u>: Let $D \in MF_K^{\phi,N}$ be 1-dimensional $/K_0$. Then:

1) If $t_N(D) = t_H(D)$ ($\iff D$ is w.a), then $V_{st}(D)$ is 1-dim'l $/\mathbb{Q}_p$.

2) If $t_N(D) > t_H(D)$, then $V_{st}(D) = 0$.
(if $t_N(D) < t_H(D)$, then $V_{st}(D)$ is ∞-ly dimensional).

Pf/ May assume WLOG that $k = \bar{k}$. Pick a K_0-basis d of D.
Then $\phi(d) = \lambda d$, $\lambda = p^{t_N(D)} \cdot u$, $u \in W(k)^\times$. Since $k = \bar{k}$, find ~~the~~ $u' \in W(k)^\times$ s.t. $\frac{\sigma u'}{u'} = u$, so by replacing ~~d~~ d with d/u', can assume $u = 1$ ($\lambda = p^{t_N(D)}$)

$\rightarrow$

(cont pf of lemma)
Define a map

recall $N = N_{B_{st}} \otimes 1 + 1 \otimes N_D$

$$V_{st}(D) \xrightarrow{\Psi} \left\{ b \in Fil^0(B_{st}),\ \phi(b) = p^{t_H(D)-t_N(D)} \cdot b,\ Nb = 0 \right\}$$

$$\| $$

$$\left\{ x \in Fil^0(B_{st} \otimes D) = (Fil^{-t_H(D)} B_{st}) \otimes D : \phi x = x,\ Nx = 0 \right\}$$

$$x \longmapsto b \text{ s.t. } b\, t^{-t_H(D)} \otimes d = x$$

This is clearly a $\mathbb{Q}_p$ - isomorphism.

In case (1) $(t_H(D) = t_N(D))$, the target of Ψ is:

$$Fil^0(B_{cris})^{\phi=1} = \mathbb{Q}_p.$$

In case (2), if $r := t_H(D) - t_N(D) < 0$, then:

$$V_{st}(D) \cong \{ b \in Fil^0(B_{cris}) : \phi(b) = p^r \cdot b \} \overset{b \mapsto bt^{-r}}{\subseteq} \{ b \in Fil^{-r} B_{cris} : \phi(b) = b \} \subseteq$$

$$\subseteq Fil^1(B_{cris})^{\phi=1} = 0 \quad (r<0).$$

■

<u>Theorem</u> (Fontaine - Colmez). Let $D \in MF_K^{\phi, N, w.a.}$. Then $V_{st}(D)$ is finite-dimensional $/\mathbb{Q}_p$, of dimension $\leq \dim_{K_0}(D)$.

Moreover, $V_{st}(D)$ is a semistable G_K-rep.;

Further, if $D' := D_{st}(V_{st}(D))$, then D' is a subobject of D, and $D' = D \iff \dim_{\mathbb{Q}_p}(V_{st}(D)) = \dim_{K_0}(D)$.

Pf (of Thm of Fontaine-Colmez).

Set $C_{st} := \mathrm{Frac}(B_{st})$, and $V := V_{st}(D)$. Let $s := \dim_{K_0}(D)$.

Since $V := V_{st}(D) \subseteq B_{st} \otimes D \subseteq C_{st} \otimes D$,

we know that V generates a C_{st}-subspace $V' \subseteq C_{st} \otimes D$, of C_{st}-dimension $\leq s$.

Since V is G_K-stable, V' is G_K-stable.

Viewing V' as a C_{st}-point of the K_0-Grassmannian classifying r-dimensional K_0-subspaces of D, we get that V' descends to a $C_{st}^{G_K}$-point. But by $(\mathbb{Q}_p, G_K)$-regularity of B_{st}, we have $C_{st}^{G_K} = K_0$. Hence $V' = C_{st} \otimes_{K_0} D'$, where $D' \subseteq D$ is a subspace of dimension r over K_0. $(r \leq s)$

By definition of V' (generated by $V = \mathrm{Fil}^0(B_{st} \otimes D)^{\phi=1, N=0}$), we have:

$(*)$ $V \subseteq V' \cap (B_{st} \otimes D) = B_{st} \otimes D'$

So since ϕ, N preserve $D' \subseteq D$ (just extend scalars to C_{st} and check it there) we can give D'_K the subspace filtration of D_K, to make D' a subobject of D (in $MF_K^{\phi, N}$).

By $(*)$, $V \subseteq (B_{st} \otimes D') \cap \mathrm{Fil}^0(B_{st} \otimes D)^{\phi=1, N=0} = V_{st}(D') \subseteq V_{st}(D) = V$

Hence $V = V_{st}(D') = V_{st}(D)$.

(cont pf of Thm)

By definition, $V' = C_{st}$-span of V, so we can choose a C_{st}-basis $\{v_1, \ldots, v_r\}$ of V', with $v_1, \ldots, v_r \in V$.

Likewise, choose a basis $\{d_1, \ldots, d_r\}$ be a K_0-basis of D', and write:

$$v_j = \sum b_{ij} d_i \quad , \quad b_{ij} \in B_{st} \qquad (b/c \ \ V = V_{st}(D')).$$

Since $\Lambda^r_{\mathbb{Q}_p} V \to \Lambda^r_{C_{st}}(V')$ carries $v_1 \wedge \cdots \wedge v_r$ to some nonzero element.

As $V' = C_{st} \otimes_{K_0} D'$, $\Lambda^r_{C_{st}} V' = C_{st} \underset{K_0}{\otimes} \overset{r}{\underset{K_0}{\Lambda}} D$, we have:

$$v_1 \wedge \cdots \wedge v_r = b \cdot d_1 \wedge \cdots \wedge d_r \ , \quad b = \det(b_{ij}) \in B_{st} \ . \ \text{So } b \neq 0.$$

On the other hand, each $v_i \in V = V_{st}(D')$ is killed by N, fixed by φ, and in Fil^0. Hence

$$v_1 \wedge \cdots \wedge v_r = b d_1 \wedge \cdots \wedge d_r \in B_{st} \otimes \Lambda^r D'$$

is also killed by N, fixed by ϕ, and is in Fil^0.

$\therefore \ V_{st}(\Lambda^r D') \neq 0$.

By the lemma, we get that $t_H(\Lambda^r D') (= t_H(D')) \geq t_N(\Lambda^r D') = t_N(D')$

However, D is w.a and $D' \subsetneq D$ in $MF_K^{\phi, N}$, so $t_N(D') \geq t_H(D')$.

Hence $t_H(D') = t_N(D')$, and thus $\underline{D' \text{ is w.a.}}$

(*) We conclude from the lemma that $V_{st}(\Lambda^r D')$ is 1-dimensional. (*)

By (*), if $w_1, \ldots, w_r \in V = V_{st}(D')$, then $w_1 \wedge \cdots \wedge w_r \in \Lambda^r_{C_{st}}(V)$, so viewing it as an element of $C_{st} \otimes \Lambda^r D'$, we have

$w_1 \wedge \cdots \wedge w_r = c \cdot v_1 \wedge \cdots \wedge v_r$, for a unique $c \in \mathbb{Q}_p$.

↓

(cont pf of thm).

So for $v \in V \subseteq V'$ arbitrary, and writing $v = \sum c_i v_i$, $c_i \in C_{st}$,

we have $v_1 \wedge \cdots v_{i-1} \wedge v \wedge v_{i+1} \wedge \cdots \wedge v_r = c_i (v_1 \wedge \cdots \wedge v_r)$, $c_i \in \mathbb{Q}_p$,

and $\{v_1, \ldots, v_r\}$ is a $\mathbb{Q}_p$-basis of V. ($\Rightarrow V$ has finite dim$/\mathbb{Q}_p$ r.)

Hence $\dim_{\mathbb{Q}_p} V \leq \dim_{K_0}(D)$.

To conclude, note also that from

$$v_1 \wedge \cdots \wedge v_r = b \cdot d_1 \wedge \cdots \wedge d_r, \quad b \in B_{st}.$$

we have G_K acting on b via a $\mathbb{Q}_p^\times$-valued character. So, since $b \neq 0$,

by $(\mathbb{Q}_p, G_K)$-regularity of B_{st} we get $b \in B_{st}^\times$, and hence:

$\{v_1, \ldots, v_r\}$ is a B_{st}-basis of $B_{st} \otimes_{K_0} D'$.

Hence the B_{st}-linear, G_K-equivariant map

$$B_{st} \otimes_{\mathbb{Q}_p} V = B_{st} \otimes V_{st}(D') \longrightarrow B_{st} \otimes_{K_0} D'$$

is an isomorphism, so as K_0-vectorspaces, we have:

$$(\dagger) \quad D_{st}(V) \simeq (B_{st} \otimes_{K_0} D')^{G_K} = B_{st}^{G_K} \otimes D' = D'$$

But finally, $(\dagger)$ respects ϕ, N, Fil, and since we already showed that

D' is w.a and $D_{st}(V)$ is w.a (as V is semistable), we conclude

that $(\dagger)$ is an iso in $MF_K^{\phi, N, w.a}$ (this category is abelian!).

Thus, $D' \cong D_{st}(V)$, so $D_{st}(V) \subseteq D$ and, $\dim_{K_0} D_{st}(V) = \dim_{\mathbb{Q}_p} V$, so

$D_{st}(V) = D$ iff $\dim_{\mathbb{Q}_p} V = \dim_{K_0} D$.

E.O.C.

p-adic Hodge Theory, MATH 726 Fall 2008

Assignment 1

1. Let I be a directed set and $\{G_i\}_{i \in I}$ an inverse system of finite groups with projection maps $\phi_{ij} : G_i \to G_j$ for all $i, j \in I$ satisfying $j \leq i$. Give each G_i the discrete topology and denote by π the product $\pi := \prod_{i \in I} G_i$ endowed with the product topology. Define

$$G := \varprojlim_{i \in I} G_i := \{(g_i)_{i \in I} \mid \phi_{ij}(g_i) = g_j \text{ for all } j \leq i\} \subseteq \pi$$

   (a) Show that G is a closed subset of π.

   (b) Give G the subspace topology. Show that G is compact and totally disconnected for this topology.

   (c) Prove that the natural projection maps $\phi_i : G \to G_i$ are continuous, and that the (open) subgroups $K_i := \ker \phi_i$ for a sub basis of open neighborhoods of the identity.

   (d) Show that a subgroup of G is open if and only if it is closed and of finite index.

2. Let $I \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the inertia subgroup and $W \subseteq I$ the wild inertia subgroup. Show that there is a non-canonical isomorphism of topological groups

$$I/W \simeq \prod_{\ell \neq p} \mathbf{Z}_\ell.$$

   What can be said if one replaces $\mathbf{Q}_p$ with a general p-adic field K?

3. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{Q}_p)$ be a continuous representation. Show that for all $\ell \neq p$, the image under ρ of any wild inertia group W_ℓ at ℓ is finite. Is the same necessarily true of the image of any I_ℓ?

4. Let F be a finite extension of $\mathbf{Q}_\ell$, and suppose $\rho : G_F \to \mathrm{GL}_n(\mathbf{Q}_p)$ is a continuous representation. Show that $\overline{F}^{\ker(\rho)}$ is infinitely (wildly) ramified if and only if the image of (wild) inertia under ρ is infinite.

5. Do Exercise 1.2.5 in the notes.

6. Do Exercise 1.3.2 in the notes.

7. Let K be a p-adic field. Show that the image of the p-adic cyclotomic character $\chi : G_K \to \mathbf{Z}_p^\times$ is closed.

8. Show that the two definitions of *continuous representation* given in Definition 1.2.1 of the notes really are equivalent.

9. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C})$ be a continuous representation.

   (a) Prove that up to conjugation by an element of $\mathrm{GL}_n(\mathbf{C})$, the representation ρ factors through $\mathrm{GL}_n(K)$ for some field K of finite degree over $\mathbf{Q}$. (You may use the fact that any compact, totally disconnected subgroup of $\mathrm{GL}_n(\mathbf{C})$ is finite).

   (b) Prove that we may take K above to be an abelian extension of $\mathbf{Q}$.

   (c) For a prime p, is it the case that any continuous $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C}_p)$ must factor through $\mathrm{GL}_n(K)$ for some $K/\mathbf{Q}_p$ of finite degree?

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Assignment 2

1. Let Γ be a profinite group and R a complete discrete valuation ring with fraction field K that is a p-adic field. We suppose that Γ acts on R via continuous automorphisms (and hence also on K). Recall that if V is a finite dimensional vector space over K, an R-lattice in V is a finite free R-submodule Λ of V with the property that $\Lambda \otimes_R K \simeq V$. Show that any V with semilinear Γ action (i.e. $g(\alpha v) = g(\alpha)g(v)$ for all $\alpha \in K$ and $v \in V$) admits a Γ-stable R-lattice Λ as follows:

   (a) Choose any R-lattice $\Lambda_0 \subseteq V$. By choosing bases, show that $\mathrm{Aut}_R(\Lambda_0)$ is an open subgroup of $\mathrm{Aut}_K(V)$.

   (b) Conclude that the preimage Γ_0 of $\mathrm{Aut}_R(\Lambda_0)$ in Γ under the representation $\rho : \Gamma \to \mathrm{Aut}_K(V)$ is of finite index in Γ.

   (c) Letting $\{\gamma_i\}$ be any finite set of coset representatives for Γ/Γ_0, show that the sum (taken inside V)
   $$\Sigma_i \rho(\gamma_i)\Lambda_0$$
   is a Γ-stable R-lattice in V.

2. Let K be a p-adic field and $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Define the dual of W by $W^* := \mathrm{Hom}_{\mathbf{C}_K-\mathrm{lin}}(W, \mathbf{C}_K)$ with G_K-action given by $g.\varphi(w) := g\varphi(g^{-1}w)$ (i.e. W^* as a $\mathbf{C}_K$-vector space is the usual $\mathbf{C}_K$-linear dual of W). Verify that indeed $W^* \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and that $W^{**} \simeq W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Show that W^* is Hodge-Tate if and only if W is. Hint: you may want to use the "concrete" characterization of Hodge-Tate representations given in class.

3. It may be helpful to know a little Galois cohomology for this exercise. I recommend looking at Tate's article `http://modular.fas.harvard.edu/Tables/Notes/tate-pcmi.html` or Serre's book.

   Let $\eta : G_K \to \mathbf{Z}_p^\times$ be any continuous character. Fix an extension
   $$0 \longrightarrow \mathbf{C}_K(\eta) \longrightarrow W \longrightarrow \mathbf{C}_K \longrightarrow 0 \tag{1}$$
   in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

   (a) By choosing a $\mathbf{C}_K$-linear vector space splitting of this exact sequence, show that we may identify W with $\mathbf{C}_K(\eta) \oplus \mathbf{C}_K$ with $g \in G_K$-acting via
   $$g(v, \alpha) = (g.v + \alpha \cdot \tau(g), \alpha)$$
   where $\tau : G_K \to \mathbf{C}_K(\eta)$ is a function satisfying $\tau(hg) = \eta(g)\tau(h) + \tau(g)$, i.e. τ is a 1-cocycle.

   (b) Prove that τ is continuous, and that making a different choice of splitting alters τ by a coboundary.

   (c) Show that the association $W \rightsquigarrow \tau$ induces a bijection between isomorphism classes of extensions of $\mathbf{C}_K$ by $\mathbf{C}_K(\eta)$ and the set $H^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta))$. If you feel energetic, show that this is even an isomorphism of abelian groups, where we add two extensions by taking their Baer sum.

   (d) Deduce from the Ax-Sen-Tate theorem that if $\eta(I_K)$ is infinite, then (1) splits (as an extension in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$!) and that this splitting is *unique*.

4. Let K be a p-adic field and fix $q \in K$ with $|q| < 1$. Then $q^{\mathbf{Z}} := \{q^n \mid n \in \mathbf{Z}\}$ is a discrete subgroup (lattice) of $\overline{K}^\times$. Consider the quotient $E_q := \overline{K}^\times / q^{\mathbf{Z}}$; this abelian group admits a natural structure of G_K-module through the action on $\overline{K}^\times$. For each $r \geq 0$, let $E_q[p^r]$ be the subgroup of E_q consisting of p^r-torsion elements.

   (a) Let ζ be a primitive p^r-th root of unity and choose a p^r-th root ξ of q in $\overline{K}^\times$. Show that the natural map $i_{\zeta,q} : (\mathbf{Z}/p^r\mathbf{Z})^2 \to E_q[p^r]$ induced by
   $$(m, n) \mapsto \xi^n \zeta^m \in \overline{K}^\times$$
   is an isomorphism of abelian groups. What happens to $\iota_{\zeta,\xi}$ if we change our choices of ζ and ξ?

   (b) Define $T_p(E_q) := \varprojlim_r E_q[p^r]$ by using the natural multiplication by p maps $E_q[p^{r+1}] \to E_q[p^r]$. Show that $T_p(E_q)$ is a free $\mathbf{Z}_p$-module of rank 2 and gives a continuous 2-dimensional representation $\rho_{E_q} : G_K \to \mathrm{GL}_2(\mathbf{Z}_p)$.

   (c) Set $V_p(E_q) := T_p(E_q) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Using (a), show that the natural maps $\mathbf{Z}/p^r\mathbf{Z} \to (\mathbf{Z}/p^r\mathbf{Z})^2$ and $(\mathbf{Z}/p^r\mathbf{Z})^2 \to \mathbf{Z}/p^r\mathbf{Z}$ given by $m \mapsto (m, 0)$ and $(m, n) \mapsto n$ realize $V_p(E_q)$ as an extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$, i.e. that we have a canonical exact sequence of continuous G_K-modules
   $$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow V_p(E_q) \longrightarrow \mathbf{Q}_p \longrightarrow 0\,. \tag{2}$$

(d) Prove that $V_p(E_q)$ is Hodge-Tate. Hint: Use Problem (3).

(e) Prove that (2) is non-split as an extension of representations of G_K, even if we extend scalars to $\overline{K}$.

5. Let K be a p-adic field with finite residue field $\mathbf{F}_q$. Pick $\alpha \in \mathrm{GL}_n(\mathbf{C}_K)$ and consider the uramified Galois representation defined by

$$G_K \twoheadrightarrow G_{\mathbf{F}_q} \simeq \widehat{\mathbf{Z}} \longrightarrow \mathrm{GL}_n(\mathbf{C}_K)$$

defined by sending $1 \in \widehat{\mathbf{Z}}$ to α. Show that this is a continuous representation if and only if all eigenvalues of the matrix α have absolute value 1. Use this to give an example of a continuous, n-dimensional G_K-representation with $\mathbf{C}_K$ coefficients that does not factor through $\mathrm{GL}_n(L)$ for any algebraic extension L/K.

✓ 6. Let K be a p-adic field containing μ_p and let $\chi : G_K \to \mathbf{Z}_p^\times$ be the cyclotomoc character.

(a) Show that χ has image in $1 + p\mathbf{Z}_p$.

(b) For any $s \in \mathbf{Z}_p$, show that the character χ^s of G_K defined by the composition of χ with the map $1+p\mathbf{Z}_p \to 1+p\mathbf{Z}_p$ given by $x \mapsto x^s$ makes sense and is continuous.

(c) Prove that χ^s is Hodge-Tate if and only if $s \in \mathbf{Z}$.

7. Fix a p-adic field and let η be a nontrivial finite order continuous character $\eta : G_K \to \mathbf{Q}_p^\times$.

✓(a) Show that η factors through the natural inclusion $\mathbf{Z}_p^\times \hookrightarrow \mathbf{Q}_p^\times$.

✓(b) Prove that there are no nonzero G_K-homomorphisms $K \to K(\eta)$.

(c) Suppose that L/K is finite Galois and the restriction of η to G_L is trivial. Show that there exists a nonzero homomorphism $L \to L(\eta)$ of L-modules with semilinear G_K-action, and hence that these two G_K-modules are isomorphic.

8. Fix a field E of characteristic p and let (M, φ_M) be an étale φ-module over E. Define $M^\vee$ to be the E-linear dual of M and let $\varphi_{M^\vee}$ be the map

$$M^\vee \longrightarrow (\varphi_E^*(M))^\vee \longrightarrow M^\vee \tag{3}$$

where the first map takes a linear functional ℓ on M to the linear functional on $\varphi_E^*(M) := M \otimes_{E,\varphi_E} E$ given by $m \otimes e \mapsto \varphi_E(\ell(m))e$, and the second map is the E-linear dual of the inverse of the E-linear isomorphism $\varphi_E^*(M) \to M$ given by the linearization of φ_M. Prove that $\varphi_{M^\vee}$ is semilinear over φ_E, and that its linearization is an isomorphism. Hint: show that the linearization of first map in (3) is the canonical isomorphism

$$\varphi_E^*(M^\vee) = \mathrm{Hom}_E(M, E) \otimes_{E,\varphi} E \simeq \mathrm{Hom}_E(M, E_\varphi) \simeq \mathrm{Hom}_{\varphi-\mathrm{sl}}(M, E) \simeq \mathrm{Hom}_E(\varphi_E^*(M), E) = \varphi_E^*(M)^\vee$$

where E_φ denotes E as an E-module via φ_E, and $\mathrm{Hom}_{\varphi-\mathrm{sl}}$ is the E-module of φ_E-semilinear E-module homomorphisms.

9. Let M be any étale φ-module over $\mathscr{O}_{\mathscr{E}}$. Show that $\mathbf{V}_{\mathscr{E}}(M)$ is continuous as a G_E-representation. (done in class)

10. Let $E = \mathbf{F}_q$, so $G_E \simeq \widehat{\mathbf{Z}}$. Let $\rho : G_E \to \mathrm{Aut}_{\mathbf{F}_p}(V)$ be a continuous representation on a d-dimensional $\mathbf{F}_p$-vector space V, and let $(M, \varphi_M) = \mathbf{D}_E(V)$ be the associated étale φ-module over $E = \mathbf{F}_q$. Identifying G_E with $\widehat{Z}$ and choosing a basis for V, show that the $d \times d$-matrix $\rho(1)$ is the inverse of the $d \times d$-matrix of the linearization of φ_M.

11. Fix a pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ as in the notes and let (M, φ_M) be a φ-module over $\mathscr{O}_{\mathscr{E}}$; i.e. a finitely generated $\mathscr{O}_{\mathscr{E}}$-module with a φ-semilinear endomorphism $\varphi_M : M \to M$. Show that φ_M is étale if and only if $\varphi_M \bmod p$ is étale. Hint: first show that M and $\varphi^*(M)$ are abstractly isomorphic as $\mathscr{O}_{\mathscr{E}}$-modules—i.e. that they have the same rank and invariant factors. Conclude that φ_M is an isomorphism if and only if it is surjective, and show that surjectivity may be checked modulo p.

$\mathcal{E}$

12. Let M be a finitely generated module over a complete discrete valuation ring R of characteristic zero with uniformizer p. Suppose that G is a monoid acting on R by ring endomorphisms and on M by semilinear module endomorphisms. Show that for each n, G acts on M/p^nM and that $\varprojlim_n (M/p^n)^G = M^G$.

13. Prove that $\mathbf{V}_{\mathscr{E}}(\mathscr{E}/\mathscr{O}_{\mathscr{E}}) = \mathbf{Q}_p/\mathbf{Z}_p$.

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Assignment 3

1. Let K be a p-adic field and set $B_{\mathrm{dR}}^{\mathrm{naive}} := \mathbf{C}_K((t))$, equipped with the $\mathbf{C}_K$-semilinear G_K-action defined by $g.t^n := \chi^n(g)t^n$ where $\chi : G_K \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character. Give $B_{\mathrm{dR}}^{\mathrm{naive}}$ the t-adic filtration, so it becomes a filtered $\mathbf{C}_K$-vector space with semilinear G_K-action. We define

$$D_{\mathrm{dR}}^{\mathrm{naive}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$$

by $D_{\mathrm{dR}}^{\mathrm{naive}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^{\mathrm{naive}})^{G_K}$ with filtration induced by the filtration on $B_{\mathrm{dR}}^{\mathrm{naive}}$, and we call $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ "naively de Rham" if $\dim_K D_{\mathrm{dR}}^{\mathrm{naive}}(V) = \dim_{\mathbf{Q}_p}(V)$. Prove that V is naively de Rham if and only if it is Hodge-Tate.

2. Let K be a 2-adic field, and consider any choice of $\epsilon = (1, \zeta_2, \zeta_4, \zeta_8, \ldots) \in R_K$, with $\{\zeta_{2^i}\}$ a collection of compatible primitive 2^i th roots of 1 in $\mathscr{O}_{\mathbf{C}_K}$. Show that $[\epsilon] - 1 \in W(R)$ is a generator of the principal ideal $\ker\theta$.

3. Do Exercise 4.4.8 in the notes.

4. Suppose $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is 1-dimensional. Show that V is Hodge-Tate if and only if it is de Rham.

5. Prove that the Frobenius automorphism of $W(R)[1/p]$ does not preserve $\ker\theta_K$, and so does not naturally extend to B_{dR}^+.

6. Prove $W(R) \cap (\ker\theta_K)^j = (\ker\theta)^j$ for all $j \geq 1$.

The next two problems are taken from Berger's article "An introduction to the theory of p-adic representations".

7. Let K be a p-adic field, fix $q \in K$ with $|q| < 1$ and set $E_q := \overline{K}^\times / q^{\mathbf{Z}}$, considered as a G_K-module through the action on $\overline{K}^\times$. We saw on Assignment 2, problem 4 that $V_p(E_q) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_r E_q[p^r]$ is 2-dimensional $\mathbf{Q}_p$-representation of G_K, and that

$$e := (\epsilon^{(r)})_{r \geq 0} \quad \text{and} \quad f := (q^{(r)})_{r \geq 0}$$

give a basis of $V_p(E_q)$ where $\epsilon^0 = 1$, $\epsilon^{(1)} \neq 1$, $q^{(0)} = q$ and for all $r \geq 1$, we have $(\epsilon^{(r+1)})^p = \epsilon^{(r)}$ and $(q^{(r+1)})^p = q^{(r)}$. Denote by $\underline{\epsilon}$ and $\underline{q}$ the elements of R defined by the p-power compatible sequences $(\epsilon^{(r)})$ and $(q^{(r)})$.

   (a) Show that $g.e = \chi(g)e$ and $g.f = f + c(g)e$ for some $c(g) \in \mathbf{Z}_p$ depending on g.

   (b) Show that the series $\sum_{n \geq 1} (-1)^{n+1} \frac{([\underline{q}]/q - 1)^n}{n}$ for $\log(\frac{1}{q}[\underline{q}])$ makes sense and converges in B_{dR}^+. We define

$$u := \log_p(q) + \log(\frac{1}{q}[\underline{q}]).$$

   Morally, $u = \log([\underline{q}])$.

   (c) Let $t = \log([\underline{\epsilon}]) \in B_{\mathrm{dR}}$. Show that $g.t = \chi(g)t$ and $g.u = u + c(g)t$ for $c(g)$ as in (1).

   (d) Prove that $V_p(E_q)$ is de Rham. Hint: all you have to show is that the K-vector space $(B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V_p(E_q))^{G_K}$ has dimension 2. Do this by using u and t to appropriately modify the B_{dR}-basis $1 \otimes e$ and $1 \otimes f$ of $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V_p(E_q)$ to be G_K-invariant.

8. We can generalize exercise (7). Let V be any extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Prove that V is de Rham as follows:

   (a) Let $\widehat{K^\times}$ be the projective limit $\varprojlim_n (K^\times/(K^\times)^{p^n})$ with transition maps the natl $K^\times/(K^\times)^{p^{n+1}} \to K^\times/(K^\times)^{p^n}$ ~~given by p-th power~~. Fix a choice $(\epsilon^{(n)})$ of a compatible system of p-power roots of unity in $\overline{K}$ and Consider the map $\delta : \widehat{K^\times} \to H^1_{\mathrm{cont}}(G_K, \mathbf{Z}_p(1))$ defined as follows: for $q = q^{(0)}$ in $\widehat{K^\times}$, choose a sequence $(q^{(n)})_{n \geq 0}$ in $\overline{K}$ with $(q^{(n+1)})^p = q^{(n)}$ for all n and let $\delta(q)$ be the cocycle c determined by $g(q^{(n)}) = (q^{(n)}) \cdot (\epsilon^{(n)})^{c(g)}$. Show that any two choices of $(q^{(n)})$ give cohomologous cycles, so δ is well-defined.

   (b) Prove that δ induces an isomorphism $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{K^\times} \simeq H^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1))$.

   (c) Look over your work on Assignment 2, problem 3 and convince yourself that $H^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1))$ classifies isomorphism classes of G_K-extensions of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$. Conclude that we can choose a basis $\{e, f\}$ of V such that $g.e = \chi(g)e$ and $g.f = f + c(g)e$ where $c(g)$ is the cocycle corresponding to $q \in \mathbf{Q}_p \otimes \widehat{K^\times}$ as above.

   (d) Defining $u =$ "$\log([\underline{q}])$" as above, show that we can appropriately modify the basis $\{1 \otimes e, 1 \otimes f\}$ of $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ so as to be G_K-invariant. Conclude that V is de Rham.

$q \in K^\times/(K^\times)^p \to H^1(G_K, \mu_p)$ $\quad q^{1/p} \in \overline{K}^\times$ $\quad \left(g \mapsto \frac{g q^{1/p}}{q^{1/p}}\right)$ $\quad \mu_p(\overline{K}^\times)$

$$\begin{tikzcd}
0 \arrow[r] & \mu_{p^{n-1}}(\bar{K}) \arrow[r] & \bar{K}^\times \arrow[r, "\cdot p^{n-1}"] & \bar{K}^\times \arrow[r] & 0 \\
0 \arrow[r] & \mu_{p^n}(\bar{K}) \arrow[r] \arrow[u, "\wedge p"] & \bar{K}^\times \arrow[r, "p^n"'] \arrow[u, "\wedge p"] & \bar{K}^\times \arrow[r] \arrow[u, "id"] & 0
\end{tikzcd}$$

$$0 \to \mu_{p^n}(\bar{K}) \to \bar{K}^\times \xrightarrow{\cdot p^n} \bar{K}^\times \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & \mu_{p^{n-1}}(K) \arrow[r] \arrow[d, hook] & K^\times \arrow[r, "\cdot p^{n-1}"] \arrow[d, "id"] & K^\times \arrow[r] \arrow[d, "\wedge p"] & H^1(G_K, \mu_{p^{n-1}}(\bar{K})) \arrow[r] \arrow[d, hook, "?"] & 0 \\
0 \arrow[r] & \mu_{p^n}(K) \arrow[r] & K^\times \arrow[r, "\cdot p^n"] & K^\times \arrow[r] & H^1(G_K, \mu_{p^n}(\bar{K}^\times)) \arrow[r] & 0
\end{tikzcd}$$

$$q \longmapsto \left(g \mapsto \frac{g q^{1/p^n}}{q^{1/p^n}}\right)$$

$$K^\times / (K^\times)^{p^n} \xrightarrow{\sim} H^1(G_K, \mu_{p^n}(\bar{K}^\times))$$

$$\varprojlim K^\times / K^{\times p^n}$$

$$\widehat{K^\times} \simeq \varprojlim K^\times / (K^\times)^{p^n} = H^1(G_K, \mathbb{Z}_p(1))$$

$$\left(q^{(n)/p^n}\right)^{p^n} = q^{(0)/p^n}$$

$$K^\times / (K^\times)^{p^n} \longrightarrow K^\times / (K^\times)^{p^{n-1}}$$

$$q \in \widehat{K^\times} = (q^{(1)}, q^{(2)}, q^{(3)}, \ldots)$$

$$\in K^\times / (K^\times)^p \quad \ldots$$

$$\left(q^{(n)}\right)^p = q^{(0)/p^n}$$

$$q^{(n)/p^n} = q^{(0)/p^{2n}}$$

$$\zeta_p \qquad \zeta_{p^2} \qquad \zeta_{p^3}$$

$$\left(\frac{g q^{(1)/p}}{q^{(1)/p}}, \frac{g q^{(2)/p^2}}{q^{(2)/p^2}}, \frac{g q^{(3)/p^3}}{q^{(3)/p^3}}, \ldots \right)$$

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Assignment 4

1. Let K be a p-adic field. This exercise gives an alternative way of seeing that $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$ is not full.

   (a) Let $V, V' \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$. Prove that $D_{\mathrm{dR}}(V)$ and $D_{\mathrm{dR}}(V')$ are isomorphic in Fil_K if and only if V and V' have the same Hodge-Tate numbers; i.e. if and only if they have the same Hodge-Tate weights and for each Hodge-Tate weight i, the multiplicities $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V))$ and $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V'))$ are equal.

   (b) Show that there exists a non-split extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ in $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$. Hint: Think back to previous assignments.

   (c) Show that D_{dR} can not be full.

2. Let F be a field. Do one (or more) of the following:

   (a) For objects D, D' of Fil_F, show that the canonical F-linear isomorphism $D \otimes_F D'^* \simeq \mathrm{Hom}_F(D', D)$ is an isomorphism in Fil_F, where the tensor product is given its usual tensor-product filtration and $\mathrm{Hom}_F(D', D)$ is given the filtration $\mathrm{Fil}^i \mathrm{Hom}_F(D', D) := \mathrm{Hom}_{\mathrm{Fil}_F}(D', D[i])$.

   (b) Show that the canonical F-linear isomorphisms

   $$\det(D^*) \simeq \det(D)^* \qquad \text{and} \qquad \det(D \otimes D') \simeq \det(D)^{\dim_F D'} \otimes \det(D')^{\dim_F D}$$

   are isomorphisms in Fil_F.

   (c) Prove that for a short exact sequence in Fil_F

   $$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

   the canonical F-linear isomorphism $\det(D') \otimes \det(D'') \simeq \det(D)$ is an isomorphism in Fil_F.

3. Let n be a positive integer and K a p-adic field. Show that if V is any extension

   $$0 \longrightarrow \mathbf{Q}_p(n) \longrightarrow V \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

   in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, then V is de Rham. Hint: adapt the argument of Example 6.3.5 in the notes.

4. Let D be a K_0-vector space with a σ-semilinear endomorphism $\phi : D \to D$. If D has finite K_0 dimension, show that ϕ is injective if and only if it is bijective. Give a counterexample to this with D of infinite dimension.

5. Let D be an isocrystal over K_0. Prove that $t_N(D) = t_N(\det D)$. Hint: first show that if $D(\alpha)$ and $D(\beta)$ are isoclinic of slopes α and β respectively, then $D(\alpha) \otimes_{K_0} D(\beta)$ is isoclinic of slope $\alpha + \beta$. Then work with a basis for D adapted to the isoclinic decomposition of D as guaranteed by Lemma 7.2.7.

6. Let D be a filtered (φ, N)-module over K. Prove that D is weakly admissible if and only if D^* is.

7. Let $h : M' \to M$ be a bijective morphism in Fil_K. Show that h is an isomorphism in Fil_K if and only if $\det(h) : \det(M') \to \det(M)$ is an isomorphism.