Representation Theory of Finite Groups

Def: Let G be a group, k a field, V a k-vectorspace. A representation of G in V is a group homomorphism $\rho: G \longrightarrow \mathrm{Aut}_k V = GL(V)$.

In other words, (assuming $\dim V < \infty$), ρ assigns a matrix $\rho(g)$ to each element $g \in G$ so that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$.

- Reasons to study representation theory:

1st) We want to realize an abstract group G (or rather $G/\ker\rho$) in terms of matrices.

Remark: To specify a representation, we have to specify V & ρ.
Sometimes ~~one~~ only gives ρ (and V is understood), or V (bad, but common)

Example: $(\mathbb{Z}, +)$, free cyclic group.
A representation of $\mathbb{Z}$ in V can be specified by giving the value at $1 \in \mathbb{Z}$ $\quad (\rho(1) = A \in GL(V), \rho(n) = A^n)$.

Def: Two representations $\rho_1, \rho_2 : G \to GL(V)$ are isomorphic if there exists $\sigma \in GL(V)$ s.t. $\sigma \rho_1(g) \sigma^{-1} = \rho_2(g) \quad \forall g \in G$.

Usually we don't distinguish isomorphic representations.

Remark: The question of classifying representations of G in V is a generalization of the question of classifying linear operators in V.

2nd) Given a representation, find what kind of representation it is (or find the canonical form of this representation).

We often find "given" representations:
if G acts on a set (manifold, etc) X, then G acts on the functions on X. (homologies of X, etc)

Example 1: $G = \mathbb{Z}/3\mathbb{Z}$.

Q: Find all 2-dim'l reps of G up to isomorphism.

A representation ρ of G is given by a 2×2-matrix, $\rho(1)=A$, s.t. $A^3 = I$.

a) If $k = \mathbb{C}$:

$$A \sim \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{s.t. } \alpha^3 = \beta^3 = 1.$$

So G acts on x and y axes independently.

b) If $k = \mathbb{R}$

$A \sim$ rotation by $\frac{2\pi}{3}$ (or I)

c) If $k = \mathbb{F}_3$

As $x^3 - 1 = (x-1)^3$, then $A \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or I

Let now G be a group, and suppose π is a V-representation. ($\pi : G \to GL(V)$)

Def:

- $W \subset V$ (a subspace) is called <u>G-invariant</u> if $\pi(g)x \in W$ $\forall x \in W$, $\forall g \in G$.

(in this case, we can restrict π to W (<u>subrepresentation</u>) and to V/W (<u>quotient representation</u>)).

Then $\pi(g) = \left(\begin{array}{c|c} \pi_W(g) & * \\ \hline 0 & \pi_{V/W}(g) \end{array}\right)$

Def: π is <u>irreducible</u> if the only G-invariant subspaces are trivial (0 & V).

In example 1:

a) not irreducible.

b) irreducible (for rotations) } want to distinguish them!

c) not irreducible.

Def: A representation on V is indecomposable if V cannot be written as a direct sum of two subrepresentations, $V = W_1 \oplus W_2$ ($W_1, W_2 \neq 0$).

[$\pi(g)$ cannot be put into the form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$ for a choice of bases (and $\forall g$)].

Rk: irreducible $\Rightarrow$ indecomposable.

In example ~~1~~:

a) not indecomposable

b) irreducible ($\Rightarrow$ indecomposable)

c) indecomposable.

From now on, we'll assume that $\dim V < +\infty$.

Then, by induction, any representation V can be written as

$$V = W_1 \oplus \cdots \oplus W_n \quad , \quad W_i \text{ indecomposable.}$$

Q: is the decomposition unique?

Krull-Schmidt Thm:

Let V be a representation of G ($\dim V < \infty$). Suppose that

$V = W_1 \oplus \cdots \oplus W_n = W_1' \oplus \cdots \oplus W_m'$, indecomposable. Then $n = m$ and

$W_i \cong W'_{s(i)}$ for some permutation $s \in S_n$.

as G-representations.

Lemma 1: Let V, W be indecomposable reps, and $\alpha \in Hom_G(V, W)$, $\beta \in Hom_G(W, V)$ and $\beta \circ \alpha$ is an isomorphism (invertible).

Then α and β are isomorphisms.

Proof (of Lemma 1):

- $\operatorname{Im}\alpha \cap \operatorname{Ker}\beta = 0$ (because $\beta\circ\alpha$ is an iso).
- Let $x \in W$. Then write $x = \alpha(\beta\alpha)^{-1}\beta x + y$.

Note that $y \in \operatorname{Ker}\beta$, and $\alpha(\beta\alpha)^{-1}\beta x \in \operatorname{Im}\alpha$.

So $W = \operatorname{Im}\alpha \oplus \operatorname{Ker}\beta$. Since W is indecomposable, $\operatorname{Ker}\beta = 0 \Rightarrow$ //.

Lemma 2 (Fitting Lemma): Assume V is indecomposable rep. Let $\alpha \in \operatorname{End}_G V$. Then either α is an isomorphism or α is nilpotent. (assume $\dim V < \infty$)

Pf Since $\dim V < \infty$, $\exists\, n \in \mathbb{Z}$ s.t. $\begin{cases} \operatorname{Im}\alpha^n = \operatorname{Im}\alpha^{n+1} = \cdots \\ \operatorname{Ker}\alpha^n = \operatorname{Ker}\alpha^{n+1} = \cdots \end{cases}$

- $\operatorname{Im}\alpha^n \cap \operatorname{Ker}\alpha^n = 0$:

(Let $x \in \operatorname{Im}\alpha^n$. So $x = \alpha^n y$. If $x \in \operatorname{Ker}\alpha^n$, then $\alpha^{2n}y = 0 \Rightarrow$
$\Rightarrow y \in \operatorname{Ker}\alpha^{2n} = \operatorname{Ker}\alpha^n \Rightarrow \alpha^n y = 0 \Rightarrow x = 0$).

- $x \in V \Rightarrow \alpha^n x = \alpha^{2n} y \Rightarrow \alpha^n(x - \alpha^n y) = 0 \Rightarrow x = \alpha^n y + z$,
where $z \in \operatorname{Ker}\alpha^n$, and $\alpha^n y \in \operatorname{Im}\alpha^n$.

So $V = \operatorname{Im}\alpha^n \oplus \operatorname{Ker}\alpha^n$. By indecomposability, either $\operatorname{Im}\alpha^n = 0$ (α nilpotent) or $\operatorname{Ker}\alpha^n = 0$ (α isomorphism). //

Lemma 3: V indecomposable rep. Write $\alpha \in \operatorname{End}_G V$ as $\alpha = \alpha_1 + \cdots + \alpha_n$, and suppose α is invertible.
Then one of the α_i's is invertible.

Proof: By induction, can assume $n = 2$, $\alpha = \alpha_1 + \alpha_2$. Then $id = \underbrace{\alpha_1\alpha^{-1}}_{\beta_1} + \underbrace{\alpha_2\alpha^{-1}}_{\beta_2}$

Assume β_1, β_2 not invertible $\Rightarrow$ nilpotent (by lemma 2).

$\Rightarrow \beta_1 + \beta_2$ is nilpotent $\Rightarrow$!! ($\beta_1 + \beta_2 = id$!) //

<u>Proof of the Theorem</u>:

$V = W_1 \oplus \cdots \oplus W_n = W_1' \oplus \cdots \oplus W_m'$

Consider projections $p_i : W_1 \to W_i'$

$q_i : W_i' \to W_1$

$\mathrm{Pr}_{W_1} = q_1 p_1 + q_2 p_2 + \cdots + q_m p_m \Rightarrow$ invertible on $W_1 \Rightarrow p_i q_i$ is invertible on $W_1 \Rightarrow$

$\Rightarrow p_i, q_i$ are isomorphisms $W_1 \cong W_i' \Rightarrow$ (induction, quotienting out) $\Rightarrow$ //

Each of the indecomposable (blocks) representation can still have invariant subspaces. So we write each of the resulting blocks in up-triangular form:

$$\pi_i(g) = \left(\begin{array}{c|c|c} \pi_{i1}(g) & & * \\ \hline 0 & \pi_{i2}(g) & \\ \hline 0 & 0 & \pi_{i3}(g) \end{array}\right)$$

So that π_{ik} is irreducible (does not have a nontrivial G-invariant subspaces).

In terms of subrepresentations, we want to write V as:

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

So that
- each V_i is G-invariant
- each V_{i+1}/V_i is irreducible.

Such a filtration is called a <u>composition series</u> of V (or π), and V_{i+1}/V_i are called <u>composition factors</u>.

<u>Jordan-Hölder Thm</u>: V a rep. of G, and $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ ~~two filtrations~~
$0 = V_0' \subset V_1' \subset \cdots \subset V_m' = V$ composition series.

Then the factors are the same, up to reordering (and $n = m$).

i.e. $V_{i+1}/V_i \cong V'_{s(i)+1}/V'_{s(i)}$ for some permutation s.

Proof (of J-H):
Refine both filtrations:

$$V_{n-1} = (V_{n-1} + V_0') \subseteq (V_{n-1} + V_1') \subseteq (V_{n-1} + V_2') \subseteq \cdots \subseteq (V_{n-1} + V_m') = V = V_n$$

$$V_{n-2} = (V_{n-2} + V_0') \cap V_{n-1} \subset \cdots \quad (V_{n-2} + V_m') \cap V_{n-1} = V_{n-1}$$

$\vdots \quad \vdots$

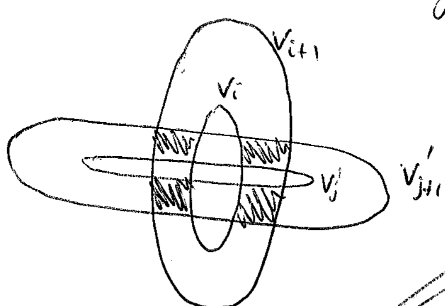
$$V_1 = (V_1 + V_0') \cap V_2 \subset (V_1 + V_1') \cap V_2 \subset \cdots \subseteq (V_1 + V_m') \cap V_2 = V_2$$

(and similarly for the second filtration).

In the first filtration, the factors are: $((V_i + V_{j+1}') \cap V_{i+1}) / (V_i + V_j') \cap V_{i+1}$

Zassenhaus lemma: this expression is $i \leftrightarrow j$ invariant.

Pf/ Can use bases (although it can be proven for arbitrary modules). Take bases that are good for all subspaces involved. Then we can use Venn diagrams:



Since the original filtrations have irreducible factors, the refined filtrations have the same factors as the original ones, so the theorem follows.

Morphisms

$\pi_1 : G \to GL(V_1)$
$\pi_2 : G \to GL(V_2)$ reps of a group G.

Def: $Hom_G(V_1, V_2) := Hom(\pi_1, \pi_2) = \{ \varphi$ linear maps $V_1 \to V_2$ s.t. $\varphi \pi_1(g) = \pi_2(g)\varphi(g)$ $\forall g \in G \}$

$End_G(V_1) := Hom_G(V_1, V_1)$.

$Aut_G(V_1) := \{ \varphi \in End_G(V_1) : \exists \varphi^{-1} \}$.

Schur's lemma:

If π_1, π_2 are irreducible and $\varphi \in Hom_G(V_1, V_2)$ $\Rightarrow \varphi = 0$ or φ is an isomorphism ($\exists \varphi^{-1}$).

In particular:

1. $V_1 \not\cong_G V_2$ & V_1, V_2 irreducible $\Rightarrow Hom_G(V_1, V_2) = 0$.
2. V irreducible $\Rightarrow End_G(V)$ is a division algebra.

Def: A ring R is a division ~~algebra~~ ring if $\underset{R}{x \neq 0} \Rightarrow \exists x^{-1} \in R$.

Pf (of Schur's lemma)

Im φ & Ker φ are subreps of (V_2 and V_1 resp.) $\overset{\text{irreducible}}{\Longrightarrow}$ $\begin{cases} \text{Im } \varphi = \{0 \text{ or } V_2 \\ \text{Ker } \varphi = \{0 \text{ or } V_1 \end{cases}$ $\Rightarrow$ ✓ //

Theorem: Let K be a field, $K = \bar{K}$. Let A be a division algebra $/K$.
Assume that $\dim_K A < \infty$.
Then $A = K$.

Pf Let $x \in A$. Let p be the minimal polynomial of x (i.e. the generator of the ideal I in $K[t]$, $I = Ker\left(\begin{matrix} K[t] \to A \\ t \mapsto x \end{matrix}\right)$).

If $p(x) = q_1(x) q_2(x) = 0$, then as A is a division algebra, either $q_1(t)$ or $q_2(t)$ is zero $\Rightarrow$!! So p is irreducible.

As $K = \bar{K}$ and p is irreducible, $p(t) = t - \lambda$, so $x - \lambda = 0 \Rightarrow x = \lambda \Rightarrow K = A$. //

Remark: There are 3 division algebras $/\mathbb{R}$: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ ← quaternions.
($\dim_{\mathbb{R}} \mathbb{H} = 4$).

Combining Schur's lemma with the previous theorem, we get:

Corollary: $\varphi \in \mathrm{End}_G(V)$, V irreducible rep of G over $k = \bar{k} \Rightarrow \varphi$ is scalar.

Remark: $\mathrm{End}_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{R}^2) \supset$ rotations of $\mathbb{R}^2$
↑ rotation representation

Def. if $\pi : G \to GL(V)$ is a G-rep $/k$, we say:

- π is stably (absolutely) irreducible (resp. indecomposable) if it remains irreducible (resp. indecomposable) after any field extension.

  In other words $V \otimes_k \bar{k}$ irreducible (resp. indecomposable).

  (i.e. $\pi \otimes id : G \to GL(V \otimes_k \bar{k})$ is irreducible (resp. indecomposable)).

Example: $\mathbb{Z}/3\mathbb{Z} \to GL(\mathbb{R}^2)$ where A = rotation by $\frac{2\pi}{3}$
$[k] \mapsto A^k$

is irreducible, but not stably irreducible.

Characteristic-0 facts.

Maschke's Thm: if $|G| < \infty$ (group), k a field w/ $\mathrm{char}(k) \nmid |G|$, then irreducible $\equiv$ indecomposable.

Or in other words, if $W \subset V$ is a G-invariant subspace, then there exists a G-invariant W' s.t. $V = W \oplus W'$.

~~pf~~ Take $\tilde{W}'$ to be any complement subspace to W

Averaging trick: Let $P_{\tilde{W}'}$ be the projection onto W along $\tilde{W}'$.

Take $P' := \frac{1}{|G|} \sum_{g \in G} \pi(g)^{-1} P_{\tilde{W}'} \pi(g)$

- if $x \in W$, $P'x = x$.
- if $x \notin W$, $P'x \in W$.

$\Rightarrow P'$ is a projection onto W along $W' := \ker P'$. //

Operation on Representations.

Def: Let V be a representation of G, $\pi : G \to GL(V)$.
Then the dual representation (on V^*) is given $\pi^* : G \to GL(V^*)$.
where $\pi^*(g) = \pi(g^{-1})^t$

Def V, W two reps of G, given by π_V, π_W
Then there exists a representation $\pi_{V \otimes W} : G \longrightarrow GL(V \otimes_k W)$ s.t.

$$\pi_{V\otimes W}(g)(v \otimes w) = \pi_V(g) v \otimes \pi_W(g) w$$

Rk: in kG, $\pi_{V\otimes W}(g_1 + g_2) \neq \underbrace{\pi_V(g_1+g_2) \otimes \pi_W(g_1+g_2)}_{\pi_V(g_1)\otimes\pi_W(g_1) + \pi_V(g_2)\otimes\pi_W(g_2) + \pi_V(g_1)\otimes\pi_W(g_2) + \pi_V(g_2)\otimes\pi_W(g_1)}$. extra!

We will prove the existence of this representation $\otimes$ later, in a more conceptual way.

Construction of Representations.

Let X be a G-set (set w/an action by G).
Let $k[X] :=$ k-valued functions on X.
Then there is a representation of G on $k[X]$ given by:

$$(\pi(g) \cdot f)(x) := f(g^{-1}x).$$

Proof: $(\pi(g_1)(\pi(g_2) f))(x) = \pi(g_2) f (g_1^{-1} x) = f(g_2^{-1} g_1^{-1} x) = f((g_1 g_2)^{-1} x) = \pi(g_1 g_2) f(x)$ //

Note, if $X = X_1 \sqcup X_2$ (as G-sets!), then $k[X] = k[X_1] \oplus k[X_2]$ as G-reps.

So we're only interested on X a single G-orbit.
So $X = G/H$ for $H \leq G$ any subgroup.

In the case we are reduced to, $k[G/H] = \{ f \in k[G] : f(gh) = f(g) \ \forall h \in H, \forall g \in G \}$.

We call the rep. on $k[G/H]$ the <u>induced representation from H.</u>

Today, we'll concentrate on $H = 1$. (later we'll treat general H)

<u>Def</u>: $k[G]$ with the left G-action is called the <u>regular representation</u> of G.

<u>Note</u>: $k[G/H]$ is a subrepresentation of $k[G]$, so it should be enough to study all $k[G]$.

Assume now G finite

A natural basis in $k[G]$ is the set of characteristic functions $\{ \chi_h : h \in G \}$.

(ie $\chi_h(g) = \begin{cases} 0 & h \neq g \\ 1 & h = g \end{cases}$).

The regular representation in this basis:

$\pi(g) \chi_h = \chi_{gh}$ ← like a permutation matrix ... not too good.

Another set of natural functions on G is given by matrix elements of G-representations:

Let V be a G-representation: $\pi_V : G \to GL(V)$. Take a basis in V.

Then $\pi_V^{ij}(g) \in k$, and so $\pi_V^{ij} \in k[G]$.

↖ the matrix entry ij.

[If $v \in V$, $\lambda \in V^*$, then $\lambda(\pi_V(g) v) \in K$, so $\pi_V^{ij}(g) = e_i^*(\pi_V(g) e_j)$]

<u>Goal</u>: Matrix elements are good for the regular representation.

More precisely,

<u>Thm</u>: V a G-rep. $\pi_V : G \to GL(V)$. Fix $v \in V$. Then the linear map $V^* \to k[G]$ sending $\lambda \mapsto \lambda(\pi(\cdot) v)$ is a G-representation homomorphism.

The previous theorem says that a column of matrix elements (given by v) maps V^* into $k[G]$.

Pf $\pi_{reg}(h)\left(\lambda(\pi(g)v)\right) = \lambda\left(\pi(h^{-1}g)v\right) = \lambda\left(\pi(h^{-1})\pi(g)\,v\right) =$

$= \pi(h^{-1})^t \lambda\left(\pi(g)v\right)$ //

Next we need to see that it's an injection:

Rk: if V is irreducible, V^* is irreducible too, so $V^* \to k[G]$ is an embedding.

So $k[G]$ contain all irreducible representations!

Rk: If $k=\mathbb{R}$ (or $\mathbb{C}$) then $V \cong V^*$ (problem set). $\pi(g)$ is orthogonal.

Assume now $k=\mathbb{C}$ (or $k=\bar{k}$, char $k=0$)

1) Maschke $\Rightarrow V \cong W_1 \oplus \cdots \oplus W_n$, W_i irreducible.

2) Schur $\to$ V, W irreducible reps $\Rightarrow \operatorname{Hom}_G(V,W) = \begin{cases} 0 & \text{if } V \not\cong W \\ \mathbb{C}\,\mathrm{Id}_V & \text{if } V \cong W \end{cases}$

• A "Strange" Construction with Schur's Lemma:

Spz V, W are two irreducible G-reps. We want an element of $\operatorname{Hom}_G(V,W)$.

Let $\varphi: V \to W$ be any linear map.

Then let $\tilde{\varphi} := \frac{1}{|G|} \sum_{g \in G} \pi_W(g^{-1})\,\varphi\,\pi_V(g)$.

Claim: $\tilde{\varphi} \in \operatorname{Hom}_G(V,W)$.

Pf $\tilde{\varphi}\pi_V(h) = \frac{1}{|G|}\sum_g \pi_W(g^{-1})\,\varphi\,\pi_V(\overset{f}{\overset{\shortparallel}{gh}}) = \frac{1}{|G|}\sum_f \pi_W(hf^{-1})\,\varphi\,\pi_V(f) = \pi_W(h)\,\tilde{\varphi}$ //

By Schur's lemma, $\tilde{\varphi} = \begin{cases} 0 & \text{if } V \not\cong W \\ \lambda \cdot id & \text{if } V \cong W \end{cases}$

In the second case, taking traces yields $\lambda \cdot \dim V = \operatorname{tr}\varphi \Rightarrow \lambda = \frac{1}{\dim V}\operatorname{tr}\varphi$.

Let's see what we have proved:

Pick orthonormal basis $\{e_i\}$ in V wrt some G-invariant hermitian product in V, and similarly for W.

Then Π is unitary: $\Pi(g^{-1}) = \overline{\Pi(g)}^t$, and we can write $\tilde{\varphi}$ as follows:

$$\tilde{\varphi}^{ij} = \frac{1}{|G|}\sum_{g\in G}\sum_{k,l} \Pi_W^{ik}(g^{-1})\,\varphi^{kl}\,\Pi_V^{lj}(g) = \frac{1}{|G|}\sum_{\substack{g\in G\\ k,l}} \overline{\Pi_W^{ki}}(g)\,\varphi^{kl}\,\Pi_V^{lj}(g) = \begin{cases} 0 & V\neq V \\ \sum_k \frac{\varphi^{kk}}{\dim V}\delta_{ij} & \text{if } V = W \end{cases}$$

- If $V \not\cong W$, as φ is arbitrary, it means that

$$\frac{1}{|G|}\sum_{g\in G} \overline{\Pi_V^{ki}}(g)\,\Pi_W^{li}(g) = 0 \quad \forall k, i, l, j$$

- If $V = W$, we get

$$\frac{1}{|G|}\sum_{g\in G} \overline{\Pi_V^{ki}(g)}\,\Pi_W^{li}(g) = \delta^{kl}\delta^{ij}\cdot\frac{1}{\dim V}$$

We can define a hermitian scalar product on $\mathbb{C}[G]$ as follows,

Given $\varphi, \psi \in \mathbb{C}[G]$, $\langle \varphi, \psi\rangle := \frac{1}{|G|}\sum_{g\in G} \overline{\varphi(g)}\,\psi(g)$

<u>Facts about</u> $\langle\cdot,\cdot\rangle$:

- $\{\chi_g\}$ form an orthonormal basis wrt $\langle\cdot,\cdot\rangle$.
- $\langle\cdot,\cdot\rangle$ is G-invariant.
- $$\begin{cases} \langle \Pi_V^{ik}, \Pi_W^{jl}\rangle = 0 \quad \forall i,j,k,l \text{ if } V\not\cong W. \\ \langle \Pi_V^{ik}, \Pi_W^{jl}\rangle = \begin{cases} 0 & \text{unless } \begin{matrix} i=j \\ k=l \end{matrix} \\ \frac{1}{\dim V} & \text{if } \begin{matrix} i=j \\ k=l\end{matrix} \end{cases} \end{cases}$$

<u>Theorem</u> (Unitary matrix elements of (G-representation) irreducible form an orthogonal system in $\mathbb{C}[G]$.

Pf above! //

In particular, $\mathbb{C}[G] = \left(\bigoplus_{\substack{\lambda \text{ irred.} \\ \text{isomorphism classes}}} V_\lambda^{\oplus \dim V_\lambda} \right) \oplus W$ (**)

$\uparrow$ rest of space..

Claim: $W = 0$.

We will see this in next lecture.

Remarks:

1) From (**), we see that $|G| \cancel{\geq} \sum_{\lambda \text{ irr}} (\dim V_\lambda)^2$ "=" if $W=0$

2) The two orthonormal bases in $\mathbb{C}[G]$ $\left(\{\chi_\lambda\}, \left\{ \frac{\pi_V^{ij}}{\sqrt{\dim V}} \right\} \right)$ are related by Fourier transform:

Let $G = S^1 = \{e^{i\varphi}\} \subseteq \mathbb{C}$ (not finite...)

Ex. The irreducible reps are all 1-dim:

$\chi_\lambda : G \to \mathbb{C}^*$, $\chi_\lambda = (e^{i\varphi})^\lambda$, $\lambda \in \mathbb{Z}$.

The matrix element of this reps (function on $G = S^1$) are

$\{ e^{i\lambda\varphi}, \lambda \in \mathbb{Z} \}$ — Fourier coefficients.

Characters.

Def: The character of a representation $\pi : G \to GL(V)$ is the function on G, given by $\chi_V(g) = \chi_\pi(g) = \mathrm{tr}_V \pi(g) \in \mathbb{C}[G]$.

Rk: Two characters corresp. to nonisomorphic irreducible reps. will be perpendicular (wrt $\langle , \rangle$).

Properties:

P1: $\chi_V(g) = \chi_V(hgh^{-1})$. So characters are function on the conjugacy classes of G.

P2: $\chi_V(e) = \dim V$.

P3: $\chi_V(g^{-1}) = \overline{\chi_V(g)}$

P4: $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$

$\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$ ← exercise.

$\chi_{V^*}(g) = \overline{\chi_V(g)}$.

~~Def: The ring of G~~

Def: The <u>representation ring</u> R_G of G is given by:

- As a set, $R_G = \{V\text{-reps of } G\}/\cong$
- Addition is $\oplus$, multiplication is $\otimes$

By Krull-Schmidt, Maschke, we have $(R_G, \oplus)$ is a free abelian group, generated by the irreducible representations. ← in $\mathbb{C}$!

Also, the properties indicated before imply that χ is a ring-homomorphism from R_G to class-functions (functions on conjugacy classes).

<u>More Properties</u>

P5. $\chi_{\underset{\text{reg.rep in } \mathbb{C}[G]}{\pi_{reg}}}(g) = \chi_{reg}(g) = \begin{cases} 0 & g \neq e_G \\ |G| & g = e_G \end{cases}$

Pf. In the basis of characteristic functions $\{\chi_g\}$, $\pi(g)$ is just a permutation matrix. So $\pi_{reg}(g)\chi_h = \chi_{gh}$ //

P6. If V, W are irreducible, $\langle \chi_V, \chi_W \rangle = \delta_{V,W} = \begin{cases} 1 & V = W \\ 0 & V \neq W \end{cases}$

Proof $\langle \chi_V, \chi_W \rangle = \sum_{i,j} \langle \pi_V^{ii}, \pi_W^{jj} \rangle = \begin{cases} 0 & V \not\cong W \\ \sum_{i,j} \langle \pi_V^{ii}, \pi_V^{jj} \rangle = \sum_i \frac{1}{\dim V} = 1 & V \cong W \end{cases}$ //

<u>Corollary 1:</u>

V is irreducible $\iff \langle \chi_V, \chi_V \rangle = 1$.

Pf Write $V = W_1^{\oplus k_1} \oplus \cdots \oplus W_n^{\oplus k_n}$, W_i irreducible $\forall i$, nonisomorphic.

$\langle \chi_V, \chi_V \rangle = \sum_{i,j} k_i k_j \langle \chi_{W_i}, \chi_{W_j} \rangle = \sum_i k_i^2$ //

<u>Corollary:</u> Write $V = W_1^{\oplus k_1} \oplus \cdots \oplus W_n^{\oplus k_n}$ (nonisomorphic irreducibles).

Then $k_i = \langle \chi_V, W_i \rangle$.

Since irreducible characters are orthogonal ($\Rightarrow$ linearly independent), we get:

Corollary 3: $V \simeq W \Leftrightarrow \chi_V = \chi_W$ (V, W not necessarily irred).

Corollary 4: $\mathbb{C}[G] = \underbrace{\bigoplus_i}_{\text{all irreducibles}} W_i^{\oplus \dim W_i}$ (so matrix elements form an orthogonal ~~system~~ basis in $\mathbb{C}[G]$.

Pf

$$\langle \chi_{reg}, \chi_{W_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \underbrace{\overline{\chi_{reg}(g)}}_{= 0 \text{ unless } g = e} \chi_{W_i}(g) = \chi_{W_i}(e_G) = \dim W_i$$

We have now a map

$$\chi : R_G \longrightarrow \{\text{Class-functions on } G\} \subset \mathbb{C}[G].$$

We can take $R_G \otimes_{\mathbb{Z}} \mathbb{C}$, and we get a map of vectorspaces.
we will prove that this is actually an isomorphism

Numerology of representations

1) $\dim_{\mathbb{C}} \{\text{class function}\} = \#\{\text{conjugacy classes}\} \xrightarrow{\chi} \boxed{\#\text{irreducible reps of } G = \#\text{ conjugacy classes of } G}$.

2) $\left\{\begin{matrix}\text{Matrix elements of} \\ \text{irred. reps}\end{matrix}\right\}$ is a basis in $\mathbb{C}[G]$.

So $$\sum_{V \text{ irred.}} (\dim V)^2 = \# G$$

Examples: S_3. Want to find R_G = class functions.

The answer will be the character table. $\chi_i(C_j)$ $1 \leq i, j \leq k$.

Recall that $$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \chi_2(g) = \frac{1}{|G|} \sum_{C \text{ conj classes}} |C| \overline{\chi_1(C)} \chi_2(C)$$

reps \ conj class	e	(12)	(123)	$\leftarrow$ conj class
	1	3	2	$\leftarrow$ # elts
trivial $\to \chi_1$	1	1	1	
sign $\to \chi_2$	1	1	-1	
χ_3	2	a	b	

$\uparrow$ $\dim V_i$

Know: if χ_1, χ_2, χ_3 are the three characters,

$$\dim V_1^2 + \dim V_2^2 + \dim V_3^2 = 6$$

so only possibility $\dim V_1 = \dim V_2 = 1$, $\dim V_3 = 2$

To complete the table, we can use orthogonality:

$\langle \chi_3, \chi_3 \rangle = 1$

$\langle \chi_3, \chi_1 \rangle = 0$

$\langle \chi_3, \chi_2 \rangle = 0$

$$\left.\begin{array}{l} 2 + 3\cdot 1a + 2\cdot 1\cdot b = 0 \\ 2 - 3\cdot a + 2\cdot b = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} a = 0 \\ b = -1 \end{array}}.$$

Now we check:

$\langle \chi_3, \chi_3 \rangle = \frac{1}{6}\left(2^2 + 0^2\cdot 3 + (-1)^2\cdot 2\right) = 1.$ ✓

What is V_3?

S_n has a natural n-dim rep in $\mathbb{C}[\{1,2,\dots,n\}]$, which we know is not irreducible. The space of constant function.

Its complement is irreducible (and is called the fundamental representation).

In computations with functions on G-sets, one often uses the following:

Theorem (fixed point formula):

Spse G acts on a set X. ($|G|, |X| < \infty$). Then G acts on $\mathbb{C}[X]$ and the character is given by:

$$\chi_{\mathbb{C}[X]}(g) = \#\ g\text{-fixed points } \#\{x \in X : gx = x\}.$$

Pf In the basis of characteristic functions, g is given by a permutation matrix. (i.e. a single one in each row & column). $\Rightarrow$ the trace is the number of ones in the diagonal. = number of fixed points!

$$\Big(\text{or } \chi_{\mathbb{C}[X]}(g) = \operatorname{tr} \pi(g) = \sum_{x \in X} \underset{\substack{\uparrow \\ \text{char } f'n}}{\eta_{gx}}(x) = \#\{x \in X : gx = x\}. \Big)$$

Back to S_3,

we have S_3 acting on $\mathbb{C}[\{1,2,3\}] = \mathbb{C}^3$, and the character is:

	e	(12)	(123)
$\chi_{\mathbb{C}^3}$	3	1	0

← fixed-point formula

<u>Note</u>: $\langle \chi_{\mathbb{C}^3}, \chi_{\mathbb{C}^3} \rangle = \frac{1}{6}\left(1\cdot 3^2 + 3\cdot 1^2 + 2\cdot 0^2\right) = 2 \Rightarrow \chi_{\mathbb{C}^3}$ not irreducible.

Also, it is the sum of two distinct irreducibles.

So $\mathbb{C}^3 = \underset{\substack{\uparrow \\ \text{constant} \\ \text{functions}}}{\text{trivial}} \oplus V$, V of dimension 2 $\Rightarrow V = V_3$ (fundamental rep).

Moreover, $\chi_{V_3} = \chi_{\mathbb{C}^3} - \chi_{\text{trivial}}$, as we can see.

<u>Example</u>: S_4.

conj classes / # elts	e	(12)	(123)	(1234)	$(12)(34)$	
	1	6	8	6	3	/24
triv. → χ_{tr}	1	1	1	1	1	
χ_{sign}	1	−1	1	−1	1	
fundamental standard χ_{fund}	3	1	0	−1	−1	
$\chi_{\bar{f}}$	3	−1	0	1	−1	
χ_W	2	0	−1	0	2	

↑ dimension

<u>Reps of S_4</u>

1) $V_f = \mathbb{C}[\{1,2,3,4\}] \ominus V_{tr}$

$$\chi_f = \underset{\text{fixed-point formula}}{\chi_{\mathbb{C}[\{1,2,3,4\}]}} - \chi_{tr} = (4,2,1,0,0) - (1,1,1,1,1)$$

As $\langle \chi_f, \chi_f \rangle = 1 \Rightarrow \chi_f$ is irreducible.

2) Try to multiply the representation we have, to get more representations.

$V_f \otimes V_{sign}$: as a vectorspace, $\simeq V_f$.

But $\pi_{V_f \otimes V_{sign}}(g) = \mathrm{sign}(g) \cdot \pi_f(g)$

$\chi_{V_f \otimes V_{sign}} = \chi_f \cdot \chi_{sign}$.

$\langle \chi_{V_f \otimes V_{sign}}, \chi_{V_f \otimes V_{sign}} \rangle = \langle \chi_{V_f}, \chi_{V_f} \rangle = 1$, so $\chi_{V_f \otimes V_{sign}}$ is irreducible.

<u>Def</u>: $V_f^- := V_f \otimes V_{sign}$.

To compute χ_W, just impose that it has norm 1, perp. to the other 4, and s.t. $\chi_W(e) > 0$.

<u>Application</u>:

We compute $V_f \otimes V_f = U_1 \oplus \cdots \oplus U_k$, U_i irreducibles. (this is easy).

A more natural tensor product would be:

$\mathbb{C}^4 \otimes \mathbb{C}^4$ (where $\mathbb{C}^4 = \mathbb{C}[\{1,2,3,4\}]$).

understood as function in two variables.

As $\mathbb{C}^4 = V_f \oplus V_{tr}$, then $\mathbb{C}^4 \otimes \mathbb{C}^4 = (V_f \otimes V_f) \oplus (V_{tr} \otimes V_f) \oplus (V_{tr} \otimes V_f) \oplus (V_{tr} \otimes V_{tr})$

$$\underset{?}{(V_f \otimes V_f)} \quad \underset{V_f}{(V_{tr} \otimes V_f)} \quad \underset{V_f}{(V_{tr} \otimes V_f)} \quad \underset{V_{tr}}{(V_{tr} \otimes V_{tr})}$$

$$V_f \otimes V_f = V_{tr}^{\oplus k_{tr}} \oplus V_{sign}^{\oplus k_{sign}} \oplus \cdots \oplus W^{\oplus k_W}$$

where, for instance, $k_{tr} = \langle \chi_{V_f \otimes V_f}, \chi_{tr} \rangle \ldots$

As $\chi_{V_f \otimes V_f} = (\chi_{V_f})^2$, then

$$\begin{cases} k_{tr} = \frac{1}{24}\left(1 \cdot 1 \cdot 9 + 6 \cdot 1 \cdot 1 + 8 \cdot 0 \cdot 0 + 6 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 1\right) = 1 \\ k_{sign} = 0 \\ k_{fund} = 1 \\ k_f^{(-)} = 1 \\ k_W = 1 \end{cases}$$

⋮

We conclude then:

$$V_f \otimes V_f = V_{tr} \oplus V_f \oplus V_f^- \oplus W.$$

This means that we have S_4-maps

$V_f \otimes V_f \to V_{tr}$ ← scalar product S_4-invariant

$V_f \otimes V_f \to V_f$ ← hard

$V_f \otimes V_f \to V_f^-$ ← map from $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathbb{C}^3$: cross-product $\begin{cases}\text{rotation-invariant} \\ \text{reflection} \\ \text{anti-invariant}\end{cases}$

$V_f \otimes V_f \to W$ ← hard

Representations & Subgroups.

① Spse $H \triangleleft G$, $\pi : G/H \to GL(V)$ a rep. of G/H. Then π can be extended to G, by $G \xrightarrow{p} G/H \xrightarrow{\pi} GL(V)$.

Examples:

Let $H := \{e, \underbrace{(12)(34), (13)(24), (14)(23)}_{C \text{ conjugacy class}}\} \triangleleft S_4$.

$S_4/H \cong S_3$ ← S_4 acts on C by conjugation.

Take V = fundamental rep. of S_3 ($\dim V = 2$) $\rightsquigarrow$ get irreducible rep. of S_4, of dimension 2. ⇒ it is W.

② Let $H \leqslant G$, not necessarily normal. If $\pi : G \to GL(V)$ is a rep of G, we get $\pi_{|H} : H \to GL(V)$. We call it $\mathrm{Res}_H^G \pi = \mathrm{Res}_H^G V$.

Proposition:

- $\mathrm{Res}_H^G V_1 \oplus V_2 = \mathrm{Res}_H^G V_1 \oplus \mathrm{Res}_H^G V_2$
- $\mathrm{Res}_H^G V_1 \otimes V_2 = \mathrm{Res}_H^G V_1 \otimes \mathrm{Res}_H^G V_2$
- Restriction is transitive: $K \subset H \subset G \Rightarrow \mathrm{Res}_K^H \mathrm{Res}_H^G \pi = \mathrm{Res}_K^G \pi$.

Rk: Res does not preserve irreducibility!

③ Induction:

$H \leq G$, $\pi : H \to GL(V)$. We want a representation of G.

Example 1:

$H = \{e\} \Rightarrow$ want a rep of G. $\left\{ \begin{array}{l} \text{trivial} \\ \text{regular in } \mathbb{C}[G]. \end{array} \right.$

Example 2: $H \subset G$, $\pi : H \to \mathbb{C}^*$ be the trivial rep.

Then we can take $\mathbb{C}[G/H]$. Then H acts trivially, so:

$$\mathbb{C}[G/H] = \{ f : G \to \mathbb{C} : f(gh) = f(g) \quad \forall h \in H, \forall g \in G \}.$$

Def: The induced rep. of G is given by:

$$\mathrm{Ind}_H^G V = \mathrm{Ind}_H^G \pi \underset{\text{as vectorspace}}{=} \{ f : G \to V \mid f(gh) = \pi(h)^{-1} f(g) \quad \forall h \in H, \forall g \in G \}.$$

The G-action is given by left multiplication:

$$\left(\left(\mathrm{Ind}_H^G \pi \right)(x) f \right)(g) = f(x^{-1} g)$$

Remark: $\dim {}_H^G V = \dim V \cdot |G/H|$.

Proposition: $\mathrm{Ind}_H^G V_1 \oplus V_2 = \mathrm{Ind}_H^G V_1 \oplus \mathrm{Ind}_H^G V_2$.

Examples:

- $\mathrm{Ind}_H^G \underset{\text{trivial}}{\pi_{tr}} = \mathbb{C}[G/H]$
- $\mathrm{Ind}_H^G \underset{\text{reg rep of } H}{\mathbb{C}[H]} = \mathbb{C}[G]$

Proof: $\mathrm{Ind}_H^G \mathbb{C}[G] = \{ f : G \to [H \to \mathbb{C}] + \text{condition} \} = \{ f : G \times H \to \mathbb{C} + \text{condition} \}$

$= \{ f : G \times H \to \mathbb{C} : \cancel{f(g,h)} \; f(gh_1, h_2) = f(g, h_1 h_2) \; \forall h_1, h_2 \in H, g \in G \} = [\text{look at } f(g, e_H)]$

$= \{ \varphi : G \to \mathbb{C} \}$.

• Frobenius Reciprocity:

Let V be a rep. of G, W a rep. of H. Then: $(H \leq G)$.

Theorem: $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.

(in other words, Ind is adjoint to Res)

Remark: For G a gp, V a G rep, U an irreducible G-rep,

$\dim \mathrm{Hom}_G(U,V) = \dim_G \mathrm{Hom}_G(V,U) = \#$ times U "enters" the direct sum decomposition of V.

Pf: $V = \bigoplus \underbrace{U_i}_{\text{distinct irreducible}}^{\oplus n_i}$

$$\mathrm{Hom}\left(U_0, \bigoplus U_i^{\oplus n_i}\right) = \bigoplus \underbrace{\mathrm{Hom}(U_0, U_i)}_{\dim \delta_{i0}}^{\oplus n_i}$$

//

~~RK~~ On the Theorem:

For V, W irreducibles, the Thm says:

#times V enters the direct sum decomp of $\mathrm{Ind}\, W$ = # times W enters the direct sum decomp. of $\mathrm{Res}\, V$.

Proof of Theorem (Frob Reciprocity):

$\mathrm{LHS} = \{\text{maps } V \to \{G \to W + \text{conditions}\} + \text{condition}\} = \{\text{maps } V \times G \xrightarrow{\varphi} W + \text{cond}\}$

The conditions are • $\varphi(v, gh) \overset{\text{def of induction}}{=} h^{-1}\varphi(v, g) \quad \forall h \in H$

• $\varphi(v, x^{-1}g) \overset{\text{because } \mathrm{Hom}_G}{=} \varphi(xv, g) \quad \forall x \in G$

Such φ's are in 1:1 corresp $\psi: V \to W$, by $\psi = \varphi(\cdot, e_G)$ (by 2^{nd} condition).

The first condition means ~~$\psi(hv) = \varphi(hv, e_G) =$~~

$$\psi(hv) = \varphi(hv, e_G) = \varphi(v, h^{-1}) = h\,\varphi(v, e_G) = h\,\psi(v) \Rightarrow \psi \text{ is } H\text{-equivariant.}$$

//

- <u>Character of $\mathrm{Ind}_H^G W$.</u>

We want an explicit presentation of $\mathrm{Ind}_H^G W$.

Let $\{x_\sigma\}_{\sigma \in G/H}$ be H-coset representatives.

Then, as a vectorspace, $\mathrm{Ind}_H^G W \simeq \bigoplus_\sigma x_\sigma W$

The G-action is given by $\quad g \cdot x_\sigma w = \cancel{x_{\sigma^g}(h(g)w)}\; x_{\sigma^g}(h(g)w)$

(and $\sigma \mapsto \sigma^g$ is the G-action on G/H).

(and $h(g) \in H$ does not depend on the choice of representatives).

↑ but <u>does</u> depend on the class (ie on σ).

(actually, $h(g) = g x_\sigma \cdot x_{\sigma^g}$).

Then, the character is given by $\mathrm{tr}_{\mathrm{Ind} W}(g) = \sum_{\substack{\sigma \in G/H \\ \text{s.t. } \sigma^g = \sigma}} \mathrm{tr}_{x_\sigma W}\, h(g)$

Note $\sigma^g = \sigma \Leftrightarrow g x_\sigma = x_\sigma h \Leftrightarrow$

$\Leftrightarrow x_\sigma^{-1} g x_\sigma \in H$

Hence $\mathrm{tr}\left(\mathrm{Ind}_W(g)\right) = \sum_{x_\sigma^{-1} g x_\sigma \in H} \mathrm{tr}_W (x_\sigma^{-1} g x_\sigma)$

Now, for $h' \in H$, $x_\sigma \mapsto x_\sigma \cdot h'$ doesn't change the trace. Hence, the trace equals:

$$\frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x)$$

We have proved!

<u>Proposition</u>: $\chi_{\mathrm{Ind} W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x g x^{-1} \in H}} \chi_W(x g x^{-1}) \left(= \sum_{\substack{x \in G/H \\ x g x^{-1} \in H}} \chi_W(x g x^{-1}) \right)$.

//

Remark: $\operatorname{supp} \chi_{\mathrm{Ind}_H^G W} \subseteq \bigcup_{x\in G} xHx^{-1}$ (normal closure of H).

Applications:

- Suppose $H \triangleleft G$.

  Q: When $\mathrm{Ind}_H^G W$ is irreducible?

  A: $\Leftrightarrow \langle \chi_{\mathrm{Ind}_H^G W}, \chi_{\mathrm{Ind}_H^G W} \rangle = 1$

set $\begin{cases} xgx^{-1} = h \\ yx^{-1} = z \end{cases}$; indep of x!

$$\frac{1}{|G||H|^2} \sum_{\substack{x\in G\\ y\in G\\ g\in G\\ xgx^{-1}\in H\\ ygy^{-1}\in H}} \overline{\chi_W(xgx^{-1})}\,\chi_W(ygy^{-1}) = \frac{1}{|G||H|^2} \sum_{\substack{x\in G\\ h\in H\\ z\in G\\ (zhz^{-1}\in H) \leftarrow H\triangleleft G!}} \overline{\chi_W(h)}\,\chi_W(zhz^{-1}) =$$

depends on conj class of z

$$= \frac{1}{|H|^2} \sum_{\substack{h\in H\\ z\in G}} \overline{\chi_W(h)}\,\chi_W(zhz^{-1}) = \frac{1}{|H|} \sum_{\substack{h\in H\\ z\in G/H}} \overline{\chi_W(h)}\,\chi_W(zhz^{-1}) =$$

$$= \sum_{\{z_i\}\in G/H} \langle \chi_W, \chi_W^{z_i} \rangle_H \quad , \text{ where } \chi_W^{z_i}(h) := \chi_W(z_i h z_i^{-1}).$$

$\chi_W^{z_i}$ is the character of the representation of H in W, given by:

$$\pi_{W^{z_i}}(h) = \pi_W(z_i h z_i^{-1})$$

Then $\mathrm{Ind}_H^G W$ irreducible $\Leftrightarrow \begin{cases} \langle \chi_W, \chi_W \rangle_H = 1 \\ \langle \chi_W, \chi_{W^{z_i}} \rangle_H = 0 \quad \forall z_i \neq 1 \end{cases}$

We state it as a theorem:

Thm: If $H \triangleleft G$, $\mathrm{Ind}_H^G W$ is irreducible $\Leftrightarrow$
- W irreducible
- $W^{z_i} \not\cong W^{z_j}$ for $i \neq j$, where the z_i's are reps of G/H

Another application

Recall R_G = free abelian group generated by irreducibles.

Let $CF(G) = \{\text{class-functions on } G\}$. We know that

$$CF(G) \underset{\chi}{\cong} \mathbb{C} \otimes_{\mathbb{Z}} R_G$$

The induction can be thought of as a map $CF(H) \xrightarrow{\text{Ind}} CF(G)$

Theorem (Artin): Let S be a set of subgroups of G. Then TFAE:

(1) G is a union of conjugates of elements of S.

(2) $\text{Coker}\left(\text{Ind}: \bigoplus_{H \in S} R_H \to R_G\right)$ is torsion (exp is finite).

Corollary: Characters of representations induced from 1-dim reps of cyclic subgroups span$/\mathbb{C}$ the space of class-functions on G.

(This is very important for algebraic number theory — cyclic Galois groups —).

Proof (of Thm):

First, (2) $\Leftrightarrow$ (2'): $\text{Ind}: \bigoplus_{H \in S} CF(H) \twoheadrightarrow CF(G)$ is surjective.

(1) $\Rightarrow$ (2'):

$$\bigoplus CF(H) \xrightarrow{\text{Ind}} CF(G) \text{ surj} \Leftrightarrow CF(G) \xrightarrow{\text{Res}} \bigoplus_{H \in S} CF(H) \text{ injective} \quad \text{(by Frobenius, adjoint.)}$$

~~Assume not.~~ So suppose $f \in CF(G)$ s.t. $f|_H = 0 \ \forall H \in S$. Want $f = 0$.

But by (1), as $f|_H = 0$ and $G = \bigcup \overline{H}$, and f is conj-invariant, $\Rightarrow$ ✓.

(2') $\Rightarrow$ (1): Let $A = \bigcup_{\substack{x \in G \\ H \in S}} xHx^{-1}$. Want to prove $A = G$.

Let $B = G \setminus A$.

Let $f \in CF(H)$. Consider $\text{Ind}_H^G f$. Then ~~$\text{Supp}^G_H \text{Ind}$~~

$$\text{Supp}\, \text{Ind}_H^G f \subseteq \bigcup_{x \in G} xHx^{-1} \subseteq A \Rightarrow \text{Supp}\, \underbrace{\text{Im}(\text{Ind})}_{CF(G)} \subseteq A \Rightarrow A = G$$

Summary of the analytic (ie $\mathbb{C}[G]$) representation theory over $\mathbb{C}$.

1) $\mathbb{C}[G] = \bigoplus_{V \text{ irred.}} V^{\oplus \dim V}$

In particular, each irreducible is contained in $\mathbb{C}[G]$.

2) $\underset{\uparrow \text{rep. ring}}{R_G} \xrightarrow{\chi} \underset{\nwarrow \text{class function on } G}{CF(G)}$

Moreover, $R_G \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} CF(G)$.

3) We have induction in functions $\mathrm{Ind}_H^G W$.

Crucial Facts used for (1), (2), (3):

- Averaging operator $\frac{1}{|G|}\sum_{g\in G}$
- $\mathbb{C}[G]$ has an G-invariant hermitian product $\langle \varphi, \psi \rangle_G = \frac{1}{|G|}\sum_{g\in G} \varphi(g)\overline{\psi(g)}$

Generalizations (for $|G| = \infty$)

Want an analogue of $\sum_{g\in G} \sim \int_G d\mu(g)$ for some μ measure on G.

This measure should be G-invariant: $d\mu(gh) = d\mu(h)$.

Theorem (Haar): Any "decent" (locally compact topological) group has a left-invariant measure (defined on the σ-algebra generated by compact subsets). This measure is unique (up to scalar multiplication).

So we can use $\int_G$ instead of $\sum_{g\in G}$. The main difference is that G can have infinite volume $\mu(G) (= \int_G d\mu = \infty)$.

Notes: $\mu(G) < \infty \Leftrightarrow G$ is compact

Compact groups behave very similarly to finite groups.

Examples:
- $S^1 = U(1) = \{z \in \mathbb{C} \mid \|z\| = 1\}$ is compact.
- $(\mathbb{Z}_p, +)$ is compact
- $(\mathbb{R}, +)$ is not compact
- $(\mathbb{Q}_p, +)$ is not compact

Example: $(\mathbb{R},+)$ abelian $\Rightarrow$ • irreducibles are 1-dimensional
• characters = matrix elements

$\pi : (\mathbb{R},+) \to \mathbb{C}^*$ a rep. so $\pi(t)\pi(s) = \pi(t+s)$.

Let $\psi := \log \pi$, so that $\psi(0) = 0$.

Assume π continuous, so ψ is now well-defined.

Then $\psi(t) + \psi(s) = \psi(t+s)$.

$\Rightarrow \psi(nt) = n\,\psi(t) \quad \forall n \in \mathbb{Z}$.

Can see also $\psi(rt) = r\,\psi(t) \quad \forall r \in \mathbb{Q}$

As ψ is continuous, $\psi(t) = t \cdot \psi(1)$. Let $\lambda := \psi(1)$.

So $\pi_\lambda(t) = e^{\lambda t}$ for some $\lambda \in \mathbb{C}$. $\leftarrow$ all reps!

Note: $\pi \in C^0(\mathbb{R}) \Rightarrow \pi \in C^\infty(\mathbb{R})$.

Remarks: • $\pi_\lambda(t) \in \mathbb{C}[\mathbb{R}]$

• $\overline{\pi_\lambda(-t)} = \pi_\lambda(t) \iff \pi_\lambda$ is unitary

So π_λ unitary $\iff \operatorname{Re}\lambda = 0$ (so not all reps are unitary, unlike what happened with finite gps)

• The natural scalar product on $\mathbb{R}$ is

$\int_{-\infty}^{\infty} \overline{\varphi(t)}\,\psi(t)\,dt$, so the natural space of functions for the regular rep. is $L_2(\mathbb{R})$

But neither of the ~~$e^{\lambda t}$~~ π_λ is contained in $L_2(\mathbb{R})$!

However, in terms of distributions, $\{\pi_\lambda\}_{\lambda \in i\mathbb{R}}$ form an orthonormal basis in $L_2(\mathbb{R})$ (Fourier basis). $\equiv$!

So unitary characters form a "basis" in the regular representation.

Example 2: $(S, \cdot)$

Reps of S = Reps of $\mathbb{R}$ s.t. $\pi(t+2\pi) = \pi(t)$.

So $\pi_\lambda(t) = e^{\lambda t}$ is a S-rep $\Leftrightarrow \lambda \in i\cdot\mathbb{Z}$

So the reps of $(S, \circ)$ are $\{\pi_n(t) = e^{int}\}_{n\in\mathbb{Z}}$

- All reps are unitary
- $L_2(S) = \bigoplus_{n\in\mathbb{Z}} \pi_n$

$\}$ ← exact same statements as for finite groups.

Algebraic Representation Theory.

Let G a finite group, k a field.

$$kG = \{ \sum_i k_i g_i : k_i \in k, g_i \in G \} \quad \text{(formal linear combinations).}$$

It is an associative algebra $/k$, with respect to the formal k-linear structure, and with product given by $(\sum k_i g_i)(\sum k'_j g_j) = \sum_{i,j} k_i k'_j (g_i g_j)$

Def: kG is called the group algebra of G.

Remark: G-representation $\equiv$ kG-module.

$$\pi : G \to V \longleftrightarrow \sum k_i g_i \mapsto \sum k_i \pi(g_i)$$

Review of associative algebras and their modules

A/k an associative algebra (A a k-linear space with k-linear product $A \otimes_k A \to A$, $a\otimes b \mapsto ab$)

(i.e. $(a+b)c = ac + bc$

$a(b+c) = ab + ac$

$(\alpha a)(b) = a(\alpha b) = \alpha ab \quad \forall \alpha \in k$.)

$(ab)c = a(bc)$ ← This is the condition added so to say that it is associative.

Def A left A-module ${}_A M$ is a k-linear space with a k-linear operation

$$A \otimes_k M \to M \qquad a\otimes m \mapsto am$$

s.t. $(ab)m = a(bm)$

(i.e. $(a+b)m = am + bm$, $a(m+n) = am + an$, $(\alpha a)m = a(\alpha m)$)

In other words, $_AM$ is a ~~k-linear~~ homomorphism of k-algebras, $A \longrightarrow \mathrm{End}_k(M)$.

Def: A right A-module M_A is a k-linear space M with a k-linear operation:

$$M \otimes_k A \longrightarrow M$$
$$(m \otimes a) \longmapsto ma$$

Such that $m(ab) = (ma)b$

In other words, M_A is a homomorphism of k-algebras,

$$A^{opp} \longrightarrow \mathrm{End}_k M$$

A with opposite product.

Def: Let A, B be two k-algebras. An A-B-bimodule $_AM_B$ is a k-linear space with ~~a~~ k-linear operations:

$$\cancel{A \otimes M \otimes B \longrightarrow M}$$
$$\cancel{a \otimes m \otimes b \longmapsto amb}$$

~~Such that~~ $\cancel{(am)b = a(mb)}$

$$A \otimes M \longrightarrow M$$
$$a \otimes m \longmapsto am$$
$$M \otimes B \longrightarrow M$$
$$m \otimes b \longmapsto mb$$

Such that $(am)b = a(mb)$. (and s.t. $(aa')m = a(a'm)$, $m(bb') = (mb)b'$).

Remark: Any k-linear space is a left, right k-module and k-k-bimodule.

(i.e. $_AM = {}_AM_k$, ...).

We will assume hereafter that all the modules are finite-dimensional $/k$.

Def M, N - modules (left, right or bi-). Then:

$M \oplus_k N$ is given by the k-linear space $M \oplus_k N$ with the natural module structure.

(for example, for left modules: $a(m \oplus n) = am \oplus an$).

Or more abstractly, want $A \otimes (M \oplus N) \to M \oplus N$

We know $A \otimes (M \oplus N) \underset{nat}{\simeq} A \otimes M \oplus A \otimes N \to M \oplus N$ module ops on summands.

<u>Remark</u>: $M \otimes_k N$ is <u>not</u> a module (if M, N are left A-modules, for instance)

$a(m \otimes n) \overset{?}{=} am \otimes an$ is not k-linear in A!

However, we can take $\otimes$ of bimodules:

<u>Def</u> If ${}_AM_B$, ${}_BN_C$, we can define ${}_AM_B \otimes_B {}_BN_C$ naturally as an A-C-bimodule (defined as usual, using balanced products)

with the action: $a(m \otimes n) = am \otimes n$
$(m \otimes n)c = m \otimes nc$

Similarly, $\mathrm{Hom}_A({}_AM_B, {}_AN_C)$ is an B-C-bimodule, with respect to the action
$(b\varphi)(m) = \varphi(mb)$
$(\varphi c)(m) = \varphi(m) \cdot c$

• <u>Hopf Algebras</u>

<u>Obs</u>: $A = kG$ is an algebra, but also $A^* = k[G]$, via:

$$\langle f, \sum k_i g_i \rangle := \sum_i k_i f(g_i)$$

But $A^* = k[G]$ is <u>also</u> an algebra, w.r.t. $(fg)(x) = f(x)g(x)$

Moreover, the pairing $\langle \cdot, \cdot \rangle : A \times A^* \to k$ is G-invariant.

<u>Remarks</u>:

· In general, the dual space to an algebra $(A, A \otimes A \xrightarrow{\mu} A)$ is a <u>coalgebra</u> $(B, B \xrightarrow{\mu^*} B \otimes B)$

Def: An *algebra* $/k$ is a k-linear space A together with two k-linear maps:

- $A\otimes A \xrightarrow{\mu} A$ (product) $\quad (\otimes = \otimes_k)$
- $\eta : k \longrightarrow A$ (unity)

Such that the following diagrams are commutative:

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] \arrow[dr, phantom, "\circlearrowright"] & A\otimes A \arrow[d, "\mu"] \\ A\otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} k\otimes A \arrow[r, "\eta\otimes id"] & A\otimes A \arrow[d, "\mu"] & A\otimes k \arrow[l, "id\otimes\eta"'] \\ & A \arrow[ul, "\wr"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

A *coalgebra* over k is a k-linear space A, together with two k-linear maps:

- $A \xrightarrow{\Delta} A\otimes A$ (coproduct)
- $A \xrightarrow{\varepsilon} k$ (counit)

Such that the following diagrams commute (just reverse the arrows)

$$\begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"'] \\ A\otimes A \arrow[u, "id\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} k\otimes A & A\otimes A \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & A\otimes k \\ & A \arrow[ul, "\wr"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

Remark: An algebra is commutative if:

$$\begin{tikzcd} A\otimes A \arrow[r, "\mu"] \arrow[d, "\tau"'] & A \\ A\otimes A \arrow[ur, "\mu"'] & \end{tikzcd} \qquad \text{where } \tau(x\otimes y) = y\otimes x.$$

(and similarly for a coalgebra).

Examples of coalgebras:

1) kG is a coalgebra (as $k[G]^* = kG$, and $k[G]$ is an algebra).

$\Delta(g)$? By duality, $\langle \Delta g, \varphi\otimes\psi\rangle = \langle g, \varphi\psi\rangle = \varphi(g)\psi(g) = \langle g\otimes g, \varphi\otimes\psi\rangle$

So $\Delta(g) = g\otimes g$.

Warning: $\Delta(2g+3h) = 2g\otimes g + 3h\otimes h$, NOT $(2g+3h)\otimes(2g+3h)$!!

Remark: In a general coalgebra A, elements x s.t. $\Delta x = x\otimes x$ are called group-like elements.

2) $k[G]$ is a coalgebra because $k[G] = (kG)^*$.

$\Delta f \in k[G]\otimes k[G] = k[G\times G]$?

$(\Delta\varphi)(g,h) = \varphi(gh)$

In particular, $(\Delta \eta_g)(h_1,h_2) = \eta_g(h_1h_2) = \begin{cases} 1 & \Leftrightarrow h_1h_2 = g. \\ 0 & \text{ow.}\end{cases}$ char. function at g.

So $\Delta \eta_g = \sum_{h\in G} \eta_h \otimes \eta_{h^{-1}g}$

3) Homology is a coalgebra:

In topology for instance:

$\Delta : H_\bullet(X) \to H_\bullet(X)\otimes H_\bullet(X)$ is obtained as induced from the diagonal map $X \to X\times X$.

4) $A = k[[x]]$ is an algebra

$A^* =: D$ is a coalgebra:

D is the space of derivations at 0: $D = k\text{-Span of } \{\frac{\partial^n}{\partial x^n} : n\geq 0\}$.

The pairing is given by $L\cdot f|_{x=0} =: \langle L, f\rangle$.

Let's compute: $\Delta\frac{\partial}{\partial x}$

$\langle \Delta\frac{\partial}{\partial x}, f\otimes g\rangle = \langle \frac{\partial}{\partial x}, fg\rangle = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x} = \langle \frac{\partial}{\partial x}\otimes 1 + 1\otimes\frac{\partial}{\partial x}, f\otimes g\rangle$

So $\Delta\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\otimes 1 + 1\otimes \frac{\partial}{\partial x}$

Remark: In a general coalgebra A, an element $x\in A$ s.t. $\Delta x = 1\otimes x + x\otimes 1$ is called a primitive element.

Def. A <u>bialgebra</u> over k is a k-linear space A together with k-linear maps ($\otimes = \otimes_k$)

$\mu : A \otimes A \to A$ (product)

$\eta : k \to A$ (unit)

$\Delta : A \to A \otimes A$ (coproduct)

$\varepsilon : A \to k$ (counit)

Such that:

1) (A, μ, η) is an algebra.
2) (A, Δ, ε) is a coalgebra
3) Δ, ε are algebra morphisms or, equivalently,

3') μ, η are coalgebra morphisms; explicitly:

3'') $\Delta\mu = (\mu \otimes \mu)(id \otimes \tau \otimes id)\Delta \otimes \Delta : A \otimes A \to A \otimes A$

$\tau(x \otimes y) = y \otimes x$ $(A \otimes A \to A \otimes A)$.

and similar axioms for units (all four pairings $\Delta\eta$, $\varepsilon\mu$, $\varepsilon\eta$ commute).

(*) $x \otimes y \in A \otimes A$

want "$\mu\Delta = \Delta\mu$"

$\Delta\mu = \Delta(xy)$

$\mu\Delta(x \otimes y) = \mu(\Delta x \otimes \Delta y) = \sum_{ij} z_i z_j' \otimes t_i t_j'$ $\left(\text{if } \begin{array}{l} \Delta x = \sum z_i \otimes t_i \\ \Delta y = \sum z_i' \otimes t_i' \end{array} \right)$.

<u>Example</u>:

kG is a bialgebra. For (3'') we check on the basis $\{g\}_{g \in G}$.

$\Delta(g_1 g_2) = g_1 g_2 \otimes g_1 g_2$

we want that it equals $\Delta g_1 \Delta g_2 = (g_1 \otimes g_1)(g_2 \otimes g_2) = (g_1 g_2) \otimes (g_1 g_2)$ ✓.

$\Rightarrow \Delta$ is an algebra morphism.

Example: $k[G]$ is a bialgebra:

$$\Delta(\varphi\psi)(g_1,g_2) = (\varphi\psi)(g_1g_2) = \varphi(g_1g_2)\,\psi(g_1g_2)$$

$$(\Delta\varphi\,\Delta\psi)(g_1,g_2) = \Delta\varphi(g_1,g_2)\,\Delta\psi(g_1,g_2) = \varphi(g_1g_2)\cdot\psi(g_1g_2) \qquad \checkmark.$$

Note: by symmetry, we didn't need this calculation: this is the same as the previous example, just replacing g_i by its characteristic function!

Note: $k[G]$ is an algebra because it is $k[\text{set}]$.

It is a coalgebra because G is a group.

(More precisely:

- product in G $\rightsquigarrow$ coproduct in $k[G]$
- unit in G $\rightsquigarrow$ counit in $k[G]$
- assoc. in G $\rightsquigarrow$ coassociativity in $k[G]$.

Q: What about inverses in G?

Def: Let A be a bialgebra. An antipode (coinverse) is a k-linear map $S: A \to A$ such that TFC:

$$\begin{tikzcd}
A\otimes A \arrow[r, "S\otimes id"] & A\otimes A \arrow[d, "\mu"] \\
A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & A \\
A\otimes A \arrow[r, "id\otimes S"] & A\otimes A \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
g\otimes g \arrow[r, mapsto] & g^{-1}\otimes g \arrow[d] \\
g \arrow[u] \arrow[r, mapsto] & 1
\end{tikzcd}$$

Examples:

1) In kG, the antipode is given by $S(g) = g^{-1}$ for $g \in G$, and extended by linearity.

2) In $k[G]$, the antipode is given by $(S(\varphi))(g) := \varphi(g^{-1})$

(the transpose to the antipode is the antipode in the dual bialgebra — i.e. the diagram is symmetric —).

Remark:
The antipode is an algebra and coalgebra- antiautomorphism

(i.e $S(ab) = S(b)S(a)$).

Also, the antipode is unique under some conditions.

Def: A Hopf algebra is a bialgebra with antipode.

Examples:

1) $k[G]$ and kG are Hopf algebras.

2) If G is a topological group, then $H_*(G)$ is a (graded) Hopf algebra (proven by Hopf).

3) If H is a commutative Hopf algebra $/k$ (mult. is commutative, comultiplication need not be so), then $\operatorname{Spec} H$ (as an algebra) is a group scheme. In particular, the set of points of $\operatorname{Spec} H$ ($\operatorname{Hom}_k(H, S) \equiv S$-points) is an abstract group with respect to the product:

$$\operatorname{Hom}(H,S) \times \operatorname{Hom}(H,S) \longrightarrow \operatorname{Hom}(H,S)$$
$$(\alpha, \beta) \longmapsto \alpha\beta \quad , \ (\alpha\beta)(a) = \mu_S((\alpha\otimes\beta)(\Delta a))$$

Note: $k[G]$ Hopf alg. $\Leftrightarrow$ G group.
kG Hopf alg $\Rightarrow$ kG-modules form a tensor category.

For a general algebra A and two left modules ${}_AM$, ${}_AN$, we can consider ${}_AM \otimes_k {}_AN$ as an $A \otimes_k A$-module.

If A is a Hopf algebra, then we can consider an $A \otimes_k A$-module as an A-module, via Δ (algebra homomorphism) $A \to A \otimes_k A$, so the tensor product of two left-modules is a left module.

Similarly, $\mathrm{Hom}_k({}_AM, {}_AN)$ is in general an A-A-bimodule or, equivalently, an $A^{opp} \otimes A$-module.

But if A is a Hopf algebra, we can use the antipode to make an A^{opp}-module an A-module and then Δ to get an A-module: for $\varphi \in \mathrm{Hom}_k({}_AM, {}_AN)$

$$(a \cdot \varphi)(m) = \sum_i c_i \cdot \varphi(S(b_i)m) \quad \text{if} \quad \Delta a = \sum b_i \otimes c_i, \text{ and } S = \text{antipode}.$$

In particular, if M is a left-module, then $M^* := \mathrm{Hom}_k(M, k)$ becomes a left A-module (via the antipode).

Example: Let $A = kG$.

- Tensor product of kG-modules: M, N:
$$g(m \otimes n) := (\Delta g)(m \otimes n) = (g \otimes g)(m \otimes n) = (gm) \otimes (gn)$$
- Duals: Given a kG-module M, get a kG-module M^* by:
$$gm^* = S(g)m^* = g^{-1}m^*.$$

- Frobenius & Symmetric algebras.

Let A be a k-algebra (always assume that all vectorspaces to be finite-dimensional)

Def: A bilinear form on A, $\beta : A \otimes A \to k$, is <u>associative</u> if $\beta(ab,c) = \beta(a,bc)$ $(\forall a,b,c)$.

A bilinear form on A, β, is <u>symmetric</u> if $\beta(a,b) = \beta(b,a)$.

There's a linear bijection between bilinear forms and linear functionals on A: $(\lambda : A \to k)$

via $\begin{cases} \beta(a,1) =: \lambda(a) \\ \beta(a,b) := \lambda(ab) \end{cases}$

β is nondegenerate $\Leftrightarrow$ Ker λ contains no nontrivial $\overset{(0)}{\ast}$ left or right ideals.

β is symmetric $\Leftrightarrow$ $\lambda(ab) = \lambda(ba)$ (we say that λ is a "trace" on A).

Def An algebra A equipped with an associative (symmetric) non-degenerate bilinear form β is called a Frobenius (symmetric) algebra.

Example: kG is symmetric (i.e. Frobenius symmetric) w.r.t. $\beta(a,b) = \lambda(ab)$,

where $\lambda\left(\sum k_g g_k\right) = k_e$ ← the coef of $e \in G$.

In other words, $\beta(g, g^{-1}) = 1$, $\beta(g,h) = 0$ if $g \neq h^{-1}$.

<u>Note</u>: β is <u>not</u> the usual bilinear form on $k[G] \overset{\langle\cdot,\cdot\rangle}{\cong} kG$ $\langle g, h \rangle = \delta_{gh}$.

because $\langle \cdot, \cdot \rangle$ is not associative.

<u>Example</u>: $\mathrm{Mat}(n,k)$ is symmetric wrt $\beta(M,N) := tr(MN)$.

Note that a non-degenerate associative bilinear form on A defines an isomorphism with A^*. ${}_AA \cong ({A_A})^*$. (as left A-modules)

In general, we know that the dual of a projective module is injective. Therefore:

Thm (ii) If A is Frobenius, then (1) ${}_AA$ is injective.

(2) For an A-module, the following are equivalent (still assuming A is Frobenius)

M projective, M injective, M^* projective, M^* injective.

Note: kG has two bilinear forms: $\langle g, h\rangle = \delta_{gh}$

$\beta(g,h) = \delta_{gh^{-1}}$

In general, if A is an algebra and $\langle\cdot,\cdot\rangle : A\otimes A \to k$ is a bilinear form, we can define a coaction $\Delta : A \to A\otimes A$.

$(kG, \langle\cdot,\cdot\rangle) \rightsquigarrow$ Hopf algebra $\Delta(ab) = \Delta a\, \Delta b$.

Q: What about β?

A: Spse (A,β) is a ~~Hopf alg~~ Frobenius algebra. We obtain $\delta : A \to A\otimes A$ coming from β, (i.e. $\beta(\delta a, b\otimes c) = \beta(a, b\cdot c)$)

$\beta(\delta(ab), c\otimes d) = \beta(ab, cd) = \beta(a, bcd) =$

$= \beta(\delta a, bc\otimes d) = \beta(\delta a, (b\otimes 1)\cdot(c\otimes d)) = \beta(\delta a\cdot(b\otimes 1), c\otimes d)$

So $\boxed{\delta(ab) = \delta a\cdot(b\otimes 1)}$ ← Frobenius condition.

(Similarly, could break it in a different way)

Conclusion: The Frobenius Algebra is NOT a Hopf algebra.

Q: What is the meaning of the Frobenius condition?

A: It appears in plane cobordisms.

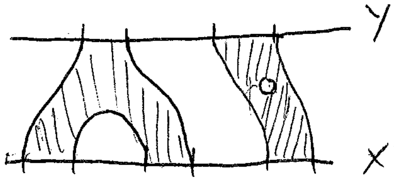
Topological definition of a Frobenius Algebra.

A Frobenius algebra is a functor $\underline{2Cob} \longrightarrow \underline{Vectorspaces/K}$ (+ more properties)

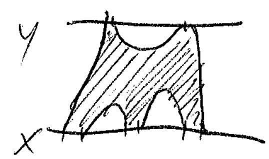
where $\underline{2Cob}$ is the category of plane cobordisms:

$\underline{\text{Objects}}$: Finite Unions of closed intervals of $\mathbb{R}$, up to homotopy ($\simeq \mathbb{Z}_{\geq 0}$) ← count # of intervals

$\underline{\text{Morphisms}}$:



(up to homotopy) $\in$ Hom(X,Y)



The composition is just by concatenation.

Now, $F: \underline{2Cob} \to \underline{Vect/K}$:

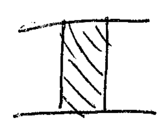
$F(\text{—}) = A$

$F(\text{— — —}) = A \otimes A \otimes A$

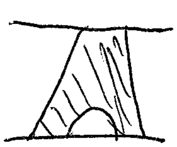
$F(\ \ ) = K$

$\underline{\text{Basic Cobordisms}}$:

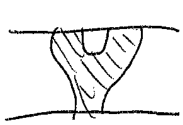
$\xrightarrow{F}$ $\lambda : A \to K$ (trace)



← identity $\xrightarrow{F} A \xrightarrow{id} A$

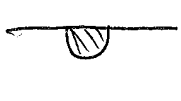


$\xrightarrow{F}$ $A \otimes A \xrightarrow{\cdot} A$ (product)



$\xrightarrow{F}$ $A \xrightarrow{\delta'} A \otimes A$

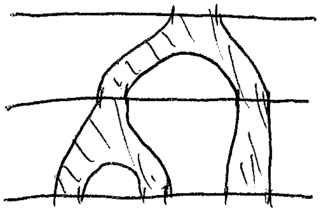
We can build any cobordism out of these five basic ones!



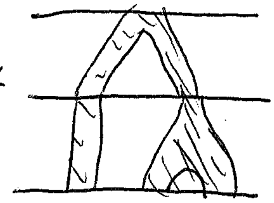
$\xrightarrow{F}$ $K \to A$ (unit)

Relations :

1) $\circ$ is associative :

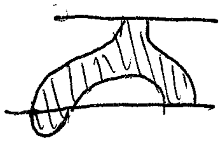


$\simeq$

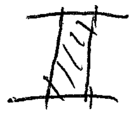


$\leftarrow$ the two are homotopic !

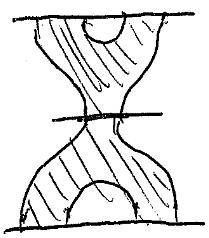
2) the "unit" is the unit :



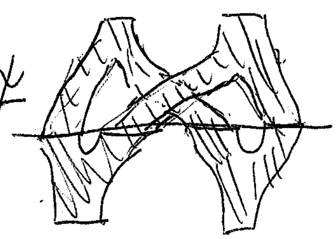
$\simeq$



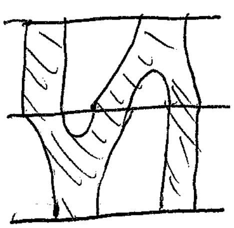
3) Relation between coproduct and product :



$\not\simeq$



But it is actually homotopic to



$\overset{F}{\rightsquigarrow} \quad \delta' a \cdot (1 \otimes b)$

So we have $\delta'(ab) = \delta'(a) \cdot (1 \otimes b)$

If we put $\delta' = \tau \circ \delta \circ \tau^{-1}$ $\quad \left(\tau(x \otimes y) = y \otimes x \right)$

So the Frobenius condition is actually quite natural!

We could in this way check all the Frobenius algebra relations.

Remarks: In particular, given a Frobenius algebra we can associate a number associated to any closed region of the plane:

 $\in \mathrm{Hom}(-,-) \xrightarrow{F} \mathrm{Hom}(k,k) = k.$

• Structure of Associative Algebras and Modules.

Let A be an associative algebra/k. Everything finite-dimensional. An A-module will mean a left A-module.

We have the same standard definitions as for representations:

Def If M is an A-module,

- M is irreducible (simple) if $[N \subsetneq M \Rightarrow N = 0.]$
- M is indecomposable if $[M = N_1 \oplus N_2 \Rightarrow N_1$ or $N_2 = 0.]$
- M is completely reducible (semisimple) if either of these 2 equivalent conditions hold:
  - (i) $\forall N \subset M$, $\exists N'$ s.t. $M = N \oplus N'$.
  - (ii) $M = \bigoplus_i S_i$, S_i simple.

Also, the standard theorems:

Krull-Schmidt: Any module is the direct sum of indecomposable modules, and the summands are uniquely determined by the module.

Jordan-Hölder: Any module M has a composition series
$$0 = N_0 \subset N_1 \subset \cdots \subset N_k = M \quad \text{s.t. } N_{i+1}/N_i \text{ is simple, and}$$
the quotients are unique up to reordering.

Schur's lemma: ~~If M, N are simple, then~~

If M, N are simple, then $\mathrm{Hom}_A(M,N) = \begin{cases} 0 & \text{if } M \neq N \\ \text{a division algebra} & \text{if } M = N \end{cases}$

- Semisimple Algebras.

Def An algebra A is <u>semi-simple</u> if any of the following two equivalent conditions hold:

i) any A-module is semisimple.

ii) A, as a left A-module, is semisimple.

Pf We need to see that any A-module is semisimple (knowing that A is so).

Recall that any A-module M is a quotient of a free A-module.

Also, the set of semisimple modules is closed under submodules, $\oplus$'s and quotients, so done! (Rk: <u>not</u> closed under extensions!!).

<u>Note</u>: in semisimple modules, submodules are "the same" as quotients!

[$\circledast$ is part of the proof of Krull-Schmidt thm

subgroups: Given $N \subset M$ and $L \subset N$, want to find L' s.t. $L \oplus L' = N$.

Know that $\exists L''$ s.t. $M = L \oplus L''$. Take $L' := L'' \cap N$.]

<u>Remark</u>: Maschke thm says that kG is semisimple. (if $\mathrm{char}(k) \nmid |G|$).

<u>Goal</u>: Classification of semisimple algebras:

- <u>Generalized Schur's lemma</u>.

Let M be a semi-simple A-module, where we write $M = \bigoplus_i M_i$, where

$M_i = \bigoplus_{k=1}^{n_i} M_{ik}$, $\underbrace{S_i}_{\text{simple}} \simeq M_{ik} \simeq M_{ik'}\ \forall k, k',\ \forall i$, and $M_{ik} \not\simeq M_{jk'}\ \forall i \neq j$.

<u>Then</u>: $\mathrm{End}_A M = \bigoplus_i \mathrm{Mat}_{n_i}(\Delta_i)$, Δ_i division ring, $\Delta_i := \mathrm{End}_A S_i$.

Pf $\mathrm{Hom}_A(M, M) = \bigoplus_{i,j} \mathrm{Hom}(M_i, M_j) \overset{\text{schur}}{=} \bigoplus_i \mathrm{Hom}(M_i, M_i)$

We compute $\mathrm{Hom}(M_i, M_i) = \mathrm{Hom}\left(\bigoplus_k M_{ik}, \bigoplus_\ell M_{i\ell}\right) = \bigoplus_{k,\ell} \overbrace{\mathrm{Hom}(M_{ik}, M_{i\ell})}^{\Delta_i} = \mathrm{Mat}_{n_i}(\Delta_i)$ //

Wedderburn-Artin Thm: If A is a semisimple algebra, then $A = \bigoplus_i A_i$, where $A_i = Mat(n_i, \Delta_i)$, where Δ_i = division ring.

Pf Write $A = (End_A({}_AA))^{op}$ // (note that $Mat(n_i, \Delta_i)^{opp} \cong Mat(n_i, \Delta_i^{opp})$)

Remark: This theorem implies that KG as algebra does not determine G!

But: as a Hopf algebra it does!

(elements of G are the group-like elements of it, ie st $\Delta g = g \otimes g$).

In the case of a semisimple algebra A, we want a description of the simple modules over A.

An example of such is, if $A = \bigoplus_{i=1}^{r} Mat(n_i, \Delta_i)$, the A-module $S_i = \Delta_i^{n_i}$, the vectorspace of $Mat(n_i, \Delta_i)$.

Fact: $\{S_i\}$ is a complete set of simple A-modules.

Observation: Submodules of ${}_AA$ = left-ideals.

Simple submodules of ${}_AA$ = minimal left-ideals.

Proposition: Every simple A-module is isomorphic to a minimal left ideal.

~~Proof~~

Remark: We don't assume A to be semisimple!

Pf Write $A = \sum L_i$ (L_i left-ideals), M = simple A-module.

Then $M = A.M = \sum L_i M \Rightarrow \exists L_i$ s.t $L_i M \neq 0 \Rightarrow \exists x \in M$ s.t $L_i\{x\} \neq 0$ $\Rightarrow L_i x = M$. (M simple)

So we have a map of A-modules $L_i \longrightarrow M$, $l \mapsto l \cdot x$.

As L_i is simple, this is an isomorphism, so done. //

So we now only need to find the left-ideals in $Mat(n,\Delta)$.

Def: Let A be any algebra. Then $e \in A$ is called an idempotent if $e^2 = e$.

Fact: if e is an idempotent, then $1-e$ is also an idempotent: $(1-e)^2 = 1-2e+e^2 = 1-e$.

Example: In $Mat(n,\Delta)$, idempotents are projectors on Δ^n: an idempotent e projects Δ^n onto $\mathrm{Im}(e)$, along $\ker(e)$.:

$$x \in \Delta^n \Rightarrow x = \underset{\underset{\mathrm{Im}\, e}{\cap}}{ex} + \underset{\underset{\ker e}{\cap}}{(1-e)x}$$

Def: Let $e_1, e_2 \in A$ be idempotents.

- e_1, e_2 are orthogonal if $e_1 e_2 = e_2 e_1 = 0$.
- e an idempotent is primitive if $\nexists\, e_1, e_2$ idempotents s.t. $e = e_1 + e_2$

Example: In the matrix example, primitive projectors are line projectors.

Prop (relation between idempotents & ideals):

(i) Let A be any algebra; then there is a 1-1 correspondence between

- direct-sum decompositions ${}_A A = L_1 \oplus \cdots \oplus L_n$ (as A-modules)
- decompositions $\underset{\underset{A}{\cap}}{1} = e_1 + \cdots + e_n$, where e_i's are orthogonal idempotents.

via $L_i = A \cdot e_i$

(ii) If A is semisimple, then any nonzero left-ideal L is generated by an idempotent (i.e $L = A \cdot e$). (then L is simple iff e is primitive).

Proof: Note that (i) $\Rightarrow$ (ii) trivially: if ${}_A A$ is semisimple, then ${}_A A = L \oplus L' \overset{(i)}{\Rightarrow} (ii)$ //

For (i), think of ${}_A A$ as a K-vectorspace. Then a direct sum decomp. as a K-vectorspace corresponds to $\{p_i \in \mathrm{End}_K A\}$ (p_i projections)

Since $\oplus$ is as A-modules, $p_i \in \mathrm{End}_A({}_A A) \Rightarrow p_i$ is given by right-multiplication by an element e_i of A. //

In the matrix example, the primitive idempotents are line projectors. $\to$ there is only one (up to isomorphism) simple $Mat(n,\Delta)$-module, namely

$$Mat(n,\Delta)\cdot \underset{\text{a line projector}}{e} = \Delta^n = S \qquad \left(\text{say } e = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\right).$$

Note: $_{Mat(n,\Delta)}Mat(n,\Delta) = \overbrace{S\oplus\cdots\oplus S}^{n}$, so S is projective, hence

any sequence $0\to M\to N\to S\to 0$ splits.

But by induction, then any sequence $0\to M\to N\to L\to 0$ splits

Hence $Mat(n,\Delta)$ is semisimple.

Theorem (Wedderburn-Artin):

A is a semisimple algebra $\iff A = \bigoplus_{i=1}^{r} Mat(n_i,\Delta_i)$

Simple modules of such A are $\{S_i = \Delta_i^{n_i}\}$.

Equivalently,

Theorem: A is a semi-simple Algebra and let $\{S_i\}_{i=1}^{r}$ be the set of (isomorphism classes) of simple A-modules.

Then, if $\Delta_i = End_A(S_i)$ (a division algebra), and $n_i = \dim_{\Delta_i} S_i$,

$$A = \bigoplus_{i=1}^{r} Mat(n_i,\Delta_i)$$

(we will prove the first statement).

Example: $k=\bar{k}$, char $k=0$. Let G be a finite group. Then kG is semisimple, and any finite division algebra $/k$ is k itself.

So $kG = \bigoplus_{i=1}^{r} Mat(\dim S_i, k)$, where S_i's are irreducible rep's of G.

The dual statement, that we proved before, is that $k[G]$ has a basis given by matrix elements of irreducible representations.

Burnside Theorem: Let $K=\bar{K}$, char $K=0$, G a finite group, and $\pi: G \to GL(V)$ a rep. of G. Assume π is irreducible.

Then: $\{\pi(g)\}_{g\in G}$ span (linearly) ~~that~~ $\mathrm{End}_K(V)$.

Proof: In terms of the group algebra KG, the statement is,

$KG \xrightarrow{\pi} \mathrm{End}_K(V)$ is surjective.

which is part of the statement of Wedderburn-Artin. //

Density Theorem (or Double centralizer Thm):
(the density thm is the statement in the ∞-dim situation)

Let A be a semi-simple algebra, M an A-module, and assume $A \subseteq \mathrm{End}_K M$ as subalgebra

Let $R' := \mathrm{End}_A(M)$, which is a K-algebra.

$R'' := \mathrm{End}_{R'}(M)$

Then: $R'' \cong A$.

Pf Easy, exercise //

Remark: If M is simple, then $R' = K$ (by Schur's lemma) (with $K=\bar{K}$), and so $R'' = \mathrm{End}_K M$, so we get $R'' = A$ by Wedderburn-Artin.

Simple Algebras

Def An algebra A is *simple* if A has no nontrivial proper two-sided ideals.

Example: $A = \mathrm{Mat}(n, \Delta)$ is simple

Proof: (could do an elementary proof, linear algebra) But we go abstract:

2-sided ideals = submodules of ${}_A A_A$ as an A-A-bimodule.

As A is semisimple, then $A \otimes A$ is semisimple, too. Write ${}_A A_A = I \oplus J$ as A-A-bim

(Suppose I a two sided ideal. Then has a complement J, because A is semisimple)

So $1 = u + v$, $u \in I$, $v \in J$. Then $uv \in I \cap J \Rightarrow uv = 0$

Same for $v \cdot u$.

But then $(u+v)^2 = 1 \Rightarrow u^2 + v^2 = 1 \Rightarrow u^2 = u,\ v^2 = v$.

Also,
$$\begin{aligned} a \cdot 1 &= a \cdot \underset{\in I}{u} + a \cdot \underset{\in J}{v} \\ \| \\ 1 \cdot a &= u \cdot a + v \cdot a \end{aligned} \Rightarrow \begin{aligned} au &= ua \\ av &= va \end{aligned} \quad \forall a \in A.$$

So $u, v \in Z(A)$. So u, v are orthogonal central idempotents.

As $A = Mat(n, \Delta)$, $u \in \Delta$. $u^2 = u \Rightarrow u = 0$ or 1. So $I = A \cdot u$ is either A or 0. //

<u>Proposition</u>: If A is simple, then A is semisimple.

Pf Next time //

Assuming this prop, then A simple $\Rightarrow A = \bigoplus_{i=1}^{r} Mat(n_i, \Delta_i)$.

Each $Mat(n_i, \Delta_i)$ is a 2-sided ideal. As A is simple, there is only one such summand, so $A = Mat(n, \Delta)$.

<u>Wedderburn's Theorem</u>

(i) Any semisimple algebra is a direct sum of simple algebras.

(ii) Any simple algebra A is a matrix algebra $A = Mat(n, \Delta)$.

(iii) A simple algebra has a unique non-trivial simple module (up to isomorphism).

<u>Pf (of Prop)</u>

Sps A is a simple algebra (no 2-sided ideals).

Let I be a minimal left-ideal of A (so I a simple A-module).

$$J := \sum_{\substack{\{a_i\} \\ \text{a basis of } A}} I a_i \quad \text{is a 2-sided ideal} \overset{A \text{ simple}}{\Rightarrow} J = A.$$

<u>Claim</u>: A sum of simple modules is semisimple.

<u>Pf</u>: It is a quotient of a direct sum of simples, and semisimples are preserved by quotients. //

· Center of A and blocks

Def. $Z(A) := \{a \in A : ab = ba \ \forall b \in A\} \cong \mathrm{End}_{A\otimes A}({}_A A_A)$ (center of A)

The isomorphism is $a \in Z(A) \longmapsto m_a$ = multiplication by a. $\in \mathrm{End}_{A\otimes A}({}_A A_A)$

If $x \in \mathrm{End}_{A\otimes A}({}_A A_A) \Rightarrow x \in \mathrm{End}_A({}_A A) \Rightarrow x$ is given by $a \mapsto a \cdot x$ (for some $x \in A$).

Also, $x \in \mathrm{End}(A_A) \Rightarrow abx = axb \ \forall b \in A \Rightarrow bx = xb$. //

Examples: • A semisimple, $A = \oplus \mathrm{Mat}(n_i, \Delta_i)$. Then $Z(A) \cong \bigoplus_i \Delta_i \subset \bigoplus_i \mathrm{Mat}(n_i, \Delta_i)$.

• $A = kG$, G a finite group.

$Z(A)$ has a basis given by $\{\sum_{g\in C} g : C \text{ conjugacy class}\}$.

In particular, $\dim Z(A) = \#$ conjugacy classes.

Pf

$$\sum_{g\in G} k_g g \in Z(G) \Leftrightarrow h\Big(\sum_{g\in G} k_g g\Big) = \Big(\sum k_g g\Big)h \quad \forall h \in G \Rightarrow$$

$$\Rightarrow \sum_g k_g(hg) = \sum_g k_g(gh) \Leftrightarrow k_{h^{-1}g} = k_{gh^{-1}} \quad \forall g,h \in G. \Leftrightarrow k_{hgh^{-1}} = k_g \quad \forall g,h \in G$$ //

Putting together the two examples, if we assume char $k = 0$, then:

$\#$ simple kG-modules $\leq \#$ conjugacy classes of G

($= \#$ irreducible G-reps $/k$) $\quad \overset{\uparrow}{(=)}$ if $k = \bar{k}$ (so Δ_i's $= k$). ← we knew that before, by using characters.

Let now A be any algebra, not necessarily semisimple.

Defs (i) $u \in A$ is a <u>central idempotent</u> if $u^2 = u$, $u \neq 0$, $u \in Z(A)$.

(ii) A central idempotent is <u>primitive</u> if it is not the sum of two orthogonal central idempotents.

Proposition: There is a 1-1 correspondence between decompositions of A into $A = B_1 \oplus \cdots \oplus B_n$, B_i two-sided ideals; and decomposition of $1 \in A$ into orthogonal central idempotents ($1 = e_1 + \cdots + e_n$, e_i's orthogonal central idempotents).

The correspondence is given by $B_i = A e_i$

Moreover, A can be written as $A = B_1 \oplus \cdots \oplus B_n$, where the B_i's are indecomposable two-sided ideals; and this decomposition is unique.

Proof:

For the first part it is similar to the proof for left ideals:

think of A as an A-A-bimodule.

Then given ${}_AA_A = \bigoplus$ $\underbrace{A\text{-}A\text{-bimodules}}_{\text{2-sided ideals}}$ $\Rightarrow \exists\, e_1, \ldots, e_n$ projectors onto the direct summands

These projectors are $A \otimes A$-invariant, so in particular $e_i \in \mathrm{End}_{A\otimes A}({}_AA_A) = Z(A)$.

For the second part, the decomposition $A = B_1 \oplus \cdots \oplus B_n$ into 2-sided ideals is in particular a decomposition of $A \otimes A$-modules. So by Krull-Schmidt, we get the irreducible one. We just need to prove that this is unique.

Assume that we have $1 = e_1 + \cdots + e_n$ want that $e_1 = f_k$ for
$1 = f_1 + \cdots + f_n$ (some k.)

$$(e_i f_j)^2 \overset{\text{central}}{=} e_i^2 f_j^2 \overset{\text{idempotents}}{=} e_i f_i \Rightarrow e_i f_j \text{ is } \begin{cases} \text{central idempotent} \\ 0 \end{cases}$$

$e_1 = e_1 \cdot 1 = e_1 f_1 + \cdots + e_1 f_m$. As ~~$e_1$ indecompos.~~ e_1 is primitive, $e_1 = e_1 \cdot f_k$, and $e_1 \cdot f_{i \neq k} = 0$. For f_k, get $f_k e_1 = f_k \Rightarrow f_k = e_1$.

<u>Def</u>: The indecomposable 2-sided ideals $\{B_i\}$ in $A = B_1 \oplus \cdots \oplus B_n$ are called <u>blocks</u> of A.

We can extend the previous theorem, as:

<u>Prop</u>: If A is an A-algebra, $A = B_1 \oplus \cdots \oplus B_n$ w/ B_i's are indecomposable (blocks) 2-sided ideals.

Then any A-module M has a unique decomposition

$M = M_1 \oplus \cdots \oplus M_n$, where M_i is a B_i-module (i.e. $B_j M_i = 0$ for $j \neq i$)

Also, the relation with the orthogonal principal central idempotents is:

$B_i = A\mu_i = \mu_i A \Rightarrow M_i = \mu_i M$.

<u>Example</u>: If A is semisimple,

- B_i's are matrix algebras,
- M_i's are isotypical components of M.

<u>Corollary</u>: If $A = B_1 \oplus \cdots \oplus B_n$ is the block decomp. of A,

a) Any indecomposable A-module is a B_i-module, for some i.

b) Any submodule and any quotient of an indecomposable B_i-module is a B_i-module.

c) If exact $0 \to M_1 \to M_2 \to M_3 \to 0$ where $\begin{cases} M_1 \text{ is a } B_i\text{-module} \\ M_3 \text{ is a } B_j\text{-module} \quad (j \neq i) \end{cases}$

Then the sequence splits.

In other words, the category of A-modules is a direct sum of categories of B_i-modules.

<u>Example</u>: (of a non-semisimple algebra)

Let G be a p-group, k a field of $\mathrm{char}(k) = p$. Let $A = kG$.

Then A has a unique simple module. (in particular, A has a unique block)

↓

(continues example).

To see that A has a unique (i.e. the trivial) simple module, it is enough to show that any G-representation $/\mathbb{K}(W)$ has an invariant subspace V.

For this, it is enough to find a nonzero G-fixed point.

So let $x \in W$ and consider the $\mathbb{F}_p$-linear span of Gx (i.e. the abelian gp generated by $G\cdot x$, call it X. Observe that $p \mid |X|$, and that $p \mid |\{\text{any non-trivial orbit}\}|$

So $p \mid \#\{\text{fixed points}\} \overset{0 \text{ is a fixed point}}{\Rightarrow}$ there are at least $p-1$ nonzero fixed points.

This implies that any G-rep $/K$ is $\pi : G \to \mathrm{End}_K V$ is given by, in some basis (which will be coming from the Jordan-Hölder thm),

by $\pi(g) = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

Q: How to decompose the category of modules into blocks?

A: Using characters.

Central Characters.

Let A be an algebra $/K$. Assume $K = \overline{K}$.

Then $Z(A)$ is a finite-dimensional commutative algebra $\Rightarrow Z(A) = \hat{\bigoplus_{i\in I}}$ local algebras (R_i)

(so R_i is the local algebra at the point $i \in \operatorname{Spec} A$).

Each R_i has a unique idempotent ~~xxxx~~ u_i s.t. $u_i \equiv 1 \bmod R_i$

(recall that $R_i/_{\mathrm{rad}\, R_i} = k$) (easy version of Hensel's lemma).

These u_i's are the central idempotents of A.

Now let S be a simple A-module. Then we have a map

$$\chi : Z(A) \longrightarrow \mathrm{End}_A S$$
$$z \longmapsto [x \mapsto zx]$$

By Schur's lemma, $\mathrm{End}_A S = k$.
Hence we have a map $\chi : Z(A) \to k$, corresponding to each simple module S.
This is called the <u>central character of S</u>.
Writing $Z(A) = \oplus R_i$, then χ is given by $\begin{cases} \chi(u_i) = 1 \\ \chi(u_{j\neq i}) = 0 \end{cases}$

(for some i, which is the i for which S belongs to the block B_i).

<u>Proposition</u>: Two simple modules belong to the same block iff their central characters are equal.
(So we can distinguish (~~modules~~ blocks) using central characters).

<u>Rk</u>: In the case $A = kG$, char $k = 0$, $k = \bar{k}$, the central characters are the usual characters (class-functions)
(because $Z(kG) = k\text{-span}\left(\sum_{g\in C} g\right)$).

Representation of $S^n / \mathbb{C}$.

Let $A := \mathbb{C}S_n$, which we know is semisimple.
We want the simple A-modules.

Simple A-modules $\overset{1:1}{\longleftrightarrow}$ Primitive idempotents.
$$Au \leftrightarrow u$$

So we want to find the primitive idempotents.

<u>Example</u>: S_3 $(A = \mathbb{C}S_3)$.

1) $f_{id} := \sum_{g\in G} g \Rightarrow h f_{id} = f_{id} h = f_{id}$ $(\forall h \in G)$; $f_{id}^2 = |G| \cdot f_{id}$.

Let now $u := \frac{1}{|G|} f_{id} \Rightarrow u$ is idempotent, and the left ideal Au_{id} is an A-module : $Au_{id} = \mathbb{C}u_{id}$, with trivial G-action.

(Example) 2)

2) $f_{sgn} := \sum_{g \in S_n} (\mathrm{sgn}(g))\, g \Rightarrow h\, f_{sgn} = f_{sgn}\, h = \mathrm{sgn}(h) \cdot f_{sgn}$

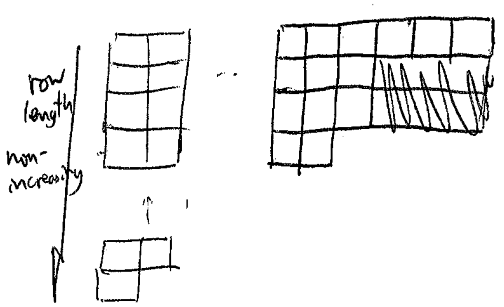
So also $f_{sgn}^2 = |S_n|\, f_{sgn} \Rightarrow u := \frac{1}{|S_n|} f_{sgn}$ is the sgn rep

$A u_{sgn} = \mathbb{C} u_{sgn}$.

We know that there's a third rep, which is 2-dimensional.
There are many idempotents that generate it, for dim > 1. We will find such an idempotent using Young theory:

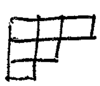
<u>Young Theory:</u> (of $\mathbb{C}S_n$ - modules)

The Young diagram of a partition $n = k_1 + \cdots + k_m$ is:



k_1 boxes
k_2 boxes
⋮
k_m boxes

$\#$ boxes $= n$.

Example:  corresponds to the partition $6 = 3 + 2 + 1$

<u>Note</u>: # simple $\mathbb{C}S_n$-modules $=$ # conj. classes in S_n $\overset{\text{cycle decomposition}}{=}$ # partitions of n $=$ # Young diagrams

<u>Hope</u>: the partitions ($=$ labels of conjugacy classes) should label (naturally) the simple $\mathbb{C}S_n$-modules.

<u>Remark</u>: For a general group, $\{$conj classes$\} = \{$simple reps$\}$ but there's no natural bijection.

For S_n, however, we do have a primitive idempotent in $\mathbb{C}S_n$ for each Young diagram λ with n boxes.

Def Let λ be a Young diagram. Then a <u>Young Tableau</u> of shape λ is a placement of numbers $1, \dots, n$ into the boxes of λ

$\left(\text{eg } \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 4 & 1 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \right)$

- Such a tableau is <u>standard</u> if the numbers in each row and in each column increase.
- A tableau is <u>canonical</u> if it is of the form e.g. $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$

Now, back to S_n, let T be a tableau of shape λ (say, the canonical one)

$P_T := \{ g \in S_n : g \text{ preserves rows of } T \}$

$Q_T := \{ g \in S_n : g \text{ preserves columns of } T \}$

Let $a_T, b_T, c_T \in \mathbb{C} S_n$ be given by:

$$a_T = \sum_{g \in P_T} g \qquad b_T := \sum_{g \in Q_T} \mathrm{sign}(g)\, g \qquad c_T := a_T b_T$$

<u>Examples</u>: In S_3, there are 3 Young diagrams: $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}$, $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$

Take canonical tableaux, $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$P_{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}} = G, \quad Q_{\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}} = 1 \implies c_{\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}} = \sum_{g \in G} g = \ell_{id}$

Similarly, $c_{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}} = c_{sgn}$, $\quad c_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}} = \underbrace{(1 + (12))}_{a_T} \underbrace{(1 - (13))}_{b_T}$

<u>Exercise</u>: $c_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}}^2 = 3\, c_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}} \implies A \cdot c_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}}$ is a simple S_3-module.

In general, c_T (up to normalization) is a principal idempotent in $\mathbb{C} S_n$.

Theorem:

1) $c_\lambda^2 = n_\lambda c_\lambda$, $n_\lambda \in \mathbb{Z} \setminus \{0\}$.

2) $V_\lambda := \mathbb{C}S_n \cdot c_\lambda$ is an irreducible representation of S_n (Specht module).

3) V_λ does not depend on the tableau (only on the Young diagram).

4) $V_\lambda \simeq V_\mu \iff \lambda = \mu$.

Moreover, "everything" about V_λ can be found from λ:

For example,

$\dim V_\lambda = \#$ standard tableaux of shape λ.

$\left(\dim V_{\text{⊞}} = \# \left\{ \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix} \right\}\right)$

• Characters of S_n

Let $R_n := \{\text{characters of } S_n\}$, an abelian group, $\mathbb{C}$-linear space with $\langle , \rangle_n$.

Put $R := \bigoplus_{n=0}^{\infty} R_n$

It has the following structure:

- R has a scalar product $\langle \cdot, \cdot \rangle := \oplus \langle \cdot, \cdot \rangle_n$
- R has a grading ($\deg f = n$ if $f \in R_n$).
- R has a graded product
$$R_n \otimes R_m \to R_{n+m}$$
$$(f, g) \longmapsto \mathrm{Ind}_{S_n \times S_m} =: f \cdot g$$

(here $S_n \times S_m \subseteq S_{n+m}$ is the subgroup preserving the partition $\{1, \dots, n\}, \{n+1, \dots, n+m\}$.

- R has a graded coproduct $\Delta : R_n \longrightarrow \bigoplus_{k=0}^{n} R_{n-k} \otimes R_k$

(By Frobenius Reciprocity, Δ is dual to $\cdot$). $f \longmapsto \mathrm{Res}^{S_n}_{S_{n-k} \times S_k} f$

<u>Proposition</u>: R is a commutative cocommutative graded Hopf algebra.

<u>Theorem</u>: There is (essentially) only one commutative cocommutative graded Hopf algebra.

(essentially means irreducible, as we could take the sum of two copies).

<u>Example</u> (of another Hopf algebra):

$\Lambda := \{$symmetric polynomials of countably many variables$\}$.

$\rightsquigarrow$ Examples of elements: $h_1(x_1, \dots) = x_1 + x_2 + \cdots = \sum x_i$

$h_2(x_1, \dots) = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_2 x_3 + \cdots = \sum x_i x_j$

(Λ = projective limit of the rings of symmetric polynomials in n variables).

Λ has a product (product of polynomials).

Λ has a coproduct
$$\begin{aligned} \Lambda &\xrightarrow{\Delta} \Lambda \otimes \Lambda \\ f &\longmapsto \Delta f \end{aligned}$$

where $(\Delta f)(x_1, \dots, y_1, \dots) := f(x_1, y_1, x_2, y_2, \dots)$

<u>Example</u>: $\Delta h_1 = h_1 \otimes 1 + 1 \otimes h_1$

$\Delta h_2 = h_2 \otimes 1 + 1 \otimes h_2 + h_1 \otimes h_1$

<u>Theorem</u>: $R \simeq \Lambda$ with $\{$trivial reps of $S_n\} \longleftrightarrow \{h_n\}$.

(for example, $\mathrm{res}^{S_2}_{S_1 \times S_1}(\text{trivial}) = 1\text{-}1$ component of Δh_2

$= h_1 \otimes h_1 \rightsquigarrow (\text{trivial of } S_1) \otimes (\text{trivial of } S_1)$)

• Non-semisimple algebras / modules.

As always, assume dim (everything) is finite.

Examples:

1) kG, char$(k) = p$, G a p-group. Then kG is not semisimple.

(because the only simple module of kG is trivial) ← and so kG is not the sum of matrix algebras.

2) $B(n) := \{ x \in Mat(n,k) : x = \begin{pmatrix} * & * & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$ (upper triangular) $\}$. is not semisimple:

The module k^n (column vectors) has a submodule without a complement.

For example, $V := \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rangle$ has no complement.

Again, if we cannot use direct summands, we can indeed use filtrations:

If M is any A-module, M has a Jordan-Hölder filtration,

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_0 = 0 \quad \text{s.t} \quad M_{i+1}/M_i \text{ is simple.}$$

Unfortunately, the Jordan-Hölder filtration is not unique (although the simple subquotients are)

A better filtration is the radical filtration..

Note: M_{n-1} in J-H is a submodule such that M/M_{n-1} is simple. ($\Leftrightarrow M_{n-1}$ is maximal)

We look for $X \subseteq M$ s.t M/X is semisimple.

Def: The radical of M is rad(M) := smallest submodule X of M s.t M/X is semisimple.

Q: Does X exist?

Prop: rad$(M) = \bigcap \{$maximal submodules of $M\}$.

Pf Let $X := \bigcap\{\text{maximals}\} = \bigcap_{i=1}^{r} M_i$ ← as dim $M < \infty$. for some $M_1, \ldots, M_r$ maximals.

First, $M/\bigcap M_i \overset{\text{CRT}}{\cong} \text{Im}\left(M \to \underbrace{\frac{M}{M_1} \oplus \cdots \oplus \frac{M}{M_r}}_{\text{semisimple}}\right) \Rightarrow M/X$ is semisimple.

Suppose M/Y is semisimple, $M/Y = L_1/Y \oplus \cdots \oplus L_s/Y$, where $Y \subset L_i \subseteq M$. ↓

(cont'd)

We have written $M/Y = L_1/Y \oplus \cdots \oplus L_s/Y$, L_i/Y simple.

Let $M_i := L_1 + \cdots + \overset{\text{removed}}{\widehat{L_i}} + \cdots + L_r \subseteq M$.

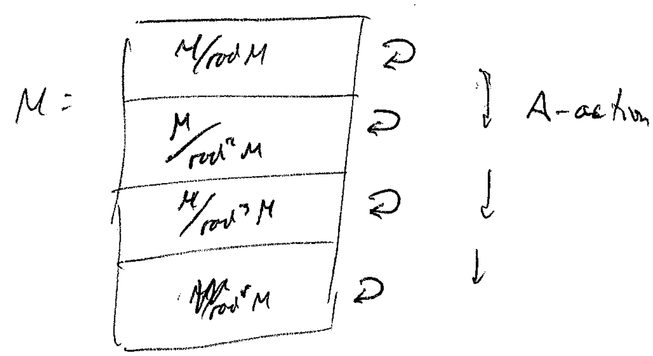
Claim: M_i is maximal, $\forall i$. (and then note that $\bigcap M_i/Y = \bigcap L_i/Y = 0 \Rightarrow \bigcap M_i = Y$)

Pf $M/M_i \cong L_i/Y$ which is simple //.

Now we can consider $M \supset \operatorname{rad} M \supset \overset{\text{radrad } M}{\operatorname{rad}^2 M} \supset \operatorname{rad}^3 M \supset \cdots \supset 0$.

This filtration is then well-defined, and "unique".

Picture:



Def $M/\operatorname{rad} M$ is called the top or the head of M.

Remark: The Jordan-Hölder filtration is not (in general) a refinement of the radical filtration.

Similarly, we can consider the largest semisimple submodule of M (called the socle of M), Again:

Proposition: $\operatorname{soc}(M) =$ sum of all the simple submodules.

Pf omitted.

We have then the socle filtration: $M \supset \cdots \supset \operatorname{soc}^2 M \supset \operatorname{soc} M \supset 0$

where $\operatorname{soc}^2 M = \operatorname{soc}(M/\operatorname{soc} M)$

In this case, the picture is:

$$\begin{array}{|c|c}
\hline
\mathrm{soc}^m M & \supseteq \\
\hline
\vdots & \supseteq \\
\hline
\mathrm{soc}^2 M & \supseteq \\
\hline
\mathrm{soc}\, M & \supseteq \\
\hline
\end{array}
\quad \Big\downarrow \text{ A-action}$$

__Warning__: The socle filtration $\neq$ the radical filtration.

• __The (Jacobson) radical of an algebra__ ← or of a ring!

"rad A is $\mathrm{rad}_A A$ (as a left A-module)."

There are equivalent definitions of rad A:

__Def / Proposition__: The __radical__ of A (rad A) is each of the following:

(i) The set of all elements of A which annihilate each simple A-module. (or semisimple)

(ii) the minimal left ideal $I \subseteq A$ s.t A/I is semi-simple (as A-module).

(iii) The intersection of all the maximal left ideals.

(iv) The maximal nilpotent ideal.

__Remarks__:

1) $a \in A$ annihilates an A-module M $\Leftrightarrow$ $aM = 0$.

2) (ii) $\Leftrightarrow$ (iii) by the last lecture.

3) The (i)-rad A is a 2-sided ideal $\Rightarrow$ $A/\mathrm{rad}\,A$ is an algebra.

By (i)-(iii), $A/\mathrm{rad}\,A$ is a semisimple A-module. But it is also a semisimple $A/\mathrm{rad}\,A$-module. So $A/\mathrm{rad}\,A$ is a semisimple algebra.

$$\Rightarrow A/\mathrm{rad}\,A = \bigoplus_{i=1}^{n} \mathrm{Mat}(n_i, A_i)$$

4) Simple A-modules = Simple $A/\mathrm{rad}\,A$-modules

5) An ideal $I \subseteq A$ is nilpotent if $\exists\, n \in \mathbb{N}$ s.t $I^n = 0$ (i.e $x_1 \cdots x_n = 0$ for any $x_1, \dots, x_n \in I$)

(in the commutative case, it is enough to consider large powers of elements).

Lemma: If I, J are nilpotent ideals, then $I+J$ is also nilpotent:

Pf If $I^n = 0$, $J^m = 0$, then $(I+J)^{n+m} = 0$. //

This implies that there exists the maximal nilpotent ideal.

If A is commutative, then $Nil(A)$ is an ideal, and it is the maximal nilpotent!

Proof (of Def/Prop):

(i) $\Leftrightarrow$ (iv)

Let $\text{rad}\, A$ be the (i)-rad, and N be the (iv)-rad.

Let S be a simple A-module.

Then $NS = \begin{cases} 0 & \text{want this} \\ S \end{cases}$: If $NS = S$, then $N^n S = S$ $\forall n \Rightarrow !!$ because $N^n = 0$ for some n.!

S is simple!

So $N \subseteq \text{rad}\, A$.

Conversely, consider the Jordan-Hölder decomp. of ${}_A A$:

$A = A_n \supset A_{n-1} \supset \cdots \supset A_0 = 0$, A_i / A_{i-1} simple

Then $\text{rad}\, A$ kills all A_i / A_{i-1} $\Rightarrow \text{rad}(A) \cdot A_i \subseteq A_{i-1}$

So $(\text{rad}\, A)^{n+1} A = 0 \Rightarrow \text{rad}\, A$ is nilpotent $\Rightarrow \text{rad}\, A \subseteq N$. // (of (i) $\Leftrightarrow$ (iv)).

We've already seen that (ii) $\Leftrightarrow$ (iii).

Now (i) $\Leftrightarrow$ (ii):

Let $\text{rad}\, A = $ (i)-rad, and let I be the (ii)-rad.

Nakayama Lemma: Let M be a (finitely generated) A-module.

Then $IM = M \Rightarrow M = 0$

Proof: Step 1: If $a \in I$, then $(1-a)$ has a left inverse:

$1 = a + (1-a) \Rightarrow A = I + A \cdot (1-a)$. If $A(1-a) \neq A$, then $A(1-a)$ is a proper left-ideal, contained in a maximal left ideal K.

As $I = \bigcap$ (maximal left ideals), $I \subset K$. Then $A = I + A(1-a) \subset K \Rightarrow !!$

(note $A(1-a) = A \Leftrightarrow 1-a$ has a left-inverse).

Step (2) Let $\{x_1, .., x_r\}$ be a minimal set of generators of M.

$$IM = M \Rightarrow x_r = \sum_{i=1}^{r} a_i x_i \Rightarrow (1 - a_r) x_r = \sum_{i=1}^{r-1} a_i x_i .$$

Since $1-a_r$ is invertible, we can express x_r in term of the other x_i's which contradicts minimality of $\{x_1, .., x_r\}$.

(of Nakayama)

Back to proving (i) ⇔ (ii):

Let $a \in \text{rad } A$, J be a maximal left-ideal. (want to show $a \in J$)

$a \cdot (A/J) = 0 \Rightarrow a \cdot A \subseteq J \Rightarrow a = a \cdot 1 \in J$. (So $\text{rad } A \subseteq I$).

simple, as J is maximal

Assume that ~~rad A ≠~~ $\text{rad } A \neq I$. So there exists $S \neq 0$ simple A-module such that $IS \neq 0$. As S is simple, $IS = S$. But by Nakayama's lemma, $S = 0$ ⇒!!

Theorem: $\text{rad}(M) = (\text{rad } A) \cdot M$

Pf/

1) $\frac{M}{\text{rad } M}$ is ~~semi~~simple. Let $X := M/(\text{rad } A)M$.

X is a $A/\text{rad } A$ -module, and $A/\text{rad } A$ is semisimple $\Rightarrow X$ is semisimple.

So $(\text{rad } A) M \supseteq \text{rad } M$.

2) ~~Assume that rad(A) M ≠ rad M. So ∃~~

Consider $\frac{M}{\text{rad } M}$. It is a semisimple A-module. As $\text{rad}(A)$ annihilates semisimple A-modules, then $(\text{rad } A) M \subseteq \text{rad } M$.

More general: $\operatorname{rad}^n M = (\operatorname{rad} A)^n \cdot M$

In particular, the length of the radical filtration of M is $\max\{n : (\operatorname{rad} A)^n M \neq 0\}$.

Theorem: $\operatorname{Soc} M = \operatorname{Ker}(\operatorname{rad} A) \left(= \{x \in M : ax = 0 \ \forall a \in \operatorname{rad} A\}\right)$

Pf Exercise (very easy)

Corollary: $\operatorname{soc}^n M = \operatorname{ker}(\operatorname{rad} A)^n$.

In particular, the length of the socle filtration is $\max\{n : (\operatorname{rad} A)^n M \neq 0\}$

So: radical length = socle length.

Examples:

1) $B(n)$ (upper-triangular matrices) $\subseteq \operatorname{Mat}(n, k)$.

$\operatorname{rad} B(n)$ = maximal nilpotent ideal $= N(n)$ ← strictly upper-triangular matrices (i.e. 0's on diagonal).

~~First, note~~

$\frac{B(n)}{N(n)}$ = diagonal matrices $= D(n) \simeq k^n$.

$\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in k^n$

Then simple modules over $D(n)$ are 1-dim and given by $\pi_i(\underline{\lambda}) = \lambda_i$.

So there are exactly n simple modules. $\pi_i \in \operatorname{End}(k)$

The simple modules over $B(n)$ are essentially the same: $\pi_i\left(\begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_n \end{pmatrix}\right) = \lambda_i$

Note: The π_i's are not submodules of $B(n)$ as a left-module over itself (except for $i = 1$). In fact, submodules of ${}_{D(n)}B(n)$ are columns $\left\{\begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \end{pmatrix}\right\}$

Remark: For a semisimple algebra A, any simple module is a submodule of ${}_A A$!

2) If $P_i := \left\{\begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}\right\}^{\updownarrow i} \subseteq k^n$, $\operatorname{rad} P_i = \left\{\begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}^{\updownarrow i}$. So $\operatorname{top} P_i = \frac{P_i}{\operatorname{rad} P_i} = S_i$,

the simple module given by π_i.

Continuing with the example, the radical filtration of P_i is:

$$\left.\begin{array}{|c|l}
\hline
S_i & \cong \pi_i \\
\hline
S_{i-1} & \cong \pi_{i-1} \\
\hline
S_{i-2} & \\
\hline
\vdots & \\
\hline
S_1 & \cong \pi_1 \\
\hline
\end{array}\right\downarrow N(n)$$

<u>Example(2)</u> kG, $|G| = p^r$, $\mathrm{char}(k) = p$, G = cyclic.

Q: What are the simple kG-modules? $\leftarrow$ we had seen this before.

A: $\pi_{trivial}$ is the only one.

$\mathrm{rad}(\underset{\text{commutative}}{kG}) = \{\text{nilpotent elements}\}$

Consider elements $(e-g) \in kG$ (e = identity in G). Then $(e-g)^{p^r} = e^{p^r} - g^{p^r} = 0$

Then $\{e-g : g \in G\}$ span $\ker(kG \xrightarrow{\pi_{triv}} k)$, so $\mathrm{rad}\, kG = \mathrm{Ker}[kG \xrightarrow{\pi_{triv}} k]$

Hence $kG/\mathrm{rad}\, kG \cong k$

We saw in the example of $B(n)$ that, given a projective P_i, we obtained a simple module S_i (by quotienting out). This will actually be true in general.

Also, note that ${}_{B(n)}B(n) = P_1 \oplus \cdots \oplus P_n$

We now generalize what we've seen ~~for~~ in $B(n)$.

<u>Projective Modules of A</u>

~~(= submodules of free modules)~~ (= direct summands of free modules).

Krull-Schmidt Thm $\Rightarrow$ indecomposable projective are direct summands of ${}_A A$

<u>Theorem</u>: The map $P \longmapsto P/\text{rad}\,P$ is a bijective from the set of isomorphism classes of indecomposable projective A-modules to the set of isomorphism classes of simple A-modules.

<u>Proof</u>: (next time.)

<u>Remark</u>: For Frobenius algebras (kG), projective = injectives.

<u>Corollary</u>: Let ${}_AA = \underbrace{P_1 \oplus \cdots \oplus P_1}_{n_1} \oplus \underbrace{P_2 + \cdot \oplus P_2 \oplus \cdots \oplus P_2}_{n_2} \oplus \cdots$ w/ P_i's are indecomposable and non isomorphic.

Then $n_i = \dim S_i$, where $S_i = P_i/\text{rad}\,P_i$ is the corresponding simple module.

Pf [Cf. semisimple case: if ${}_AA$ is s/simple, ${}_AA \cong \oplus S_i^{n_i}$, and then $n_i = \dim S_i$]

Pf Multiply the decomp. of ${}_AA$ by rad A, & get

$\text{rad}\,A = \bigoplus (\text{rad}\,P_i)^{n_i}$. Taking the quotient,

$A/\text{rad}\,A = \bigoplus \left(\frac{P_i}{\text{rad}\,P_i}\right)^{n_i}$. Now $A/\text{rad}\,A$ is semi-simple, and by the theorem $\frac{P_i}{\text{rad}\,P_i}$ are simple modules of $A/\text{rad}\,A$. Then $n_i = \dim \frac{P_i}{\text{rad}\,P_i}$ //

<u>Example</u>: kG, $|G| = p^r$, char $k = p$ $\overset{\text{know}}{\Longrightarrow}$ The only simple kG-module is trivial. (S_{tr})

Our corollary, as $\dim S_{tr} = 1$, says that kG is indecomposable as a ~~left~~ kG-module.

For instance, if $G = \langle g \rangle$ is cyclic, then the action of g on kG is given by $\begin{pmatrix} 1 & 1 & & 0 \\ & & 1 & \\ & & & 1 \\ 0 & & & 1 \end{pmatrix}$ (exactly one Jordan block).

In general (for a general group algebra) projective modules have a lot to do with p-Sylow subgroups of G.

For example, there is the following theorem ~~(which we don't prove)~~:

<u>Thm</u>: If a p-Sylow subgroup of G has order p^r (char $k = p$), then every projective kG-module has dimension divisible by p^r.

Pf/ Let P be a kG-projective module, $H \subset G$ a p-Sylow's subgroup.

<u>Claim</u>: $\mathrm{res}^G_H P$ is projective

Pf/ It is enough to prove it for free modules $(\mathrm{res}(\oplus) = \oplus(\mathrm{res}))$.

$$\mathrm{res}^G_H\, kG = \bigoplus kH x_i \quad \text{where } \{x_i\} \text{ are reps of left } H\text{-cosets.}$$

So $\mathrm{res}^G_H\, kG \simeq (kH)^{\oplus(G/H)}$, which is free. ($\Rightarrow$ projective).

By the claim & example for p-groups, we get the result. //

We now restate and proof the pending theorem:

<u>Thm</u>: The map $P \to P/\mathrm{rad}\,P$ is a bijection from the set of iso classes of ~~projective modules~~ indecomposable projective modules to the set of isomorphism classes of simple modules.

Pf/ We need to prove

① P indecomposable projective $\Rightarrow$ $P/\mathrm{rad}\,P$ simple.

② $\dfrac{P}{\mathrm{rad}\,P} \cong \dfrac{Q}{\mathrm{rad}\,Q} \Rightarrow P \simeq Q$ (for P, Q proj. indecomp.).

③ S simple $\rightarrow \exists\, P$ projective indecomp. s.t. $\dfrac{P}{\mathrm{rad}\,P} \cong S$.

Proof #1 (internal):

Indecomposable projective of A = indecomposable left-ideal of A. $= Au$, where u is a primitive idempotent.

For the Thm, we want a correspondence indecomposable left ideals in $A \Leftrightarrow$ the ones in $A/\mathrm{rad}A$

In other words, we need to "lift" idempotents of $A/\mathrm{rad}\,A$ to A.

i.e. $\quad (u+x)^2 = u+x \qquad (u \in A/\mathrm{rad}A,\ x \in \mathrm{rad}A)$.

Can be solved because $\mathrm{rad}\,A$ is nilpotent. (Hensel's lemma).

Proof #2 (external):

We look at $\mathrm{End}_A(M)$ (M an A-module) instead of M.

We want to identify indecomposable modules "from the outside".

Lemma: A an algebra, M an A-module.

Then M is indecomposable $\Longleftrightarrow \mathrm{End}_A M$ is local.

(Def: An algebra B is local if one of the equivalent statements hold:

- $B/\mathrm{rad}\,B$ is a division algebra.
- B has a unique maximal left-ideal.
- non-invertible elements of B form a left-ideal.
- any element of B is either invertible or nilpotent.)

Pf of Lemma:

$\Rightarrow$] Let $f \in \mathrm{End}_A(M)$. $\Rightarrow f$ is a K-linear operator in M, commuting w/ the action of A.

So any generalized eigenspace of f is an A-direct summand.

Since M is indecomposable, there's only one generalized eigenspace. (i.e. f has only one eigenvalue) $\begin{cases} =0 \Rightarrow f \text{ is nilpotent} \\ \neq 0 \Rightarrow f \text{ is invertible} \end{cases}$

$\Leftarrow$] Assume M is not indecomposable, $M = M_1 \oplus M_2$. Then the projection onto M_1 along M_2 belongs to $\mathrm{End}_A M$, but it is neither nilpotent nor invertible!

Proof of the Thm.

Let P, Q be A-modules, $\varphi \in \mathrm{Hom}_A(P,Q)$. Then $\exists\ \tilde{\varphi} \in \mathrm{Hom}_{A/\mathrm{rad}A}\left(P/\mathrm{rad}P,\ Q/\mathrm{rad}Q\right)$ s.t. the following commutes:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ {\scriptstyle \pi_P}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\ P/\mathrm{rad}P & \xrightarrow{\tilde{\varphi}} & Q/\mathrm{rad}Q \end{array}$$

(because $\varphi(\mathrm{rad}P) = \varphi((\mathrm{rad}A)\cdot P) = (\mathrm{rad}A)\cdot\varphi(P) \subseteq (\mathrm{rad}A)\cdot Q \subseteq \mathrm{rad}Q$.)

(we get a functor $\tilde{}$ from one Category to the other).

Assume P, Q are projective A-modules and $\tilde{\varphi} \in \mathrm{Hom}_{A/\mathrm{rad}A}\left(P/\mathrm{rad}P,\ Q/\mathrm{rad}Q\right)$.

Then there exists a lift $\varphi \in \mathrm{Hom}_A(P,Q)$, because π_Q is surjective and P is projective.

In other words, we have a surjection

$$\mathrm{Hom}_A(P,Q) \twoheadrightarrow \mathrm{Hom}_{A/\mathrm{rad}A}\left(P/\mathrm{rad}P,\ Q/\mathrm{rad}Q\right).$$

Back to the proof of the Thm,

Let $P = Q$. Have a surjection $\mathrm{End}_A(P) \twoheadrightarrow \mathrm{End}_{A/\mathrm{rad}A}(P/\mathrm{rad}P)$

As $\mathrm{End}_A(P)$ is local, then the other is so (very easy).

This proves that P indecomp. proj. $\Rightarrow \dfrac{P}{\mathrm{rad}P}$ is indecomposable $\overset{\text{semisimple}}{=}$ simple. (1)

To prove (2), let $\tilde{\rho}: P/\mathrm{rad}P \xrightarrow{\sim} Q/\mathrm{rad}Q$. $\tilde{\rho}$ lifts to $\rho: P \to Q$.

$\mathrm{Im}\,\tilde{\rho} \not\subset \mathrm{rad}\,Q$. As $\mathrm{Im}\,\tilde{\rho}$ is a submodule and $\mathrm{rad}\,Q$ is the only maximal such, we deduce that $\mathrm{Im}\,\rho = Q \Rightarrow P \xrightarrow{\rho} Q \to 0$ is exact.
as $Q/\mathrm{rad}Q$ is simple

Since Q is projective, $P = (\mathrm{Ker}\,\rho) \oplus Q$. But P is indecomposable, so $P \simeq Q$.

Finally, S simple $\Rightarrow S = Ax = (I_1 \oplus \cdots \oplus I_n)x \Rightarrow S = I_1x + \cdots + I_nx \Rightarrow$ one of $I_ix = S$ and I_ix is projective cover.
indecomp. left ideals

Examples:

1) A an algebra.

All indecomposable modules we have seen so far (simples, projectives ($=$ injectives for kG)) are submodules of ${}_AA$. In particular, their dimensions are $\leq \dim_k A$.

Warning: there could be arbitrarily large A-modules.

For example, let $G := C_p \times C_p$ (cyclic $\times$ cyclic) $\{= \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}\}$, char$(k) = p$.

Then kG has an indecomposable module in any even dimension.

A rep. of G ($= kG$-module) is given by two linear operators E_1, E_2 ($= \pi$(generators of G)), such that $E_1E_2 = E_2E_1$ and $E_i^p = 1$.

Let n be an integer ($2n = \dim$ of rep). The rep V_n is as follows:

Fix a basis $\{v_1, .., v_n, w_1, .., w_n\}$ of V_n ($\dim 2n$).

Let X be the linear operator $\{v_i \mapsto w_i$, $w_i \mapsto 0$

Let Y " ——— " — " $\begin{cases} v_i \mapsto w_{i+1} \\ v_n \mapsto 0 \end{cases}$, $w_i \mapsto 0$

Note: $X^2 = Y^2 = 0$.
$XY = YX = 0$

Set $E_1 := I + X$
$E_2 := I + Y$

Note that $E_1E_2 = E_2E_1$. Also, as char$(k) = p$, $\begin{cases} E_1^p = X^p + I^p = I^p = I. \\ E_2^p = Y^p + I^p = I^p = I \end{cases}$

Claim: This rep V_n is indecomposable.

Pf: We will prove that $\mathrm{End}_{kG}(V_n)$ is local ($\Leftrightarrow V_n$ indecomposable).

Let $Z \in \mathrm{End}_{kG} V_n$. Z is a linear operator commuting with X and Y.

$\downarrow$

(cont pf of example)

In the basis v_i, w_j's, we have:

$$X = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \begin{matrix} v's \\ w's \end{matrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \quad \left(N = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix} \right) \quad Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

$XZ = ZX \iff B = 0, A = D$. So $Z = \begin{pmatrix} A & 0 \\ C & A \end{pmatrix}$

$YZ = ZY \iff \begin{pmatrix} 0 & 0 \\ NA & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ AN & 0 \end{pmatrix} \iff NA = AN$

Doing the calculation, $A = \begin{pmatrix} \lambda_1 & & & 0 \\ \lambda_2 & \ddots & & \\ \lambda_3 & \ddots & \ddots & \\ \cdots & \lambda_3 & \lambda_2 & \lambda_1 \end{pmatrix}$. (no condition for C).

Z has the form $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_1 \end{pmatrix}$. If $\lambda_1 = 0$, Z is ~~invertible~~ nilpotent

$\lambda_1 \neq 0$, Z is invertible.

So End V_n is local.

<u>Remark</u>: A very similar problem consists in finding indecomposable $k[x,y]$-modules (= coherent sheaves on $\mathbb{A}^2$).

A $k[x,y]$-module = a pair of matrices X, Y s.t $XY = YX$

$X^n = Y^n = 0$ means that the sheaf is supported at 0.

<u>Example</u> 2) kS_3-modules, char $(k) = 3$.

<u>Q</u>: How many simple modules are there?

$\#$ simples $= \#$ projectives.

dim (indec projective module) is divisible by 3 (the $\overset{3}{p}$-order of $\overset{S_3}{G}$)

So $\#$ indecomp. projectives ≤ 2 (as $kG = \oplus$ indecomp. projectives).

As there are ≥ 2 simple modules (trivial, sign). So this is all we have.

So if P_{triv}, P_{sign} are projective covers of S_{triv}, S_{sign}, then

${}_{kG}kG = P_{tr} \oplus P_{sign}$; and $\dim P_{triv} = \dim P_{sign} = 3$.

Q: What happened to the 3rd rep. (the 2-dimensional one) that we had in 0-char?

Recall it was the complement to S_{triv} in the natural 3-dim rep of S_3.

S_{triv} is spanned by $(1,1,1)$ in $k[\{1,2,3\}] = V$, and in char 0, the complement of $k\langle(1,1,1)\rangle$ is ~~...~~ simple.

But in char $k = 3$, $\dfrac{V}{\langle(1,1,1)\rangle}$ is not simple:

The line $\langle(1,-1,0)\rangle$ mod $\langle(1,1,1)\rangle$ is a subrepresentation:

$$(12)(1,-1,0) = (-1,1,0) = -(1,-1,0) \quad \checkmark$$

$$(13)(1,-1,0) = (0,-1,1) = -(1,-1,0) + (1,1,1) \quad \checkmark$$

$$(23)(1,-1,0) = (1,0,-1) = -(1,-1,0) - (1,1,1) \quad \checkmark$$

$\leftarrow$ not true in char 0!

Q: What is the radical?: $\operatorname{rad} kS_3$?

A: $\operatorname{rad} kS_3 = \operatorname{ann}(S_{tr}) \cap \operatorname{ann}(S_{sign}) = \{ \sum_{g \in S_3} a_g g : \sum a_g = \sum \operatorname{sign}(g) \cdot a_g = 0 \}$

Q: Describe P_{tr} and P_{sign}. (both are either $\begin{smallmatrix}\square\\ \square\square\end{smallmatrix}$ or $\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}$ 1-dim, 2-dim).

Q': Find the radical filtration of P_{tr} and P_{sign}.

Lemma: Suppose that A is symmetric ($\exists$ tr-form λ s.t. $\lambda(ab) = \lambda(ba)$, $\lambda: A \to k$).
Suppose that P is an indecomposable projective A-module.
(in particular, $P/\operatorname{rad} P$ is simple).
Then $\operatorname{soc} P$ is simple.

Pf/ A-symmetric $\Rightarrow P^*$ is projective + indecomposable. $\operatorname{soc} P^* = (P/\operatorname{rad} P)^*$ $\Rightarrow$ it is simple. //

<u>Remark.</u> In fact, it is also true that if P is indecomposable projective and A is symmetric, then ~~Soc~~ $\mathrm{Soc}\, P \simeq \frac{P}{\mathrm{rad}\, P}$.

In the kS_3 case, the lemma implies that the radical filtration of P_{tr} looks like $\left.\begin{array}{|c|}\hline S_{tr} \\ \hline S_1 \\ \hline \end{array}\right\} X$ $\begin{array}{|c|} S_2 \\ \hline \end{array}$ (S_1, S_2 are either $\simeq S_{tr}$, S_{sgn}).

Let's look at $X := \frac{P_{tr}}{\mathrm{rad}^2 P_{tr}}$. (So X looks like $\begin{array}{|c|}\hline S_{tr} \\ \hline S_1 \\ \hline \end{array}$)

In other words, X is a nontrivial <u>extension</u> of S_1 by S_{tr}.

<u>Def</u>: Let A be an algebra, M, N be A-modules. Then an <u>extension</u> of M by N is an exact sequence (or E)

$$0 \to N \to E \to M \to 0$$

Two extensions E_1 & E_2 are <u>equivalent</u> if there exists an isomorphism $\eta : E_1 \to E_2$ s.t.:

$$\begin{tikzcd} & E_1 \arrow[dd, "\eta"] \arrow[dr] & \\ N \arrow[ur] \arrow[dr] & & M \\ & E_2 \arrow[ur] & \end{tikzcd}$$

commutes.

$\mathrm{Ext}^1(M,N)$ is the set of extensions of M by N, up to equivalence.

Let's use the representation notation for modules (A-module M is a vectorspace M with the map $\pi : A \to \mathrm{End}_k M$).

In an extension $0 \to N \to E \to M \to 0$, then $E = N \oplus M$ as vectorspace, and the action is given by
$$\pi(x) = \left(\begin{array}{c|c} \pi_N(x) & \varepsilon(x) \\ \hline 0 & \pi_M(x) \end{array}\right) \begin{array}{l} \updownarrow N \\ \updownarrow M \end{array}$$

where also
- $\pi(\alpha x + \beta y) = \alpha \pi(x) + \beta \pi(y)$
  $\varepsilon(\alpha x + \beta y) = \alpha \varepsilon(x) + \beta \varepsilon(y)$
- $\pi(xy) = \pi(x)\pi(y) \rightsquigarrow \pi_N(x)\varepsilon(y) + \pi_M(y)\varepsilon(x) = \varepsilon(xy)$ &

Note: Choosing a basis $\{e_i\}$ of A, $\mathcal{E}$ can be defined only by giving $\mathcal{E}(e_i)$ for all i, and the second condition is a <u>linear condition</u> on the set $\{\mathcal{E}(e_i)\}$.

<u>Therefore</u>, extensions form a K-vectorspace.

Also, we can see what equivalence means:

$\mathcal{E}, \mathcal{E}'$ are equivalent (via $\eta : E \to E'$) if:

$$\underbrace{\begin{pmatrix} 1 & \delta \\ 0 & 0 \end{pmatrix}}_{\eta} \begin{pmatrix} \pi_M(x) & \mathcal{E}(x) \\ 0 & \pi_M(x) \end{pmatrix} = \begin{pmatrix} \pi_N(x) & \mathcal{E}'(x) \\ 0 & \pi_M(x) \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}.$$

i.e. $\mathcal{E}(x) + \delta\pi_M(x) = \pi_N(x)\delta + \mathcal{E}'(x)$, so

$$\mathcal{E}'(x) - \mathcal{E}(x) = \delta\pi_M(x) - \pi_N(x)\delta$$

Again, equivalence is a linear condition

So $\mathrm{Ext}^1(M,N)$ is a linear K-vectorspace.

<u>Back to S_3:</u>

we look for $\boxed{\begin{smallmatrix} S_{tr} \\ S_1 \end{smallmatrix}}$ which is <u>not</u> semisimple.

In other words, we ~~look for~~ need $\mathrm{Ext}^1(S_{tr}, S_1)$ to be nontrivial.

<u>claim</u>: $\mathrm{Ext}^1_{KS_3}(S_{tr}, S_{tr}) = 0$.

Pf: $\mathrm{Ext}(S_{tr}, S_{tr})$ is given by matrices $\pi((ij)) = \begin{pmatrix} 1 & \mathcal{E}(i,j) \\ 0 & 1 \end{pmatrix}$

satisfying some relations.

In particular, $\pi((ij))^2 = id$. So want $2\mathcal{E}(ij) = 0$. Since $\mathrm{char}\, K \neq 2$,

$\mathcal{E}(i,j) = 0 \Rightarrow$ only the trivial extension.

From this computation, the only non-semisimple modules having a radical filtration $\begin{array}{|c|}\hline S_{tr} \\ \hline S_1 \\ \hline\end{array}$ is when $S_1 = S_{sign}$.

Now, P_{tr} looks like $\begin{array}{|c|}\hline S_{tr} \\ \hline S_{sign} \\ \hline S_2 \\ \hline\end{array}\Big\} X$. Looking at X (bottom part),

and again using that $\mathrm{Ext}^1(S_{sign}, S_{sign}) = 0$, we conclude that

$S_2 = S_{tr}$. $\Rightarrow P_{tr} = \begin{array}{|c|}\hline S_{tr} \\ \hline S_{sign} \\ \hline S_{tr} \\ \hline\end{array}$

A similar argument would prove that $P_{sign} = \begin{array}{|c|}\hline S_{sign} \\ \hline S_{tr} \\ \hline S_{sign} \\ \hline\end{array}$

<u>Remarks about extensions:</u>

- As $\mathrm{Ext}^1(M,N)$ is k-linear space, $0 \in \mathrm{Ext}^1(M,N)$. It corresponds to the trivial extension $(0 \to N \to M \oplus N \to M \to 0)$.

- M projective $\Leftrightarrow$ any ext. splits $\Leftrightarrow$ $\mathrm{Ext}^1(M,N) = 0$

  So $\mathrm{Ext}^1(M,N)$ measures how far is M from being projective:

  Consider $0 \to K \xrightarrow{\delta} P \xrightarrow{\rho} M \to 0$ (choose a proj. P surjecting onto M).

  <u>Claim:</u> $\mathrm{Ext}^1(M,N)$ can be identified with $\mathrm{Hom}(K,N)\big/\delta^*(\mathrm{Hom}(P,N))$

  via
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\delta"] \arrow[d] & P \arrow[d, "\eta"] \arrow[dr, "\rho"] & & \\ 0 \arrow[r] & N \arrow[r, "\beta"'] & E \arrow[r, "\alpha"'] & M \arrow[r] & 0 \end{tikzcd}$$

pf/ ① Given $0 \to N \to E \to M \to 0$, we want to find a map $K \to N$:

- P projective $\Rightarrow$ can lift ρ to η (not uniquely).
- $\alpha \circ (\eta \circ \delta) = \rho \circ \delta = 0 \Rightarrow \eta \circ \delta \in \mathrm{Im}\,\beta \Rightarrow$ can find an arrow $K \to N$.

Note that this does not depend on the extension in the class.

② η is not unique. So suppose η_1, η_2 are lifts of ρ.

Then $\alpha\circ(\eta_1 + \eta_2) = 0 \Rightarrow \eta_1 - \eta_2 \in \operatorname{Im}\beta \Rightarrow \eta_1 - \eta_2$ maps $P \to N$

So we get a well-defined map

$$\operatorname{Ext}^1(M,N) \longrightarrow \frac{\operatorname{Hom}(K,N)}{\delta^* \operatorname{Hom}(P,N)}$$

③ Now, suppose we are given $[\mu] \in \dfrac{\operatorname{Hom}(K,N)}{\delta^* \operatorname{Hom}(P,N)}$.

We want to construct $0 \to N \to E \to M \to 0$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\delta"] \arrow[d, "\mu"'] & P \arrow[d, dashed] \arrow[dr, "\rho"] & & \\
0 \arrow[r] & N \arrow[r, dashed, "\alpha"] & E \arrow[r, dashed, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

Define E to be the push-forward of
$$\begin{tikzcd}
K \arrow[r, "\delta"] \arrow[d, "\mu"'] & P \arrow[d, dashed] \\
N \arrow[r, dashed] & E
\end{tikzcd}$$

Actually, $E = \dfrac{N \oplus P}{J}$ where

$$J = \{ (\mu(x), -\delta(x)) : x \in K \}.$$

and so α is the inclusion $N \overset{\alpha}{\hookrightarrow} E$.

Now,
$$\begin{tikzcd}
K \arrow[r, "\delta"] \arrow[d] & P \arrow[d] \arrow[dd, bend left, "\rho"] \\
N \arrow[r] \arrow[dr, "0"'] & E \\
 & M
\end{tikzcd}
\Rightarrow \exists!\ \beta : E \to M.$$

(actually, β is the projection $\dfrac{N \oplus P}{J} \to P$ composed with ρ)

Easily, check that this forms an exact sequence.

Claim: • these two maps are inverse to each other.

• these bijection is k-linear.

Radical & Extensions

Let A be an algebra. List the simple and projective modules:

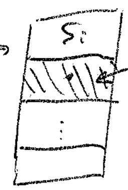
$S_1, \dots, S_n$ simples

$P_1, \dots, P_n$ ind. projectives, $\frac{P_i}{\operatorname{rad} P_i} \simeq S_i$

$\underline{Q}$: $\operatorname{Ext}^1(S_i, S_j)$?

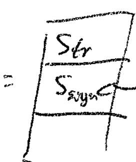
Prop: $\operatorname{Ext}^1(S_i, S_j) \simeq \operatorname{Hom}_A\left(\frac{\operatorname{Rad} P_i}{\operatorname{Rad}^2 P_i}, S_j\right)$

it's semisimple!

Remark: $P_i \to$ 

So $\operatorname{Ext}^1(S_i, S_j)$ "counts" how many times S_j occurs in the semisimple module $\frac{\operatorname{Rad} P_i}{\operatorname{Rad}^2 P_i}$.

Example: $A = kS_3$, char $k = 3$.

Then $\dim \operatorname{Ext}^1(S_{tr}, S_{sign}) = 1 \Rightarrow P_{tr} =$ 

exactly one copy of S_{sign}.

Proof: Have the projective resolution $0 \to \operatorname{rad} P_i \xrightarrow{\eta} P_i \to S_i \to 0$

$\Rightarrow$ (last result) $\Rightarrow \operatorname{Ext}^1(S_i, S_j) \simeq \operatorname{Hom}(\operatorname{rad} P_i, S_j) \big/ \eta^* \operatorname{Hom}(P_i, S_j)$ $=: (*)$

General Remark: Let $f \in \operatorname{Hom}(M, S)$ (S simple).

Then $f(\operatorname{rad} M) = f((\operatorname{rad} A) M) = (\operatorname{rad} A) \cdot f(M) \in \operatorname{rad} S = 0$.

In particular, $\operatorname{Hom}(M, S) \simeq \operatorname{Hom}(M/\operatorname{rad} M, S)$.

$(*) = \operatorname{Hom}\left(\frac{\operatorname{rad} P_i}{\operatorname{rad}^2 P_i}, S_j\right) \big/ (0)$ because $\eta^* \operatorname{Hom}(P_i, S_j) = \{0\}$.

Assume now that $k = \bar{k}$, so that it is easier to compute dimensions.

We want to compute $\dim_k \mathrm{Ext}^1(S_i, S_j)$ in terms of A.

$$\dim_k \mathrm{Ext}^1(S_i, S_j) = \dim_k \mathrm{Hom}_A\left(\frac{\mathrm{rad}\, P_i}{\mathrm{rad}^2 P_i}, S_j\right) = \#\ S_j\text{'s in } \frac{\mathrm{rad}\, P_i}{\mathrm{rad}^2 P_i} =$$

$$= \dim_k \mathrm{Hom}_A\left(S_j, \frac{\mathrm{rad}\, P_i}{\mathrm{rad}^2 P_i}\right) = \left(\text{general remark, previous page}\right) \left(\frac{P_j}{\mathrm{rad}\, P_j} = S_j\right)$$

$$= \dim_k \mathrm{Hom}_A\left(P_j, \frac{\mathrm{rad}\, P_i}{\mathrm{rad}^2 P_i}\right) = [P_j \text{ is projective}] =$$

$$= \dim_k \left(\frac{\mathrm{Hom}(P_j, \mathrm{rad}\, P_i)}{\mathrm{Hom}(P_j, \mathrm{rad}^2 P_i)}\right) = (*)$$

Remarks:

- $P_i = Au_i$ where u_i = idempotent.
- $\mathrm{Hom}(Au, M)$ (where u = idempotent, M any module) is $\cong uM$

With this, $(*) = \dim_k \left(\frac{\mathrm{Hom}(Au_j, \mathrm{rad}\, Au_i)}{\mathrm{Hom}(Au_j, \mathrm{rad}^2 Au_i)}\right) =$

$$= \dim_k \left(\frac{u_j\, \mathrm{rad}\, Au_i}{u_j\, \mathrm{rad}^2 Au_i}\right).$$

So $\dim_k \mathrm{Ext}^1(S_i, S_j) = \dim_k \left(\frac{u_j\, \mathrm{rad}\, Au_i}{u_j\, \mathrm{rad}^2 Au_i}\right)$

Example: $B(n)$ - upper-triangular matrices.

We have $u_i = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} - i$. $\frac{\mathrm{rad}\, A}{\mathrm{rad}^2 A} \cong \left\{\begin{pmatrix} 0 & * & 0 \\ & & * \\ 0 & & 0 \end{pmatrix}\right\}$ ← 2nd up-diagonal.

Hence $\dim u_i \left(\mathrm{rad}\, A / \mathrm{rad}^2 A\right) u_j = \begin{cases} 0 & i \neq j-1 \\ 1 & i = j-1 \end{cases}$

Ext-quiver:

quiver = directed graph. (still assume $k = \bar{k}$)

Def. The <u>Ext-quiver</u> of an algebra A is given by:

- vertices = simple A-modules.
- # arrows from S_i to S_j $= \dim_k \mathrm{Ext}^1(S_i, S_j)$

Examples:

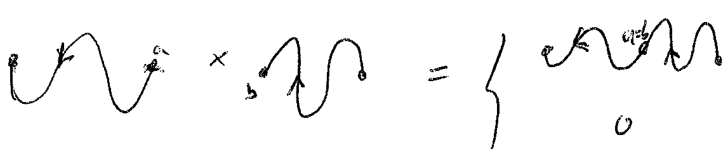
The Ext-quiver for $B(n)$:

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \; \cdots \; \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$

(by the prev. example).

Let Q be a quiver (#vertices $< \infty$, # arrows $< \infty$).

Def: The <u>path algebra</u> of Q (kQ) is generated by paths as a k-linear space, with multiplication given by concatenation:



if $a = b$

0 if $a \neq b$

Remark: a vertex is a path of length 0.

Remark. $\dim_k kQ \iff \nexists$ oriented cycles.

We want to study the kQ-modules.

Facts:

- Paths of length 0 are idempotents in kQ (<u>notation</u>: u_i, where i is a vertex)
- In kQ, $1 = u_1 + \cdots + u_n$ ($\{1, \ldots, n\}$ the vertices).
- $u_i \cdot \underset{j}{\bullet} \xrightarrow{t} \bullet = \begin{cases} \bullet \xrightarrow{t} \bullet & i = j \\ 0 & i \neq j \end{cases}$ (and similarly on the right).

From this, we see that:

$$\mu_i (kQ) \mu_j = \{\text{paths from } j \text{ to } i\}.$$

<u>Facts:</u>

- kQ is generated by $\{\mu_i\}$ and by arrows $\{\circ \to \circ\}$ with the only relation $\mu_i \times \underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ} = \begin{cases} \underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ} & i=j \\ 0 & i\neq j \end{cases}$ and $\underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ} \times \mu_i = \begin{cases} \underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ} & k=i \\ 0 & k\neq i \end{cases}$

  and $\mu_i \mu_j = \delta_{ij} \mu_i$

From this, if M is a kQ-module,

- $\mu_i M =: V_i \xRightarrow{1=\mu_1+\cdots+\mu_n} M = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ (as a k-linear space).
- $\underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ}\, V_i \subseteq V_j$ if $i=k$

  $\underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ}\, V_i = 0$ if $i \neq k$

In other words, a kQ-module is the same as a <u>representation of Q</u> ?

Def A <u>representation of a quiver Q</u> is:

- A vectorspace at each vertex
- A linear map $V_j \to V_k$ for every edge $\underset{k}{\circ}\!\leftarrow\!\underset{j}{\circ}$

<u>Examples:</u>

$V \circlearrowleft = Q$. A rep of Q in V = linear operator in V.

In particular, if $k = \bar{k}$, $\left\{ \begin{matrix} \text{indecomposable} \\ \bar{k}Q\text{-modules} \end{matrix} \right\} \Big/ \text{iso} = \{\text{Jordan blocks}\}$

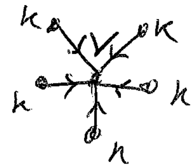
Example:

$Q: \underset{V}{\bullet} \longrightarrow \underset{W}{\bullet}$. Reps of Q are linear operators from V to W.

And indecomposable kQ-modules are $\underset{}{\overset{k}{\bullet}} \longrightarrow \overset{k}{\bullet}$

$\underset{k}{\bullet} \longrightarrow \underset{0}{\bullet}$

because such a linear operator has matrix conjugate to $\begin{pmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix}$.

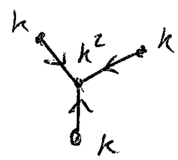
Example: $Q =$ .

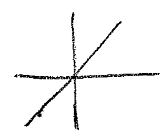
Indecomposable modules of Q:

$\overset{k}{\bullet} \longrightarrow \overset{V}{\bullet}$ are injective (in this case, however) (otherwise $\ker(\bullet\!\to\!\bullet)$ would be a direct summand)

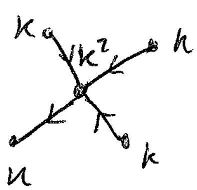
(i.e. kQ would be $\bullet \oplus$ ).

So indecomposable representations of Q are collections of lines in V.

For example,  $\rightsquigarrow$ 3 lines in k^2 $\Rightarrow$ only one indecomposable module

(as, by a linear transformation, they can be moved to  .

Another example:

 Indecomp. modules = 4 lines in k^2

There are many indecomposable modules (in fact, a 1-parameter family).

In general, a "starred" quiver corresponds to looking at collection of subspaces of a vectorspace.

Observation: kQ is $\mathbb{Z}$-graded by path length:

$$kQ = \bigoplus_{n=0}^{\infty} kQ_{(n)} \quad \text{where} \quad kQ_{(n)} = \text{span of paths of length } n.$$

and $kQ_{(n)} \cdot kQ_{(m)} \subseteq kQ_{(n+m)}$.

Also, $kQ_{(0)}$ = linear combination of idempotents $= ku_1 \oplus \cdots \oplus ku_r$, u_i corresp. to vertex i.

To compute the radical, let $T := \bigoplus_{n>0} kQ_{(n)}$.

- T is a nilpotent ideal, if there are no cycles in Q.

  This is clear, as $T^k = \bigoplus_{n \geq k} kQ_{(n)}$ and $kQ_{(k)} = 0$ for $k >> 0$ ∎.

- $kQ/T \underset{\text{alg}}{\cong} kQ_{(0)} \underset{\text{alg}}{\cong} k^{\oplus r}$ ← semisimple

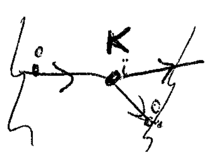
So T is maximal nilpotent $\Rightarrow T = \operatorname{rad}(kQ)$.

Prop: $\operatorname{rad} kQ$ is spanned by paths of length ≥ 1

$\operatorname{rad}^s kQ$ is spanned by paths of length $\geq s$.

To find the simple kQ-modules, we look for simple $\frac{kQ}{\operatorname{rad} kQ}$-modules

= simple modules of $ku_1 \oplus \cdots \oplus ku_r = \{S_i\}_{i=1}^r$. S_i corresponds to the Q-representation:



$\left\{ \begin{array}{l} K \text{ at vertex } i \\ 0 \text{ at other vertices} \\ \text{arrows} = 0 \end{array} \right.$

In particular, each u_i is principal.

Also, $P_i = kQ u_i$ is an indecomposable projective module over kQ, with $P_i / \operatorname{rad} P_i = S_i$.

Q: What is P_i?

A: P_i consists of the span of paths starting at the vertex i

Hence it is clear that $\dfrac{\{\text{paths starting at } i\} \leftarrow P_i}{\{\text{those of length} \geq 1\} \leftarrow \operatorname{Rad} P_i}$

$= \{\text{paths of length } 0 \text{ starting at } i\} = S_i$.

Q: What is $\operatorname{Ext}^1(S_i, S_j)$?

Note that we don't need to assume $K = \bar{K}$, because the simple modules are just K and so $\operatorname{End}_K S_i \simeq K$, even if $K \neq \bar{K}$.

$$\dim_K \operatorname{Ext}^1(S_i, S_j) = \dim_K \left(\frac{u_j (\operatorname{Rad} KQ) u_i}{u_j (\operatorname{Rad}^2 KQ) u_i} \right) = \dim_K \left(\frac{\{\text{paths of length} \geq 1 \text{ from } i \text{ to } j\}}{\{\text{paths of length} \geq 2 \text{ from } i \text{ to } j\}} \right) = \cancel{\#\text{ paths from } i \to j}$$

$$= \#\text{ edges}\left(\underset{i}{\bullet} \longrightarrow \underset{j}{\bullet} \right)$$

<u>Example</u>:  $\Rightarrow \dim \operatorname{Ext}^1(S_1, S_2) = 3$

$\dim \operatorname{Ext}^1(S_2, S_1) = 0$

<u>Rk</u>: Since there are no cycles, either $\operatorname{Ext}^1(S_i, S_j)$ or $\operatorname{Ext}^1(S_j, S_i)$ is 0.

<u>Example</u>:  Q Reps of Q = pairs of $n \times n$ matrices

There is no way to classify those. ← it's crucial that we assume that there are no cycles.

<u>Computing extensions of KQ-modules</u>: $(\operatorname{Ext}^1(S_i, S_j))$

We want E s.t. $0 \to S_j \to E \to S_i \to 0$

Then E is a KQ-module $\underset{i}{\overset{K}{\bullet}}$ $\underset{j}{\overset{K}{\bullet}}$ + linear operators on arrows.

↓

$\underset{i}{\overset{k}{\circ}} \longrightarrow \underset{j}{\overset{k}{\circ}}$ $\rightsquigarrow$ assign any number to such an arrow.

$\underset{j}{\circ} \longrightarrow \underset{i}{\circ}$ $\rightarrow$ all 0.

So $\dim_k \mathrm{Ext}^1(S_i, S_j) = \#$arrows from i to j — as before.

For a general algebra A over $k = \bar{k}$, we have:

Def The Ext-quiver is defined as:

- vertices: simple A-modules.
- arrows: as many as $\dim_k \mathrm{Ext}^1(\text{vertices})$

Observation: The Ext-quiver of kQ is Q itself (for a quiver Q)

Q: Given A, construct its Ext-Quiver Q and then construct the path-algebra of Q. What is the relation to A?

From now on, we assume $k = \bar{k}$, and that A is basic:

Def: An algebra A is basic if all simple modules are 1-dimensional$/k$.

$\left(\Leftrightarrow {}_A A = P_1 \oplus \cdots \oplus P_n \right.$, P_i indecomp. projectives with $\left.\dfrac{P_i}{\mathrm{rad}\, P_i} = S_i \text{ simples}\right)$.

$(\Leftrightarrow 1_A = u_1 + \cdots + u_n$, u_i principal idempotents s.t $P_i = Au_i$).

We want this condition because the path algebra of a Quiver is always basic.

In this case,

$$\dim_k\left(\mathrm{Ext}^1(S_i, S_j)\right) = \dim_k\left(\frac{u_j(\mathrm{rad}\, A)u_i}{u_j(\mathrm{rad}^2 A)u_i}\right)$$

$\downarrow$

Choose a complement B_{ij} to the space $u_j(\mathrm{Rad}^2 A)u_i$ in $u_j(\mathrm{Rad} A)u_i$.
This wouldn't be too smart because $B_{ij}B_{jk} \not\subset B_{ik}$.

<u>But</u>: $u_k \cdot x = \delta_{jk} x \qquad (x \in B_{ij})$

$x u_k = \delta_{ik} x \qquad (x \in B_{ij})$. Q is the Ext-quiver of A

Therefore, the map $\alpha : kQ \longrightarrow A$ given by a choice of a basis in B_{ij} is an algebra homomorphism.

Explicitly, if u_i are idempotents in A, and $\{e_{ij}^k\}$ is a basis in B_{ij}, then ~~$\alpha(x\alpha)$~~

- α (i-vertex of Q) $= u_i$
- α (k^{th} arrow in Q from i to j.) $= e_{ij}^k$

Why it is a homomorphism:
Just check that the (few) relation in kQ are mapped to relations in A.

<u>Claim</u>: $\mathrm{Coker}\, \alpha \subseteq \mathrm{rad}^2 A$ and $\ker \alpha \subseteq \mathrm{rad}^2 kQ$.

<u>Lemma</u>: Let A, B be algebras, and $\alpha : A \to B$ s.t. α is surjective mod $\mathrm{Rad}^2 B$. Then α is <u>surjective</u>.

<u>Proof</u>: Let $C := \mathrm{Im}\, \alpha$. We know that $C + \mathrm{Rad}^2 B = B$. Want $C = B$.
We do induction:

<u>Claim</u>, $C + \mathrm{Rad}^n B = C + \mathrm{Rad}^{n+1} B$

Pf/ Spz $z \in \mathrm{Rad}^n B \Rightarrow z = xy$, $x \in \mathrm{Rad}^{n-1} B$, $y \in \mathrm{Rad}\, B$

By induction, $x = x' + x''$, with $x' \in C \cap (\mathrm{Rad}^{n-1} B)$, $x'' \in \mathrm{Rad}^n B$.
$y = y' + y''$, with $y' \in C \cap \mathrm{Rad}\, B$, $y'' \in \mathrm{Rad}^2 B$.

Then $z = (x' + x'')(y' + y'') = \underbrace{x'y'}_{\in C} + x'y'' + x''y' + x''y'' \in C + \mathrm{Rad}^{n+1}B + \mathrm{Rad}^{n+1}B + \mathrm{Rad}^{n+2}B$ //

From this claim, $C + \mathrm{Rad}^2 B = B \Rightarrow C = B$.

<u>Theorem</u>: Let A = basic algebra $/k=\bar{k}$; Q the ext-quiver of A.

Then A is a quotient of kQ by an ideal contained in paths of length ≥ 2.

<u>Examples</u>: kG, G a cyclic of order p, char $k = p$.

Then kG is a quotient of $k\left(\bullet\circlearrowleft\right)$ (as only rep is S_{tr}, $\dim \mathrm{Ext}^1(S_{tr}, S_{tr}) = 1$)

In fact, $kG = k\left(\bullet\circlearrowleft\right) / x^p = 0$.

<u>Example</u>: $B(n)$ = upper-triangular matrices.

$S_1, \ldots, S_n$ simples.

$\dim \mathrm{Ext}^1(S_i, S_j) = \begin{cases} 1 & i = j-1 \\ 0 & \text{else} \end{cases}$

So $B(n)$ is a quotient of $k\left(\overset{1}{\bullet} \to \bullet \to \bullet \to \cdots \overset{n-1}{\bullet} \to \overset{}{\bullet}_{n}\right)$

By dimension count, this is actually an isomorphism.

(actually, the matrix $\begin{array}{cc} \begin{array}{ccc} & i & \end{array} & \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & & 0 \end{pmatrix} & -j \end{array}$ corresponds to $\overset{j}{\bullet} \to \bullet \to \bullet \to \overset{i}{\bullet}$).

Morita Theory

Observation: It could happen that the category of A-modules and the category of B-modules "look the same", even though $A \not\cong B$.

Example: A/k a simple algebra, $k = \bar{k}$.

Then • S := (unique) simple module, $\dim_k S = n$.

• $A \simeq Mat_n(k)$, $S \underset{k}{\simeq} k^n$.

Obs: $\{A\text{-modules}\}$ "=" $\{k\text{-modules (= vectorspaces)}\}$

i.e. any A-module M is isomorphic to $M \simeq \bigoplus_{i=1}^{m} S$

and can assign $M \rightsquigarrow k^m$

$$Hom(M, E) \xrightarrow[\text{Schur's Lemma}]{} Mat_{\ell \times m}(k)$$

because $Hom(M,E)$ looks like a matrix

$$\begin{array}{c} \begin{array}{cccc} S & S & S & S \end{array} \\ \underbrace{\begin{pmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{pmatrix}}_{m} \end{array} \begin{array}{c} S \\ S \\ S \end{array} \Big\} \ell$$

and each block is $k \cdot \mathbb{I}_n$ (scalar matrix).

In general, if A is semisimple with r simple modules, then the category $\{A\text{-modules}\}$ looks like the category of $\mathbb{Z}^r$-graded vectorspaces (i.e. the category of $\underbrace{k \oplus k \oplus \cdots \oplus k}_{r}$-modules.

because $M \simeq S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ corresponds to the graded vectorspace

$V = V_1 \oplus \cdots \oplus V_r$, $V_i = k^{n_i}$.

So Suppose $\mathcal{C}, \mathcal{D}$ are two categories.

- It may happen that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic:

  $\exists$ F, G functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ s.t. $F \circ G = id$, $G \circ F = id$.

- Another, more relaxed condition is: that $\mathcal{C}$ and $\mathcal{D}$ are equivalent;

<u>Def</u> $\mathcal{C}$ is <u>equivalent</u> to $\mathcal{D}$ if $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ such that

$$F \circ G \simeq id \quad , \quad G \circ F \simeq id$$

where $\simeq$ means that there is an invertible natural transformation.

In other words, there is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- any object of $\mathcal{D}$ is isomorphic to an object in $\operatorname{im}(F)$, say $F(X)$, $X \in Ob(\mathcal{C})$.
- F is fully faithful, i.e. $Hom(F(A), F(B)) \simeq Hom(A, B)$

<u>Observation</u>:

It could happen that for two algebras A and B, the categories of modules are equivalent.

(i.e. "iso classes of A-modules" $\simeq$ "iso classes of B-modules" and morphisms are the same).

Assume for now that A modules is equivalent to B-modules, via an equivalence $A\text{mod} \underset{G}{\overset{F}{\rightleftarrows}} B\text{mod}$. (and assume F, G preserve addition of morphisms, i.e preserve the ring structure of $Hom(M,M)$)

<u>Then</u>:

1) $F(\text{Simple})$ is simple (S simple $\Leftrightarrow$ $End(S)$ is a division ring)
   where $End(S) = Hom(S,S)$

2) $F(\text{indecomposable})$ is indecomposable (M indecomp $\Leftrightarrow$ $Hom(M,M)$ is local)

3) $F\left(\begin{matrix}\text{projective}\\ \text{injective}\end{matrix}\right)$ is $\begin{matrix}\text{projective}\\ \text{injective}\end{matrix}$ (because they are defined by diagrams).

4) $F(\text{free})$ is <u>not</u> necessarily free !

In particular, $F({}_AA)$ is <u>not</u> ${}_BB$ in general.

But at least $F({}_AA)$ is projective (as ${}_AA$ is free $\Rightarrow$ proj)

For example, under $Mat_n(k) \longrightarrow k$-modules,

${}_AA = Mat_n(k)$ goes to k^n, which is <u>not</u> k.

5) F(block) is a block (parts of the category without morphisms among them).

So we can see how many 2-sided ideals A has from its category of A-modules!

6) we can also find $Z(A)$ (center of A) from A-modules:

<u>Theorem</u>: Let A be a k-algebra. Then $Z(A)$, as a k-algebra, is the algebra of endomorphisms of the identity functor on the category of A-modules.

Pf/ An endomorphism of the identity functor is a natural transformation $id_{Amod} \to id_{Amod}$, i.e. f is a k-linear transformation f_M on each A-module, and should satisfy:

- commutes with the A-action.
- Commutes with A-module morphisms

$$\begin{array}{ccc} M & \xrightarrow{f_M} & M \\ \varphi\downarrow & \circlearrowleft & \downarrow\varphi \\ N & \xrightarrow[f_M]{} & N \end{array}$$

<u>Claim</u>: f_M's $\overset{1:1}{\longleftrightarrow} Z(A)$. via $f \longleftrightarrow z := f_{{}_AA}(1_A) \in A$

Pf/ $z \in Z(A)$: because $f_{AA}(a \cdot 1) = a \cdot f_{AA}(1) = az$. If $\varphi_b : A \to A$ maps $1 \mapsto b$, then

$$f_A(\varphi_b(1)) = f_A(b) = b \cdot z$$
$$\parallel \qquad\qquad\qquad\qquad \parallel$$
$$\varphi_b(f_A(1)) = \varphi_b(z) = z \cdot \varphi_b(1) = z \cdot b \qquad \checkmark$$

Now, any other module is a sum of cyclic modules, which are surjective images of ${}_AA$, on which f acts by multiplication by z.

So $f_M(m) = z \cdot m$

Q: When do we have that A-modules $\sim$ B-modules?

Suppose $F : A\text{mod} \longrightarrow B\text{mod}$

$$A \longmapsto F(A)$$

We have seen that $F(A)$ is projective (but not necessarily free)

Observations:

- Let $P := F(A)$, a projective B-module.
- P contains (as direct summands) each left-ideal of B

in other words,

P contains all indecomposable projective B-modules.

or:

Any projective B-module is a direct summand of a direct sum of several copies of P.

or:

For any B-module M, we have a surjection $\bigoplus_{i=1}^{r} P_i^{\simeq P} \longrightarrow_B M$.

Def: A projective generator of a module category is a module P s.t.:

i) P is projective

ii) direct sums of copies of P cover all modules.

(So if $F : A\text{mod} \to B\text{-mod}$, then $F(A)$ is a projective generator)

We want to recover A from $P := F(A)$.

Observe that $\mathrm{End}_A\, {}_AA \simeq A^{opp}$ (right multiplications)

As F is an equivalence, $\mathrm{End}_B\, P \simeq A^{opp}$.

In other words, $A \simeq (\mathrm{End}_B P)^{opp}$

This means that P is a B-A-bimodule $({}_BP_A)$.

Now, let M be an A-module.

Claim: $F(M) = {}_BP_A \otimes_A {}_AM$

Pf. It is true for $M = {}_AA$. So it is also true for free-modules, and so it is true for any module, by considering a surjection from a free module.

~~Theorem.~~

Example: Let B be an algebra, and $P_1, \dots, P_r$ indecomposable projective (non-isomorphic)

Consider $P := P_1 \oplus \cdots \oplus P_r$ (the smallest projective generator).

Suppose we want to find an equivalent category.

So let $A := (\mathrm{End}\, P)^{opp}$.

We will see then that A-modules $\overset{equiv}{\simeq}$ B-modules.

We hope that A is "simpler" than B". But to find P and End P is as hard as to describe B-modules, so it is not as nice as we'd like. (from the practical point of view).

A more explicit way to produce equivalent algebras is given by "Morita contexts". In general, there's a very simple class of functors Amod $\to$ Bmod :

Given a bimodule ${}_BM_A$, we get a functor F: Amod $\to$ Bmod as follows: $F(-) = {}_BM_A \otimes_A -$

- Morita Context (pairs of bimodules)

Example: K-modules $\simeq Mat(n,K)$-modules :

Take $V = K^n$. Then V is a $K - Mat(n,K)$ bimodule (by multiplying a row vector in V by the right by a matrix, by the left by scalars).

In fact, V is a projective generator of K-modules, and $End_K V \simeq Mat(n,K)^{opp}$ and (from last time) the equivalence is given by $V \otimes_{Mat(n,K)} (-)$.

What's the inverse?

Consider V^* (column vectors) $= Hom_K(V,K)$. Note that V^* is a $Mat(n,K) - K$ - bimodule, and so we have a functor $V^* \otimes_K (-)$

Claim: this is the inverse (up to natural transformation) to $V \otimes_{Mat(n,K)} (-)$

Pf ${}_{Mat}V^*_K \otimes_K {}_KV_{Mat} \otimes_{Mat} (-) \simeq (-)$:

Because $\begin{pmatrix} \vdots \end{pmatrix} \otimes_K (\ \cdots\ ) \underset{[\cdot,\cdot]}{\simeq} Mat$, $(\ \cdots\ ) \otimes_{Mat} \begin{pmatrix} \vdots \end{pmatrix} \underset{(\cdot,\cdot)}{\simeq} K$ evaluation map

<u>A more general example:</u> Let A be an algebra $(/k)$.

- P an A-module
- $B := (\mathrm{End}_A P)^{opp}$.

Then P is a A-B-bimodule.

- $Q := P^* := \mathrm{Hom}_A({}_AP_B, {}_AA_A)$. So Q is a B-A-bimodule.

P and Q produce functors $A\mathrm{mod} \underset{P\otimes_B -}{\overset{Q\otimes_A -}{\rightleftarrows}} B\mathrm{mod}$

We have maps

$$ {}_AP_B \underset{B}{\otimes} {}_BQ_A \xrightarrow{(\,,\,)} {}_AA_A $$

$$ {}_BQ_A \underset{A}{\otimes} {}_AP_B \xrightarrow{[\,,\,]} {}_BB_B $$

given by: $(p, q) := q(p) =: \overset{\text{notation.}}{(p)q}$

$$ (p')[q, p] := ((p')q)\cdot p = (p', q)\cdot p $$

and we have:

$$ [q, p]\cdot q' = q\cdot(p, q') $$

<u>Claim</u>: The two functors $A\mathrm{mod} \underset{P\otimes_B -}{\overset{Q\otimes_A -}{\rightleftarrows}} B\mathrm{mod}$ provide an equivalence if the maps $[\cdot,\cdot]$, $(\cdot,\cdot)$ are isomorphisms.

<u>Def</u> (Morita Context): A <u>Morita Context</u> for a pair of rings A and B is:

- a pair of bimodules ${}_AP_B$ and ${}_BQ_A$
- a pair of maps $[\cdot,\cdot] : {}_BQ \otimes_A P_B \longrightarrow {}_BB_B$ and $(\cdot,\cdot) : {}_AP_B \underset{B}{\otimes} {}_BQ_A \longrightarrow {}_AA_A$
- Such that $p\cdot[q, p'] = (p, q)\cdot p'$ (in ${}_AP_B$)

  $[q, p]\cdot q' = q\cdot(p, q')$ (in ${}_BQ_A$).

Note: if $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are isomorphisms, then ~~$P\otimes$~~

$P\otimes -$ and $Q\otimes -$ provide an equivalence of categories of A and B-modules.

Def: A perfect Morita context is a Morita context in which the maps $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are surjective.

Theorem: If $(\cdot,\cdot)$ is surjective, then it is an isomorphism.
($\Rightarrow$ we have an equivalence of categories in a perfect Morita context)

Pf/ We just need to see that $\ker(\cdot,\cdot)=0$.

Spse $\sum_i p_i\otimes q_i \in \ker(\ )$. As $(\cdot,\cdot)$ is surjective,

$$1_A = \sum_j (p_j', q_j')$$

So

$$\sum_j (p_j', q_j') \sum_i p_i\otimes q_i = \sum_{i,j} (p_j', q_j')\, p_i\otimes q_i =$$

$$= \sum_{i,j} p_j' \underbrace{[q_j', p_i]}_{\in B} \otimes_B q_i = \sum_{i,j} p_j'\otimes [q_j', p_i]\, q_i =$$

$$= \sum_{i,j} p_j'\otimes q_j' (p_i, q_i) = \sum_{i,j} (p_i, q_i) = 0.$$

~~Remark~~ Remark: A pair of module categories with a perfect Morita context are called Morita-equivalent.

Note: in principle, could have two equivalent module categories that are not Morita equivalent.

Example:

From the last lecture, if A is an algebra, ${}_AP$ an A-module and $B := (\mathrm{End}_A P)^{opp}$, then ${}_BQ_A = \mathrm{Hom}_A({}_AP_B, {}_AA_A)$ ($\Rightarrow$ B is a ${}_AP_B$-bimodule) and the natural maps $(\cdot,\cdot)$ and $[\cdot,\cdot]$ form a Morita context for A, B

Claim: If ${}_AP$ is a projective generator (of A-mod), then this Morita context is perfect. (The converse is also true)

Pf/ $(\cdot,\cdot) : P \otimes \mathrm{Hom}(P,A) \to A$ being surjective $\Leftrightarrow$ images of maps $P \to A$ cover A, which is true for a proj. generator!

Exercise: prove it for $[\cdot,\cdot]$.

This example shows that for any equivalence of module categories there is a Morita equivalence of the same module categories.

Example:

A an algebra, u an idempotent in A. Let $B := uAu$, which is another algebra. We have a Morita context:

$${}_A(Au)_{uAu} \qquad \qquad Au \otimes uA \xrightarrow{(\cdot,\cdot)} A$$

$\leftarrow$ bimodules $+$ given by multiplication in A.

$${}_{uAu}(uA)_A \qquad \qquad uA \otimes Au \xrightarrow{[\cdot,\cdot]} uAu$$

This is a Morita context because A is associative.

$[,]$ is clearly surjective, but $(,)$ is surjective iff

$$AuA = A \qquad (*)$$

AuA: minimal two-sided ideal containing u.

$(*) \Rightarrow$ Morita equivalence between A-mod and uAu-mod.

Example: B an algebra, $A = Mat(n, B)$.

$A \ni u$ an idempotent, given by $u = \left(\begin{array}{c|c} 1_B & 0 \\ \hline 0 & 0 \end{array}\right)$.

$$uAu = \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & 0 \end{array}\right) \right\} \simeq B$$

$AuA = A \implies$ condition $(*)$.

So we have proved that B is Morita-equiv to $Mat(n, B)$.

Skew-algebras & Invariant Theory.

Let Γ be a finite group, and let A be any algebra, such that Γ acts on A (by algebra automorphisms).

We can consider $A^\Gamma \subseteq A$, $A^\Gamma := \{a \in A : \Gamma a = a\}$ (Γ-invariants).

Geometric example (invariant theory).

Let $\Gamma \circlearrowright M \leftarrow$ a manifold or a variety.

Q: What is M/Γ?

For instance, let $M := K^2$ ($K = \mathbb{R}, \mathbb{C}$). and $\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\Gamma \times M \to M$, $-1 \cdot x \mapsto -x$

then M/Γ is a cone (w/singularities!).

In algebraic geometry we consider invariant functions.

Let M be an affine variety. Let $\mathcal{O}(M)$ = algebra of functions of M.

(for $M = K^2$, $\mathcal{O}(M) = k[X, Y]$).

We define $M/\Gamma = \mathrm{Spec}\left(\mathcal{O}(M)^\Gamma\right)$.

In other words, $\mathcal{O}(M/\Gamma) = \mathcal{O}(M)^\Gamma$

In our example, $\mathcal{O}(M/\Gamma) = k[X, Y]^\Gamma = k[X^2, Y^2, XY] = k[t, s, u]/(ts = u^2)$

which is the algebra of functions of the cone (in K^3) $(ts = u^2)$.

In the general case, we want to understand A^Γ (for instance, find its modules).

It is hard to find A^Γ explicitly, but instead we can consider the skew algebra (of A and Γ)

Def: The skew product of A and Γ, $A \# K\Gamma$ (or $A\#\Gamma$) is an algebra given by $A \otimes_K K\Gamma$ as a K-vectorspace, with the multiplication given by: $(a \otimes g)\cdot(b\otimes h) := a \cdot b^g \otimes gh$ (b^g = action of g on b).

The modules of $A\#\Gamma$ are A-modules and Γ-modules simultaneously, compatible with the multiplication.

We want to know the relation between A^Γ and $A\#\Gamma$. We would like a Morita equivalence between them.

We can try to find an idempotent to set the Morita equivalence up:

On $K\Gamma$, it is easy to find an idempotent: $u = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g$.

So consider $1_A \otimes u \in A\#\Gamma$.

Note: $uh = hu = u$, for $h \in \Gamma$.
$ua = a^u$, so that $A^u = A^\Gamma$.

Then, $u(A\#\Gamma)u = uA^\Gamma u = A^\Gamma u^2 = A^\Gamma u \cong A^\Gamma$

So u produces a pair of $A\#\Gamma - A^\Gamma$-bimodules, which produce a Morita equivalence, provided that it is perfect, i.e.:

$(A\#\Gamma)u(A\#\Gamma) \overset{?}{=} (A\#\Gamma)$.

This is a condition on the action of Γ on A and it is not always satisfied.

An example: in the algebraic geometry context, $\Gamma \curvearrowright M$, $\Gamma \curvearrowright \mathcal{O}(M)$.

Then $\left(\mathcal{O}(M) \# \Gamma\right) \mathcal{M} \left(\mathcal{O}(M) \# \Gamma\right) = \left(\mathcal{O}(M) \# \Gamma\right)$

iff the Γ-action on M is free (orbits are of size $|\Gamma|$).

E.O.C

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Math506 Representation Theory

Instructor: Anton Malkin
Place/Time: MWF 10:00-10:50 in 347AH
Office Hours: Wed 15:00-16:00 in 334IH

Syllabus: Representation theory and cohomology of finite groups and associative algebras. We start with finite group representations in complex vector spaces and study structures of the group algebra and the ring of characters. We then discuss basic concepts in the theory of associative algebras and their modules, and finally apply this general theory in the case of the group algebra of a finite group over a field of positive characteristic.

Books: The course roughly covers the content of J.-P. Serre, *Linear Representations of Finite Groups*. However we will not follow Serre's book very closely. The first half of the course is covered in any basic algebra textbook such as S. Lang, *Algebra*, or N. Jacobson, *Basic Algebra* (which also has a bit of associative algebras). The references for the second half are D.J. Benson, *Representations and Cohomology*, and Ch.W. Curtis, I. Reiner, *Methods of Representation Theory* (we'll only study a small part of the content of these last two books).

Final grade: based on homework.

Lectures/Homework: