Riemann Surfaces

Non-compactness

Let $f(x,y) = \sum_{m+n \leq d} A_{mn} x^m y^n$ of degree d.

Claim: $f^{-1}(0)$ is never compact (if it's nonempty).

Pf Look at the intersection with $x=a$ and $y=b$.

$x=a$: $f(a,y) = \sum_{m+n \leq d} A_{mn} a^m y^n = 0 \iff \sum_{n \leq d} P_n(a) y^n$

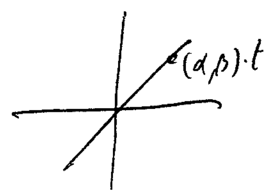
If all $P_n(a)$ are 0, then has solution for any y.

If some of these are $\neq 0$, then it's a poly. in y of deg $\leq d \Rightarrow$ has zeros.

Be careful with details, but the proof essentially works.

Tricky intersection theory

Look at $\{f(x,y)=0\} \cap \{$lines through origin$\}$.



$f(t\alpha, t\beta) = 0 = \sum A_{mn} \alpha^m \beta^n t^{n+m} = 0$

is a poly. in t $\left(= \sum_{k \leq d} P_k(\alpha,\beta) t^k\right)$.

Note that for some α, β, $P_d(\alpha,\beta) = P_{d-1}(\alpha,\beta) = \dots = P_{e+1}(\alpha,\beta) = 0$

and so will have only e points of intersection ($e < d$).

These are the reasons why we work in compact settings: $\mathbb{P}^2$.

• $\mathbb{P}^2$ (or $\mathbb{CP}^2$)

We define $\mathbb{P}^2 = \{\text{lines through } 0 \text{ in } \mathbb{C}^3\}$.

Label a line by $[X,Y,Z] = [\lambda X, \lambda Y, \lambda Z]$ for any $\lambda \neq 0$.

So $\mathbb{P}^2 = \mathbb{C}^3 \setminus \{0\} / \sim \; = \mathbb{C}^3 \setminus \{0\} / \mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}^3 \setminus \{0\}$ by scaling.

We can give $\mathbb{P}^2$ the quotient topology, coming from $\mathbb{C}^3 \setminus \{0\}$.

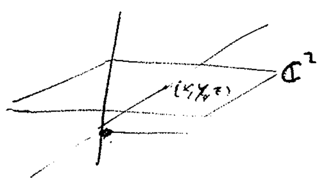
Compactness

Using the norm in $\mathbb{C}^3$, each class in $\mathbb{P}^2$ has a representative $[X,Y,Z]$ s.t. $|X|^2+|Y|^2+|Z|^2=1$.

So get a $\overset{\text{cont.}}{\text{map}}$ $\underset{\cap \mathbb{R}^6}{S^5} \xrightarrow{\pi} \mathbb{P}^2 \Rightarrow \mathbb{P}^2$ is compact (as S^5 is).

$\pi^{-1}([X,Y,Z]) = S^1$, and so $\mathbb{P}^2 \simeq S^5/S^1$

Also, $i: \mathbb{C}^2 \hookrightarrow \mathbb{P}^2$, by



and $i(\mathbb{C}^2) = \{[X,Y,Z] \mid Z \neq 0\}$

Note also $\mathbb{P}^2 \setminus i(\mathbb{C}^2) = \{[X,Y,0] \mid (X,Y) \in \mathbb{C}^2 \setminus \{0\}\} \simeq \mathbb{P}^1$. ← called the "line at infinity"

Also, if we name $\ell_{[\alpha,\beta]}$ the line in $\mathbb{C}^2$ through 0 passing by (α,β), then in $i(\mathbb{C}^2)$, we get the line $[t\alpha, t\beta, 1] = [\alpha, \beta, \frac{1}{t}] \xrightarrow[t\to\infty]{} [\alpha,\beta,0]$. (point at infinity)

Key observation:

If $F(X,Y,Z) = \sum_{\ell+m+n} A_{\ell m n} X^\ell Y^m Z^n$ is homog. of degree d, then $F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X,Y,Z)$,

so $F^{-1}(0) \subseteq \mathbb{C}^3$ is invariant under the action of $\mathbb{C}^*$, and hence we can use it to define a subspace of $\mathbb{P}^2$.

So define $C_F := \{[x,y,z] \in \mathbb{P}^2 \mid F(x,y,z) = 0\}$.

Of course, we can define a subspace of $\mathbb{C}^2$ by a polynomial $f(x,y)$ of degree d (not necessarily homogeneous).

Given $f(x,y)$, define $F(X,Y,Z)$ s.t. $f(x,y) = F(X,Y,1)$. (homogenization)

(hom.)

Conversely, define $f(x,y) := F(X,Y,1)$, which will be of degree d provided that Z is not a factor of F.

So get: $\left\{ \begin{array}{c} \text{polynomials } f(x,y) \\ \text{of deg } d \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{hom polynomials } F(X,Y,Z) \\ \text{of deg } d, \text{ s.t. } Z \text{ is } \underline{\text{not}} \text{ a factor} \end{array} \right\}$

$$C_f = \{(x,y) \mid f(x,y) = 0\} \Rightarrow i(C_f) = \{[X,Y,1] \mid f(x,y) = 0\} \underset{f(x,y) = F(X,Y,1)}{=} \{[X,Y,1] \mid F(X,Y,1) = 0\}$$

$$\Rightarrow \boxed{i(C_f) = i(\mathbb{C}^2) \cap C_F}$$

What's missing is $C_F \cap \mathbb{P}^1$ (line at infinity) $= \{[X,Y,0] \mid F(X,Y,0) = 0\}$.

(~~Homework~~ Set 1): a) $\#(C_F \cap \mathbb{P}^1) \leq d$ (or else $C_F \subset \mathbb{P}^1$).

b) $[\alpha,\beta,0] \in C_F \cap \mathbb{P}^1 \Leftrightarrow \#(\ell_{[\alpha,\beta]} \cap C_f) < d$.

Can we have $C_f = C_g$?

(a) If $f(x,y) = \lambda g(x,y)$

(b) If $f(x,y) = g(x,y)^n$ $n > 0$

More generally:

(b') if $f(x,y) = f_1(x,y)^{n_1} \cdots f_\ell(x,y)^{n_\ell}$, $g(x,y) = f_1(x,y)^{m_1} \cdots f_\ell(x,y)^{m_\ell}$

then their zero-sets are the same.

• Hilbert Nullstellensatz:

Says that (a), (b') are the only ways to achieve $C_f = C_g$.

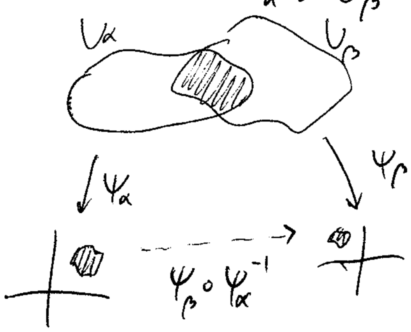
(i.e. if $g(x,y)$ vanishes everywhere on $f(x,y)=0$ and $f(x,y)$ is irreducible, then $f \mid g$)

Also, note that if $f = f_1 \cdots f_\ell$, then $C_f = \bigcup_{i=1}^{\ell} C_{f_i}$.

Riemann Surfaces (complex manifolds of $\dim_{\mathbb{C}} = 1$).

Def: A Riemann surface Σ is the giving of:

- A topological space. (Hausdorff, connected, 2AA).
- An open cover $\{U_\alpha\}$ for Σ, such that $\varphi_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto its image.
- Such that on $U_\alpha \cap U_\beta$



$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ ($\subset \mathbb{C}$, $\subset \mathbb{C}$)

is a biholomorphism (holomorphic w/ holomorphic inverse).

Examples

1) $\mathbb{C}$.

2) $\mathbb{P}^1 = \{\text{lines through } 0 \text{ in } \mathbb{C}^2\} = \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^\times = S^3 / S^1$

Open cover:

$U_0 = \{[z_0, z_1] : z_0 \neq 0\}$ $\quad U_1 = \{[z_0, z_1] : z_1 \neq 0\}$. $\quad (U_0 \cup U_1 = \mathbb{P}^1)$

$\varphi_0 : U_0 \to \mathbb{C}$ $\quad$ $\varphi_1 : U_1 \to \mathbb{C}$ $\quad$ (well-defined!)

$[z_0, z_1] \mapsto \frac{z_1}{z_0}$ $\quad$ $[z_0, z_1] \mapsto \frac{z_0}{z_1}$

$[1, z] \leftarrow z$ $\quad$ $[z, 1] \leftarrow z$

$\varphi_1 \circ \varphi_0^{-1}(z) = \varphi_1([1, z]) = \frac{1}{z}$ is a biholomorphism in $\mathbb{C} \setminus \{0\} = \varphi_\alpha(U_0 \cap U_1)$ $= \varphi_\beta(U_0 \cap U_1)$.

//

We will see in the exercises that, as a real manifold, $\mathbb{P}^1 \simeq S^2$.

We can generalize this construction to n dimensions.

In $\mathbb{P}^n = \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^\times$, $U_i = \{[z_0,\dots,z_n] \mid z_i \neq 0\}$.

$\varphi_i : U_i \longrightarrow \mathbb{C}^n$

$[z_0,\dots,z_n] \longmapsto \left(\frac{z_0}{z_i},\dots,\hat{1},\dots,\frac{z_n}{z_i}\right)$ and so on.

- Complex tori: Let $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$,

Let $\Lambda = \{m\omega_1 + n\omega_2 \mid m,n \in \mathbb{Z}\}$.

Let then $\Sigma := \mathbb{C}/\Lambda$, i.e. $z \sim z' \iff z - z' \in \Lambda$

In Σ we have a topology induced by that of $\mathbb{C}$ (quotient topology)

It can be given a structure of a Riemann surface (see problems).

As a real smooth manifold, $\Sigma \simeq T^1 = S^1 \times S^1$

Def We say that M is a smooth manifold of $\dim_{\mathbb{R}} = n$ if:

a) M is a topological space, with

b) Cover $\{U_\alpha\}$ and maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ (homeo onto their images).

c) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are $\mathcal{C}^\infty$-diffeomorphism.

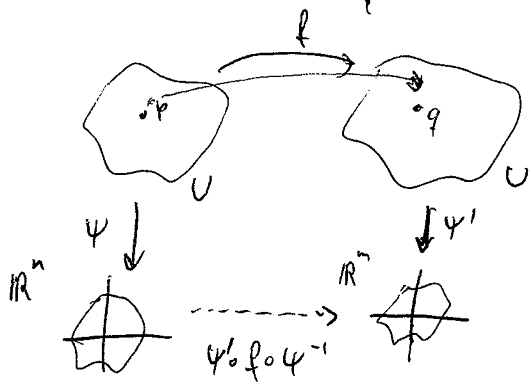
Maps between manifolds: ($\mathcal{C}^\infty$).

$f : M \longrightarrow M'$

a) f continuous map.

b) $\varphi' \circ f \circ \varphi^{-1}$ is $\mathcal{C}^\infty$

(for any choice of local coordinates).



When we say that $\Sigma \sim M$ we mean that they are diffeomorphic as smooth manifolds.

- Properties of Σ as a real manifold

Σ is a complex curve (surface). To see it as a real manifold, we have to:

- first: identify $\mathbb{C} \cong \mathbb{R}^2$ (as sets) $x+iy \mapsto (x,y)$.
- second: check that the new atlas is really an atlas, i.e. the change of coordinates are $\mathcal{C}^\infty$. (In fact, we get real analytic maps, which in particular are smooth).

Moreover, if $f(z)$ is holomorphic, $f(z) = u(z) + iv(z)$

then $f(x,y) = u(x,y) + i\,v(x,y)$ and have f hol $\Leftrightarrow$ $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$ (C-R eq's).

So $\mathrm{Jac}(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}$

$\Rightarrow |J(f)| = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \geq 0$. In fact, $|J(f)| > 0$ because otherwise it wouldn't be invertible. This is a special feature then that the coordinate transformations have. Define $\Psi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$, and

$\mathbb{R}^2 \xrightarrow{J(\Psi_{\beta\alpha})} \mathbb{R}^2$ is a linear + positive det $\Rightarrow$ orientation preserving.

<u>Conclusion</u>: Σ <u>as a real manifold</u> is <u>orientable</u>.

Hence, Closed Riemann Surfaces $\sim$ closed, orientable real surfaces $\leftarrow$ classified by their genus.

On a R.S Σ, $J(\Psi_{\beta\alpha}) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ as a 2×2 matrix, which is in $GL(2,\mathbb{R})$. Moreover, it's a homothety followed by a rotation.

In fact $\overset{\mathbb{C}^\times}{GL(1,\mathbb{C})} \hookrightarrow GL(2,\mathbb{R})$ under the identification, and $J(\Psi_{\beta\alpha})$ belongs actually to the image of $\mathbb{C}^\times$.

- Geometric significance.

~~We say~~ Suppose that $J(\varphi_{\beta\alpha}) \in SO(2) = \{ M \mid MM^T = I, \det M = +1 \}$

(note that in fact $J(\varphi_{\beta\alpha}) \, J(\varphi_{\beta\alpha})^T = (A^2+B^2) \cdot I$, so it's close to that)

$M \in SO(2) \Leftrightarrow M$ preserves the inner product (i.e. $\forall v, w \in \mathbb{R}^2$, $\langle Mv, Mw \rangle = \langle v, w \rangle$).

So if ~~on~~ a manifold all coordinate transformations lie in $SO(2)$, then one can define an inner product on the whole manifold.

What we have instead is that

$$J(\varphi_{\beta\alpha}) \in \{ M \mid MM^T = \lambda^2 I \text{ for some } \lambda \in \mathbb{R}^{\times} \} = CO(2) \quad \text{(Conformal group)}.$$

These matrices preserve angles: $\dfrac{\langle Mv, Mw \rangle}{|Mv|\,|Mw|} = \dfrac{\langle v, w \rangle}{|v|\,|w|}$.

This is called a conformal structure (can measure angles on Σ, but not lengths).

The tori $\mathbb{C}/\Lambda$ differ then in their conformal structure.

Holomorphic/Meromorphic functions/maps:

Recall that in $\Omega \subseteq \mathbb{C}$ a domain, f hol $\Leftrightarrow \dfrac{\partial f}{\partial \bar z} = 0$. Also,

f hol $\Leftrightarrow$ f is complex analytic (i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$).

Recall also the maximum principle: $\max_{\Omega} |f| = \max_{\partial \Omega} |f|$.

Consequence: Suppose Σ is a closed Riemann ~~[illegible]~~ surface, $f : \Sigma \to \mathbb{C}$ hol function. Σ being compact, f attains a maximum at some $p \in \Sigma$, then f is constant! So $Hol(\Sigma, \mathbb{C}) \simeq \mathbb{C}$. $\leftarrow$ (if Σ is connected)

Define: $Hol(\Omega) = \{$hol. functions on $\Omega\}$.

$K(\Omega) = \{$meromorphic functions on $\Omega\}$.

• Meromorphic functions:

Def $f: \Omega \to \mathbb{C} \cup \{\infty\}$ such that

a) $f|_{\Omega \setminus f^{-1}(\infty)}$ is holomorphic. (in $\Omega \setminus f^{-1}(\infty)$).

b) if $a \in f^{-1}(\infty)$, then $\exists$ nbhd $U \ni a$ s.t $f(z) = \frac{g(z)}{(z-a)^N}$ where $\begin{cases} g \in Hol(U) \\ g(a) \neq 0 \\ N > 0 \end{cases}$.

we say that a is a pole of order N.

Note: (b) $\Rightarrow$ $f^{-1}(\infty)$ is a discrete set (without any accumulation points).

(in particular, if Ω is compact, then $f^{-1}(\infty)$ is finite).

Meromorphic functions on Σ.

Say $\{U_\alpha\}$ is an open cover of Σ (atlas), $\psi_\alpha : U_\alpha \to \mathbb{C}$ local coordinates.

Def A map $f: \Sigma \to \mathbb{C}$ is holomorphic if all local representatives are. $(f \circ \psi_\alpha^{-1})$

Equivalently: A holomorphic function $f \in Hol(\Sigma)$ is a collection $\{f_\alpha\}_\alpha$

where $f_\alpha \in Hol(\psi_\alpha(U_\alpha))$ s.t $f_\alpha \circ \psi_\alpha = f_\beta \circ \psi_\beta$ on overlaps.

(because we can define $f(p) = f_\alpha \circ \psi_\alpha(p)$ for $p \in U_\alpha$).

Def $f \in K(\Sigma)$ (is a meromorphic function) is a collection of $\{f_\alpha\}_{\alpha}$

st $f_\alpha \in K(\psi_\alpha(U_\alpha))$ and $f_\alpha \circ \psi_\alpha = f_\beta \circ \psi_\beta$

Note: $K(\Sigma)$ is a field (under usual $+, \cdot$ —pointwise—).

• If Σ is closed and connected then $Hol(\Sigma) \simeq \mathbb{C}$

Example: $K(\mathbb{P}^1)$. Recall $\mathbb{P}^1 = U_0 \cup U_1$, $\begin{cases} U_0 = \{[z_0 : z_1] : z_0 \neq 0\} \\ U_1 = \{[z_0 : z_1] : z_1 \neq 0\} \end{cases}$

Take $\frac{p(z)}{q(z)}$ (a rational function). Let $F_0(z) = \frac{p(z)}{q(z)}$. Then $F_1(z) = F_0(1/z) =$

$= \frac{p(1/z)}{q(1/z)}$ is also meromorphic.

Prop: All meromorphic functions on $\mathbb{P}^1$ are of this form, i.e. $K(\mathbb{P}^1) \simeq \mathbb{C}(z)$.

Pf

Let $F_0(z)$ be meromorphic in $\mathbb{C}$. Say with zeros and poles $\{a_1, \dots, a_N\}$ with multiplicities $\{m_1, \dots, m_N\}$ ($m_i > 0 \Leftrightarrow a_i$ zero, $m_i < 0 \Leftrightarrow a_i$ pole).

Let $G_0(z) = \prod (z-a_i)^{m_i}$. Then $\frac{F_0(z)}{G_0(z)}$ has no zeros nor poles

(if $\frac{F_0(z)}{G_0(z)}$ is bounded, then Liouville $\rightarrow$ constant $\Rightarrow F_0(z) = c\prod (z-a_i)^{m_i}$).

The corresponding function $F(z) \in K(\mathbb{P}^1)$ is $F_0(z)$ in one patch, and

$F_1(z) = F_0(1/z)$ in the other.

Define now $G_1(z) = G_0\left(\frac{1}{z}\right)$ which is also a rat. function on $\mathbb{C}$.

So $G(z) \in K(\mathbb{P}^1)$.

Looking now at $\frac{F(z)}{G(z)} \in K(\mathbb{P}^1)$

Claim: $\frac{F(z)}{G(z)} = ct.$

Pf

Let $h_0(z) = \frac{F_0(z)}{G_0(z)}$, $h_1(z) = \frac{F_1(z)}{G_1(z)} = \frac{F_1(\frac{1}{z})}{G_1(\frac{1}{z})} = h_0\left(\frac{1}{z}\right)$.

Now $h_0(z)$ is hol. and has no zeros. Let $h_0(z) = \sum_{n\geq 0} c_n z^n$.

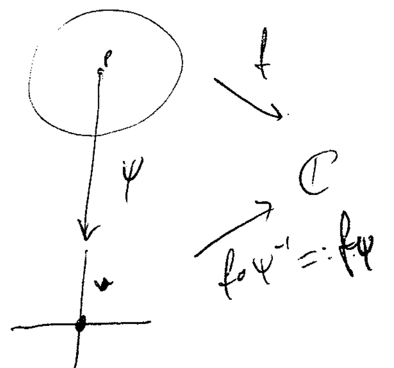
But $h_1(z) = \sum_{n=0}^{\infty} c_n z^{-n}$ is meromorphic, so it has a pole of finite order at 0,

so actually $c_n = 0$ for $n \geq n_0$. Hence $h_0(z)$ is a polynomial.

If $\deg h_0(z) \geq 1$, then $h(z)$ has a zero $\Rightarrow$!!. So $h_0(z)$ is constant. //

- **Poles of $f \in K(\Sigma)$** (and zeros!).

Suppose f has a pole at $p \in \Sigma$.



WLOG suppose $\psi(p) = 0 \in \mathbb{C}$.

Then $f_\psi(z) = z^{\nu} h(z)$ for some $\nu < 0$, and some holomorphic $h(z)$, $h(0) \neq 0$.

Claim: ν is independent of the coordinates (st $\psi(p)=0$).

Pf

$f_\psi(z) = z^{\nu} h(z)$

$f_{\tilde\psi}(\tilde z) = \tilde z^{\tilde \nu} \tilde h(\tilde z)$

$\tilde z(z) = \tilde\psi \circ \psi^{-1}(z)$ is holomorphic, invertible and $\tilde z(0) = 0$. Moreover, $\tilde z'(0) \neq 0$ (invertible)

So $\tilde z(z) = z \cdot \varphi(z)$, φ hol, $\varphi(0) \neq 0$.

So $f_{\tilde\psi}(\tilde z(z)) = f \circ \tilde\psi^{-1} \circ \tilde z(z) = (z\varphi(z))^{\tilde\nu}\, \tilde h(z\varphi(z)) =$

$$= z^{\tilde\nu} \cdot \underbrace{\varphi(z)^{\tilde\nu}\, \tilde h(z\varphi(z))}_{= h(z)} \quad //$$

Similarly if f has a zero at $p \in \Sigma$.

Def If f has a zero or pole at $p \in \Sigma$, then $\nu_p(f)$ is called the order (or multiplicity) of f at p and is st $f_\psi = z^{\nu_p(f)} h(z)$.

Also, if f has no zero or pole at p, $\nu_p(f) = 0$.

We will be considering, for $f \in K(\Sigma)$, the quantity $\sum_{p \in \Sigma} \nu_p(f)$ (finite sum!)

• Maps $f:\Sigma\to\Sigma'$

$Hol(\Sigma,\Sigma')$: ?

Def: $f:\Sigma\to\Sigma'$ is holomorphic at $p\in\Sigma$ if $f_{\psi\psi'}$ is holomorphic at $\psi(p)$.

$$\begin{tikzcd}
\Sigma \supset U \ni p \arrow[r, "f"] \arrow[d, "\psi"] & U' \ni f(p) \subset \Sigma' \arrow[d, "\psi'"] \\
\mathbb{C} \arrow[r, dashed, "\underbrace{\psi'\circ f\circ\psi^{-1}}_{=\, f_{\psi\psi'}}"'] & \mathbb{C}
\end{tikzcd}$$

As we did before, we could think of $f\in Hol(\Sigma,\Sigma')$ to be a collection of local maps $\{f_{\psi\psi'}:\psi(U)\to\psi'(U')\}$ s.t. they agree on overlaps.

Example: $Hol(\Sigma,\mathbb{C}) = Hol(\Sigma)$. ($\mathbb{C}$ thought as a R.S.).

Example: $Hol(\mathbb{C},\mathbb{P}^1) = K(\mathbb{C})$. In fact, we will see that $Hol(\Sigma,\mathbb{P}^1) = K(\Sigma)$.

Use the coord patches U_0, U_1.

$F\in Hol(\mathbb{C},\mathbb{P}^1)$. F is F_0, F_1. So $F^{-1}(U_i)$ is an open in $\mathbb{C}$.

So
$$\begin{tikzcd}
F^{-1}(U_i) \arrow[r, "F"] \arrow[rr, bend right, "F_i"'] & U_i \arrow[r, "\psi_i"] & \mathbb{C}
\end{tikzcd}$$
and $\cancel{F_1(z)=F_0(1/z)}$

$$F_1(z)=\psi_1\circ\psi_0^{-1}(F_0(z)) = \frac{1}{F_0(z)}$$

Let $P := \{z : F(z)=[0,1]\}$. So $F^{-1}(U_0)=\mathbb{C}\setminus P$, and in there,

$\psi_0\circ F_1$ is holomorphic

If $a\in P$, $F(a)=[0,1]\in U_1$, hence $F_1(z)$ is hol. at $z=a$, with value ~~$F(a)$~~

$F_1(a)=\psi_1(F(a))=\psi_1([0:1])=0$. Hence $F_1(z)=(z-a)^{\nu}h(z)$ with $h(a)\neq 0$.
(in a nbhd of a, at least)

So $F_0(z)=\dfrac{1}{F_1(z)}=(z-a)^{-\nu}\dfrac{1}{h(z)}$. Hence $F_0\in K(\mathbb{C})$.

Conversely, if $f\in K(\mathbb{C})$, define $F:\mathbb{C}\to\mathbb{P}^1$ by $F_0(z)=f(z)$, $F_1(z)=\dfrac{1}{f(z)}$ //

Exercise: Generalize this to $K(\Sigma) \leftrightarrow Hol(\Sigma, \mathbb{P}^1)$.

Now, let $f \in Hol(\Sigma, \Sigma')$.

Say $f(p) = q$. Taking φ, φ' charts with $\varphi(p) = 0$, $\varphi'(q) = 0$,

then $f_{\varphi\varphi'}$ is a hol. map taking $0 \mapsto 0$. So $f_{\varphi\varphi'}(z) = z^\nu h(z)$, $h(z) \neq 0$, h hol.

We can check that ν is still well-defined:

Prop: • ν is independent of choices ($\varphi(p) = 0 = \varphi(q)$).

• $\nu \geq 1$

Def: We define $\nu - 1$ to be the ramification index $R_p(f)$.

We want to compare $R_p(F)$ for $F \in Hol(\Sigma, \mathbb{P}^1)$ with $\nu_p(f)$ for $f \in K(\Sigma)$.

$$\Sigma \xrightarrow{F} \mathbb{P}^1 = \overset{\ni 0}{\mathbb{C}} \cup \{\overset{[0,1]}{\infty}\}$$

$$R_p(F) = \begin{cases} -\nu_p(f) - 1 & \text{if } F(p) = \infty \text{ (i.e. } p \text{ is a pole)} \\ \nu_p(f) - 1 & \text{if } F(p) = 0 \text{ (ie } p \text{ is a zero)} \\ 0 \ (= \nu_p(f)) & \text{o.w.} \end{cases}$$

Note: $\{p \in \Sigma : \nu_p(f) \neq 0\}$ is discrete. So if Σ compact, it is finite.

So $\sum_{p \in \Sigma} \nu_p(f)$ is a finite sum (in fact, it's 0).

Special case:

$\Sigma = \mathbb{P}^1$. $K(\mathbb{P}^1) = \left\{ \frac{p(z)}{q(z)} : p, q \text{ polynomials (with no common factor)} \right\}$.

Claim: $K(\mathbb{P}^1) = \left\{ \frac{p(z_0, z_1)}{q(z_0, z_1)} \text{ where } p, q \text{ are polys. homogeneous of the same degree (with no common factor)} \right\}$.

The corresponding function they define is $F([z_0, z_1]) = \dfrac{p(z_0, z_1)}{q(z_0, z_1)}$

Suppose now that F is given by $\frac{p(z_0,z_1)}{q(z_0,z_1)}$.

So zeros of F = zeros of p.
poles of F = ~~poles~~ of q.

Recall (from HW 1) if p is hom. of deg d, $p(z_0,z_1) = \prod_{i=1}^{N}(b_i z_0 - a_i z_1)^{e_i}$ $\left(\sum_{i=1}^{N} e_i = d\right)$.

So the zeros of p are at $[a_i, b_i]$.

The multiplicities are: $\nu_{[a_i;b_i]}(f) = e_i$ (exercise).

Doing the same with q (and the poles), get $\sum_p \nu_p(f) = 0$.

Now we go for the general case:

Recall that $K(\Sigma) = Hol(\Sigma, \mathbb{P}^1)$.

Under this correspondence, $f \in K(\Sigma) \sim F \in Hol(\Sigma, \mathbb{P}^1)$.

$\{$zeros of $f\}$ = $\{$zeros of $F\}$ ∈ ~~$F([1,$~~ $F(p) = [1,0]$ $\Rightarrow$ $\{F^{-1}([1,0])$.

$\{$poles of $f\}$ = $F^{-1}([0,1])$.

Also, $\nu_p(f) > 0 \sim R_p(F) = \nu_p(f) - 1$.
$\nu_p(f) < 0 \sim R_p(F) = -\nu_p(f) - 1$.

So $\sum \nu_p(f) = \sum_{p\in F^{-1}([1:0])} (R_p(F)+1) - \sum_{p \in F^{-1}([0:1])} (R_p(F)+1)$. Everything follows from:

<u>Theorem</u>: For $F \in Hol(\Sigma, \Sigma')$, let for $q \in \Sigma'$, $N(q) = \sum_{F(p)=q} (R_p(F)+1)$

(note $N(q)$ well defined because the sum is finite).

<u>Then</u> $N(q)$ is <u>constant</u>.

<u>Rk</u>: If $R_p(F) = 0$ $\forall p$ s.t. $F(p)=q$, $N(q)$ is counting the number of points in $F^{-1}(q)$.
In general, it counts this with multiplicities.

Rk: This proves then that $\sum_p \nu_p(f) = 0$

Rk: For any $F \in Hol(\Sigma, \Sigma')$, if F is nonconstant then it is surjective.

Pf of theorem:

Given $F: \Sigma \to \Sigma'$, nonconstant consider $\Sigma'_n := \{q \in \Sigma' \mid N(q) \geq n\}$.

$\Sigma'_0 = \Sigma' \supseteq \Sigma'_1 \supseteq \Sigma'_2 \supseteq \cdots$

We will show that Σ'_n is both open and closed (and hence $\Sigma'_n = \begin{cases}\Sigma \\ \emptyset\end{cases}$ or.

This will imply that if, for some N, $\Sigma'_{N+1} = \emptyset$, then $\Sigma' = \Sigma'_N$ and the theorem will follow.

Open: Let $q \in \Sigma'_n$.

Claim: $F^{-1}(q)$ is discrete: at any $p \in F^{-1}(q)$, can pick coordinates on Σ, Σ', s.t. $z(p) = 0 = z'(q)$.

In these coords, F is given by $z \mapsto z^\mu$ (in exercises will see that can assume $h(z) = 1$).

(and $R_p(F) = \mu - 1$).

On these neighborhoods, $z=0$ is the only preimage of $z'=0$. // claim

So $F^{-1}(q) = \{p_1, \dots, p_\ell\}$ (Σ, Σ' are assumed to be connected + closed)

Then $\sum_{i=1}^{\ell} (R_{p_i}(F) + 1) \geq n$.

Pick U_i nbhds of p_i, V_i nbhds of q s.t. on these F is given by $z \mapsto z^{\mu_i}$.

Take $V = \bigcap V_i$ (still an open nbhd, because finite int).

Claim: $\forall q' \in V$, $N(q') \geq n$.

Pf For each U_i, the map $z \mapsto z^{\mu_i}$ is μ_i-to-1 outside 0, so for $q' \neq 0$, $F^{-1}(q') \cap U_i$ has ~~at~~ exactly μ_i points. (exercise).

So if $a^{\mu} = z'$, $R_a(F) = 0$. Summing over all p_i, get the claim.

↓

(cont proof):

So in $F^{-1}(q') \cap \left(\bigcup_{i=1}^{\ell} U_i \right)$, $\sum_{F(p')=q'} \left(R_p(F)+1 \right) \geq \sum_{i=1}^{\ell} \left(R_{p_i}(F)+1 \right) \geq n$.

Hence, for $q' \in V$, $N(q') \geq n \Rightarrow$ open.

Closed:

Take $q \in \overline{\Sigma_n'}$. (closure). want to show that $q \in \Sigma_n'$.

Take $\{q_i\}$ a sequence in Σ_n', $\lim q_i = q$. (we know $N(q_i) \geq n$).

Want that $N(q) \geq n$.

As there are only finitely many ramified points, we can assume that the q_i's are s.t. $R_p(F) = 0$ $\forall\, p \in F^{-1}(q_i)$.

Take a sequence $\{p_i\}$ s.t. $F(p_i) = q_i$. Passing to a subsequence, we get $\lim_i p_i = p$, ~~&~~ $F(p) = q$ by continuity of F.

We can do this at least n times getting n different sequences.

But what if $\lim_i p_i = \lim \tilde{p}_i'$?

Suppose then that $\tilde{p}_1 = \dots = \tilde{p}_m = \tilde{p}$ $\left(\tilde{p}_i =: \lim_k p_{ik},\ P_i = \{p_{ik}\}_k,\ P_i \cap P_j = \emptyset \right)$.

Want to show that then $R_{\tilde{p}}(f) + 1 \geq m$.

Let U be a nbhd of $\tilde{p}$ and local coordinates s.t. the map is $z \mapsto z^{\mu}$ (for some μ). Will show that $\mu \geq m$.

But eventually all the sequences will be contained in U, and $z \mapsto z^{\mu}$ is (outside 0) a μ-to-1 map, so $\mu \geq m$.

- Holomorphic & Meromorphic differentials. (1-forms) on Σ.

On $\mathbb{C}$: expressions of the form $f(z)\cdot dz$ where f is either hol. or meromorphic.

Key features:

- appear in integrals: $\int f(z)\,dz$. If $\gamma:[a,b]\to\mathbb{C}$ is a path, $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$.

- change of variables: $z=\phi(w)$ with ϕ biholomorphic:

$$\int f(z)\,dz = \int f(\phi(w))\left(\frac{\partial\phi}{\partial w}\right)dw = \int \tilde{f}(w)\,dw$$

with the relation $\tilde{f}(w) = f(\phi(w))\,\frac{\partial\phi}{\partial w}$

(Recall: for functions, we required that $\tilde{f}(w) = f(\phi(w))$, only.)

Define: A global 1-form on Σ:

- Fix an open cover $\{(U_\alpha,\psi_\alpha)\}$ for Σ.
- local 1-forms $\omega_\alpha = f_\alpha(z_\alpha)\,dz_\alpha$ (z_α are the coords in $\psi_\alpha(U_\alpha)$).

Such that (compatibility):

- On overlapping charts $U_\alpha\cap U_\beta$ where $z_\beta = \psi_{\beta\alpha}(z_\alpha)$ $\left(\psi_{\beta\alpha} = \psi_\beta\circ\psi_\alpha^{-1}\right)$

$$f_\alpha(z_\alpha) = f_\beta\left(\psi_{\beta\alpha}(z_\alpha)\right)\cdot\left(\frac{\partial\psi_{\beta\alpha}}{\partial z_\alpha}\right)(z_\alpha).$$

Interpretation: this local data glues together to give a global section of a (nontrivial) line-bundle over Σ. (The holomorphic cotangent bundle).

Suppose ω is a global 1-form on Σ (defined locally by ω_α).
We can define $\int_\gamma \omega$, where γ is a path in Σ:

- Easy case: a path γ contained in <u>one</u> coord. chart:

  $\gamma : [a,b] \longrightarrow U_\alpha \subseteq \Sigma$.

  Then define $\displaystyle\int_\gamma \omega := \int_{\psi_\alpha \circ \gamma} \omega_\alpha$

  <u>Exercise</u>, does not depend on the choice of the U_α.

- Harder case

  $\gamma([a,b])$ is not contained in any U_α.

  Partition $[a,b]$ as $t_0 = a < t_1 < \dots < t_{n+1} = b$ such that $\gamma([t_i, t_{i+1}]) \subseteq U_i$

  ($\gamma([a,b])$ is compact!).

  Then $\displaystyle\int_\gamma \omega = \sum_i \int_{\gamma_i} \omega$ // (indep of the choices, again).

<u>One Source of examples</u>

Take $f \in K(\Sigma)$, say $f = \{(U_\alpha, \psi_\alpha, f_\alpha)\}$ ← local data $\quad (f_\alpha \in K(\psi_\alpha(U_\alpha)))$.

Define $\omega_\alpha := \left(\frac{\partial f_\alpha}{\partial z_\alpha}\right) d z_\alpha =: d f_\alpha$.

<u>Claim</u>: $\{(U_\alpha, \psi_\alpha, \omega_\alpha)\}$ defines a global meromorphic 1-form ω (on Σ)

$$\frac{\partial f_\alpha}{\partial z_\alpha} = \frac{\partial f_\beta}{\partial z_\beta}(\psi_{\beta\alpha}(z)) \cdot \frac{\partial \psi_{\beta\alpha}}{\partial z_\alpha} \quad : \text{because } RHS = \frac{\partial}{\partial z_\alpha}\left(f_\beta(\psi_{\beta\alpha}(z_\alpha))\right) = \frac{\partial}{\partial z_\alpha}\left(f_\alpha(z_\alpha)\right) \;/\!/$$

We write $\omega = df$.

<u>Notation</u>: $K^1(\Sigma) = \{1\text{-forms on } \Sigma\}$. (1-dim manifolds ⇒ there are <u>no</u> 2-forms, 3-forms, ⋮)

Ex: on $\mathbb{P}^1$, write out explicit formulae for at least one $\omega \in \mathcal{K}^1(\mathbb{P}^1)$.

Remark: 1) in general, not all $\omega \in \mathcal{K}^1(\Sigma)$ are of this form (df).

2) Given $\omega \in \mathcal{K}^1(\Sigma)$ and $f \in \mathcal{K}(\Sigma)$, then can define $f\omega$:

Locally, if f is represented by f_α, and ω by ω_α,

define $(f\omega)_\alpha = f_\alpha \omega_\alpha$ (i.e. $(f\omega)_\alpha(z_\alpha) = f_\alpha(z_\alpha)\cdot\omega_\alpha(z_\alpha)$).

Exercise: $f\omega$ is in fact in $\mathcal{K}^1(\Sigma)$.

3) Suppose $\omega, \mu \in \mathcal{K}^1(\Sigma)$, then one can define $\frac{\mu}{\omega} \in \mathcal{K}(\Sigma)$.

Locally, if $\omega_\alpha = f_\alpha(z_\alpha)\,dz_\alpha$, $\mu_\alpha = g_\alpha(z_\alpha)\,dz_\alpha$, define $\left(\frac{\mu}{\omega}\right)_\alpha(z_\alpha) := \frac{g_\alpha(z_\alpha)}{f_\alpha(z_\alpha)}$

well defined in $\mathcal{K}(\Sigma)$!

• Important Special case:

$f \in \mathcal{K}(\Sigma) \rightsquigarrow \frac{df}{f} \in \mathcal{K}^1(\Sigma)$.

• Zeros/poles & multiplicities for $\omega \in \mathcal{K}^1(\Sigma)$:

If locally ω is $\{\omega_\alpha\}$, $\omega_\alpha(z_\alpha) = f_\alpha(z_\alpha)\cdot dz_\alpha$, then $f_\alpha(z_\alpha) \in \mathcal{K}^1(\psi_\alpha(U_\alpha))$

so it has zeros and poles.

Def: $\nu_p(\omega) := \nu_{\psi_\alpha(p)}(f_\alpha)$

Lemma: it is well-defined: $\nu_{\psi_\alpha(p)}(f_\alpha) = \nu_{\psi_\beta(p)}(f_\beta)$ if $p \in U_\alpha \cap U_\beta$.

Pf: $\psi_{\beta\alpha}(z)$ is biholomorphic $\Rightarrow \frac{\partial \psi_{\beta\alpha}}{\partial z_\alpha}$ is nonzero $\Rightarrow$ ✓.

//

Note: $\nu_p(\omega) = \begin{cases} <0 & \text{at poles} \\ >0 & \text{at zeros} \\ 0 & \text{o.w.} \end{cases}$ and $\nu_p(\omega) = 0$ almost everywhere.

Rk: We'll see that $\sum_{p \in \Sigma} \nu_p(\omega) \neq 0$!!.

Examples:

1) Recall if $f \in K(\Sigma)$. Then $\omega = \frac{df}{f}$ is a 1-form.

Exercise: show $\nu_p(\omega) \geq -1 \ \forall p \in \Sigma$

2) Take $f \in K(\mathbb{P}^1)$, say $f_0(z) = z^n$ (in U_0). (i.e. $f([z_0 : z_1]) = \frac{z_1^n}{z_0^n}$).

We compute $\sum_{p \in \Sigma} \nu_p(df)$.

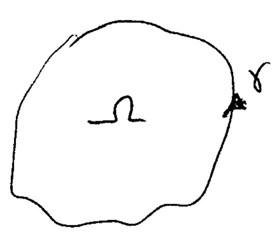
On U_0: $df_0 = n z^{n-1} dz$ has $\begin{cases} \text{no poles} \\ 1 \text{ zero with multiplicity } n-1. \\ (z=0) \end{cases}$ $\boxed{\nu_{[1:0]}(df) = n-1}$

on U_1: $f_1(z) = z^{-n} \Rightarrow df_1 = -n z^{-n-1} dz \Rightarrow \begin{cases} \text{no zeros} \\ \text{pole at } z=0, \ \nu_{[0:1]}(df) = -(n+1) \end{cases}$

So $\sum_{p \in \Sigma} \nu_p(df) = -2$ (indep. of n).

(We will see that, in general, $\sum_{p \in \mathbb{P}^1} \nu_p(df) = -\chi(\underset{S^2 \cong \mathbb{P}^1}{S^2})$ (Euler char.).)

On $\Omega \subset \mathbb{C}$, $\overline{\Omega}$ compact, Ω open:



Let $\gamma = \partial\Omega$ (with orientation).

Let $\omega \in K(\Omega) \cap Hol(\partial\Omega)$)

Then $\int_\gamma \omega = 2\pi i \cdot \sum_{p \in \Omega} Res_p(f)$

In particular, if $\Omega \in Hol(\overline{\Omega})$, then $\int_\gamma \omega = 0$.

For $\Omega \in K'(\Sigma)$:

Easy case: if $\Omega \subset \Sigma$, $\bar{\Omega}$ compact and $\bar{\Omega} \subset U$ (U one of the coord patches).

Assume ω is holomorphic nbhd of $\bar{\Omega}$.

Then $\int_{\partial\bar{\Omega}} \omega = 0$

General: For any $\Omega \subset \Sigma$, $\bar{\Omega}$ compact, can take a finite cover of coord. patches, and get the same result.

Residues: Suppose $\omega \in K'(\Sigma)$ has a pole at $p \in \Sigma$.

Take a loop around p, small enough such that $\gamma \subseteq U$ ← coord patch.

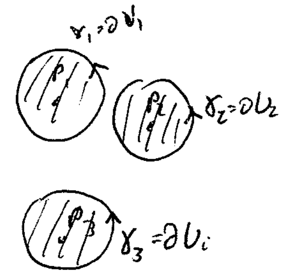
and s.t. p is the only pole inside the region.

Define $Res_p(\omega) = \frac{1}{2\pi i}\int_\gamma \omega$.

Rk: This does not depend on γ: $\int_{\gamma_1}\omega - \int_{\gamma_2}\omega = \int_{\gamma_1-\gamma_2}\omega = 0$ ← hol in there.

Then we get that, for any region $\Omega \subset \Sigma$, $\gamma = \partial\bar{\Omega}$,

$$\int_\gamma \omega = 2\pi i \sum_{p\in\Omega} Res_p(\omega).$$



Suppose now that in Σ, $\omega \in K'(\Sigma)$ with poles at $\{p_1, \dots, p_N\}$.

$(\gamma_i = \partial U_i)$.

Let $\Omega = \Sigma - \bigcup_{i=1}^N \bar{U_i}$. Ω is open, $\partial\bar{\Omega} = -\sum_{i=1}^N \gamma_i$

and ω is holomorphic on Ω. So $0 = \int_{\partial\bar{\Omega}} \omega = -\sum_{i=1}^N \int_{\gamma_i}\omega = -2\pi i \sum_{i=1}^N Res_{p_i}(\omega)$.

Conclusion: $\sum_{p\in\Sigma} Res_p(\omega) = 0$. (for Σ closed!)

Special case: $\omega = \frac{df}{f}$, $f \in K^*(\Sigma)$.

The poles of ω: at $p \in \Sigma$ where $\nu_p(f) \neq 0$ (the only candidates).

i.e. $z \mapsto z^{\nu} h(z)$, $h(0) \neq 0$. $(\nu \neq 0)$.

Then $df = \nu z^{\nu-1} h(z) + z^{\nu} h'(z)$, and $\frac{df}{f} = \frac{\nu}{z} + \left(\frac{h'(z)}{h(z)}\right)$ holomorphic

i.e. p is a pole (simple), and $\mathrm{Res}_p(f) = \nu$.

So $\sum_{p \in \Sigma} \mathrm{Res}_p\left(\frac{df}{f}\right) = \sum_{f(p)=0} \nu_p(f) + \sum_{f(p)=\infty} \nu_p(f) = \sum_{p \in \Sigma} \nu_p(f)$.

The previous result implies that $\sum_{p \in \Sigma} \nu_p(f) = 0$. (we knew that before, actually).

Holomorphic differentials on $\mathbb{P}^1$:

Say $\omega \in K'(\mathbb{P}^1)$ is actually <u>holomorphic</u>, and $\omega \neq 0$.

Define $F: \mathbb{P}^1 \to \mathbb{C}$ by $F(p) := \int_{C(p_0,p)} \omega$ $\quad C(p_0,p)$ = curve from p_0 to p

It does not depend on the curve chosen, because $\mathbb{P}^1$ is simply connected.

F is well-defined, holomorphic, and $F'(z) = f(z)$ if $\omega(z) = f(z)\,dz$

(in fact, $dF = \omega$).

As F hol. in $\mathbb{P}^1 \Rightarrow F = ct. \Rightarrow \omega = d(ct) = 0$. $\Rightarrow$!!

Conclusion: there are no (nonzero) holomorphic differentials on $\mathbb{P}^1$.

Recall:

$\omega \in K'(\Sigma)$:	$f \in K(\Sigma)$	$F \in \mathrm{Hol}(\Sigma, \Sigma')$
$\sum \mathrm{Res}_p(\omega) = 0$ $\sum \nu_p(\omega) = -\chi(\Sigma) = 2-2g$	$\sum \nu_p(f) = 0$	$\sum_{f(p)=q} (R_p(f)+1) = N(q) = \mathrm{const.}$

Definition: For $F \in Hol(\Sigma, \Sigma')$, degree of F, $\deg F := N(q)$ (at any q).

It's the number of points in $F^{-1}(q)$ for a generic point.

Riemann-Hurwitz formula:

$$2g(\Sigma) - 2 = (\deg F)(2g(\Sigma') - 2) + \sum_{p \in \Sigma} R_p(F)$$

We want to prove this formula, and understand better the $\left(\sum_{p \in \Sigma} \nu_p(\omega) = -\chi(\Sigma) \right)$.

Differential Forms:

On $\mathbb{R}^n$, let $x_1, \ldots, x_n$ be the coordinates.

Def A 1-form on $\mathbb{R}^n$ is an expression of the type $\alpha = \sum_{i=1}^n \alpha_i(x) dx_i$,

with $\alpha_i(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Example: if $f \in \mathcal{E}^\infty(\mathbb{R}^n)$, define $df(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p dx_i$.

These are linear maps on the space of tangent vectors:

$\alpha(p) : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$ linear

where $T_p\mathbb{R}^n = \{$ tangent/direction vectors at $p \}$.

To v we associate the directional derivative D_v which is a derivation on

$\mathcal{C}^\infty(U)$, U a nbhd of p. $\left(D_v(f) = \frac{d}{dt} f(p+tv) \Big|_{t=0} \right) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} v_i$

So $D_v(f) = \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\Big|_p \right)(f)$. And thus $T_p\mathbb{R}^n = \mathbb{R}\left\{ \frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p \right\}$.

If $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$, then $dx_i = \left(\frac{\partial}{\partial x_i} \right)^*$.

Behavior under $\phi: \mathbb{R}^n \to \mathbb{R}^m$.

- induces a map $\phi^*: \mathcal{C}^\infty(\mathbb{R}^m) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$
$$f \longmapsto f \circ \phi$$

- induces a map $\phi_*: T_p\mathbb{R}^n \longrightarrow T_{\phi(p)}\mathbb{R}^m$ (linear map)

  wrt $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ and $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\}$ has matrix: $\left(\frac{\partial \phi_i}{\partial x_j}\right)$

  Without coordinates: if $\gamma(t)$ is a path giving $[\gamma]_p$ a directional derivative, then know that $[\gamma]_p(f) = \frac{d}{dt}\left(f(\gamma(t))\right)_{|t=0}$.

  So $\phi_*[\gamma]_p(f) = \frac{d}{dt} f\left(\phi(\gamma(t))\right)_{|t=0}$.

- induces a map (by dualizing): $\phi^*: T^*_{\phi(p)}\mathbb{R}^m \longrightarrow T^*_p\mathbb{R}^n$.

  where $\phi^*(\alpha)(v) = \alpha(\phi_* v)$.

  If $\phi_* = \mathrm{Jac}(\phi)$ wrt $\{\frac{\partial}{\partial x_i}\}$, $\{\frac{\partial}{\partial y_j}\}$

  then $\phi^* = \mathrm{Jac}(\phi)^T$ wrt $\{dx_i\}$, $\{dy_j\}$.

  (i.e. $\phi^*(dy_i) = \sum \frac{\partial \phi_i}{\partial x_j} dx_j$).

  For $\alpha(y) = \sum \alpha_i(y)\, dy_i$, then $(\phi^*\alpha)(x) = \sum \alpha_i(\phi(x))\, \phi^*(dy_i)$

1-forms on a manifold M:

Given an open cover $\{U_\alpha\}_{\alpha \in I}$ with $\psi_\alpha: U_\alpha \to \mathbb{R}^n$ homeo onto image s.t.

$\psi_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}: \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ are diffeomorphisms.

A 1-form on M can be given as a collection $\{(U_\alpha, \psi_\alpha, \omega_\alpha)\}$ where ω_α is a 1-form on $\psi_\alpha(U_\alpha)$: $\omega_\alpha(x_\alpha) = \sum_{i=1}^n f_\alpha^i(x_\alpha)\, dx_\alpha^i$, s.t. $\psi_{\beta\alpha}^* \omega_\beta = \omega_\alpha \ \forall \alpha, \beta$.

Exercise: Write this in coordinates, and recover our earlier formula.

If we replace $\mathbb{R}^{2n}$ by $\mathbb{C}^n$:

Call the coords. in $\mathbb{R}^{2n}$ by $(x_1, \dots, x_n, y_1, \dots, y_n) \longleftrightarrow (\underbrace{x_1+iy_1}_{z_1}, \dots, \underbrace{x_n+iy_n}_{z_n})$

$(\mathbb{R}^2 \simeq \mathbb{C}, \ (x,y) \longleftrightarrow x+iy = z)$.

Let $\Omega^1(\mathbb{R}^2, \mathbb{R}) = \{\mathbb{R}\text{-valued } 1 \text{ forms on } \mathbb{R}^2\}$.

At $p \in \mathbb{R}^2$, $\Omega^1_p(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}\{dx, dy\} = T_p^*\mathbb{R}^2$.

Take the complexification $\Omega^1_p(\mathbb{R}^2, \mathbb{C}) = \Omega^1_p(\mathbb{R}^2, \mathbb{R}) \otimes \mathbb{C} = \mathbb{C}\{dx, dy\}$.

So $\omega \in \Omega^1(\mathbb{R}^2, \mathbb{C})$ is given by: $f(x,y)\,dx + g(x,y)\,dy$, $f, g : \mathbb{R}^2 \to \mathbb{C}$.

Let now $z := x+iy$, i.e. then $dx + i\,dy =: dz$, $dx - i\,dy = d\bar{z}$.

In $\Omega^1_p(\mathbb{R}^2, \mathbb{C})$ we can change the basis to $\{dz, d\bar{z}\}$ (instead of $\{dx, dy\}$).

Write $f\,dx + g\,dy = \lambda\,dz + \mu\,d\bar{z}$, where

$$\lambda = \frac{f - ig}{2}, \quad \mu = \frac{f+ig}{2}.$$

Write the form as $\underbrace{\lambda(z,\bar{z})\,dz}_{\text{type } (1,0)} + \underbrace{\mu(z,\bar{z})\,d\bar{z}}_{\text{type } (0,1)}$

<u>Note</u>: $\lambda(z,\bar{z}) = \dfrac{f(x,y)}{2} - i\,\dfrac{g(x,y)}{2} = u(x,y) + i\,v(x,y)$

λ is holomorphic $\Leftrightarrow$ (u,v) satisfy CR, $\Leftrightarrow$ $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

then $\dfrac{\partial \lambda}{\partial \bar{z}} = 0$

In that case, get a hol. 1-form $\lambda(z)\,dz$ i.e.

a <u>Complex</u> form of type $(1,0)$ with holomorphic coefficient function.

Real 1-forms

$\phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\phi(x,y) = (\phi_1(x,y), \phi_2(x,y)) = (x', y')$.

This gives:

$\phi^* : T^*_{\phi(p)}\mathbb{R}^2 \to T^*_p \mathbb{R}^2$, given by $\begin{pmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} \end{pmatrix}$ w.r.t. $\{dx', dy'\}$, $\{dx, dy\}$

Complexify: $\phi^* : (T^*_{\phi(p)}\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow (T^*_p \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$

which is given by the same matrix, regarded as a 2x2 ~~w~~ complex entries.

Change basis to $dz = dx + i dy$, $d\bar{z} = dx - i dy$.

Then $(T^*_{\phi(p)}\mathbb{R}^2) \otimes \mathbb{C}$ decomposes in $T^{*(1,0)}_{\phi(p)} \oplus T^{*(0,1)}_{\phi(p)}$

In this new basis, ϕ^* is given by $\begin{pmatrix} \frac{\partial \Phi}{\partial z} & \frac{\partial \Phi}{\partial \bar{z}} \\ \frac{\partial \bar{\Phi}}{\partial z} & \overline{\left(\frac{\partial \Phi}{\partial z}\right)} \end{pmatrix}$ where $\Phi = \phi_1 + i\phi_2$, $\bar{\Phi} = \phi_1 - i\phi_2$

If ϕ is <u>holomorphic</u> (as a map from $\mathbb{C} \to \mathbb{C}$), then $\frac{\partial \Phi}{\partial \bar{z}} = 0$.

Also, $\frac{\partial \bar{\Phi}}{\partial z}$ is so as well (it's the conjugate of it) and ~~in~~ this case

$\phi^* = \begin{pmatrix} \frac{\partial \Phi}{\partial z} & 0 \\ 0 & \overline{\left(\frac{\partial \Phi}{\partial z}\right)} \end{pmatrix}$ and so it preserves the (1,0) and (0,1) parts.

If $\omega(z') = \lambda(z')\, dz'$ is holomorphic, then

$\phi^*(\omega) = \lambda(\phi(z)) \frac{\partial \Phi}{\partial z} dz$

and so we recover the rule we had before: $\omega_\alpha(z_\alpha) = \omega_\beta(\psi_{\beta\alpha}(z_\alpha)) \frac{\partial \psi_{\beta\alpha}}{\partial z_\alpha}$.

Path integrals

Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$, $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$

and let $\omega \in \Omega^1(\mathbb{R}^n, \mathbb{R})$, say $\omega(\underline{x}) = \sum_{i=1}^{n} f_i(\underline{x})\, dx_i$

Then $\gamma^* \omega = \Big(\sum f_i(\gamma(t)) \underbrace{\frac{d\gamma_i}{dt}}_{\dot{\gamma}_i} \Big) dt$

Hence can define:

$$\int_{\gamma([a,b])} \omega = \int_a^b \gamma^* \omega .$$

Now, given $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\omega \in \Omega^1(\mathbb{R}^m)$, $\gamma : [a,b] \to \mathbb{R}^n$.

$$\int_\gamma \phi^* \omega = \int_{\phi(\gamma)} \omega$$

We wanted to show that $\sum_{p \in \Sigma} \nu_p(\omega) = -\chi(\Sigma)$

For that, we will interpret $\nu_p(\omega)$ in terms of real 1-forms, on the underlying closed oriented surface, S $\quad (S \text{ “}=\text{” } \Sigma)$.

Given $\{(U_\alpha, \psi_\alpha, \omega_\alpha)\}$, $\omega_\alpha(z_\alpha) = f_\alpha(z_\alpha)\, dz_\alpha$, then <u>define</u> $\mathrm{Re}(\omega)$ given by $\{(U_\alpha, \psi_\alpha^{\mathbb{R}}, \mathrm{Re}(\omega_\alpha))\}$ $\quad \left(\psi_\alpha^{\mathbb{R}} = \text{composition } U \xrightarrow{\psi_\alpha} \mathbb{C} \longrightarrow \mathbb{R}^2, \; \underset{x+iy}{z} \mapsto (x,y) \right)$.

If $\omega_\alpha(z_\alpha) = f_\alpha(z_\alpha)\, dz_\alpha$ and $f_\alpha = u_\alpha + i v_\alpha$, $dz_\alpha = dx_\alpha + i\, dy_\alpha$,

then $\mathrm{Re}(\omega_\alpha) = u_\alpha\, dx_\alpha - v_\alpha\, dy_\alpha$.

Suppose $\omega_\alpha(z_\alpha) = z_\alpha^{\nu}$ in the vicinity of a zero or pole. $\left(\begin{matrix}\nu = \nu_p(\omega) \\ p \in U_\alpha \\ z_\alpha(p) = 0\end{matrix}\right)$

Then $z_\alpha = r_\alpha e^{i\theta_\alpha}$ $(z_\alpha \neq 0)$.

and $\mathrm{Re}(\omega_\alpha) = r_\alpha^{\nu}\left(\cos(\nu\theta_\alpha)\, dx_\alpha - \sin(\nu\theta_\alpha)\, dy_\alpha\right)$ $(*)$

Notice $\mathrm{Re}(\omega_\alpha)$ is smooth everywhere except at ~~a finite number of points~~. $z_\alpha = 0$,

so $\mathrm{Re}(\omega)$ is smooth everywhere except at a finite number of points.

(In particular, isolated set of singularities).

Def: At each of the singularities (0 or ∞) p of a 1-form η, the <u>index</u> of η at p is:

Choose coordinates s.t. $x(p) = y(p) = 0$.

$\eta(x,y) = a(x,y)\, dx + b(x,y)\, dy$

On a suitable neighborhood U, of p (i.e. 0 of $\mathbb{R}^2$), get a map:

$$U \setminus \{p\} \longrightarrow \mathbb{R}^2 \setminus \{0\}.$$
$$q \longmapsto (a(q), b(q))$$

which leads to
$$U \setminus \{p\} \xrightarrow{\psi} S^1$$
$$q \longmapsto \left(\frac{a(q)}{\sqrt{a+b}}, \frac{b(q)}{\sqrt{a+b}}\right)$$

Can take $\psi^{-1}(S^1)$, and restrict ψ to it. Then get

$S_r \longrightarrow S^1$ $\rightsquigarrow$ can define the <u>winding number</u>. ← index.

(To compute it for $g: S^1 \to S^1$, the winding number

is ~~defined~~ computed by $\dfrac{1}{2\pi}\displaystyle\int_{S^1} g^*(d\theta)$)

Claim: For $\eta = \mathrm{Re}(\omega)$, with $\omega(z) = z^\nu dz$, then its winding number is $-\nu$.

Pf. Note that $\eta(r,\theta) = r^\nu((\cos\nu\theta)dx - (\sin\nu\theta)dy) = r^\nu(\cos(\nu(-\theta))dx + \sin(\nu(-\theta))dy)$

and so we will get a "$-$" sign.

Def. If η has a singularity at 0, then the index at 0 of η is its winding number. ~~(and otherwise it is 0)~~

Hence:

$$\sum_{p\in\Sigma} \nu_p(\omega) = -\sum_{\substack{\text{singularities} \\ \text{(zeros or poles)}}} \mathrm{Index}_p(\mathrm{Re}(\omega))$$

Poincare-Hopf Theorem: (no proof in this course).

For S a real orientable surface which is compact and closed, and for η a 1-form with isolated singularities at $\{p_1, \ldots, p_n\}$, then $\sum_{i=1}^n \mathrm{Index}_{p_i}(\omega) = \chi(S)$.

Riemann-Hurwitz formula: $F : \Sigma \to \Sigma'$

$$2g(\Sigma) - 2 = (\deg F)(2g(\Sigma') - 2) + \sum_{p\in\Sigma} R_p(F).$$

Rk: For any closed oriented real surface S, we can compute $\chi(S)$ from a triangulation of S (i.e. a decomposition of S into triangular regions s.t. any two triangles are either disjoint, or they intersect on a single vertex, or they intersect on an edge).

Fact: 1) Any Riemann surface admits a triangulation.

2) $\chi(\Sigma) = \#v - \#e + \#f$ is independent of the chosen triangulation.

• Proof of Riemann-Hurwitz:

Def A branch point of F is $q \in \Sigma'$ s.t $\sum_{f(p)=q} R_p(F) > 0$.

Construct a triangulation of Σ' such that all branch points are vertices.

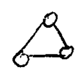
Say the set of vertices is $\{P_1', \dots, P_N'\} \cup \{P_{N+1}', \dots, P_M'\}$ $\quad (\# V' = V_b' + V_0')$

$\uparrow$ branch points

Say also that there are e' edges and f' faces.

Consider the preimages under F:

Except at $\{P_1', \dots, P_N'\}$, F is a $(\deg F)$-fold cover.

So for each $\Delta \setminus$ vertices , $F^{-1}(\Delta)$ consists of $(\deg F)$ copies of the same.

In the resulting triangulation of Σ (F is continuous), $e = (\deg F)\cdot e'$, $f = (\deg F) f'$.

At each of the P_i' $\quad i = 1..N$ (branch points):

$$\deg F = \sum_{f(p)=P_i'} \left(R_p(F) + 1 \right) = \sum_{f(p)=P_i'} R_p(F) + \# F^{-1}(P_i').$$

Hence $\# F^{-1}(P_i') = \deg F - \sum_{f(p)=P_i'} R_p(F)$

So $$v = \sum_{i=1}^{N} \left(\deg F - \sum_{f(p)=P_i'} R_p(F) \right) + \sum_{i=N+1}^{M} \deg F = (\deg F)\cdot v - \sum_{p \in \Sigma} R_p(F)$$

And so the result follows from $2g(\Sigma) - 2 = -\chi(\Sigma)$.

Application: $F : \Sigma \to \mathbb{P}^1$, $g(\mathbb{P}^1) = 0$. As $d = \deg F \geq 1$,

$$2g(\Sigma) - 2 = d(-2) + \sum_p R_p(F) \Rightarrow \sum_p R_p(F) = 2(d + g - 1) \geq 2g.$$

In particular, for $g > 0$ there are always branch points.

- **The space $\mathbb{P}^n$ (revisited).**

$\mathbb{P}^n = \{\text{lines thru } 0 \text{ in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1}\setminus\{0\} / \mathbb{C}^\times$.

Let $U_i = \{[z_0 : \dots, z_n] : z_i \neq 0\}$. $\quad \psi_i : U_i \longrightarrow \mathbb{C}^n$

$$[z_0, \dots, z_n] \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right).$$

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_j"] & \\ \mathbb{C}^n\setminus\{0\} \arrow[rr, "\psi_j\circ\psi_i^{-1}", "\frac{z_i}{z_j}"'] & & \mathbb{C}^n\setminus\{0\} \end{tikzcd}$$

Claim: $\psi_j \circ \psi_i^{-1}(w_1, \dots, w_n)$ is biholomorphic in $w_1, \dots, w_n$

(i.e. in each of the variables separately).

(i.e. $\frac{\partial}{\partial \overline{w_i}}(\psi_j \circ \psi_i^{-1}) = 0$)

This gives $\mathbb{P}^n$ the structure of a complex manifold of (complex) dimension n.

Def $F : \mathbb{P}^n \to \mathbb{P}^n$ is holomorphic at $p \in \mathbb{P}^n$ if it is locally.

$$\begin{tikzcd} U \ni p \arrow[r, "F"] \arrow[d, "\psi"'] & U' \ni F(p) \arrow[d, "\psi'"] \\ \psi(U) \subseteq \mathbb{C}^n \arrow[r, "\psi'\circ F\circ\psi^{-1}"'] & \psi(U') \subseteq \mathbb{C}^n \end{tikzcd}$$

Example:

$$\begin{tikzcd} \mathbb{C}^{n+1}\setminus\{0\} \arrow[r, "T"] \arrow[d] & \mathbb{C}^{n+1}\setminus\{0\} \arrow[d] \\ \mathbb{P}^n \arrow[r, dashed, "?"] & \mathbb{P}^n \end{tikzcd}$$

if T is linear, i.e. $T(\lambda p) = \lambda T(p)$, then can define $T([z_0 : \dots : z_n]) = [T(z_0, \dots, z_n)]$.

Locally: Say $[z_0 : \dots : z_n] \in U_0$ $(z_0 \neq 0)$.

Q: When is $[T(z_0, \dots, z_n)] \in U_0$? . Suppose $T = (T_{ij})_{i,j}$.

Then $T(z_0, \dots, z_n) = (T_{ij})\begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix} = \left(\sum_{i=1}^{n} T_{ki} z_i\right)_{k=0\dots n}$. So $[T(z_0, \dots, z_n)] \in U_0$

$$\Updownarrow$$

$$\sum_{i=0}^{n} T_{0i} z_i \neq 0.$$

(cont example)

Assume that $T([z_0,..,z_n]) \in U_0$, actually. Use local coordinates $\psi_0 : U \to \mathbb{C}^n$ on both sides.

Then
$$\begin{array}{ccc} [z_0,\dots,z_n] & \longrightarrow & \left[\sum T_{0i} z_i, \dots, \sum T_{ni} z_i\right] \\ \psi_0 \downarrow & & \downarrow \psi_0 \\ \left(\frac{z_1}{z_0},\dots,\frac{z_n}{z_0}\right) & \dashrightarrow & \left(\frac{\sum T_{1i} z_i}{\sum T_{0i} z_i}, \dots, \frac{\sum T_{ni} z_i}{\sum T_{0i} z_i}\right) \end{array}$$
(sums from $0..n$).

Call $w_i := \frac{z_i}{z_0}$. Then the local map is:

$$\frac{\sum T_{ki} z_i}{\sum T_{0i} z_i} = \frac{T_{k0} + \sum_{i=1}^{n} T_{ki} w_i}{T_{00} + \sum_{i=1}^{n} T_{0i} w_i}$$

Special case: $n=1$ ($\mathbb{P}^1 \to \mathbb{P}^1$). Get $w \longmapsto \left(\frac{T_{10} + T_{11} w}{T_{00} + T_{01} w}\right)$.

Def One such map is called a linear transformation of $\mathbb{P}^n$.

($F : \mathbb{P}^n \to \mathbb{P}^n$ s.t. F comes from a linear $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$).

If T is invertible, then F has an inverse, so F is a linear automorphism of $\mathbb{P}^n$.

Fact: Every automorphism of $\mathbb{P}^n$ is of this form.

We'll see the case $n=1$: $F : \mathbb{P}^1 \to \mathbb{P}^1$ hol, invertible.

Claim: $F([z_0,z_1]) = [T(z_0,z_1)]$ where $T : \mathbb{C}^2 \to \mathbb{C}^2$ is invertible linear transformation.

Pf

1) Reduce to the case that F fixes $[1,0]$ and $[0,1]$:

Let $P_0 = F([0,1])$. $P_0 = [P_{00}, P_{01}]$
$P_1 = F([1,0])$. $P_1 = [P_{10}, P_{11}]$

As F is injective, $P_0 \neq P_1 \Rightarrow \{(P_{00}, P_{01}), (P_{10}, P_{11})\}$ are l.i. in $\mathbb{C}^2$.

$$\mathbb{P}^1 \xrightarrow{F} \mathbb{P}^1 \xrightarrow{S} \mathbb{P}^1$$
$$[0,1] \longmapsto P_0 \longmapsto [0,1]$$
$$[1,0] \longmapsto P_1 \longmapsto [1,0]$$

If S comes from $S: \mathbb{C}^2 \to \mathbb{C}^2$ we're done and S is invertible linear.

$$(P_{00}, P_{01}) \to (0,1)$$
$$(P_{10}, P_{11}) \to (1,0)$$

2) Prove the result for G fixing $[1,0]$ and $[0,1]$.

$G: \mathbb{P}^1 \to \mathbb{P}^1$. $\deg G = 1$ (injective)

Note that $G(U_0) \subseteq U_0$. In local coordinates,

$g(z) \left(= \varphi_0 \circ G \circ \varphi_0^{-1}(z)\right) = \dfrac{p(z)}{q(z)}$ because g has to be meromorphic on $\mathbb{C}$.

But g cannot have any poles, because they correspond to points in U_0 where the image is not in U_0. Hence $q(z) = ct$.

Also, 0 is a zero of $p(z)$ (because G fixes $[1,0]$). The degree of this is 1, so $p(z) = c \cdot z$. Hence $g(z) = c \cdot z$.

(i.e. $G([1,z]) = [1, cz] \Rightarrow G([z_0, z_1]) = [z_0, c z_1] = \left[\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix}\right]$. //

<u>Proof of Riemann-Hurwitz, second version</u>

Using meromorphic differentials: $\omega \in K'(\Sigma')$. Take $\omega \in K'(\Sigma)$ ($\exists$ guaranteed by Riemann-Roch, but not proven so far).

Get $F^*\omega \in K'(\Sigma)$. Recall that for $\omega \in K'(\Sigma')$, $\sum_{q \in \Sigma'} \nu_q(\omega) = -\chi(\Sigma')$.
↑ Poincaré-Hopf.

Say $F(p) = q$. Fix coordinates z, z' s.t. $z(p) = 0 = z'(q)$.

Then $F(z) = z^\mu$, $R_p(F) = \mu - 1$.

Write $\omega(z') = g(z')\,dz'$. Then $(F^*\omega)(z) = g(F(z)) \cdot \frac{\partial F}{\partial z}\,dz = \mu g(z^\mu) z^{\mu-1}\,dz$

From this, ~~$\nu_{F^*\omega}(p) = \mu \cdot \nu_\omega(q) + (\mu - 1) = R$~~

$$\nu_p(F^*\omega) = \mu\,\nu_q(\omega) + (\mu - 1) = (R_p(F) + 1)\,\nu_q(\omega) + R_p(F).$$

Summing for all $p \in \Sigma$:

$$2g(\Sigma) - 2 = \sum_{p \in \Sigma} \nu_p(F^*\omega) = \sum_{F(p)=q}\ \sum_{q \in \Sigma'} (R_p(F) + 1)\,\nu_q(\omega) + \sum_{p \in \Sigma} R_p(F) = (\deg F) \cdot \sum_{q \in \Sigma'} \nu_q(\omega) + \sum_{p \in \Sigma} R_p(F)$$ //

Terminology: $F: \Sigma \to \Sigma'$.

- $p \in \Sigma$ s.t. $R_p(F) > 0$ is called a Ramification Point.
- $q \in \Sigma'$ s.t. $\exists p \in F^{-1}(q)$ with $R_p(F) > 0$ is called a branch point.

If F has no branch points, then it is a $(\deg F)$-fold cover.

Ex: $g(\Sigma) = g(\Sigma') = 1 \Rightarrow \sum R_p(F) = 0$.

Ex: $\Sigma' = \mathbb{P}^1$, $g(\Sigma') = 0$. Then $\sum R_p(F) = 2(\deg F + g - 1) \geq 2g > 0$ (unless $g = 0$).

$\deg F = 1$: isom $\Sigma \simeq \mathbb{P}^1$.

$\deg F = 2$: If Σ admits $F: \Sigma \to \mathbb{P}^1$, $\deg F = 2$ then Σ is called hyperelliptic.

(Algebraic) Geometry of $\mathbb{P}^n$:

1) Given $\{P_0, \dots, P_n\}$ $n+1$ points in $\mathbb{P}^n$, under which conditions can one find $T: \mathbb{P}^n \to \mathbb{P}^n$ linear (i.e. coming from a linear transformation $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$) s.t. $P_i \mapsto [0, \dots, \underset{\hat{i}}{1}, \dots, 0]$.

We need that the points $\{P_0, \dots, P_n\}$ are independent. (i.e the vectors in $\mathbb{C}^{n+1}$ corresponding to them are l.i.). (check: does not depend on representatives).

2) Given $\{P_0, \dots, P_{n+1}\}$ $n+2$ points in $\mathbb{P}^n$, we can often find $T: \mathbb{P}^n \to \mathbb{P}^n$ which takes $P_i \mapsto [0, \dots, \underset{\hat{i}}{1}, \dots, 0]$ and $P_{n+1} \mapsto [1, 1, \dots, 1]$.

If the T from (1) takes $P_{n+1} \mapsto [z_0, \dots, z_n]$, then if we look for S another transformation fixing $[0, \dots, \underset{\hat{i}}{1}, \dots, 0]$ $\forall i$, and taking $[z_0, \dots, z_n]$ to $[1, \dots, 1]$, we will find it iff $z_i \neq 0$ $\forall i$.

- Projective geometry.

Def $V \subseteq \mathbb{P}^n$ is called a linear subspace of $\mathbb{P}^n$ if

$$V = \{p \in \mathbb{P}^n : L_1(p) = \dots = L_k(p) = 0\} \text{ where } L_i : \mathbb{C}^{n+1} \to \mathbb{C} \text{ are linear.}$$

$$= \{p \in \mathbb{P}^n : L_i(\tilde{p}) = 0 \quad i = 1..k \text{ for } \tilde{p} \text{ any representative of } p\}.$$

$$= \mathbb{P}(\tilde{V}) \text{ where } \tilde{V} \subseteq \mathbb{C}^{n+1} \text{ is linear.}$$

Note: $\tilde{V}$ is the kernel of $\mathbb{C}^{n+1} \longrightarrow \mathbb{C}^k$ given by $\mathbb{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_k \end{bmatrix}$.

If $\mathrm{rk}\, \mathbb{L} = k$ (surjection), then $\dim(\ker \mathbb{L}) = n+1-k$.

and so $\dim V = \dim \mathbb{P}(\tilde{V}) = n - k$.

- $k=1$, V is a hyperplane in $\mathbb{P}^n$, of $\dim V = n-1$.
- if $\dim V = 0$ then V is a point in $\mathbb{P}^n$.
- if $\dim V = 1$ then V is a line in $\mathbb{P}^n$.

Algebraic Subsets:

$$Z = \{p \in \mathbb{P}^n : F_\alpha(p) = 0 \text{ for } \alpha \in \underset{\text{indexing set.}}{I}\}$$

where F_α are homogeneous polynomials of deg ≥ 1 on $\mathbb{C}^{n+1}$.

Exercise: Any intersection of algebraic subsets is algebraic.
Any finite union of algebraic subsets is algebraic.

This defines a topology (Zariski top.) where the algebraic subsets are the closed subsets.

Linear independence

Def: For any $Z \subseteq \mathbb{P}^n$ subset, let $\mathrm{Span}(Z) := \bigcap \{\text{linear spaces containing } Z\}$.

Def If $Z = \{P_1, \dots, P_r\}$ is a finite collection of points in $\mathbb{P}^n$, then say that Z is <u>linearly independent</u> if $\mathrm{Span}(Z)$ has $\dim = r-1$.

<u>Alternatively</u>, if they are not all contained in any linear subspace of dimension $< r-1$.

Exercise:

(a) $\{P_1, P_2\}$ distinct points in $\mathbb{P}^n$ are always linearly independent.

(b) if L, M are linear subspaces of $\mathbb{P}^n$, then $\mathrm{span}(L \cup M)$ has dimension $\dim L + \dim M - \dim L \cap M$.

Hence, given points $\{P_0, \dots, P_n\}$ in $\mathbb{P}^n$, exists $T : \mathbb{P}^n \to \mathbb{P}^n$ taking $P_i \mapsto [0, \dots 1, \dots 0]$ iff the points are linearly independent.

General Position

Def: Say that $\{P_1, \dots, P_d\}$ in $\mathbb{P}^n$ are in <u>general position</u> if any subset of $(n+1)$ or fewer is linearly independent.

<u>Lemma</u>: Say $\{P_i = [z_0^i, \dots, z_n^i] : i = 1, \dots, n+1\}$ and all of them have $z_j^i = 0$ (some j). Then they are <u>not</u> in general position.

Pf Let $V = \{[z_0, \dots, z_n] : z_j = 0\}$ defines a hyperplane in $\mathbb{P}^n$ //.

<u>Corollary</u>: $\{e_i = [0, \dots 1, \dots, 0] \quad i = 0, \dots, n$ and $[z_0, \dots, z_n]\}$ are in general position if and only if, $z_j \neq 0 \ \forall j$.

Algebraic / affine varieties in $\mathbb{P}^n$.

$X \subseteq \mathbb{P}^n$ is an algebraic variety if $X = \{ p \in \mathbb{P}^n \mid F_i(p) = 0 \quad i = 1..k \}$

where the F_i are <u>homogeneous</u> polynomials in $(z_0, \ldots, z_n)$ of deg. d_i.

$\{F_1, \ldots, F_p\}$ generate an ideal in $\mathbb{C}[z_0, \ldots, z_n]$, call it I.

Then $X = \{ p \in \mathbb{P}^n : h(p) = 0 \ \forall h \in I \} =: X(I)$

There's a correspondence (1-1) $\{ \text{alg varieties } X \} \longleftrightarrow \{ \text{homogeneous radical ideals } I \subseteq \mathbb{C}[z_0, \ldots, z_n] \}$.

Affine varieties in $\mathbb{C}^n$:

$\tilde{X} = \{ z \in \mathbb{C}^n \mid f_1(z) = \cdots = f_k(z) = 0 \}$ where f_i is a polynomial in $(z_0, \ldots, z_n)$

Given $X \subset \mathbb{P}^n$ defined by $F_1, \ldots, F_k$,

$X \cap U_0 = \{ [z_0, \ldots, z_n] : F_i[z_0, \ldots, z_n] = 0 \text{ and } z_0 \neq 0 \} = \{ [1, x_1, \ldots, x_n] : F_i(1, x_1, \ldots, x_n) = 0 \}$

Let $f_i(x_1, \ldots, x_n) = F_i(1, x_1, \ldots, x_n)$, then $\tilde{X} = \{ z \in \mathbb{C}^n : f_i(z) = 0 \ i = 1, \ldots, k \}$

If $\varphi_0 : U_0 \to \mathbb{C}^n \quad [z_0, \ldots, z_n] \mapsto \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right)$, then

$\tilde{X} = \varphi_0 (X \cap U_0)$.

Conversely, given $\tilde{X} \subset \mathbb{C}^n$ defined by $f_1, \ldots, f_k$, define:

$$F_i(z_0, \ldots, z_n) = z_0^{d_i} f_i\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right) \quad \text{if } d_i = \deg f_i.$$

Then if $X := \{ p \in \mathbb{P}^n : F_i(p) = 0 \}$, have $\tilde{X} = \varphi_0 (X \cap U_0)$.

We get a 1-1 correspondence:

$$\left\{ \begin{matrix} X \subseteq \mathbb{P}^n \\ \text{s.t. } X \cap U_0 \neq \emptyset \end{matrix} \right\} \longleftrightarrow \{ \tilde{X} \subseteq \mathbb{C}^n \}.$$

Hypersurfaces:

$X = \{ p \in \mathbb{P}^n : F(p) = 0 \}$, $\deg F = d$

($d=1$: hyperplane)

Write $F = F_1^{m_1} \cdots F_k^{m_k}$, F_i irreducible homogeneous, $\deg F_i = d_i < d$.

Then $X = \bigcup_{i=1}^{k} C_i$, where $C_i := X((F_i))$. (as sets).

Moreover, we write $X = m_1 C_1 + \cdots + m_k C_k$.

Extreme case: $\deg F = d$, $F = \prod_{i=1}^{d} L_i$, L_i distinct linear factors.

Then $C_i = \mathbb{P}(\ker L_i : \mathbb{C}^{n+1} \to \mathbb{C}) \Rightarrow C_i$ are hyperplanes in $\mathbb{P}^n$.

Special case ($\mathbb{P}^2$)

$X = \{ p \in \mathbb{P}^2 : F(p) = 0 \}$, $\deg F = d$.

$d=1$: Then $F = aZ_0 + bZ_1 + cZ_2$, $X = \mathbb{P}(\text{hyperplane in } \mathbb{C}^3) \cong \mathbb{P}^1$

$d=2$: either $F = F_1 F_2$ or F irreducible. In the second case, $\cancel{X}(F)$ is called a conic.

Ex: $F(Z) = Z_1^2 - Z_0 Z_1$. Claim: $X \cong \mathbb{P}^1$, given by:

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ [X_0, X_1] &\mapsto [X_0^2, X_0 X_1, X_1^2] \end{aligned} \qquad ((X_0 X_1)^2 - X_0^2 - X_1^2 = 0).$$

This is called the rational normal curve.

Claim: For any irreducible F of deg 2, $\exists\, T : \mathbb{P}^2 \to \mathbb{P}^2$ linear s.t.

$T(C) = \{ p \mid Z_1^2 - Z_0 Z_2 = 0 \}$ $\quad (C = X(F))$.

For $\deg F = d \geq 2$:

if $F = \prod L_i$ each L_i distinct, $X = C_1 + \dots + C_d$, each $C_i \simeq \mathbb{P}^1$

<u>Lemma</u>: if $i \neq j$ then $C_i \cap C_j = \{pt\}$.

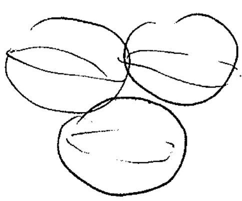
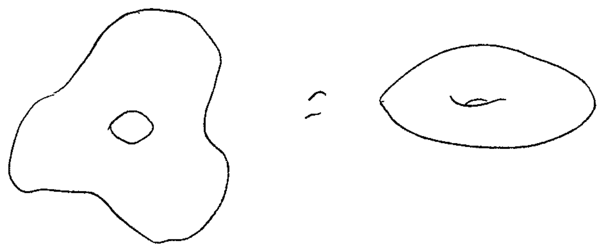
Pf Say $C_i = \mathbb{P}(\tilde{V}_i)$, $\tilde{V}_i \subset \mathbb{C}^3$ linear of dim 2.

Then $\tilde{V}_1 \cap \tilde{V}_2$ must be a <u>line</u> in $\mathbb{C}^3$.

<u>$d=1$</u>:  genus 0

<u>$d=2$</u>: 

$g = 0$.

<u>$d=3$</u>:  $=$  $g = 1$

<u>d general</u>: $\binom{d}{2}$ intersection points, i.e. $\frac{d(d-1)}{2}$.

End up getting $g = \frac{(d-1)(d-2)}{2} \left(= \frac{d(d-1)}{2} - (d-1) \right)$.

<u>Differential Geometry of Varieties.</u>

Let $X \subset \mathbb{C}^n$, $X = \{ z \in \mathbb{C}^n : f_i(z) = 0 \quad i = 1..k \}$.

<u>Q</u>: When is X a <u>submanifold</u> of $\mathbb{C}^n$?

$T_p X \subseteq T_p \mathbb{C}^n$?

We want the same also for $\mathbb{P}^n$.

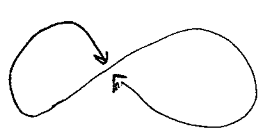
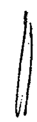
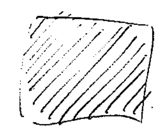
- <u>Submanifolds.</u>

Let $Y \subset X$, X a manifold.

<u>Def</u> Y is a submanifold if Y is a manifold, with the underlying topology being the subspace topology of the one in X.

(i.e. $U \subset Y$ is open $\Leftrightarrow U = V \cap Y$, V open in X).

<u>Non-examples:</u>

 $\mathbb{R} \hookrightarrow \mathbb{R}^2$  $\mathbb{R} \to \mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$
$t \mapsto [(t, t\alpha)]$ α irrational .

<u>Example:</u>
$X \subseteq \mathbb{R}^n$, X = graph of $F: \mathbb{R}^{n-k} \to \mathbb{R}^k$

$$(x_{k+1}, \dots, x_n) \longmapsto (y_1(\underline{x}), \dots, y_k(\underline{x}))$$

So $X = \{ (y_1(x_{k+1}, \dots, x_n), \dots, y_k(x_{k+1}, \dots, x_n), x_{k+1}, \dots, x_n) : (x_{k+1}, \dots, x_n) \in U \subseteq \mathbb{R}^{n-k} \}$.

X is a submanifold, with charts given by $x_{k+1}, \dots, x_n$. (only one open, U)
and $\varphi: U \to \mathbb{R}^{n-k}$ is given by F^{-1}.

Note that $U \subseteq X$ open $\Leftrightarrow U = (\tilde{U} \times \mathbb{R}^k) \cap X$.

In the complex case, we want $F: \mathbb{C}^{n-k} \to \mathbb{C}^k$ with <u>holomorphic</u> component functions.

Given $\phi: \mathbb{R}^n \to \mathbb{R}^k$. When is $\phi^{-1}(0)$ a submanifold of $\mathbb{R}^n$?
(can ask for $\phi^{-1}(y)$, any $y \in \mathbb{R}^k$).

<u>Main tool</u>: I.F.T. (Inverse/implicit function theorem).

Write $\phi : (x_1, \dots, x_n) \longmapsto (\phi_1(\underline{x}), \dots, \phi_k(\underline{x}))$, assume now $n > k$.

Then define $\mathrm{Jac}(\phi)(p) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1}\big|_p & \cdots & \frac{\partial \phi_1}{\partial x_n}\big|_p \\ \vdots & & \\ \frac{\partial \phi_k}{\partial x_1}\big|_p & \cdots & \frac{\partial \phi_k}{\partial x_n}\big|_p \end{bmatrix}$

<u>Thm</u>: If $\phi(p) = 0$ and $\mathrm{Rank}\left(\mathrm{Jac}(p)\right) = k$,

then in a nbhd of p, $\phi^{-1}(0)$ is a submanifold.

<u>More precisely</u>: suppose that, after renaming x's:

$$\begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_k}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial \phi_k}{\partial x_1} & \cdots & \frac{\partial \phi_k}{\partial x_k} \end{vmatrix} \neq 0.$$

Then $\exists$ nbhd in $\mathbb{R}^{n-k}$ of $\pi_{n-k}(p)$ where ($p \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, and π_{n-k} projects onto $\mathbb{R}^{n-k}$)

and smooth functions $y_1(x_{k+1}, \dots, x_n), \dots, y_k(x_{k+1}, \dots, x_n)$ s.t

$$\phi\left(y_1(x_{k+1}, \dots, x_n), \dots, y_k(x_{k+1}, \dots, x_n), x_{k+1}, \dots, x_n\right) \equiv 0.$$

Such points as in the <u>Thm</u> are called <u>smooth points</u>.

At a smooth point $p \in \phi^{-1}(0)$, we compute T_pX:

$$T_pX = \{\vec{v} : \vec{v} - \vec{p} = \frac{d}{dt}\Big|_{t=0} \gamma(t), \text{ where } \gamma \text{ lies on } X\}.$$

$\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ s.t $\forall t$ $\phi(\gamma(t)) = 0$.
and $\gamma(0) = p$.

Taking $\frac{d}{dt}\big|_{t=0}$ gives $0 = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(p) \frac{d}{dt}\Big|_0 \gamma_j(t)$ $\quad i = 1..k$.

So $T_pX = \{\vec{v} \mid \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(p) \cdot (v_j - p_j) = 0 \quad i = 1 \dots k\}$

<u>Note</u>: $\mathrm{Jac}(\phi)(p) : T_p\mathbb{R}^n \longrightarrow T_{\phi(p)}\mathbb{R}^k$. Then $T_pX \subseteq T_p\mathbb{R}^n$ is

the kernel of $\mathrm{Jac}(\phi)(p)$.

• Holomorphic version

$\phi: \mathbb{C}^n \to \mathbb{C}^k$ with $\phi(z_1,\dots,z_n) = (\phi_1(z_1,\dots,z_n),\dots,\phi_k(z_1,\dots,z_n))$.
with each ϕ_i holomorphic.

Theorem: If $\phi(p)=0$ and Rank $\begin{pmatrix} \frac{\partial \phi_1}{\partial z_1} & \cdots & \frac{\partial \phi_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial \phi_k}{\partial z_1} & \cdots & \frac{\partial \phi_k}{\partial z_n} \end{pmatrix}\Bigg|_p = k$.

Then: $\exists\ W_1(z_{k+1},\dots,z_n),\dots,W_k(z_{k+1},\dots,z_n)$ s.t. $\phi(W_1(\underline{z}),\dots,W_k(\underline{z}),z_{k+1},\dots,z_n)=0$

and W_j are holomorphic (in a nbhd of $\pi_{n-k}(p)$).

Rk. From the real case, we know that if $\phi^{\mathbb{R}}: \mathbb{R}^{2n} \to \mathbb{R}^{2k}$ is the corresponding real map given by: If $\phi_j(\underline{z}) = u_j(\underline{z}) + i\, v_j(\underline{z})$. ($u_j, v_j$ real-valued)

and write $z_j = x_j + i y_j$.

we can either identify $\begin{cases} (x_1,\dots,x_n,y_1,\dots,y_n) \leftrightarrow (z_1,\dots,z_n) \\ \text{or} \\ (x_1,y_1,\dots,x_n,y_n) \leftrightarrow (z_1,\dots,z_n) \end{cases}$

$$\phi(x_1,y_1,\dots,x_n,y_n) = (u_1(\underline{x},\underline{y}), v_1(\underline{x},\underline{y}),\dots,u_k(\underline{x},\underline{y}), v_k(\underline{x},\underline{y})).$$

Special case: $\phi: \mathbb{C}^2 \to \mathbb{C}$, $\mathrm{Jac}(\phi) = \left(\frac{\partial \phi}{\partial z_1}, \frac{\partial \phi}{\partial z_2}\right)$.

Assume that $\phi(0)=0$, and $\frac{\partial \phi}{\partial z_1}(0) \neq 0$.

We will see that $\exists$ hol $w(z_2)$ s.t. $\phi(w(z_2), z_2) = 0$.

By the real version of IFT,

$\phi(x_1,y_1,x_2,y_2) = (u(x_1,y_1,x_2,y_2), v(x_1,y_1,x_2,y_2))$ s.t.

a) u, v satisfy CR w.r.t. (x_1,y_1) and (x_2,y_2).

b) $\begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial y_1} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial y_1} \end{vmatrix} = \left|\frac{\partial \phi}{\partial z}\right|^2 \neq 0.$

The theorem says that $\exists\ g_1(x_1,y_2),\ g_2(x_1,y_2)\ \mathcal{C}^\infty$ s.t.

$\mu(g_1,g_2,x_2,y_2)=0=v(g_1,g_2,x_2,y_2)$.

Set $w(z_2,\bar{z}_2)=g_1(x_2,y_2)+i\,g_2(x_2,y_2)$, $(\bar{w}=g_1-ig_2)$

Claim: $\frac{\partial w}{\partial \bar{z}_2}=0$.

ϕ is hol wrt 2^{nd} var as wrt 1^{st} var as well!

Pf/ We know $\phi=\mu+iv$, $\phi(w(z_2,\bar{z}_2),z_2)=0$.

ϕ hol wrt z_2!

Hence, $0=\frac{\partial}{\partial\bar{z}_2}\phi=\frac{\partial\phi}{\partial z_1}\cdot\frac{\partial w}{\partial\bar{z}_2}+0$. As $\frac{\partial\phi}{\partial z_1}\neq 0$, we get $\frac{\partial w}{\partial\bar{z}_2}=0$, so w hol. //

For $X\subseteq\mathbb{C}^n$, $X=\{\underline{z}: f_i(\underline{z})=0,\ i=1..k\}$.

Note: May not need $\underline{k}$ functions in the nbhd of any point:

eg: $X\subseteq\mathbb{C}^4$ def. by:

$f_1(x_0,..,x_3)=x_0x_3-x_1^2$

$f_2(x_0-x_3)=x_1^2-x_0x_2$

$f_3(x_0..,x_3)=x_2^2-x_1x_3$

• if $x_0\neq 0$:

if $x_0x_3=x_1x_2$, and $x_1^2=x_0x_2$, then $x_3=\frac{x_1x_2}{x_0}$, $\frac{x_1^2}{x_0}=x_2$.

and $x_1x_3=\frac{x_1^2x_2}{x_0}$ whereas $x_2^2=x_2\cdot x_2=x_2\cdot\frac{x_1x_2}{x_0}=\frac{x_1x_2^2}{x_0}$. //

• if $x_1\neq 0$, similar, . .

Definition: For $X=\{z\in\mathbb{C}^n: f_1(z)=\cdots=f_k(z)=0\}$, we say that a point $p\in X$ is a smooth point if $\exists$ nbhd $V\ni p$, $V=W\cap X$ ($W\subseteq\mathbb{C}^n$ open), on which:

• V is cut out by $f_{i_1},..,f_{i_\ell}$ for $\ell\leq k$.

• $Rk\begin{bmatrix}\frac{\partial f_{i_1}}{\partial z_1} & \cdots & \frac{\partial f_{i_1}}{\partial z_n}\\ \vdots & & \vdots\\ \frac{\partial f_{i_\ell}}{\partial z_1} & \cdots & \frac{\partial f_{i_\ell}}{\partial z_n}\end{bmatrix}_p=\ell$

On a smooth point p,

$$T_p X = \left\{ z \;\middle|\; \sum_{b=1}^{n} \frac{\partial f_{i_a}}{\partial z_b}(z_b - p_b),\ a = 1 \ldots l \right\}.$$

If p is ~~not~~ a smooth point, then we say p is a singular point.

Special Case: Hypersurfaces: $f : \mathbb{C}^n \to \mathbb{C}$.

$X = \{ z : f(z) = 0 \}$. Then p is smooth $\Leftrightarrow$ $f'(z)|_{z=p} \neq 0$ (i.e. $\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) \neq (0, \ldots 0)$)

So p is singular $\Leftrightarrow$ $\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_n} = 0$.

Example: $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$

$(z_1, z_2) \longmapsto z_2^2 - 4z_1^3 - a z_1 - b$

Singular points:

$f = 0$: $z_2^2 - 4z_1^3 - a z_1 - b$

$\frac{\partial f}{\partial z_1} = 0$: $-12 z_1^2 - a = 0 \Rightarrow z_1 = \pm\sqrt{\frac{-a}{12}}$

$\frac{\partial f}{\partial z_2} = 0$: $2 z_2 = 0 \Rightarrow z_1 = 0$

$b = \frac{-2a}{3}\left(\frac{-a}{12}\right)^{\frac{1}{2}}$.

So if $b \neq -\frac{2a}{3}\left(\frac{-a}{12}\right)^{1/2}$ then the given curve is smooth everywhere.

Now, let $X \subseteq \mathbb{P}^n$: $X = \{ p \in \mathbb{P}^n : F_1(p) = \cdots = F_K(p) = 0 \}$, F_i hom. of degree d_i

For $p \in X$, say $p \in X \cap U_0$. So $p = [1, p_1, \ldots, p_n]$.

Locally:
$$\begin{array}{ccc} [1,z] \; U_0 & \xrightarrow{F_i} & \mathbb{C} \\ \downarrow \;\; \wr\wr & \nearrow_{f_i} & \\ z \;\; \mathbb{C}^n & & \end{array}$$

where $f_i(\underline{z}) = F_i(1, \underline{z})$

So we get $\tilde{X} \subseteq \mathbb{C}^n$, defined by $f_1(z) = \cdots = f_K(p) = 0$, s.t. under $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$, $z \mapsto [1, z]$

$i(\tilde{X}) = X \cap U_0$.

Then we say that p is smooth in $X \Leftrightarrow \psi_0(p)$ is smooth in $\tilde{X}$.

Def: $p \in X$ is smooth if $\exists$ nbhd of p in X on which:

(1) X is cut out by $F_{i_1}, \dots, F_{i\ell}$ ← check that this is the correct definition coming from the local coordinates!

(2) $\mathrm{Rank}\left(\left[\frac{\partial F_{ia}}{\partial z_j}\right]_{j=0..k}(p)\right) = \ell$.

i.e: $rk\left(\frac{\partial f_{ia}}{\partial z_j}\right) = k \Leftrightarrow rk\left(\frac{\partial F_{ia}}{\partial z_j}\right) = k$

For a homogeneous polynomial $F(x_0, \dots, x_n)$ of degree d,

Prop (Euler's formula): $\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = d \cdot F(x_0, \dots, x_n)$ where $d = \deg F$.

Pf As $F(\lambda \underline{x}) = \lambda^d F(\underline{x})$, take $\frac{\partial}{\partial \lambda}$: $\sum x_i \frac{\partial F}{\partial x_i}\Big|_{(\lambda x_1, \dots, \lambda x_n)} = d \lambda^{d-1} F(\underline{x})$.

Now take $\lambda = 1$, and done.

Tangent space to a projective variety. $X = \{ F_1(p) = \dots = F_k(p) = 0 \}$

Def: If $p \in X$ is a smooth point, p given by $F_{i_1}(p) = \dots = F_{i\ell}(p) = 0$.

$$T_p X := \left\{ [w_0, \dots, w_n] \in \mathbb{P}^n \;\middle|\; \sum_{j=0}^{n} \frac{\partial F_{ia}}{\partial z_j}(p) \cdot w_j = 0 \;\; \forall a = 1..\ell \right\}$$

Rk: At $p = [1, p_1, \dots, p_n]$,

$$T_p X \cap U_0 = \left\{ [1, w_1, \dots, w_n] : \frac{\partial F_{ia}}{\partial z_0} \cdot 1 + \sum \frac{\partial F_{ia}}{\partial z_j} w_j = 0 \quad a = 1..\ell \right\}$$

Also, $T_{(p_1, \dots, p_n)} \tilde{X} = \left\{ (w_1, \dots, w_n) : \sum \frac{\partial f_{ia}}{\partial z_j}(w_j - p_j) = 0 \right\}$ and they are equal

by Euler's formula $\left(\text{as } \frac{\partial F_{ia}}{\partial z_0} \cdot 1 + \sum \frac{\partial F_{ia}}{\partial z_j}(p_j) = 0 \right)$.

and because $\frac{\partial F_{ia}}{\partial z_j}(1, p_1, \dots, p_n) = \frac{\partial f_{ia}}{\partial z_j}(p_1, \dots, p_n)$.

Example:

$F_1(x_0,..,x_3) = x_0 x_3 - x_1 x_2$

$F_2(x_0,..,x_3) = x_1^2 - x_0 x_2$

$F_3(x_0,..,x_3) = x_2^2 - x_1 x_3$

Smooth: $\left(\frac{\partial F_i}{\partial x_j}\right) = \begin{pmatrix} x_3 & -x_2 & -x_1 & x_0 \\ -x_2 & 2x_1 & -x_0 & 0 \\ 0 & -x_3 & 2x_2 & -x_1 \end{pmatrix} = \mathrm{Jac}.$

At $\underset{\underset{U_3}{\Uparrow}}{[0,0,0,1]}$, $\mathrm{Jac} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \rightsquigarrow \mathrm{rk}\ 2.$

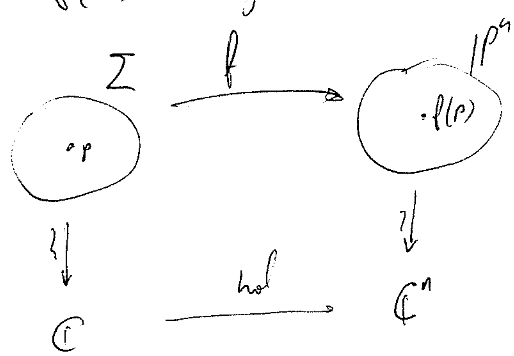
Exercise: at this point, X is cut out by F_1, F_3 (so rk condition OK).

Tangent line at $p = [x_0, x_1, x_2, x_3] \in U_0$ $(x_0 \neq 0)$ X is defined by F_1, F_2 there.

$$T_pX = \left\{ [w_0,w_1,w_2,w_3] : \sum_{j=0}^{3} \frac{\partial F_i}{\partial x_j}\Big|_p w_j = 0 \quad i=1,2 \right\} = \left\{ \begin{pmatrix} x_3 & -x_2 & -x_1 & x_0 \\ -x_2 & 2x_1 & -x_0 & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

• Holomorphic maps $\Sigma \to \mathbb{P}^n$.

Note that the correspondence R.S $\leftrightarrow$ alg-curves will be given by $f : \Sigma \to \mathbb{P}^n$, with $f(\Sigma)$ an algebraic curve.



the (local) condition defines an immersion.

We would like f to be an embedding:
(a) (locally): the local representative $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$, $z \mapsto (f_1(z),..,f_n(z))$ has to have $\mathrm{rk}\left(\frac{\partial \tilde{f}_1}{\partial z}, .., \frac{\partial \tilde{f}_n}{\partial z}\right) = 1$ at $p \in \mathbb{C}$. (i.e. not all of the components vanish)
(b) (globally): f must be 1-to-1. (injective)

Example: $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$

$$[z_0, z_1] \longmapsto [z_0^n, z_0^{n-1} z_1, \dots, z_1^n].$$

The image of f is called the Rational Normal Curve.

If $[z_0, z_1] \in U_0$ ($z_0 \neq 0$) then $f([z_0, z_1]) \in V_0$ ($V_i \subseteq \mathbb{P}^n$, $w_i \neq 0$).

The local representative is then,

$$z \mapsto [1, z] \mapsto [1, z, z^2, \dots, z^n] \mapsto (z, z^2, \dots, z^n) \quad \text{holomorphic.}$$

$$\frac{\partial f}{\partial z}(z) = (1, 2z, 3z^2, \dots, nz^{n-1}) \quad \leftarrow \text{immersion.}$$

(Can repeat it with $[z_0, z_1] \in U_1$, and get the same).

Note: in U_0-coords $f(z)$ is given by $z \mapsto [\underset{f_0(z)}{1}, z, z^2, \dots, \underset{f_n(z)}{z^n}]$ where $f_j(z) = z^j$ is actually a rational function on $\mathbb{C}$ (i.e. meromorphic function on $\mathbb{P}^1$).

So $f_j \in K(\mathbb{P}^1)$ are given by $f_j([z_0, z_1]) = \left(\frac{z_1}{z_0}\right)^j$

Claim: For any Σ (compact R.S.), for any $(n+1)$ meromorphic functions on Σ, say $f_0, \dots, f_n \in K(\Sigma)$, they define a holomorphic map $f : \Sigma \to \mathbb{P}^n$.

Pf Define $\Sigma \longrightarrow \mathbb{P}^n$

$$p \longmapsto \begin{cases} [f_0(p), \dots, f_n(p)] & \text{if } p \text{ is not a pole of any } f_j, \text{ and not a common } 0. \\ ? & \end{cases}$$

There are finitely many points where f is still undefined, say $p_1, \dots, p_N$.

At p_i: Pick coords on a nbhd of p_i in Σ, s.t $z(p_i) \equiv 0$.

Then $f_j(z) = z^{\nu_j} h_j(z)$. $h_j(z)$ holomorphic, $h_j(0) \neq 0$

Suppose ν_k is the minimal exponent, i.e $\nu_j \geq \nu_k$, $j = 0..n$.

So $(f_0(z), \dots, f_n(z)) = z^{\nu_k} (z^{\nu_0 - \nu_k} h_0(z), \dots, z^{\nu_n - \nu_k} h_n(z))$

$\nu_j - \nu_k \geq 0$ and at least one of the exponents is 0

(cont pf of claim)

So $\left(z^{\nu_0-\nu_k} h_0(z), \dots, z^{\nu_n-\nu_k} h_n(z)\right) \in \mathbb{C}^{n+1} \setminus \{0\}$ $\Rightarrow$ defines a point in $\mathbb{P}^n$.

This extends f to p_i.

<u>Note</u>: Suppose $\{f_0, \dots, f_n\}$ are linearly-~~independent~~ dependent, i.e. $\sum a_i f_i \equiv 0$ with not all $a_i = 0$.

Then $f(\Sigma) \subseteq \mathbb{P}^n$ lies in the projective linear subspace $\{\sum a_i z_i = 0\} \subsetneq \mathbb{P}^{n-1}$

<u>Def</u>: We say that $f: \Sigma \to \mathbb{P}^n$ is <u>non-degenerate</u> if $f(\Sigma)$ is not contained in any projective linear subspace of $\mathbb{P}^n$.

<u>Theorem</u>: If $f: \Sigma \to \mathbb{P}^n$ is non-degenerate and holomorphic, then f is defined by $(n+1)$ linearly-independent meromorphic functions on Σ, i.e. $f_j \in K(\Sigma)$, $j = 0..n$.

<u>Pf</u> We want $f_j \in K(\Sigma)$ s.t. $f(p) = [f_0(p), \dots, f_n(p)]$. $f: \Sigma \to \mathbb{P}^n$.

We want to use the coord function: $X_j : \mathbb{P}^n \longrightarrow \mathbb{C}$ $\leftarrow$ on U_0, works:

$[z_0, \dots, z_n] \mapsto \dfrac{z_j}{z_0}$

<u>In U_0</u>, $[z_0, \dots, z_n] \longrightarrow \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \xrightarrow{\hat{X}_j} \frac{z_j}{z_0}$

So X_j defines the j^{th} coordinate.

<u>On $\mathbb{P}^n \setminus U_0$</u>: X_j is <u>not</u> defined (can't even assign the value ∞)

This is an example of a meromorphic function on $\mathbb{P}^n$.

<u>Def</u>: Let M be a complex manifold of $\mathbb{C}$-dim n.

Given an open cover of M, $\{(U_i)\}$, then the data $\{(U_i, f_i, g_i)\}_{i \in I}$, where $f_i, g_i \in \mathcal{H}ol(U_i)$, $g_i \not\equiv 0$ in U_i (not identically 0), together with the compatibility condition $f_i g_j = f_j g_i$ $\forall i,j$, in $U_i \cap U_j \neq \phi$.

This defines a <u>meromorphic "function" on M</u> $\varphi = \left\{\frac{f_i}{g_i}\right\}$ (formal expression)

We can pull-back "functions" on $\mathbb{P}^n$ (the X_j's) to Σ:

$f^* X_j(p) = X_j(f(p))$.

<u>Fact</u>: Given $f: Y \to M$, and $\{(U_i, f_i, g_i)\}$ defining $\varphi = \frac{f_i}{g_i}$ on M,

then the data $\{(f^{-1}(U_i), f^* f_i, f^* g_i)\}$ defines also a meromorphic function on Y,

<u>provided that</u> $h(h^{-1}(U_i))$ is not contained in $g_i^{-1}(\{0\})$.

In our setting, we require $f(\Sigma)$ is not contained in $\{Z_0 = 0\}$.

As f is non-degenerate, this cannot happen. ($\{Z_0 = 0\}$ is a hyperplane).

<u>Exercise</u>: f is given by $f(p) = [f^* X_0(p), .., f^* X_n(p)]$.

<u>Example</u>: $\Sigma = \mathbb{C}/\Lambda$, $\Lambda = \{m_1\omega_1 + m_2\omega_2 ; m_1, m_2 \in \mathbb{Z}\}$, $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ $\mathbb{R}$-indep.

Then topologically, $\Sigma \simeq T^1$, i.e. $g(\Sigma) = 1$.

Then $\exists$ meromorphic function $\wp_\Lambda \in K(\mathbb{C})$

a) $\wp_\Lambda(z + m_1\omega_1 + m_2\omega_2) = \wp_\Lambda(z)$ $\forall m_1, m_2 \in \mathbb{Z}$.

b) has a pole of order 2 at $z=0$, and:

$$\wp_\Lambda(z) = \frac{1}{z^2} + h(z), \quad h(z) \text{ holomorphic}.$$

In fact, $\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$

$\wp$ defines a meromorphic function on Σ, i.e. $\wp \in K(\Sigma)$.

Also, $\wp_\Lambda' = \frac{d\wp}{dz}$ is meromorphic on Σ.

<u>Claim</u>: $\wp, \wp'$ are linearly-independent.

Pf If $\wp' = \lambda \wp$ for some $\lambda \in \mathbb{C}$, then (as $(\wp')^2 = 4\wp^3 + a\wp + b$),

we'd have $\lambda^2 \wp^2 - 4\wp^3 + a\wp + b = 0$. $(*)$

But $\wp^3 = \frac{1}{z^6} + \dots$, $\wp^2 = \frac{1}{z^4} + \dots$ $\Rightarrow$!!

//

(continues example):

Hence we get a map $\Sigma \longrightarrow \mathbb{P}^2$ which is non-degenerate.

$p \longmapsto [1, \wp_\Lambda(p), \wp_\Lambda'(p)]$

The image is an algebraic curve:

$z_0 z_2^2 - 4z_1^3 - a z_0^2 z_1 + b z_0^3 = 0 \quad \leftarrow$ hom. poly. of degree 3.

According to the "degree-genus" heuristic formula, $g = \frac{(d-1)(d-2)}{2} = 1$.

Chapter II:

Given a curve in $\mathbb{P}^2$, $C = F^{-1}(0)$ where $F(z_0, z_1, z_2)$ is homogeneous of deg d.
We want to find Σ a RS and $f : \Sigma \to \mathbb{P}^2$ so that $C = f(\Sigma)$.

At best, f is an embedding on the <u>smooth part of C</u>.

- $\rightarrow$ Where are the smooth points? (How many?)
- $\rightarrow$ What happens in a nbhd of a singular point?

<u>Lemma</u>: $C = F^{-1}(0) \subseteq \mathbb{P}^2$. $p \in C$ is singular $\iff \frac{\partial F}{\partial z_i}(p) = 0 \quad i = 0, 1, 2.$ (no need to check $F(p)=0$).

Pf By Euler's formula, if $p = [p_0, p_1, p_2]$,

$$\sum_{i=0}^{2} \frac{\partial F}{\partial z_i}(p)\, p_i = (\deg F) . F(p). \quad //$$

<u>Example</u>: $F(x,y,z) = x^3 + y^3 + z^3 + 3\lambda xyz$. $\checkmark$ do it!

If $\lambda \neq -1$, C is smooth everywhere.

If $\lambda = -1$, then $[1,1,1]$ is the only singular point.

Now, suppose $p \in C = F^{-1}(0)$ is a singular point.

WLOG, can assume $p = [1,0,0] \in U_0$.

Consider $C \cap U_0$, $\mathbb{C} \xrightarrow{i} \mathbb{P}^2$
$(x,y) \mapsto [1,x,y]$

Let $f(x,y) := F(1,x,y)$, which defines $\tilde{C} = f^{-1}(0)$ in $\mathbb{C}^2$. $\left(i(\tilde{C}) = C \cap U_0\right)$.

Also, $[1,0,0] = i(0,0)$.

Claim: $(0,0)$ is a singular point in $\tilde{C} \iff [1,0,0]$ singular in C.

Pf/ $\frac{\partial f}{\partial x}(0,0) = \frac{\partial F}{\partial z_1}(p)$; $\frac{\partial f}{\partial y}(0,0) = \frac{\partial F}{\partial z_2}(1,0,0)$. So $\boxed{\Leftarrow}$ ✓

$\Rightarrow$ By Euler's formula, if $p = [1,0,0] \in C$, then

$$0 = (\deg F)\, F(1,0,0) = \frac{\partial F}{\partial z_0}(p) + 0 + 0 \Rightarrow \frac{\partial F}{\partial z_0}(p) = 0 \text{ also.}$$

Now, write $f(x,y) = f_0(x,y) + f_1(x,y) + \cdots + f_\ell(x,y)$, $f_j(x,y)$ is the homogeneous piece of deg j.

and $\ell \leq d$.

If $f(0,0) = 0$, then $f_0(x,y) \equiv 0$ (constant term!). So $f(x,y) = ax + by + f_2(x,y) + \cdots + f_\ell(x,y)$.

So $\frac{\partial f}{\partial x}(0,0) = a$, $\frac{\partial f}{\partial y}(0,0) = b$.

* So $(0,0)$ is a <u>singular point</u> $\iff f(x,y) = f_k(x,y) + \cdots + f_\ell(x,y)$, $\underline{k \geq 2}$.

<u>Def</u>: · if $k=2$, then $(0,0)$ is a <u>double point</u> (or p is a double point of C)

· if $k=3$, triple point

· in general k, k-tuple point.

- Smooth points:

If $(0,0)$ is smooth, $f(x,y) = ax + by + f_2(x,y) + \cdots$

The tangent space is $T_{(0,0)}\tilde{C} = \left\{ \frac{\partial f}{\partial x}(0,0)\cdot X + \frac{\partial f}{\partial y}(0,0)\cdot Y = 0 \right\} = \{ax + by = 0\} = f_1^{-1}(0)$.

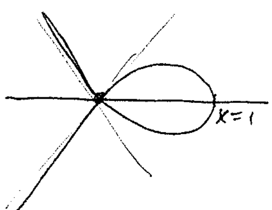
- At a k-tuple point:

$f(x,y) = f_k(x,y) + \cdots$, $\quad f_k(x,y) = \prod_{i=1}^{k} (\alpha_i x + \beta_i y)$

Then $f_k^{-1}(0)$ = union of k-lines (not necessarily distinct).

Example: $f(x,y) = x^3 - x^2 + y^2 = (y^2 - x^2) + x^3 \Rightarrow$ origin a singular (double) point.

tangent lines determined by $y^2 - x^2 = 0$, i.e. $y = \pm x$.



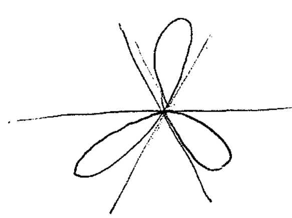
← real picture of the curve.

Def: An ordinary point $p \in C$ is a k-tuple point where the k tangent lines at p are distinct. $(k \geq 2)$

Example: $(x^2+y^2)^2 + 3x^2y - y^3 = 0$ has a triple point at the origin,

determined by $3x^2y - y^3 = 0 = y(3x^2 - y^2) = y(\sqrt{3}x - y)(\sqrt{3}x + y)$.

So the point is ordinary.



Note: if $L_\infty = \mathbb{P}^2 \setminus i(\mathbb{C}^2)$, $i : \mathbb{C}^2 \to \mathbb{P}^2$ is the line at infinity, $\{[0,x,y] \in \mathbb{P}^2\}$, then $C \cap L_\infty$ has at most d distinct points (corresponding to "missing" points of intersection on lines in $\mathbb{C}^2$.

Claim, can assume (after a change of coordinates):

$$f(x,y) = y^d + a_1(x)\,y^{d-1} + \dots + a_d(x), \quad \deg a_i(x) \le d.$$

Pf Given $f(x,y) = F(1,x,y) = \sum_{m+n\le d} A_{mn}\, x^m y^n$.

Make a substitution:

$$\left.\begin{aligned} x &= x' + \lambda y' \\ y &= y' \end{aligned}\right\} \to \text{get } \sum_{m+n\le d} A_{mn}(x'+\lambda y')^m y'^n.$$

Now the coeff. of y'^d is $\sum_{m=0}^{d} \lambda^m A_{m(d-m)} = b(\lambda)$

Notice $b(\lambda) \not\equiv 0$ because F is of degree d. Hence can pick a λ s.t. $b(\lambda) = 1$ (if $b(\lambda) \equiv A_{0d}$, then rescale the coefficients so that $A_{0d} = 1$).

In $\mathbb{C}^2$,

$R = \{(x,y) : f(x,y) = 0 = f_y(x,y)\}$ contains all singular points (it's much bigger). The fact is that R is actually already finite, which will prove that there are only finitely many singular points.

Look at the projection onto 1st coordinate:

$$\begin{array}{ccc} \mathbb{C}^2 & (x,y) & R \\ \pi \downarrow & \downarrow & \downarrow \\ \mathbb{C} & x & D = \{x : \exists y \text{ s.t. } (x,y) \in R\}. \end{array}$$

We'll show that D is finite, and that $\forall x \in D$, $\pi^{-1}(x)$ is finite.

Note: For fixed x, $\pi^{-1}(x) \cong \{ y \mid f(x,y) = 0 = f_y(x,y) \}$ ← finite # points - as it's the set of common roots of two polynomials in y.

Now, $x \in D \iff f^{(x)}(y)\ (= f(x,y))$, and $f_y^{(x)}(y)$ have a common root.

As $f^{(x)}(y) = y^d + a_1(x) y^{d-1} + \dots + a_d(x)$

then $f_y^{(x)}(y) = d\, y^{d-1} + (d-1)\, a_1(x)\, y^{d-2} + \dots + a_{d-1}(x)$.

Aside:

Given $f(x) = a_0 x^m + \dots + a_m$ $a_0 \neq 0$ when do they have a common factor?
$g(x) = b_0 x^n + \dots + b_n$ $b_0 \neq 0$.

$\iff \exists$ polys $F(x), G(x)$, $\deg F < \deg f$, $\deg G < \deg g$ s.t. $f \cdot G = g \cdot F$.

$\left(\begin{array}{l} f = c(x) f_1 \\ g = c(x) g_1 \end{array} \right. \Rightarrow f \cdot g_1 = g \cdot f_1 \Rightarrow \checkmark$. Conversely, if $f G = g F$.
As $\deg f > \deg F$, not all factors of f can be of F, so f, g have a common factor$)$ ✓

We seek $F = A_0 x^{m-1} + \dots + A_{m-1}$ s.t. $f G + g F = 0$ (changed sign, it's ok.)
$G = B_0 x^{n-1} + \dots + B_{n-1}$

writing out the condition, we want, for each k, $0 \leq k \leq m+n-1$,

$$\left\{ \begin{array}{l} a_0 B_0 + b_0 A_0 = 0 \\ \vdots \end{array} \right\}$$ ← $m+n$ linear conditions on the A's and B's. ($m+n$ variables).

Hence $\exists$ solution $\Leftrightarrow$ det [matrix on coeffs] $= 0$.

The transpose of this matrix is:

$$\left|\begin{bmatrix} a_0 & a_1 & \cdots & a_m & & \\ 0 & a_0 & \cdots & \cdots & a_m & \\ & & \ddots & & & \\ b_0 & b_1 & \cdots & b_n & & \\ 0 & b_0 & \cdots & \cdots & b_n & \\ 0 & & \ddots & & & \end{bmatrix}\begin{matrix} \Big\updownarrow n \\ \\ \Big\updownarrow m \end{matrix}\right| =: R(f,g) = \text{resultant of } f \text{ and } g.$$

Thm: f, g have a common factor $\Leftrightarrow$ $R(f,g) = 0$

Note: Define

$\alpha(x) = R_1 x^{n-1} + \cdots + R_n$ where $R_j = (j, m+n)$ cofactor of $R = (-1)^{j+m+n} |\;|$

$\beta(x) = R_{n+1} x^{m-1} + \cdots + R_{n+m}$

Lemma: $\alpha(x) f(x) + \beta(x) g(x) = R(f,g)$

Pf/ For any $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $\sum_{j=1}^{n} a_{ij} A_{kj}$ where A_{kj} is the (k,j) cofactor.

$= \begin{cases} \det A & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ //

Special case $g = f'$: $D(f) := R(f, f')$ is the discriminant of f.

Corollary: $f(x)$ has a multiple factor $\Leftrightarrow D(f) = 0$.

Pf/ if $f = f_1^{n_1}(x) \cdot h(x)$, $n_1 > 1$, then $f' = f_1^{n_1 - 1}(f_1 h'(x) + n h(x)) \Rightarrow f, f'$ have a common factor.

Spse now $f = f_1 \cdot h$, $f' = f_1 \cdot k$ (common factor).

$f' = f_1' h + f_1 h' = f_1 \cdot k \Rightarrow f_1(k - h') = f_1' h \Rightarrow f_1, h$ have a common factor by Lemma

provided that $k - h' \neq 0$. But if $k - h' = 0$, then $f_1' \cdot h = 0 \Rightarrow !!$ (f_1 has deg ≥ 1, $h \neq 0$).

//

Back to our case:

Let $\mathcal{D}(x) = \mathcal{D}(f^{(x)})$ = determinant of a matrix with entries in $\mathbb{C}[X]$.

So $\mathcal{D}(x)$ is a polynomial in x. Either $\mathcal{D}(x) \equiv 0$ or $\mathcal{D}(x) = 0$ for finitely many x.

Pf:
__Claim__: $\mathcal{D}(x) \not\equiv 0$

Pf ~~$\mathcal{D}(x) \equiv 0 \Leftrightarrow f^{(x)}(y)$ has a multiple factor $\forall x$: $f^{(x)}(y) = (f_1^{(x)}(y))^{n_1} h^{(x)}(y)$.~~

For two polynomials $f, g \in K[Y]$, f and g have a common zero iff $R(f,g) = 0$ for __K any UFD.__ Take $K = \mathbb{C}[X]$.

In our case, $\mathcal{D}(x) \in K$, and this says that

$f^{(x)}(y)$ has a repeated factor $\Leftrightarrow \mathcal{D}(x) = 0$ in $\mathbb{C}[X]$.

So if $\mathcal{D}(x) \equiv 0$, then $f^{(x)}(y)$ is reducible $\Rightarrow f(x,y)$ is reducible.

So __if the curve is irreducible,__ this cannot happen. //

So given $C = F^{-1}(0) \subset \mathbb{P}^2$ irreducible, and $S := \{\text{singular points on } C\}$, let $C^* := C \setminus S$

__Theorem__: C^* is connected. Hence C^* is a connected submanifold of $\mathbb{P}^2$.

Pf Look at $C \cap \mathbb{C}^2 = \{(x,y) \mid f(x,y) = 0\}$, $f(x,y) = y^d + a_1(x) y^{d-1} + \dots + a_d(x)$.

Then $S \cap \mathbb{C}^2 = \{(x,y) \mid f(x,y) = f_x(x,y) = f_y(x,y) = 0\}$

$\mathcal{D}(x) = 0 \Leftrightarrow f^{(x)}$ has a repeated root.

So if $(x,y) \in C^* \cap \mathbb{C}^2 =: \tilde{C}^*$, then $f^{(x)}(y)$ has __distinct__ roots (d of them).

Say $y_1(x), \dots, y_d(x)$ are s.t. $f(x, y_i(x)) = 0$

C

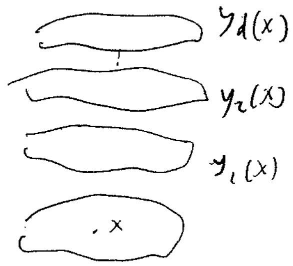
(cont pf)

Consider $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$

$(x,y) \longmapsto f(x,y)$

If $(x,y) \in \tilde{C}^*$, then by IFTh, $f(x,y(x)) = 0$ defines $y(x)$ as a holomorphic function of x (could be x as a fn of y, but ~~…~~ thanks to having chosen $f(x,y) = y^d + \dots$ we can guarantee that is $y(x)$ g.)

<u>Locally</u>:



d-fold cover

$$\begin{array}{cc} \mathbb{C}^2 & (x,y) \\ \downarrow & \downarrow \\ \mathbb{C} & x \end{array}$$

We will see that those d components are actually part of only one.

We will in fact see that it is <u>path-connected</u> (stronger).

So given $(x, y_i(x))$, $(x', y_j(x')) \in \tilde{C}^*$, $\exists$ path $\gamma(t)$ in $\tilde{C}^*$ joining them

We will take a path $\tilde{\gamma}(t)$ from x to x', and then we will lift it to $\tilde{C}^*$.

We must pick $\tilde{\gamma}(t)$ in $\mathbb{C} \setminus D$

<u>To show</u>: • can lift $\tilde{\gamma}(t)$ to some $\gamma(t)$ in C^*

• there are enough path

<u>Path lifting</u>: (Analytic continuation) $\Leftarrow$ sheaf property..

Given any $x \in \mathbb{C} \setminus D$ and d distinct roots $y_1(x), .., y_d(x) \in Hol(\Delta)$ (Δ a nbhd of x). Then each analytic function element $(\Delta, y_i(x))$ can be analytically continued along any path in $\mathbb{C} \setminus D$.

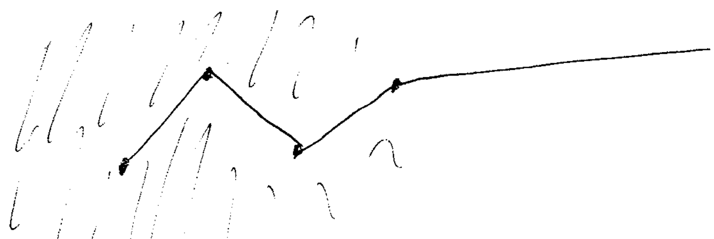
Riemann Monodromy Theorem

<u>Thm</u>: Spse $\Omega \subset \mathbb{C}$ is simply connected.

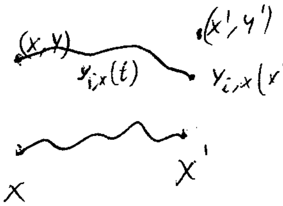
If (Δ, f) is an analytic function element on $\Delta \subset \Omega$ can be analytically continued along any path in Ω,

<u>then</u>: f can be extended to $f \in Hol(\Omega)$

To apply this thm, let Λ be a cut in $\mathbb{C}$ containing all points in D:



Now the domain in $\mathbb{C} \setminus \Lambda$ is simply connected, so we get a holomorphic function on $\mathbb{C} \setminus \Lambda$ that restricts to any of the lifts we were constructing.



$y_{i,x}(x') = y_j(x')$ but in general $y_{i,x}(x') \neq y'$.

So now we need to go from $y_{i,x}(x')$ to y'.

Note that we can take paths that cross Λ (provided that they don't touch D).

Consider so loops pointed at x', i.e. $\gamma(t)$ with $\gamma(0) = \gamma(1) = x'$.

Lift them to paths at $(x', y_{i,x}(x'))$ (at $t=0$).

At $t=1$, we get $\{y_k(x')$ for some $k = 1..d\}$.

We want to get all the $y_i(x')$ by this procedure. If this wasn't the case,

then $\{y_1(x), \dots, y_d(x)\} = E_1 \cup \dots \cup E_\ell$ where $E_\kappa = \{y_{i_1}(x), \dots, y_{i_\kappa}(x)\}$

and $y_i(x), y_j(x) \in E_\kappa \Leftrightarrow \exists$ path (loop) connecting the two.

Consider $\prod_{j=1}^{\kappa} (y - y_{i_j,x}) =: f_i(x,y)$. (holomorphic).

Then $f(x,y) = \prod_{i=1}^{l} f_i$ and if f is irreducible, <u>and</u> we can show that $f_i \in \mathbb{C}[x,y]$ then this will imply that $l = 1$, and so the result will follow.

<u>Note</u>: As $C^* = C \setminus \{p_1, \ldots, p_n\}$, $\overline{C^*} = C$, and hence C^* connected $\Rightarrow C$ connected.

So any irr algebraic curve is connected.

Look at $f_i(x,y) = y^k + b_1(x) y^{k-1} + \cdots + b_k(x)$ (it's polynomial in y, by construction).

where $b_1(x) = -\sum_{i=1}^{k} y_{i,x}$ and $b_t(x) = (-1)^t \underbrace{S_t}_{\text{elementary symmetric poly's}}(y_{1,x}, \ldots, y_{k,x})$

<u>Note</u>: the analytic continuations of $y_{i,x}^{\gamma}$ depend on γ, but the set $\{y_{i_1,x}^{\gamma}, \ldots, y_{i_k,x}^{\gamma}\}$ does <u>not</u>.

Hence the symmetric combinations of them <u>do not</u> depend on γ (ie $f_i(x,y)$ do not depend on γ).

<u>Claim</u>: $b_i(x)$ defines a hol. function on $\mathbb{C}$, and in fact $b_i(x) \in \mathbb{C}[x]$.

<u>pf</u> First, note that $b_i(x)$ is hol. in $\mathbb{C} \setminus D$. So we only need to prove that $b_i(x)$ is bounded in a nbhd of any point in D.

Each $b_i(x)$ is a sym. combination of some roots of $f(x,y)$.

As if $f(x,y) = y^d + a_1(x) y^{d-1} + \cdots + a_d(x)$, then $a_k(x) \in \mathbb{C}[x] \Rightarrow a_k(x)$ bounded in a nbhd of $p \in D$; then its roots (and hence the sym. combinations of them) are bounded, too. So $b_i(x) \in \mathcal{H}ol(\mathbb{C})$. see the note after the Thm.

To see that $b_i(x) \in \mathbb{C}[x]$, it's enough to see that $b_i\left(\frac{1}{x}\right)$ has a pole of finite order at $x = 0$, say N.

In $f(x,y) = y^d + a_1(x) y^{d-1} + \cdots + a_d(x)$, replace $\begin{matrix} x = \frac{1}{x'} \\ y = y'/x' \end{matrix}$ and consider

$\tilde{f}(x', y') := x'^d f(x', y')$ (reciprocal poly of f)

$= y'^d + x' a_1\left(\frac{1}{x'}\right) y'^{d-1} + \cdots + x'^d a_d\left(\frac{1}{x'}\right).$

Note that $a_i(x) \in \mathbb{C}[x]$ of degree $\leq i$. Hence $(x')^i a_i(\frac{1}{x'})$ is also a polynomial.

So $\tilde{f}(x',y') \in \mathbb{C}[x',y']$. Let $y_j'(x)$ be the roots of $\tilde{f}(x',y')$, and construct $b_j'(x')$, symmetric combinations of them.

Then $y_i'(x') = x' y_j(\frac{1}{x'})$. From this, then $b_a'(x) = (x')^a b_a(\frac{1}{x'})$.

Now it's enough to see that $b_a'(x)$ is bounded in a nbhd of $x=0$, but it is by a previous argument.

Hence $b_a(\frac{1}{x})$ has pole of finite order at $x=0$, so $b_a(x) \in \mathbb{C}[x]$.

We've proven:

<u>Thm</u>: if f is <u>irreducible</u>, then $\tilde{C}^*$ (and hence $\tilde{C}$) is connected.

<u>Cor</u>: $C \subseteq \mathbb{P}^2$ is connected (as $\tilde{C} = C \setminus \{\text{fin set}\}$).

<u>Note</u>: let $f(y) = y^d$, $g(y) = a_1 y^{d-1} + \dots + a_d$ ($f+g = y^d + a_1 y^{d-1} + \dots + a_d$).

if $|a_i| < M$ and $C = S^1_R$, then

for R large enough (det. by M), $|f| > |g|$ on C. So by Rouché's Theorem, f and $f+g$ have the same number of zeros inside C. But f has exactly d (all at 0), so the result follows.

We want to look now at singular points.

We will show: for every plane algebraic curve $C \subseteq \mathbb{P}^2$, $\exists$ smooth closed R.S $\Sigma(C)$, and a map $\sigma: \Sigma(C) \to \mathbb{P}^2$ holomorphic s.t. $\sigma(\Sigma(C)) = C$, and if $S \subseteq C$ is the set of singular points, then $\sigma^{-1}(S)$ is a finite set in $\Sigma(C)$, and $\sigma|: \Sigma(C) \setminus \sigma^{-1}(S) \to C^*$ is an injection (so a bijection).

[See the book for a discussion of uniqueness].

Take $P \in C \cap \mathbb{C}^2$ ← take the corresponding patch, p a singular point; with coordinates (x, y), $x \in D$.

Assume WLOG $(x,y) = (0,0)$, and $f(x,y) = y^d + a_1(x) y^{d-1} + \dots + a_d(x)$ $\quad a_i(x) \in \mathbb{C}[x]$ of deg $\leq i$.

Also assume $f(x,y) \in \mathbb{C}[x,y]$ is irreducible.

Note: $f(x,y)$ will factor locally in analytic functions of x, y (write in $\mathbb{C}\{x,y\}$).

Example: $f(x,y) = y^2 - x^2 + x^3 = y^2 + a_2(x)$

$f(0,0) = f_x(0,0) = f_y(0,0) = 0$ so $(0,0)$ is singular.

$y^2 = x^2 - x^3 = x^2(1-x)$. In a nbhd of 0, can define $\sqrt{1-x}$, so

$y_1(x) = x\sqrt{1-x}$, $y_2(x) = -x\sqrt{1-x}$ and $f(x,y) = (y - y_1(x))(y - y_2(x))$.

On a nbhd of p, $f(x,y) = f_1 \cdots f_{d'}$. As $f(0,y)$ is not identically 0, then $f_i(0,y) \not\equiv 0$

We will use: (Weierstrass Preparation Thm).

$$\begin{cases} f(x,y) \in \mathbb{C}\{x,y\} \\ f(0,y) \not\equiv 0, \ f(0,0) = 0 \end{cases}$$

Then we can write, on a suitable nbhd of $(0,0)$,

$f(x,y) = u(x,y)\, w(x,y)$ where a) $u(0,0) \neq 0$ (so $u^{-1}(x,y) \in \mathbb{C}\{x,y\}$).

b) $w(x,y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x)$ ↙ Weierstrass polynomial.

where $a_i(x) \in \mathbb{C}\{x\}$ with $a_i(0) = 0$.

Applications:

1) $\mathbb{C}\{x,y\}$ is a UFD. (i.e. can factor $f = f_1 \cdots f_k$, with f_i irreducible in $\mathbb{C}\{x,y\}$, unique up to units).

2) Get a local picture for each irreducible factor $f_i(x,y) = y^n + a_1(x) y^{n-1} + \dots$

We can assume that for each $a = 1, \dots, L$ $(f = f_1 \cdots f_L$ in a nbhd of $(0,0))$.
that $f_a(x,y) = u_a(x,y)\, \omega_a(x,y)$ where $\begin{cases} u_a(0,0) \neq 0 \\ \omega_a(x,y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x) \end{cases}$; $\begin{cases} a_i(x) \in \mathbb{C}\{x\} \\ a_i(0) = 0 \end{cases}$

Let $V_a = f_a^{-1}(0) \cap V$, so that $f^{-1}(0) \cap V = \bigcup_{a=1}^{L} V_a$.

<u>Claim</u>: $\exists$ hol map $g_a : \Delta \longrightarrow \mathbb{C}^2$ s.t. (where Δ = disc in $\mathbb{C}$)

- $g_a(\Delta) = V_a$
- $g_a : \Delta \setminus \{0\} \longrightarrow V_a - \{0,0\}$ is injective (i.e. biholomorphic with its image).
- $g_a(0) = (0,0)$.

Pf For each f_a, for every $x \neq (0,0)$ in V, f_a has n distinct roots $y_1(x), \dots, y_n(x)$.
So $f_a(x,y) = \prod_{i=1}^{n} (y - y_i(x))$ in a nbhd of x.

Take any one of the $y_i(x)$:

- defines an analytic fn element in a nbhd of x
- can analytically continue along any path in $\Delta \setminus \{0\}$.

Given $\gamma : [0,1] \longrightarrow \Delta \setminus \{0\}$, $\gamma(0) = 0$, $\gamma(1) = t$,
denote $y_i^x(s)$ for $s \in [0,1]$. Then $y_i^x(1)$ (= value at $\gamma(1) = t$) $\in \{y_1(t), \dots, y_n(t)\}$

~~Now write $\gamma^n : [0,1] \to \Delta \setminus \{0\}$ for $\gamma^n(s) := (\gamma(s))^n$~~.

Think of $\gamma : S^1 \to \Delta \setminus \{0\}$ and denote by $\gamma^n : S^1 \to \Delta \setminus \{0\}$ the new loop $\gamma(\lambda) = \gamma(\lambda^n)$

If $\pi_1(\Delta \setminus \{0\}) \simeq \mathbb{Z}$ and $[\gamma] = m$ (homotopy class), then $[\gamma^n] = n \cdot m$.

Lemma: If $\gamma : [0,1] \to \Delta \setminus \{0\}$ is any loop at x, and $y_i(x)$ is any root of f_a, then: $y_i^{\gamma^n}(1) = y_i(x)$.

Pf

Rk: we know that $y_i^{\gamma^n}(1) \in \{y_1(x), \dots, y_n(x)\}$. This just says that it's the one we started with.

Let η be an n^{th} root of 1, and consider $T := \{ y_i^{\gamma^n}(\eta^\ell) : \ell = 0, \dots, n-1 \}$.

Compare T to $S = \{ y_1(x), \dots, y_n(x) \}$

we must have $T = S$ since f_a is irreducible.

It follows that $y_i^{\gamma^n}(\eta^n) = y_i(x)$. //

Consider then $h : \Delta \setminus \{0\} \to \mathbb{C}$
$t \longmapsto y_i^{\gamma^n}(1)$ where $\gamma : [0,1] \to \Delta \setminus \{0\}$ is any path from x to t.
(what is γ^n if γ is not a loop??)

Then this should be well-defined.

Example: $f(x,y) = y^2 - x^3$. Irreducible in $\mathbb{C}\{x,y\}$, $n = 2$ (degree in y).

Away from $x = 0$, $y_i(x) = \pm x^{3/2}$. Pick any one, say $y_1(x) = + x^{3/2}$

Then the map would be $t \longmapsto (t^2, y_1(t^2)) = (t^2, t^3)$.

Also, $g_a : \Delta \setminus \{0\} \to \mathbb{C}^2$ extends to a hol. function on Δ, with $g_a(0) = (0,0)$.

Picture: D a nbhd of $(0,0)$,

$f = f_1 \cdots f_L$, f_a irred in $\mathbb{C}\{x,y\}$, $\deg f_a = n_a$

For each a, we have $g_a : \Delta_a \to \mathbb{C}^2$
$t \longmapsto (t^{n_a}, y_i(t^{n_a}))$.

Using C^* = smooth part of C, and Δ_a, one for each a at each singular pt, we want to construct a smooth R.S Σ, ~~such that~~ with a map

$g : \Sigma \longrightarrow \mathbb{P}^2$ and such that.

1) $g(\Sigma) = C$

2) if $S \subset C$ is the set of singular points, $g^{-1}(S) \subset \Sigma$ is a finite set of points

3) $g : \Sigma \setminus g^{-1}(S) \hookrightarrow\!\!\!\!\rightarrow C^*$

Suppose for simplicity that C has one singular point, and f is irreducible in a nbhd of this point.

Define then $\Sigma = \left(C^* \cup \Delta\right)/\sim$ where $p \in C^*$, $t \in \Delta_a$, $p \sim t \Leftrightarrow g_a(t) = p$.

$$\begin{tikzcd} \Delta_a\setminus\{0\} \arrow[r, "g_a"] \arrow[d] & C^* \arrow[d] \\ \Delta_a \arrow[r] & \Sigma \end{tikzcd}$$

(g_a is called a local normalization).

If we have many Δ_a, then we add them one at a time : $C^* \cup \Delta/\sim = C^* \cup_g \Delta$

Intersection of Curves in $\mathbb{P}^2$

Let $C_1 = F_1^{-1}(0)$, $C_2 = F_2^{-1}(0)$.

Q: Do they intersect ?
If they do, what is their intersection ?

A: Yes, always.
$C_1 \cap C_2$ is a set of points, with "$\#(C_1 \cap C_2) = (\deg F_1)(\deg F_2)$".

First, let's write:

$F_1(X,Y,Z) = a_0(Y,Z)\cancel{X}^n + \cdots + a_n(Y,Z)$, $\deg F_1 = n$, $\deg(a_j(Y,Z)) = j$ or $-\infty$
i.e. $a_j(Y,Z)$ can be $\equiv 0$.

$F_2(X,Y,Z) = b_0(Y,Z)X^m + \cdots + b_m(Y,Z)$, $\deg F_2 = m$, $\deg b_j(Y,Z) = j$ ....

Claim: $C_1 \cap C_2 \neq \emptyset$ ~~iff~~ (i.e. F_1, F_2 have a common root).

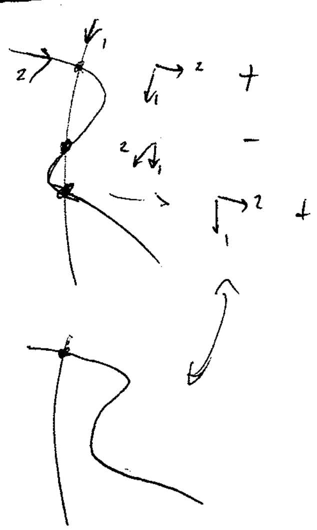
Pf If for some (y_0, z_0), $a_j(y_0,z_0) \equiv 0$ (i.e. $F_1(X, y_0, z_0) = 0 \ \forall X$).
~~then~~ but $F_2(X, y_0, z_0)$ has roots, say X_0, then done.

If there's no (y_0, z_0) s.t. $a_j(y_0, z_0) \equiv 0 \ \forall j$, then
$F_1(X, y_0, z_0)$, $F_2(X, y_0, z_0) \in \mathbb{C}[X]$ will have a common root if
$R(F_1, F_2) = 0$.

Claim: $R(F_1, F_2) \in \mathbb{C}[Y,Z]$ is homogeneous. (check it) of degree $m \cdot n$.

So $R(F_1, F_2) = \prod(\alpha_a Y + \beta_a Z)$

How to count intersections:



If we fix an orientation on the plane, then one can count the intersections with the sign, and then this is topologically robust:

1) For cpx submanifolds $X \subset Y$ of a space Y, we can use cx structure to define orientations

2) $X_1 \cap X_2 \ni p$ and intersect transversally, otherwise have no orientation.

We also have to define the multiplicity of an intersection:

<u>Recall</u> (case of $C \cap L$, L a line):

$$C \cap L \subset \mathbb{C}^2 \cap \mathbb{P}^2, \quad C : f(x,y) = \sum_{m+n \le d} A_{mn} x^m y^n$$

$$L : g(x,y) = \alpha x - \beta y \quad (\beta \neq 0).$$

The intersection point is $y = \frac{\alpha}{\beta} x$,

$$0 = f\left(x, \frac{\beta}{\alpha} x\right) = \sum A_{mn} \left(\frac{\alpha}{\beta}\right)^n x^{m+n} = \sum_{k=0}^{d} a_k\left(\frac{\alpha}{\beta}\right) x^k$$

The intersection points are the roots of $f(x, \frac{\alpha}{\beta} x) = 0$, but they are not necessarily distinct.

Then, counting w/ multiplicity, $\# C \cap L = d$.

Now, if $\deg C_1 = k$, $\deg C_2 = \ell$, and $p \in C_1 \cap C_2$.

We define the <u>multiplicity</u> of p, i.e. $(C_1 \cdot C_2)_p$

<u>Use local pictures</u>.

pick coords, so $p = (0,0)$ in $\mathbb{C}^2$.

$C_1 \cap \mathbb{C}^2$ given by $f(x,y) = 0$

$C_2 \cap \mathbb{C}^2$ given by $g(x,y) = 0$

We can choose the coordinates so that
$$\begin{cases} f(x,y) = y^k + a_1(x) y^{k-1} + \cdots + a_k(x) \\ g(x,y) = y^\ell + b_1(x) y^{\ell-1} + \cdots + b_\ell(x). \end{cases}$$

Local picture for C_1:

$f = f_1 \cdots f_L$ with $f_a \in \mathbb{C}\{x,y\}$ irreducible

For each f_a; $y^{k_a} + a_1(x) y^{k_a - 1} + \cdots + a_{k_a}(x)$

For each $x \neq 0$, $f_a(x,y) = \prod_{\nu=1}^{k_a} \left(y - y_\nu^{(a)}(x)\right)$.

<u>Normalization</u>: $\Delta \xrightarrow{\varphi_a} C_1^{(a)}$

$t \longmapsto \left(t^{k_a}, y_\nu^{(a)}(t^{k_a})\right)$

$$\Phi_a : \Delta \xrightarrow{\varphi_a} C_1^{(a)} \xrightarrow{g} \mathbb{C}$$

$C_1^{(a)} \subseteq \mathbb{C}^2$

$g(\varphi_a(t)) = g\left(t^{k_a}, y_2^{(a)}(t^{k_a})\right) = 0$

Note that $g \circ \varphi_a \in Hol(\Delta)$, and one can look at the order of the 0 at $t=0$.

i.e. $g(t) = t^{m_a} \cdot \tilde{g}(t)$, $\tilde{g}(0) \neq 0$, $\tilde{g}$ hol.

Def: $(C_1^{(a)} \cdot C_2)_p := m_a$.

$$(C_1 \cdot C_2)_p := \sum_{a=1}^{L} m_a .$$

Claim: $(C_1 \cdot C_2)_p = (C_2 \cdot C_1)_p$. ← won't prove it for now.

Example: $\begin{cases} C_1 = L_1 \\ C_2 = L_2 \end{cases}$ $(\deg f = \deg g = 1)$

$f(x,y) = y + \alpha_1 x$
$g(x,y) = y + \alpha_2 x$

$$\Delta \xrightarrow{\varphi} C_1$$
$$t \longmapsto (t, y_1(t)) = (t, -\alpha_1 t)$$

So $g(\varphi(t)) = g(t, -\alpha_1 t) = (\alpha_2 - \alpha_1) \cdot t$

At $t=0$, if $\alpha_1 \neq \alpha_2$, have $m = 1$

Example: Look again at $C_1 = L$, $C_2 = C$ with degree >1, and will recover the special case from the beginning.

If $C_1 \cap C_2 = \{p_1, \ldots, p_N\}$ and $(C_1 \cdot C_2)_{p_i} = m_i$.

Define: $D = m_1 p_1 + \cdots + m_N p_N$ is the divisor on $\mathbb{P}^2$ associated to $C_1 \cap C_2$.

Bezout's Theorem: If C_1, C_2 have no common component, then

$$\sum_{p \in C_1} (C_1 \cdot C_2)_p = \deg C_1 \cdot \deg C_2$$

if $p \notin C_1 \cap C_2$, $(C_1 \cdot C_2)_p = 0$

Pf Spz one curve, C, is reducible, defined by $F = F_1^{n_1} \cdots F_L^{n_L}$, $C_i = F_i^{-1}(0)$.

Claim: if E is any other curve, and $p \in C \cap E$, then $(C \cdot E)_p = \sum_{a=1}^{L} m_a (C_a \cdot E)_p$

So then, $\sum_p (C \cdot E)_p = \sum_{a=1}^{L} m_a \sum_p (C_a \cdot E)_p = \sum_{a=1}^{L} m_a \cdot \deg C_a \cdot \deg E = \deg E \cdot \deg C$ //

So WLOG can assume C and E are irreducible.

So now assume C is irreducible of $\deg C = \ell$, E of deg k; and C not a component of E.

Easy case: E is a union of k lines; i.e. if $E = G^{-1}(0)$, then $G(x,y,z) = \prod_{i=1}^{k} G_a(x,y,z)$ ($\deg G_a = 1$).

write $E = L_1 + \cdots + L_k$. So $\sum_p (C \cdot E)_p = \sum_p \sum_{a=1}^{k} (C \cdot L_a)_p = \sum_{a=1}^{k} \sum_p (C \cdot L_a)_p = \sum_{a=1}^{k} \ell = k \cdot \ell$

General Case: E defined by $G(x,y,z)$ homogeneous of degree k.

Idea: Pick k lines $L_1, \dots, L_k$, defined by $G_1, \dots, G_k$.

Let $E_k = L_1 + \cdots + L_k$ (E_k deg-k curve def by $\prod G_a$).

Compare now $\sum_p (C \cdot E)_p$ with $\sum_p (C \cdot E_k)_p$.

Note that $\dfrac{G}{G_1 \cdots G_k}$ defines a meromorphic function on $\mathbb{P}^2$

Take the normalization of C, i.e. $\varphi : \Sigma \longrightarrow \mathbb{P}^2$, $\varphi(\Sigma) = C$.

$$\Big\downarrow \frac{G}{G_1 \cdots G_k}$$

So get a meromorphic function $f \in K(\Sigma)$, $f = \varphi^*\left(\frac{G}{G_1 \cdots G_k}\right)$ $\mathbb{C} \cup \{\infty\}$

So we have $f = \varphi^*\left(\frac{G}{G_1\cdots G_n}\right) \in K(\Sigma)$.

Consider the divisor associated to f (of zeros/poles)

$$(f) = (f)_0 - (f)_\infty = \sum_{f(q_i)=0} m_i q_i - \sum_{f(q_i)=\infty} m_i q_i$$

Claim: $\sum_p (C\cdot E)_p - \sum_p (C\cdot E_n)_p = \deg(f) = 0 \Rightarrow$ theorem follows.

Exercise: can assume (by choosing wisely the E_n) that $C \cap E \cap E_n = \phi$.

Then $p \in C\cap E \Leftrightarrow p = \varphi(q)$ and $G(p) = 0$

$p \in C \cap E_n \Leftrightarrow p = \varphi(q)$ and $G_1(p)\cdots G_n(p) = 0$

Under our assumption, we can't have $G(p)=0$ and $G_1(p)\cdots G_n(p)=0$

Hence $\begin{cases} f(q) = 0 \Leftrightarrow G(\varphi(q)) = 0 \\ f(q) = \infty \Leftrightarrow G_1\cdots G_n(\varphi(q)) = 0 \end{cases}$

Exercise: $(C\cdot E)_p = \mathrm{ord}_q(f)$, where $\varphi(q) = p$. if $f(q)=0$

$(C\cdot E_n)_p = \mathrm{ord}_q(f)$ — — if $f(q) = \infty$

Simple Consequences

1) Any two curves have nonempty intersection.

2) If C is reducible, then it has singular points. (i.o.w smooth $\Rightarrow$ irreducible).

Pf If C is defined by $F = F_1F_2$, $C = C_1 \cup C_2$, by (1) $C_1\cap C_2$ have nonempty intersection

Let $p \in C_1 \cap C_2$: $F_1(p) = F_2(p) = 0$.

For $a = 0,1,2$; $\frac{\partial F}{\partial z_a}(p) = \frac{\partial F_1}{\partial z_a}(p)F_2(p) + F_1(p)\frac{\partial F_2}{\partial z_a}(p) = 0 + 0 = 0 \Rightarrow p$ singular.

$d=1$, ok!

3) # singular points on curve C is finite. In fact, for a curve C with $\deg C > 1$,

(cont obs):

For one of $a = 0,1,2$, $\frac{\partial F}{\partial z_a} \not\equiv 0$, so it defines a curve E, of $\deg E \leq n-1$ $\deg C$.

By Bézout, $\#\{C \cap E\} \leq n\cdot(n-1)$, and $\mathrm{Sing}(C) \subseteq C \cap E$.

4) Let C_1, C_2 both of degree n, with $(C_1 \cdot C_2)_p = 1$ $\forall p \in C_1 \cap C_2$.

(So that $\#\{C_1 \cap C_2\} = n^2$).

Suppose that for some $m < n$, $m\cdot n$ of the points $\left(\{P_1,\dots,P_{nm}\} \subset \{C_1 \cap C_2\}\right)$ lie on an irreducible curve of degree m, call it E.

<u>Then</u> the remaining points $(n^2 - nm = n(n-m))$ lie on a curve of degree $n-m$.

<u>Rk</u> (special case):

Start with a conic C ($\deg C = 2$).

Take 6 distinct points on C, $\{P_1, \dots, P_6\}$.

Take lines $L_1 = \overline{P_1P_2}$, $L_2 = \overline{P_2P_3}$, $\dots$ $L_6 = \overline{P_6P_1}$ (<u>exercise</u>: check that they are distinct)

Form 2 degree-3 curves:

$$C_1 = L_1 + L_3 + L_5$$

$$C_2 = L_2 + L_4 + L_6$$

Then $\deg C_1 = \deg C_2 = 3$; $C_1 \cap C_2 \supseteq \{P_1,\dots,P_6\}$ but $C_1 \cap C_2$ also contains $L_1 \cap L_4$, $L_2 \cap L_5$, $L_3 \cap L_6$ (3 more points, call them $\{P_7, P_8, P_9\}$).

So $\#\{C_1 \cap C_2\} = 9 = n^2$.

Then the previous result says that P_7, P_8, P_9 are colinear!

This ~~result~~ is <u>Pascal's Mystic Hexagon.</u>

<u>Pf of (4)</u>: Pick a point $[a,b,c]$ on E, but <u>not</u> in $C_1 \cap C_2$.

Define $F(x,y,z) := \lambda F_1 - \mu F_2$, $\lambda = F_2(a,b,c)$, $\mu = F_1(a,b,c)$.

Look at $F^{-1}(0) \cap E$ contains $n\cdot m$ points from $C_1 \cap C_2 \cap E$, plus $[a,b,c]$.

$\Rightarrow$ at least $nm+1$ points. By Bézout, E is a component of $F^{-1}(0)$ &

If we knit $E = R^{-1}(0)$, then F will factor as $F = R \cdot S$, and $S^{-1}(0)$ is the curve we want.

• Degree-Genus Formula

(for irreducible $C \subset \mathbb{P}^2$ with ordinary double point singularities) ← (if F comes from a smooth R-S, this is what we'll find)

(p is an ordinary double point if there are <u>two distinct</u> tangent lines at p).

This is that C is given locally by $f(x,y)$ $(p \sim (0,0))$

s.t $f(x,y) = f_2(x,y) + \cdots = a x^2 + 2bxy + cy^2 + \cdots$ with $\underline{b^2 - ac \neq 0}$.

Say that the ordinary double points are $p_1, \ldots, p_\delta$.

Let Σ be the normalization of C, i.e. a compact, smooth R.S with

$\varphi : \Sigma \to \mathbb{P}^2$, $\varphi(\Sigma) = C$ etc.

Let $g := \text{genus}(\Sigma)$.

<u>Theorem</u>: If $\deg(C) = d$, then $g = \dfrac{(d-1)(d-2)}{2} - \delta$.

Pf

Let C be defined by $F(x,y,z) = 0$, $\deg F = d$.

At least one of F_x, F_y, F_z is non-trivial.

Can assume $F_z \not\equiv 0$ and $\deg F_z = d-1$

Let E be the curve defined by $F_z = 0$.

Examine $C \cap E$:

a) C irred and $\deg C > \deg E \Rightarrow$ they do not share any component. → apply Bézout.

Get $\displaystyle\sum_{p \in C} (C \cdot E)_p = d \cdot (d-1)$

b) Will count differently ↓

$$\sum_p (C\cdot E)_p = \sum_{i=1}^{\delta} (C\cdot E)_{p_i} + \sum_{\substack{p \text{ smooth} \\ \text{in } C}} (C\cdot E)_p$$

We will show that $(C\cdot E)_{p_i} = 2$, and that $\sum_{\substack{p \text{ smooth} \\ \text{in } C}} (C\cdot E)_p = 2(g+d-1)$ $(C\cap E)'$

1) From the definition and local form $\Rightarrow (C\cdot E)_{p_i} = 2$.

2) Recall that Riemann-Hurwitz says that, if $f: \Sigma \to \Sigma'$,

and $R = \sum_p (\nu_p(f) - 1)\cdot p$ (ramification divisor), then

$\deg R = (2g(\Sigma)-2) - (\deg f)(2g(\Sigma')-2)$.

So if $\Sigma' = \mathbb{P}^1$ and $\deg f = d$, get $\deg R = 2g - 2 + 2d = 2(g+d-1)$.

Hence we want to identify $\sum_{p\in (C\cap E)'} (C\cdot E)_p$ as the degree of R, for some suitable $\varphi: \Sigma \to \mathbb{P}^1$ of degree d.

(1) $(C\cdot E)_{p_i}$:

Fix coordinates s.t. $p_i \sim (0,0)$. So $\begin{cases} C: 0 = f(x,y) = ax^2 + 2bxy + cy^2 + \cdots \\ E: 0 = f_y(x,y) = 2bx + 2cy + \cdots \end{cases}$

Note that $\frac{\partial f_y}{\partial y}(0,0) = 2c$, so if $c \neq 0$ then p_i is smooth of E.

If $c=0$, $f(x,y) = ax^2 + 2bxy + \cdots \Rightarrow$ tangent lines at $\begin{cases}\{x=0\} \\ \{ax+2by=0\}\end{cases}$

Assume $c\neq 0$. Then use the local coordinates on E to compute $(C\cdot E)_{p_i}$:

Since $\frac{\partial f_y}{\partial y} \neq 0$ the Implicit function thm says that $f_y(x,y)=0$ defines $y(x)$.

Hence we have a local normalization for E is $t \longmapsto (t, y(t))$

To compute $(C\cdot E)_{p_i}$, we need to compute the multiplicity of $f(t, y(t))$ at $t=0$.

Lemma: $y(t) = \frac{-b}{c} t + O(t^2)$. $\quad (f(x,y) = ax^2 + 2bxy + cy^2$

Pf $f_y(t, y(t)) = 2bt + 2cy(t) + O(t^2) = 0$. So $0 = 2b + 2c\dot{y}(0) \Rightarrow \dot{y}(0) = \frac{-b}{c}$ //

Hence $f(t, y(t)) = t^2\left(a - \frac{2b^2}{c} + \frac{b^2}{c}\right) + \cdots = t^2\left(\frac{ac - b^2}{c}\right) + \cdots$

$= (C \cdot E)_{p_c}$! //

(2) Compute $\sum_{p \in (C \cap E)'} (C \cdot E)_p = \deg R$ for some R: $\quad (\Psi: \Sigma \to \mathbb{P}^1$ ~~[illegible]~~$)$

$(C \cap \mathbb{C}^2) = f^{-1}(0)$, $\quad (E \cap \mathbb{C}^2) = f_y^{-1}(0)$.

Consider the map $\begin{array}{c} \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C} \\ (x,y) \longmapsto x \end{array}$ and compose it with $\tilde{\Sigma} \to \mathbb{C}^2$

which is the map restricted. (i.e. $\tilde{\Sigma} = \varphi^{-1}(C \cap \mathbb{C}^2)$, where φ is the normalizing map).

Claim: $\pi \circ \varphi =: \tilde{\Psi}$ extends to a map $\Psi: \Sigma \to \mathbb{P}^1$ with:

* $\deg \Psi = d$
* $R = \sum_{q \in \varphi^{-1}((C \cap E)')} \left((C \cdot E)_{\varphi(q)}\right) \cdot q$

Pf At $(x,y) \in f^{-1}(0)$ s.t. $f_y(x,y) \neq 0$, then by i.f.t have $y(x)$ in a nbhd of x.

So again can use x as a local coord. in C: φ is given by

$t \longmapsto (t, y(t))$.

So $\pi \circ \varphi(t) = t$ is unramified at those points.

We still must examine the preimages of the singular points and those in $(C \cap E)'$.

(Subclaim): $\varphi^{-1}(p_j) \notin \operatorname{Supp} R$.

- at $p \in (C \cap E)'$, if $\pi \circ \varphi(t) = t^\nu \tilde{\varphi}(t)$, then $\nu = (C \cdot E)_p + 1$.

we will ommit pt (1) of the claim, and prove the second part:

If $p \in (C \cap E)'$, choose coordinates s.t $p \sim (0,0)$.

So $f(x,y) = 0 = f_y(0,0)$ but $f_x(0,0) \neq 0$ (smooth points).

Hence ifT $\Rightarrow$ $f(x,y) = 0$ defines $x(y)$.

Using y as a local coordinate in C:

$$t \mapsto (x(t), t) \xrightarrow{\pi} x(t)$$

So we need to calculate $\nu = \text{mult}_{t=0}\, x(t)$.

Also, computing $(C \cdot E)_p$ from the defining function f_y using local coords on C,

(get $f_y(x(t), t)$). Have $(C \cdot E)_p = \text{mult}_{t=0}\, f_y(x(t), t)$.

Recall $f(x(t), t) = 0 \Rightarrow$ ~~$f_x(0,0) \cdot \dot{x}(0) + f_y(0,0) = 0 \Rightarrow f_y = -f_x \cdot \dot{x}$~~

$$f_x(x,y) \cdot \dot{x}(y) + f_y(x,y) = 0 \quad \Rightarrow f_y(t) = - f_x(t) \cdot \dot{x}(t).$$

As $f_x(0,0) \neq 0$, $\text{mult}_{t=0}\, f_y = \text{mult}_{t=0}\, \dot{x} = \text{mult}\, x - 1 \Rightarrow \checkmark$.

<u>Remains to do</u>: part 1 of subclaim, and that $\pi \circ \varphi$ can be extended globally to $Z \to \mathbb{P}^1$.

Also, compute the degree of ψ:

At smooth points (i.e away from $\text{Supp}\, R$), we just need to count the number of preimages: $x \mapsto (x, y(x)) \xrightarrow{\pi} x$. So we need to look at $\pi^{-1}(x)$ (as the other part is injective).

So $\pi^{-1}(x) = \{(x,y) : f(x,y) = 0\}$ = roots of $f(x, \cdot) \Rightarrow d$ roots.

We prove the following:

<u>Claim</u>: $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}$ extends to $\pi : \mathbb{P}^2 \setminus \{[0,0,1]\} \longrightarrow \mathbb{P}^1$.

Pf

$$\begin{array}{ccc} \mathbb{C}^2 \cong U_0 & (x,y) \longmapsto x & \\ & \wr \qquad\quad \wr & \\ & [1,x,y] \qquad [1,x] & \end{array} \quad \rightsquigarrow \quad [X,Y,Z] \longmapsto [X,Y]$$

defines a map $\mathbb{P}^2 \setminus \{[0,0,1]\} \longrightarrow \mathbb{P}^1$.

Let $L = \{Z=0\} \subseteq \mathbb{P}^2$, and $p = [a,b,c] \neq [0,0,1]$, then let L_p = line containing $[0,0,1]$ and p.

Then $L_p \cap L$ ~~will~~ define a map $\mathbb{P}^2 - \{[0,0,1]\} \longrightarrow L \simeq \mathbb{P}^1$, which is the map we are looking at: L_p defined by $-bx+ay=0$.

So the intersection with L is $[a,b,0]$. //

Riemann-Roch

We'll count the number of meromorphic $\left\{\begin{array}{l}\text{functions}\\ \text{differentials}\end{array}\right.$ on a compact Riemann Surface, subject to constraints on zeros/poles.

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of Σ.

$f \in K(\Sigma)$, $\omega \in K'(\Sigma)$

a) $f \in K(\Sigma)$ defined by local meromorphic $\{f_\alpha \in K(U_\alpha)\}$

s.t. $z_\beta = z_\beta(z_\alpha)$, then $f_\alpha(z_\alpha) = f_\beta(z_\beta(z_\alpha))$.

b) $\omega \in K'(\Sigma)$ defined by local meromorphic $\{\omega_\alpha \in K^*(U_\alpha)\}$.

where, ~~if~~ we ask $\omega_\alpha(z_\alpha) = \omega_\beta(z_\beta(z_\alpha)) \dfrac{\partial z_\beta}{\partial z_\alpha}$.

At any $p \in \Sigma$, with $z(p)=0$, can write

$f(z) = z^\nu \tilde{f}(z)$, $\tilde{f}(0) \neq 0$ and define $\nu_p(f)$, $\nu_p(\omega)$.

$\omega(z) = z^\nu \tilde{\omega}(z)$, $\tilde{\omega}(0) \neq 0$

Hence we get a divisor.

$$(f) = \sum \nu_p(f)\cdot p = \underbrace{(f)_0}_{\text{zeros}} - \underbrace{(f)_\infty}_{\text{poles}} \qquad \deg (f) = \sum_p \nu_p(f) = 0$$

$$(\omega) = \sum \nu_p(\omega)\cdot p = (\omega)_0 - (\omega)_\infty \qquad \deg (\omega) = -\chi(\Sigma) = 2-2g$$

Now, for any divisor $D = \sum n_i P_i$, let $\mathcal{L}(D) = \{f \in K(\Sigma) : (f)+D \geq 0\} \cup \{0\}$

($(f)+D \geq 0$ means, if $D = \sum n_i P_i$, that for P_i s.t. $n_i > 0$, f can have a pole of order $\leq n_i$; for P_i s.t. $n_i < 0$, f has to have a zero of at least order n_i; f has no other poles.

Ex: $\mathcal{L}(D)$ is a vectorspace.

Let $\ell(D) := \dim_{\mathbb{C}} \mathcal{L}(D)$.

Special case: $D=0$. Then $(f) \geq 0 \Leftrightarrow f \in Hol(\Sigma) \Rightarrow \ell(D) = 1$.

Now, for $\omega \in K'(\Sigma)$, define $K'(D) := \{\omega \in K'(\Sigma) : (\omega) \underset{\substack{\uparrow\\ \text{careful!}}}{-} D \geq 0\} \cup \{0\}$.

Let $i(D) := \dim_{\mathbb{C}} K'(D)$

If genus$(\Sigma) = g$ and $\deg D = \sum n_i = d$, then R.R. says:

$$\boxed{\ell(D) - i(D) = d + 1 - g}$$

Special case: $D=0$. Then we get that $i(0) = g$.

Note that $i(0) = \cancel{\dim \Omega'(\Sigma)\ \text{hol}}\ \dim \Omega'(\Sigma)$ (holomorphic differentials)

So R.R. says that $\dim_{\mathbb{C}} \Omega'(\Sigma) = g$.

Special case: $g=0$. Then $\Sigma = \mathbb{P}^1$, Given $\omega \in \Omega'(\mathbb{P}^1)$, we can define

$f : \mathbb{P}^1 \to \mathbb{C}$ holomorphic by $f(p) = \int_{p_0}^{p} \omega$ (any fixed p_0).

i.e. $\omega = df = 0$. So $\Omega'(\mathbb{P}^1) = \{0\}$.

So for $g=0$, $D=0$, $\left[\begin{matrix} \ell(0)=1 & d=0 \\ i(0)=0 & g=0 \end{matrix}\right] \Rightarrow$ RR true in this case.

In fact, on $\mathbb{P}^1$, if $D \geq 0$. $K^1(D) = \{\omega \in K^1(Z) : (\omega) - D \geq 0\} \subseteq \Omega^1(Z)$

so on $\mathbb{P}^1$, $K^1(D) = \{0\}$, i.e. $i(D) = 0$.

In this case, RR says $\ell(D) = d + 1$.

We know that $K(\mathbb{P}^1) \leftrightarrow$ rational functions on $\mathbb{C}$

$$f \rightsquigarrow g(z) = \frac{P(z)}{Q(z)} \qquad f[1, z] = g(z).$$

Assume that $D \subseteq U_0$, and on $\mathbb{C}$ is given by $D = \sum n_i P_i$, $P_i \in \mathbb{C}$, $n_i > 0$

Then $f \in \mathcal{L}(D)$ is of the form:

$$\frac{P(z)}{(z - P_1)^{m_1} \cdots (z - P_N)^{m_N}}, \quad m_i \leq n_i$$

Moreover, as f has no pole at ∞, $\deg P(z) \leq m_1 + \cdots + m_N$

We use partial fractions:

$$g(z) = \sum_{j=0}^{m_1} \frac{A_{1j}}{(z - P_1)^{j}} + \sum_{j=0}^{m_2} \frac{A_{2j}}{(z - P_2)^j} + \cdots$$

So $\mathcal{L}(D) = \mathrm{Span}\left\{ 1, \frac{1}{(z - P_1)}, \dots, \frac{1}{(z - P_1)^{n_1}}, \dots \right\} \Leftarrow \ell(D) = d + 1$.

Now if $D \geq 0$ but $g > 0$,

<u>Proposition</u>: $\ell(D) \leq d + 1$ (if $D \geq 0$ ~~and $g>0$~~).

Pf Construct maps in an exact sequence: $0 \to \mathbb{C} \xrightarrow{a} \mathcal{L}(D) \xrightarrow{b} \mathbb{C}^d$

and so $\dim \mathcal{L}(D) + 1 \leq d$.

The map $\underline{a}$ is clear: sends a constant to the constant function.

Let $D = \sum n_i P_i \geq 0$.

Pick local coordinates z_i on a nbhd of P_i.

For any $f \in \mathcal{L}(D)$, $f(z) = \frac{A_{i,n_i}}{z_i^{n_i}} + \cdots + \frac{A_{i,1}}{z_i} + \overbrace{\tilde{f}(z)}^{\text{hol.}}$,

(cont pf of prop.)

Take the coefficients $\{A_{1,1}, \ldots, A_{1,n_1}, A_{2,1}, \ldots, A_{2,n_2}, \ldots, A_{N,1}, \ldots, A_{N,n_N}\}$ to construct the map $\mathcal{L}(D) \to \mathbb{C}^d$.

Note that the kernel of the map is exactly $\mathcal{H}ol(\Sigma)$ = constant functions.

Next, note that $\Omega^1(\Sigma) = K^1(0)$
$\supseteq K^1(D)$ if $D \geq 0$ $\Leftarrow$ so $i(D) \leq g$.

We will show that $\dim \Omega^1(\Sigma) = g$.

If $\omega \in \Omega^1(\Sigma)$ is a holomorphic 1-form,

We know that for any 1-form $\lambda(z,\bar{z}) = \alpha(z,\bar{z})\,dz + \beta(z,\bar{z})\,d\bar{z}$ and any path γ in Σ, can compute $\int_\gamma \lambda$.

__For loops__: if γ, γ' are two loops in Σ, then $\int_\gamma \lambda - \int_{\gamma'} \lambda = \int_{\gamma-\gamma'} \lambda$

If $\gamma - \gamma' = \partial\Omega$, then $\int_{\gamma-\gamma'} \lambda = \int_{\partial\Omega} \lambda = \iint_\Omega d\lambda$. So if $d\lambda = 0$, $\int_\gamma \lambda = \int_{\gamma'} \lambda$ whenever $\gamma - \gamma' = \partial\Omega$.

So $H_1(\Sigma, \mathbb{C}) = \{\text{loops in } \Sigma\}/\sim$ where $\gamma \sim \gamma' \Leftrightarrow \gamma - \gamma' = \partial\Omega$.

and we know that (Alg Top) $\dim_{\mathbb{C}} H_1(\Sigma, \mathbb{C}) = 2g$.

If λ satisfies $d\lambda = 0$, then we get $H_1(\Sigma, \mathbb{C}) \xrightarrow{\lambda} \mathbb{C}$ (trivial map if $\lambda = df$)

$[\alpha] \rightsquigarrow \int_\alpha \lambda$

Conversely, we have:

__Fact__: If $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a basis for the homology and $\int_{\gamma_i} \lambda = 0 \;\; \forall i$,

then $\lambda = df$ for a function $f \in \mathcal{C}^\infty(\Sigma, \mathbb{C})$.

(define $f(p) = \int_{p_0}^{p} \lambda$ for any path $p_0 \rightsquigarrow p$, for any fixed p_0).

Note that if $\omega = f(z)\,dz$ is holomorphic, then $d\omega = \frac{\partial f}{\partial z} \cancel{dz \wedge dz} + \cancel{\frac{\partial f}{\partial \bar z}} dz \wedge d\bar z = 0$

so all holomorphic 1-forms are closed.

To see that $\dim \Omega'(C) \leq g$, suppose that $\exists$ lin. indep $\omega_1, \ldots, \omega_{g+1}$.

(i.e. $\sum c_i \omega_i = 0 \Rightarrow c_i = 0 \ \forall i$).

Consider the system of $2g$ linear equations for $\alpha_1, \ldots, \alpha_{g+1}, \beta_1, \ldots, \beta_{g+1}$:

$$\int_{\gamma_j} \left(\sum \alpha_i \omega_i + \sum \beta_i \bar\omega_i \right) = 0 \quad \text{for each loop } \gamma_j \text{ in a basis for } H_1(\Sigma, \mathbb{Z}).$$

The coefficients in the equation are the $\int_{\gamma_j} \omega_i$ and $\int_{\gamma_j} \bar\omega_i$, which defines a map $\mathbb{C}^{2g+2} \to \mathbb{C}^{2g}$.

Let $\alpha_1, \ldots, \alpha_{g+1}, \beta_1, \ldots, \beta_{g+1}$ be a nontrivial solution (i.e. in the kernel of that map).

Let $\varphi := \sum \alpha_i \omega_i$, $\psi := \sum \bar\beta_i \omega_i$ $\quad (\varphi, \psi \in \Omega'(C))$, and $\int_{\gamma_j} \varphi + \bar\psi = 0 \quad j = 1..2g$

Hence $\varphi + \bar\psi$ is exact. $\Rightarrow \varphi + \bar\psi = df$ (some f).

<u>Lemma</u>: If $\varphi, \psi \in \Omega'(C)$ and $\varphi + \bar\psi = df$, then $\varphi = \psi = 0$.

Pf ommited.

So by the lemma, $\sum \alpha_i \omega_i = 0 = \sum \bar\beta_i \omega_i$, and at least one of the two linear combinations is nontrivial $\Rightarrow$ ✓.

<u>Note</u>: $\dim \Omega'(C) \geq g$: as:

we want to construct at least g lin. indep $\omega_i \in \Omega'(C)$.

we can embed $\Sigma \hookrightarrow \mathbb{P}^2$ as a plane algebraic curve, with at worst ordinary double points say $p_1, \ldots, p_\delta$.

We will find a linear space $S^{d-3}(-\Gamma)$ of dimension g, and an injection $S^{d-3}(-\Gamma) \hookrightarrow \Omega^1(C)$.

Clues: $g = \frac{(d-1)(d-2)}{2} - \delta$ ($d = \deg F$, the defining polynomial).

$\frac{(d-1)(d-2)}{2} = \dim S^{d-3}$ where $S^k = \{$hom poly of deg k in $(x,y,z)\}$.

From $\Sigma \hookrightarrow \mathbb{P}^2$, consider the affine piece, i.e. $F=0$ on $U_0 \subset \mathbb{P}^2$, defined by $f(x,y)=0$.

Note that $f_x\,dx + f_y\,dy = 0$ on the curve $C_0 = f^{-1}(0)$.

On $\{(x,y) \in C_0 \mid f_y(x,y) \neq 0\}$, we can use x as a local coordinate (imp. F.T.).

So one can define $\frac{dx}{f_y(x,y(x))}$

Similarly, on $\{(x,y) \in C_0 \mid f_x(x,y) \neq 0\}$ we can define $\frac{-dy}{f_x(x(y),y)}$ ← to make the two compatible.

On the overlaps, they patch together to define $\omega \in \Omega^1(C_0^*)$.

Pf follows from $f_x\,dx + f_y\,dy = 0$ //.

But ω has singularities at $p_1, \dots, p_\delta$ and is only defined on C_0^* ← need to extend it!

To extend ω, we multiply by a suitable polynomial $g(x,y) = G[1,x,y]$, where $G[x,y,z]$ is homogeneous, not identically zero on C, with zeros at $p_1, \dots, p_\delta$ (so to cancel the poles of ω). (see Griffiths, pg 104).

We get enough such holomorphic differentials if $\deg G = d-3$.

Let $\Gamma := p_1 + \dots + p_\delta$ (as a divisor). And so $S^{d-3}(-\Gamma) = \{$hom pol. of deg $d-3$ s.t. vanish at $p_i\ \forall i\}$.

Note that $\dim S^{d-3}(-\Gamma) \geq \dim S^{d-3} - \delta = g$.

We still have to check that this makes an injection. $S^{d-3}(-\Gamma) \hookrightarrow \Omega^1(C)$.

If ω_G is a one form defined by G is $\omega_G \equiv 0$, this would mean that $C \subset G^{-1}(0)$.

However, $\deg C = d$ and $\deg G^{-1}(0) = d-3 \Rightarrow$ can't happen. //

Prop: if $D \geq 0$, then $\ell(D) \leq d+1-g+i(D)$ (note that $i(D) \leq g$, so this is an improvement).

Note: If $D \geq 0$ and $i(D) \neq 0$, then $\ell(D) \leq d+1-g+i(D)$ is enough to prove R.R:

For if $i(D) \neq 0$, then $\exists\, \omega \in K'(D)$ (i.e. $(\omega) - D \geq 0$). Let $E := (\omega) - D$.

Then $E \geq 0$, $\deg E = \deg(\omega) - \deg D = 2g-2-d$ $(d = \deg D)$.

Apply the estimate to E, to get $\ell(E) \leq (2g-2-d)+1-g+i(E) = g-1-d+i(E)$

Claim: $\ell(E) = i(D)$ and $i(E) = \ell(D)$.

Pf/ Given D, E s.t. $D + E = (\omega)$, for some $\omega \in K'(\Sigma)$, we get an isomorphism $\mathcal{L}(D) \xrightarrow{\sim} K'(E)$:

$$f \longmapsto f\cdot\omega$$

it is a map $K(\Sigma) \to K'(\Sigma)$, we need to check that it restricts properly:

a) $(f) + D \geq 0 \Rightarrow (f\omega) - E \geq 0$ (✓).

b) Inverse: $\varphi \in K'(E) \longmapsto \varphi/\omega \in \mathcal{L}(D)$.

With this, then $\ell(D) \geq d+1-g+i(D)$ which is the converse inequality!

Pf of Prop:

We construct a map $\mathbb{C}^d \times \Omega'(C) \to \mathbb{C}$

Fix local coordinates at each p_i, say z_i ← defined on U_i s.t. $z_i(p_i) = 0$.

Look at the Laurent series at each p_i in z_i, of order n_i.

$$\left\{ \eta_i = \frac{A_{i,n_i}}{z_i^{n_i}} + \cdots + \frac{A_{i,1}}{z_i} \right\} \simeq \mathbb{C}^{n_i}. \text{ So think of } \mathbb{C}^d \simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i}$$

Given $\omega \in \Omega'(C)$, define $\eta_i \cdot \omega \in K'(U_i)$. Compute $\mathrm{Res}_{p_i}(\eta_i \omega)$: $\omega = \sum^{\nu} h(z)\,dz$

Exercise. $\mathrm{Res}_{p_i}(\eta_i \omega) = 0$ if $\nu \geq n_i$

Define finally the map $\mathbb{C}^d \times \Omega^1(C) \xrightarrow{\mathrm{Res}} \mathbb{C}$ by

$$((\eta_1, \dots, \eta_N), \omega) \longmapsto \sum_{i=1}^{N} \mathrm{Res}_{p_i}(\eta_i \omega) \qquad \left(\text{think } \mathbb{C}^d = \bigoplus_{i=1}^{N} \mathbb{C}^{n_i}\right)$$

Check: a) $\mathrm{Res}(-,-)$ is bilinear

b) $\mathrm{Res}(\vec{\eta}, \omega) = 0 \;\; \forall \vec{\eta} \Leftrightarrow (\omega) - D \geq 0$ (i.e. $\omega \in K^1(D)$).

So one can define $\mathbb{C}^d \times \left(\Omega^1(C) / K^1(D)\right) \longrightarrow \mathbb{C}$ which is a non-degenerate bilinear pairing.

Note: if the η_i are the local versions of a $f \in K^1(\Sigma)$, then $\sum \mathrm{Res}_{p_i}(\eta_i \omega) = \sum \mathrm{Res}(f\omega) = 0$

So $\mathrm{Res}(-, \omega)$ is zero on $\mathcal{L}(D)/\mathbb{C} \subseteq \mathbb{C}^d$ ← the embedding defined previously.

Linear Algebra fact: given $B: V \times W \to \mathbb{C}$ bilinear and non-degenerate, with a subspace $L \subset V$ s.t. $B|_{L \times W} \equiv 0$.

Then: $\dim L \leq \dim V - \dim W$.

In our case, as $L = \mathcal{L}(D)/\mathbb{C}$ $(\dim = \ell(D) - 1)$, $V = \mathbb{C}^d$ ($\dim d$) and $W = \Omega^1(C)/\Omega^1(D)$ ($\dim g - i(D)$) the result follows.

The remaining thing to prove is that, for any D, $\ell(D) \geq d + 1 - g$ (this will prove the whole R.R, as we will see later).

Idea: Use that $\Sigma \to \mathbb{P}^2$ as a plane alg. curve with ordinary double points.

Construct then $f \in K^1(\Sigma)$ as the pullbacks of $\frac{H}{G}$, H, G homogeneous polynomials of the same degree in (x, y, z).

In order to get $f \in \mathcal{L}(D)$ $\left(f = i^*\left(\frac{H}{G}\right)\right)$, suppose that $i(\Sigma) = C \subseteq \mathbb{P}^2$ is defined by $\{F(x,y,z) = 0\}$.

Denote $\underset{\substack{\uparrow \\ \text{divisor on } \Sigma}}{H \cdot \Sigma} = i^{-1}\big(\underset{\substack{\uparrow \\ \text{divisor on } C}}{H^{-1}(0) \cdot F^{-1}(0)}\big)$ and $G \cdot \Sigma$ similarly.

Let finally $\Delta := i^{-1}(\underbrace{p_1 + \dots + p_\delta}_{\text{ordinary double points.}})$

<u>One can show</u>: if $D = D' - D''$ where $D' \geq 0$, $D'' \geq 0$, then we need:

a) $G \cdot \Sigma \geq D' + \Delta$

b) $H \cdot \Sigma \geq D'' + G \cdot \Sigma - D'$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow H \cdot \Sigma \geq D'' + \Delta$

c) $\deg G = \deg H$

d) $F \nmid G$.

We will ~~not~~ see it, by one can show that, for large enough degree of G, one can find at least $d + 1 - g$ l.i. G/H. $\Rightarrow \ell(D) \geq d + 1 - g$ //.

So far, we know:

a) $\ell(D) \leq d + 1 - g + i(D) \leq d + 1$ if $D \geq 0$.

b) $\ell(D) \geq d + 1 - g$

c) if $D + E = (\omega)$ where $\omega \in K'(\Sigma)$, then $\begin{cases} i(D) = \ell(E) \\ \ell(D) = i(E) \end{cases}$

From this we already get RR, ie for any D, $\ell(D) - i(D) = d + 1 - g$.

<u>Ex</u>: Consider the following cases:

a) $\ell(D) = i(D) = 0$

b) $\ell(D) \neq 0$, $i(D) = 0$

c) $\ell(D) = 0$, $i(D) \neq 0$

d) $\ell(D) \cdot i(D) \neq 0$.

Consequences of Riemann-Roch.

- Special case $D = p$. $(D \geq 0,\ d = 1)$.

  $(\omega) - D \geq 0 \iff \omega \in \Omega^1(\Sigma)$ and $\omega(p) = 0$.

Case 1 $g = 0$ Then $i(D) = 0$, and so $RR \Rightarrow \ell(p) = 2 = \dim_{\mathbb{C}} \mathcal{L}(p)$.

But $(f) + p \geq 0$ is satisfied by the constants $(\simeq \mathbb{C})$.

So $\exists f \in K'(\Sigma)$, $f \neq$ constant, with a single pole at p.

We can use f to define a map $f : \Sigma \to \mathbb{P}^1$.

As $f^{-1}(\{\infty\}) = \{p\}$ with multiplicity one, $\deg f = 1$, so f is a biholomorphism $\Rightarrow \Sigma \simeq \mathbb{P}^1$.

Case 2: $g > 0$

claim: $i(p) < g$ (we know $0 \leq i(p) \leq g$)

Pf If $i(p) = g$, then $\ell(p) = 1 + 1 - g + g = 2$. As before, we would get that $\Sigma \simeq \mathbb{P}^1$ but they have different genus!!

If $g \geq 1$:

This means then that $\Omega^1(p) \subsetneq \Omega^1(\Sigma)$, so $\exists\, \omega \in \Omega^1(\Sigma) \setminus \Omega^1(p)$,

i.e. $(\omega) - p$ is not effective (ie $\omega(p) \neq 0$)

So for any $p \in \Sigma$, not all hol differentials vanish at p

Take a basis $\{\omega_1, \dots, \omega_g\}$ for $\Omega^1(\Sigma)$, and define a map:

$$\varphi : \Sigma \longrightarrow \mathbb{P}^{g-1}$$
$$p \longmapsto [\tilde{\omega}_1(p), \dots, \tilde{\omega}_g(p)]$$

(in local coordinates $\omega_i(z) = \tilde{\omega}_i(z)\,dz$).

which is well-defined by the previous observation.

Q: is φ 1-1 ?

A: Yes! (usually)

If $\varphi : \Sigma \to \mathbb{P}^{g-1}$ is not 1-1, then $\exists\, p, q \in \Sigma$ s.t. $\varphi(p) = \varphi(q)$,

i.e. $\omega_i(p) = \lambda\, \omega_i(q)$ for some $\lambda \in \mathbb{C}^*$ $(i = 1..g)$.

Set $D := p + q$, a degree-2 divisor on Σ.

Claim: $\Omega^1(D) = \Omega^1(p)$ ✓.

(i.e. $\omega(p) = 0 \Rightarrow \omega(q) = 0$)

Lemma: $i(p) = g - 1\ (= i(D))$ by the previous claim

pf/ $\Omega^1(p) = \{ \omega = \sum \lambda_i \omega_i \;:\; \omega(p) = 0 \}$.

Think of $\sum \lambda_i \omega_i(p) = 0$ as a linear equation on the λ_i's.

and then it is obvious that $\dim \Omega^1(p) = g - 1$. //

Apply now R-R to the divisor $D = p + q$:

$\ell(p+q) = 2 + 1 - g + g - 1 = 2$.

Again we get $f : \Sigma \to \mathbb{P}^1$ of degree 2:

$f^{-1}(\infty) = \{p, q\}$ with multiplicity 1.

Definition: Σ is called hyperelliptic if $\exists$ a degree-2 map $f : \Sigma \to \mathbb{P}^1$.

So if Σ is not hyperelliptic, we get what is called the canonical embedding

$\Sigma \hookrightarrow \mathbb{P}^{g-1}$.

Note: if $g = 2$, then we get $f : \Sigma \to \mathbb{P}^{2-1} = \mathbb{P}^1$. This cannot be injective

~~Claim, this map has degree 2~~ (as genus$(\Sigma) = 2 \neq$ genus$(\mathbb{P}^1) = 0$).

So all genus-2 curves are hyperelliptic.

E.O.C.