Understanding Coleman's Theory of Integration

Marc Masdeu-Sabaté

April 28, 2008

1 Results in p-adic analytic geometry

1.1 Affinoids

Consider $\mathbb{C}_p$, and fix an absolute value $|\cdot|$ on it. Fix $K \subseteq \mathbb{C}_p$ a complete subfield. Write R for the maximal order of K, and $\mathfrak{p}$ for the maximal ideal of R. Let $\mathbb{F} = R/\mathfrak{p}$ be the residue field of K ($\mathbb{F}$ is an algebraic extension of $\mathbb{F}_p$).

Given an affinoid X over K, denote by $A(X)$ the *algebra of rigid analytic functions on X over K*. We have that $\operatorname{Sp} A(X) = X$.

For $f \in A(X)$, and $x \in X$, write $|f(x)|$ for the absolute value of the image of f in $A(X)/x$. Set also:

$$\|f\|_X \stackrel{\text{def}}{=} \sup_{x \in X}\{|f(x)|\}$$

and define $A_0(X) \stackrel{\text{def}}{=} \{f \in A(X) \mid \|f\|_X \leq 1\}$.

We have that $\|f\|_X$ is a seminorm (called the spectral norm) on $A(X)$, and that $A_0(X)$ is a sub-R-algebra of $A(X)$. The spectral norm is a norm if X is reduced, and then $A(X)$ is complete with respect to this norm.

Set also $\tilde{A}(X) \stackrel{\text{def}}{=} A_0(X)/\mathfrak{p}A_0(X)$, and $\tilde{X} \stackrel{\text{def}}{=} \operatorname{Spec} \tilde{A}(X)$. Then $\tilde{X}$ is a scheme of finite type over $\mathbb{F}$ if $A_0(X)$ is of topological finite type over R (true if $K = \mathbb{C}_p$, or K a DVR). In general, $\tilde{X}^{\text{red}}$ is of finite type.

Definition 1.1. We say that X has **good reduction** if $A_0(X)$ is of topological finite type over R and $\tilde{X}$ is smooth over $\mathbb{F}$.

Lemma 1.2. *Suppose that $r\colon Y \to X$ is a morphism of affinoids over K, such that the image of $\tilde{Y}$ is a dense open subset of $\tilde{X}$. Let $f \in A(X)$. Then $\|f\|_X = \|f \circ r\|_Y$.*

Definition 1.3. A **Tate R-algebra** is an R-algebra of the form

$$R\langle x_1, \cdots, x_n\rangle / I$$

for some finitely-generated ideal I of $R\langle x_1, \cdots, x_n\rangle$, the ring of restricted power series in $x_1, \cdots, x_n$ (which is actually the completion of $R[x_1, \cdots, x_n]$ over R).

Definition 1.4. The **annihilator** in A of $r \in R$ is:

$$\operatorname{Ann}_A(r) \stackrel{\text{def}}{=} \{a \in A \mid ra = 0\}$$

Definition 1.5. Given a homomorphism $A \to B$ of Tate R-algebras, we say that B **is R-torsion free over** A if:

$$\operatorname{Ann}_B(r) = \operatorname{Ann}_A(r) \cdot B$$

for all $r \in R$.

Let any Tate R-algebra A, we set $\tilde{A} \overset{\text{def}}{=} A/\mathfrak{p}A$.

Definition 1.6. We say that B **is formally smooth over** A if $\tilde{B}$ is smooth over $\tilde{A}$ and B is R-torsion free over A.

The proof of the following statement is ommited, as it is not needed in the sequel.

Proposition 1.7. *The following are equivalent:*

1. *B is formally smooth over A,*
2. *$\tilde{B}$ is smooth over $\tilde{A}$ and B is flat over A,*
3. *B/rB is smooth over A/rA for all $r \in R$.*

The following theorem is what is needed to prove Theorem 1.12.

Theorem 1.8. *Suppose that there is a commutative diagram of Tate R-algebras*

$$\begin{array}{ccc} D & \longleftarrow & B \\ \uparrow & \swarrow & \uparrow \\ C & \longleftarrow & A \end{array}$$

such that $\tilde{C} \to \tilde{D}$ is surjective and B is formally smooth over A. Suppose that there is a homomorphism $s \colon \tilde{B} \to \tilde{C}$ making the reduction of the diagram commutative. Then there is a lifting $\overline{s} \colon B \to C$ of s which makes the original diagram commutative.

Proof. We will proceed by proving several lemmas that will patch together to get our result.

Lemma 1.9. *Suppose that $A \to B$ is a surjective homomorphism of Tate R-algebras. Then its kernel is finitely generated.*

Proof. First, note that WLOG can assume that $A = R_m$ (in general, A is a quotient of it, so there is no harm in replacing it). The hypothesis says that B is a quotient of R_n (for some $n \geq 0$), with finitely generated kernel J:

$$0 \to J \to R_n \to B \to 0$$

Let now $h \colon R_n \to R_m$ be a homomorphism such that te following commutes:

$$\begin{array}{ccc} & & R_n \\ & \overset{h}{\swarrow} & \downarrow \\ R_m & \longrightarrow & B \end{array}$$

Take now $x'_1, \cdots x'_m$ lifts (in R_n) of the images of $x_1, \cdots x_m \in R_n$ in B. Then the kernel of $R_m \to B$ is generated by $h(J)$, together with the set of $\{x_i - h(x'_i) \mid 1 \leq i \leq m\}$, so it's finitely generated. □

As B is *topologically of finite type* over R, a fortiori it is so over A. Hence there is a surjection $A_n \twoheadrightarrow B$, for some n. By the previous lemma, there exist $G_1 \cdots, G_m \in A_n$, such that

$$B \simeq A_n/\langle G_1, \cdots, G_m \rangle \text{ as an } A\text{-algebra.}$$

Let $G \overset{\text{def}}{=} (G_1, \cdots, G_m) \in A_n^m$, and let g denote the composition $A_n \to B \to D$, and $\tilde{V}$ the composition $\tilde{A}_n \to \tilde{B} \overset{s}{\to} \tilde{C}$. The fact that $\tilde{C} \twoheadrightarrow \tilde{D}$ is surjective implies that $C \twoheadrightarrow D$, and so $D \simeq C/I$, for some ideal I.

Lemma 1.10. *There exists a* $V\colon A_n \to C$ *which lifts* $\tilde{V}$*, and such that* $V \equiv g \pmod{I}$.

$$\begin{array}{ccccc} & & \overset{g}{\frown} & & \\ A_n & \longrightarrow & B & \longrightarrow & D = C/I \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{A}_n & \longrightarrow & \tilde{B} & \longrightarrow & \tilde{D} \\ & & \underset{\tilde{V}}{\smile} & & \end{array}$$

Proof. As $C \twoheadrightarrow D$, there exists a hom. $g'\colon A_n \to C$ such that

$$g' \equiv g \pmod{I}$$

(just take, if $X = (x_1, \cdots, x_n)$, $g'(X)$ to be any lift of $g(X)$, and extend to all A_n in the natural way). In the same way, $\tilde{V}$ can be lifted to $V'\colon A_n \to C$. Then one has:

$$V'(X) - g'(X) \in (\mathfrak{p} + I)^n \subseteq C^n$$

Now let $a \in \mathfrak{p}^n \subseteq C^n$, and $b \in I^n \subseteq C^n$ be such that:

$$V'(X) - g'(X) = a - b$$

Set then $d \overset{\text{def}}{=} V'(X) - a = g'(X) - b$, and clearly $\tilde{d} = \tilde{V}$, and $d \equiv g \mod I$.

Hence we may take V to be the unique homomorphism $A_n \to C$ such that $V(X) = d$. □

The homomorphism V is a first approximation to the lifting we are after.

We need to construct a sequence of approximations that tends to our desired lift. As our algebra is complete, we will be then get the lift by taking the limit.

Lemma 1.11. *There exists an* $n \times m$ *matrix* N*, and* $m \times m$ *matrices* M *and* Q *over* A_n *such that:*

$$G\left(X + NG\right) = G^t M G + QG$$

where the coordinates of Q *are in* $\mathfrak{p}A_n$*.*

Let now $V_0 = V$, and define recursively V_k by setting

$$V_{k+1}(X) \overset{\text{def}}{=} V_k(X) + N\left(V_k(X)\right) G\left(V_k(X)\right)$$

As $V_{k+1}(X) \in C^m$, it determines a unique homomorphism $V_{k+1}\colon A_n \to C$. From the previous lemma, $V_{k+1} - V_k \to 0$. As Tate algebras are complete, the limit of these will do. □

1.2 Lifting Morphisms

If $h\colon X \to Y$ is a morphism of affinoids over K, denote by $\tilde{h}\colon \tilde{X} \to \tilde{Y}$ its reduction. Given $\tilde{h}$, we say that h lifts $\tilde{h}$.

Theorem 1.12. *Suppose* $K = \mathbb{C}_p$ *or* K *a DVR. Suppose that there is a commutative diagram of* ***reduced*** *affinoids over* K*:*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

such that $W \to Y$ is a closed immersion, and $\tilde{X}$ is smooth over $\tilde{Z}$. Suppose that $h\colon \tilde{Y} \to \tilde{X}$ is a morphism commuting with the reduction of the given diagram. Then there is a lifting $\overline{h}\colon Y \to X$ of h commuting with the diagram:

$$\begin{array}{ccc} W & \longrightarrow & X \\ \big\downarrow & \overset{\overline{h}}{\nearrow} & \big\downarrow \\ Y & \longrightarrow & Z \end{array}$$

Suppose now that S is a scheme over a field F, and $\sigma\colon F \to F$ is an automorphism of F. Let S^σ be the scheme over F obtained by base change via σ. We have a commutative diagram:

$$\begin{array}{ccc} S^\sigma & \xrightarrow{\quad\sigma\quad} & S \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(F) & \xrightarrow{\sigma} & \mathrm{Spec}(F) \end{array}$$

Given a form (a function or a differential) f on S, one denotes by f^σ its pullback via σ to S^σ. The resulting map $f \mapsto f^\sigma$ is σ-linear (but not linear, in general).

Let now X be an affinoid over $F = K$, and let $S = \mathrm{Spec}(A(X))$ (over $\mathrm{Spec}(K)$), and let σ be a continous automorphism of K. Let then X^σ be the affinoid characterized by $\mathrm{Spec}(A(X^\sigma)) = S^\sigma$, as schemes over K.

Next, consider the case $F = \mathbb{F}$, and let σ be the Frobenius automorphism of $\mathbb{F}$. For each $n \in \mathbb{Z}^+$, there is a canonical morphism $\phi\colon S \to S^{\sigma^n}$, called the Frobenius morphism, and characterized by the equation $\phi^* f^{\sigma^n} = f^{p^n}$ (for $f \in \mathcal{O}_S(U)$). If S is of finite type over $\mathbb{F}$, then there exists some positive integer n such that $S \simeq S^{\sigma^n}$. Fix an F-isomorphism $\rho\colon S^{\sigma^n} \to S$, and then the morphism $\rho \circ \phi\colon S \to S$ is called a **Frobenius endomorphism** of S.

Now suppose that X is an affinoid over K, and $\overline{\sigma}$ is a continuous automorphism of K, which restricts to the Frobenius automorphism σ on $\mathbb{F}$. As $\tilde{X}$ is of finite type over $\mathbb{F}$, then $\tilde{X}$ has Frobenius endomorphisms, and an endomorphism of X lifting one of those is called a **Frobenius endomorphism** of X. Such an endomorphism is actually K-linear (and not just σ-linear). We have the following corollary:

Corollary 1.13. *Suppose that X is a reduced affinoid over K with good reduction. Then:*

1. *X possesses a Frobenius endomorphism.*
2. *There is a morphism from X to $X^{\overline{\sigma}}$ lifting the robenius morphism $\tilde{X} \to \tilde{X}^\sigma$.*
3. *$X \simeq X^{\overline{\sigma}^n}$ for some positive integer n.*

Let X be as in the previous corollary, and let ϕ be a Frobenius endomorphism of X. In each residue class U of X there is a unique point ε_U such that

$$\phi^m(\varepsilon_U) = \varepsilon_U$$

for some positive integer m. It can be computed/defined as: think of U as a point in $\tilde{X}(\mathbb{F}^a)$ (over the algebraic closure of $\mathbb{F}$). Then there is some m such that $\tilde{\phi}^m(U) = U$, because U is defined over some finite extension of $\mathbb{F}_p$. Then

$$\varepsilon_U = \lim_{n\to\infty} \phi^{mn}(x)$$

for any $x \in U$. This point ε_U is called a **Teichmüller** point of ϕ.

1.3 Differentials

Suppose that X is an affinoid over K. Let $\Omega^1_{X/K}$ be the module of rigid differentials on X, and $d\colon A(X) \to \Omega^1_{X/K}$ the natural derivation. We define $\Omega^i_{X/K}$ as the i-th exterior power of $\Omega^1_{X/K}$.

If W is any rigid space over K, we can make a natural complex of rigid sheaves $(\Omega^\bullet_{W/K}, d)$ on W. A **closed differential** will then be an element $\omega \in H^0(W, \Omega^1_{W/K})$ such that $d\omega = 0$.

Proposition 1.14. *Let X be a connected reduced affinoid with good reduction over K. Let $D = \mathrm{red}^{-1}\tilde{\Delta}$, where $\tilde{\Delta}$ is the diagonal in $\tilde{X} \times \tilde{X}$. Then D has a natural structure of rigid analytic space, and we let $A(D)$ be the ring of rigid analytic functions on D. Consider $p_1, p_2\colon D \to X$ the two natural projections. Suppose that ω is closed on X. Then:*

$$p_1^*\omega - p_2^*\omega \in dA(D)$$

Proof. Let $\mathcal{C}$ be a cover of $\tilde{X}$ by affine opens such that $Y \in \mathcal{C}$ may be expressed as a finite unramified covering of an affine open subset of $\mathbb{A}^d_{\mathbb{F}}$, where d is the dimension of X. For each $Y \in \mathcal{C}$, the inverse image $\overline{Y} = \mathrm{red}^{-1} Y$ has a natural structure of an affinoid over K such that $\tilde{\overline{Y}} = Y$.

Fix $Y \in \mathcal{C}$. There exist functions $\tilde{x}_1, \cdots, \tilde{x}_n$ on Y, which are local parameters at each point of Y (by how we are taking our Y). Let $x_1, \cdots, x_n$ be liftings to $\overline{Y}$. Then $x_1, \cdots, x_n$ are also local parameters everywhere on $\overline{Y}$. So we may write:

$$\omega = f_1 dx_1 + \cdots + f_n dx_n$$

for some $f_i \in A(Y)$.

The idea of the proof can be seen in the case $n = 1$. In that case, write $\omega = f(x)dx$, and then $p_1^*\omega - p_2^*\omega = f(x)dx - f(y)dy$. We want to "integrate" this. So let $h \stackrel{\text{def}}{=} x - y$, and rewrite the previous expression as $f(y+h)d(y+h) - f(y)dy$. Now expand f around y, noting that h is divisible by p (because $h = x - y$ vanishes on the diagonal). Write then:

$$p_1^*\omega - p_2^*\omega = \sum_{i=1}^{\infty} \frac{f^{(i)}(y)}{n!} h^n dy + \sum_{i=0}^{\infty} \frac{f^{(i)}(y)}{n!} h^n dh$$

Now, just check that if one defines $F \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \frac{f^{(i-1)}}{n!} h^i$, then dF is the desired expression.

Following we write the general case, wich is just the same, but messier. So let now $x \stackrel{\text{def}}{=} (x_1, \cdots, x_n)$, and let $C \stackrel{\text{def}}{=} p_1^*x - p_2^*x$. Clearly $C \in T^n$, where T is the ideal of $A_0(X) \otimes A_0(X) \subseteq A(D)$ consisting of functions which vanish on Δ (the diagonal on $X \times X$). Set then

$$F_Y \stackrel{\text{def}}{=} \sum_I \frac{1}{I!} (p_2^* F_I) C^I$$

where $I = (i_1, \cdots, i_n) \in \mathbb{Z}^n$, $I > 0, I \neq 0$, $I! = i_1! \cdots i_n!$, and

$$F_I \stackrel{\text{def}}{=} \frac{d^{i_1}}{dx_1^{x_1}} \cdots \frac{d^{i_k - 1}}{dx_k^{i_k - 1}} f_k$$

where k is such that $i_k > 0$ and $i_j = 0$ for $j > k$. It is a well-known fact that, if $f \in A(Y)$ and $J \in \mathbb{Z}^n$, $J \geq 0$, then:

$$\left| \frac{1}{J!} \frac{d^J}{dx^J} f \right|_{\overline{Y}} \leq |f|_{\overline{Y}}$$

Hence:

$$\left|\frac{F_I}{I!}\right|_{\overline{Y}} \leq \frac{\max_j |f_j|_{\overline{Y}}}{|i_k|}$$

and so $F_Y \in A\left(D_{\overline{Y}}\right)$.

Now we can compute dF_Y and prove that, on $\overline{Y}$,

$$dF_Y = p_1^*\omega - p_2^*\omega$$

Next, we check that the F_Y glue together into a function $F \in B(D)$ as required. □

Corollary 1.15. *Suppose that $f_1, f_2 \colon X' \to X$ are morphisms of reduced connected affinoids with good reduction, such that $\tilde{f}_1 = \tilde{f}_2$. Let ω be a closed 1-form on X. Then:*

1. $f_1^*\omega - f_2^*\omega \in dA(X')$
2. *Suppose that λ is a function on $X(\mathbb{C}_p)$, analytic on each residue class of X, and such that $d\lambda = \omega$. Then:*

$$f_1^*\lambda - f_2^*\lambda \in A(X')$$

Let now V be a proper scheme of finite type over R, and let $\tilde{V}$ be its special fiber. Let $W \subseteq \tilde{V}$ be an affine open set. Consider:

$$\overline{W} \stackrel{\text{def}}{=} \{x \in V_K \mid x \text{ is closed and } \tilde{x} \in W\}$$

Then $\overline{W}$ has a natural structure of affinoid over K. If V is smooth, then $\overline{W}$ has good reduction, and $\tilde{\overline{W}} = W$. The set $\overline{W}$ is called a **Zariski affinoid open set** of V.

Definition 1.16. Suppose that V_K is smooth. A **differential of the second kind on** V_K is an element $\omega \in \Omega^1_{V_K/K}(U)$, for some dense open U of V_K, such that:

1. $d\omega = 0$.
2. there exists a Zariski open covering $\mathcal{C}$ of V_K such that for each $W \in \mathcal{C}$,

$$Res^U_{U\cap W}(W) \in Res^W_{U\cap W}(\Omega^1_{V_K/K}(W)) + d\mathcal{O}_{V_K}(U \cap W)$$

.

Definition 1.17. Supppose that V is smooth and proper over R. We say that **Frobenius acts properly on** V if for each Frobenius endomorphism ϕ of $\tilde{V}$ there is a polynomial $Z(T) \in \mathbb{C}_p[T]$ such that:

1. No root of $Z(T)$ in $\mathbb{C}_p$ is a root of unity.
2. For each Zariski affinoid open W of V such that $\phi\tilde{W} = \tilde{W}$, there is a lifting $\overline{\phi} \colon W \to W$ of the restriction of ϕ to $\tilde{W}$ such that

$$Z(\overline{\phi^*})\omega \in dA(W)$$

for each algebraic differential of the second kind ω on V_K regular on W.

Remark. If (ii) holds for one lifting $\overline{\phi}$, then it holds for all, thanks to the previous corollary.

Theorem 1.18. *Suppose that K is a DVR and that V is a smoot projective scheme over R. Then any Frobenius endomorphism acts properly on V.*

2 p-adic Abelian Integrals

Here the integrals are constructed, following the Dwork principle.

Theorem 2.1. *Let X be a smooth connected affinoid over K with good reduction $\tilde{X}$. Let ω be a closed one-form on X. Let ϕ be a Frobenius endomorphism of X, and suppose that $P(T)$ is a polynomial over $\mathbb{C}_p$ such that*

$$P(\phi^*)\omega \in dA(X)$$

and such that no root of $P(T)$ is a root of unity. Then there exists a locally analytic function f_ω on $X(\mathbb{C}_p)$ unique up to an additive constant such that:

1. *$df_\omega = \omega$*
2. *$P(\phi^*)f_\omega \in A(X)$.*

Proof. We copy the proof in the original paper, but for the special calse of $P(T) = T - a$, with $a \in \mathbb{C}_p$ (the degree of $P(T)$ is $n = 1$). This is (hopefully) enough to see the ideas behind it.

So assume that $\phi^*\omega - a\omega \in dA(X)$.

We first prove uniqueness: suppose that one has two solutions to the problem. Then their difference would be a locally analytic function g satisfying $dg = 0$ and $\phi^*g - ag \in A(X)$. We will see that g is constant. As $dg = 0$, g must be locally constant, and thus $\phi^*g - ag$ is locally constant as well. As X is connected, then $\phi^*g - ag = C$, for some constant $C \in \mathbb{C}_p$. We will prove that $g(x) = C/(1-a)$ for all $x \in X$.

Let U be a residue class of X, and let $\varepsilon = \varepsilon_U$ be a Teichmüller point of ϕ in U, of period m (so that $\phi^m(\varepsilon) = \varepsilon$). Then one can check (by induction, for example) that:

$$(\phi^*)^k g - a^k g = C\frac{1-a^k}{1-a} \tag{1}$$

Take now $k = m$ and evaluate at ε, to get:

$$(1-a^m)g(\varepsilon) = C\frac{1-a^m}{1-a}$$

As $1 - a^m$ is invertible, this implies that $g(\varepsilon) = C/(1-a)$ as we want.

Now let $x \in U$ be arbitrary. As g is locally constant and $\phi^{mk}(x) \to \varepsilon$, there is some integer k such that:

$$g(\phi^{mk}(x)) = g(\varepsilon)$$

Again by Equation 1, we get:

$$a^{mk}g(x) = a^{mk}C/(1-a)$$

So $g(x) - C/(1-a)$ is in the kernel of a^{mk}, thought as acting on $\mathbb{C}_p$. But here is the trick to cancel this: for each integer r, there is some element $y_r \in U$ such that $\phi^{mr}(y_r) = x$. Using Equation 1 again, we deduce that:

$$g(x) - C/(1-a) = a^r\left(g(y_r) - C/(1-a)\right)$$

so $g(x) - C/(1-a)$ is both in the kernel of a^{mk} and in the image of a^{mr} for all $r \geq 0$, and only 0 is there:

$$\ker(a^{mk}) \cap (\cap_{r\geq 0} a^{mr}(\mathbb{C}_p)) = \{0\}$$

so g is constant.

Next, we prove existence: for this, write first $\phi^*\omega = a\omega + dh$ for some $h \in A(X)$. We can surely integrate ω locally, but we need to do it in a coherent way so that the second condition in the theorem is satisfied. Once again, the Teichmüller points will save the day. If U is a residue class of X and $\varepsilon \in U$ is the corresponding Teichmüller point for ϕ, write m for the minimal positive integer such that $\phi^m(\varepsilon) = \varepsilon$. Then define f_U to be the local integral to ω, normalized such that:

$$f_U(\varepsilon) = \frac{1}{1-a^m} \sum_{i=0}^{m-1} a^i h\left(\phi^{m-(i+1)}(\varepsilon)\right)$$

and define f by $f|_U \stackrel{\text{def}}{=} f_U$. We can then compute $(\phi^* f)(\varepsilon)$ and show that it equals $af(\varepsilon) + h(\varepsilon)$ as we wanted. □

Corollary 2.2. *The function f_ω is analytic on each residue class of X.*

Corollary 2.3. *The function f_ω depends modulo constants only on ω and not on the choice of P.*

Corollary 2.4. *Let ω' be a closed one-form on X such that $P(\phi^*)\omega' \in dA(X)$. Then:*

1. *$f_{\omega+\omega'} = f_\omega + f_{\omega'}$ (modulo constants in $\mathbb{C}_p$),*
2. *if ω is exact, then $f_\omega \in A(X)$.*

Corollary 2.5. *The function f_ω is independent (up to constants) of the choice of ϕ.*

Let now σ be a continuous automorphism of $\mathbb{C}_p$. Let ω^σ denote the pullback of ω to X^σ. Let f_ω^σ be the function on $X^\sigma(\mathbb{C}_p)$ defined by:

$$f_\omega^\sigma(x) \stackrel{\text{def}}{=} \sigma f_\omega(\sigma^{-1}(x))$$

Corollary 2.6. *The differential ω^σ satisfies the hypotheses of the theorem over X^σ, and $f_\omega^\sigma = f_{\omega^\sigma}$ up to constants. In particular, if σ fixes K, then $f_\omega^\sigma = f_\omega$ up to constants.*

Proposition 2.7. *Suppose that $F\colon X' \to X$ is a morphism of smooth affinoids with good reduction over K. Let $\omega' = F^*\omega$. Then there exists a Frobenius endomorphism ϕ' of X' and a polynomial $P'(T)$ in $\mathbb{C}_p[T]$ such that*

$$P'(\phi'^*)\omega' \in dA(X')$$

and such that no root of $P'(T)$ is a root of unity. Moreover, if $f_{\omega'}$ is a solution of the Theorem with ω' in place of ω, then $f_{\omega'} = F^* f_\omega$ up to constants.

Proof. The key observation to be made is that there exists Frobenius endomorphisms $\phi\colon X \to X$ and $\phi'\colon X' \to X'$ compatible with F on the reductions. That is, such that the following commutes:

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{F}} & \tilde{X} \\ \downarrow{\scriptstyle \tilde{\phi}'} & & \downarrow{\scriptstyle \tilde{\phi}} \\ \tilde{X}' & \xrightarrow{\tilde{F}} & \tilde{X} \end{array}$$

Then there is (by what we have seen so far) a polynomial $P(T)$, without roots of unity, such that:

$$P(\phi^*)\omega \in dA(X)$$

From this, we deduce that:

$$P(\phi;^*)\omega' \in dA(X')$$

Also, we deduce (because $\phi \circ F$ and $F \circ \phi'$ to the same morphism) that:

$$F^*(\phi^*)^k f_\omega - (\phi'^*)^k F^* f_\omega \in A(X') \quad \text{for all } k > 0$$

and hence:

$$P(\phi'^*)F^* - F^* P(\phi^*) \in A(X')$$

Now apply the uniqueness of $f_{\omega'}$ to conclude the result. □

Next, we will describe how to integrate differentials ω of the second kind on V_K, where V is a smooth, proper, connected scheme of finite type over R.

Let $\mathcal{D}$ be the collection of Zariski affinoid opens X in V_K such that, on X,

$$\omega - dg_X \in \Omega^1_K(X)$$

for some g_X in $K(V_K)$ (the function field of V_K). Note that $\mathcal{D}$ is a covering, because ω is of the second kind. Let $(\omega)_\infty$ be the support of the polar divisor of ω on V_K, and write $V'_K \stackrel{\text{def}}{=} V_K - (\omega)_\infty$.

Fix a Frobenius endomorphism ϕ of $\tilde{V}$. Write $\mathcal{D}'$ for the subcollection of $\mathcal{D}$ consisting of those X such that $\phi\tilde{X} = \tilde{X}$ (note that $\mathcal{D}'$ is also a covering of V_K (why??). Let $Z(T)$ be a polynomial associated to V and ϕ (as in Definition 1.17).

Fix now $X \in \mathcal{D}'$. Write for short $g = g_X$, and set $\nu = \nu_X = \omega - dg$. Let $\overline{\phi} = \overline{\phi_X}$ be a lifting of the restriction of ϕ to $\tilde{X}$. Hence:

$$Z(\overline{\phi}^*)\nu \in dA(X)$$

By Theorem 2.1, there exists $f = f_X$, locally analytic on X and unique up to an additive constant such that $df = \nu$, and $Z(\overline{\phi}^*)f \in A(X)$.

Now, set $h_X \stackrel{\text{def}}{=} f + g$, as a function on $X - (\omega)_\infty$.

Claim. *The function h_X is independent of the choices of f and g, up to an additive constant.*

Proof. Suppose that $g' \in K(V_K)$ is such that $\omega - dg' = \nu' \in \Omega^1_K(X)$. It follows then that $\nu' = \nu + d(g - g')$, and so in particular $g - g' \in A(X)$. If now f' is a solution of $df' = \nu'$, and $Z(\overline{\phi}^*)f' \in A(X)$, then $f' = f + (g - g')$, from a previous corollary (up to constants). This finishes the proof. □

Finally, we need to patch together the local integrals h_X:

Lemma 2.8. *Let $X, X' \in \mathcal{D}'$. Then $h_X - h_{X'}$ is constant on $X \cap X'$.*

Proof. Note first that $X \cap X' \in \mathcal{D}$, so it suffices to prove it in the case $X' \subseteq X$. In this case, we may take $g_X = g_{X'}$. Then $\nu_{X'}$ is the restriction of ν_X to X', and if we restrict f_X to X' we get a solution for our problem, hence $h_{X'} = h_X|$, as we wanted. □

This makes the map $(X, X') \mapsto h_X - h_{X'}$ into a 1-cocycle wrt the covering $\mathcal{D}'$ and the constant sheaf $\mathbb{C}_p$. It is actually a coboundary, since any finite subcollection of $\mathcal{D}'$ has non-empty intersection.

We have proved:

Theorem 2.9. *There exists a function f_ω on $V'_K(\mathbb{C}_p)$, unique up to an additive constant, such that:*

1. *$df_\omega = \omega$,*
2. *For each $X \in \mathcal{D}'$, there exists a $g \in K(V_K)$ such that $f_\omega - g$ extends to ta locally analytic function on X, and*

$$Z(\overline{\phi_X}^*)(f_\omega - g) \in A(X)$$

Definition 2.10. Given ω and f_ω as above, and given two points $P, Q \in V'_K(\mathbb{C}_p)$, the integral of ω from P to Q is defined as:

$$\int_P^Q \omega \stackrel{\text{def}}{=} f_\omega(Q) - f_\omega(P)$$

Proposition 2.11. *Let ω and ω' be two differentials of the second kind on V_K. Then:*

- *If $P, Q \notin (\omega)_\infty \cup (\omega')_\infty$, we have:*

$$\int_P^Q (\omega + \omega') = \int_P^Q \omega + \int_P^Q \omega'$$

- *If $\omega = dg$ for a meromorphic function g on V_K, then:*

$$\int_P^Q \omega = g(Q) - g(P)$$

- *Let $g \colon W \to V$ be a morphism of smooth proper schemes over R, on which Frobenius acts properly. Then, if $g(Q), g(P) \notin (\omega)_\infty$, we have:*

$$\int_P^Q g^*\omega = \int_{g(P)}^{g(Q)} \omega$$

- *If $P, Q \notin (\omega)_\infty$, then:*

$$\left(\int_P^Q \omega\right)^\sigma = \int_{\sigma(P)}^{\sigma(Q)} \omega^\sigma$$

where the second integral is taken on V^σ

The following theorem, whose proof ommit for now, is a strenghtening of the change of variable formula from the previous proposition:

Theorem 2.12 (Change of Variablles). *Suppose that V and W are smooth proper schemes of finite type over a ring R on which Frobenius acts properly. Suppose $f \colon V_K \to W_K$ is a* rational *map. Let ω be a differential of the second kind on W_k. Then:*

$$\int_P^Q f^*\omega = \int_{f(P)}^{f(Q)} \omega$$

for any $P, Q \in V(\mathbb{C}_p)$ in the domain of regularity of f such that $f(P), f(Q) \notin (\omega)_\infty$.

Corollary 2.13. *The integral* $\int_P^Q \omega$ *doesn't depend on the model* V *for* V_K.

Corollary 2.14. *Suppose that* V_K *is a variety over* K *which may be completed to a smooth proper scheme* V *of finite type over* R *on which Frobenius acts properly. Let* ω *be a regular differential on* V_K *of the second kind. Then for* $P, Q \in V_K(\mathbb{C}_p)$ *the integral* $\int_P^Q \omega$ *depends only on* V_K *and not on its completion.*

Let now G be a *connected commutative group scheme* over R, which is an extension of an abelian scheme A by a vector group B:

$$0 \to B \to G \to A \to 0$$

Let $\mathcal{O}$ be the origin on G.

Theorem 2.15. *Let* ω *be an invariant differential on* G, *and let*

$$\lambda_\omega(Q) \stackrel{def}{=} \int_{\mathcal{O}}^{Q} \omega$$

where $Q \in G_K(\mathbb{C}_p)$. *(this is well defined by a previous corollary). Then:*

1. λ_ω *is a homomorphism from* $G_K(\mathbb{C}_p)$ *into* $\mathbb{C}_p$
2. λ_ω *is locally analytic, and* $d\lambda_\omega = \omega$.

Proof. Let $T_a \colon G \to G$ denote translation by $a \in G_K(\mathbb{C}_p)$. Then $T_a^*\omega = \omega$, and so by the change of variables formula:

$$\int_{\mathcal{O}}^{P} \omega = \int_{\mathcal{O}}^{P} T_Q^*\omega = \int_{Q}^{P+Q} \omega = \int_{\mathcal{O}}^{P+Q} \omega - \int_{\mathcal{O}}^{Q} \omega$$

which implies $\lambda(P) = \lambda(P+Q) - \lambda(Q)$.

The second statement was known already from the previous results. □

In particular, we get the addition theorem:

Theorem 2.16. *Let* C *be a complete curve over* K *with a smooth proper model over* R, *on which Frobenius acts properly. Consider* D_1, D_2, D_3 *three divisors on* C, *such that* $D_1 + D_2 \equiv D_3 + n[P]$. *Then, for any differential* ω *of the first kind on* C, *we have:*

$$\sum_{i=1}^{n} \int_{P}^{P_i} \omega + \sum_{i=1}^{n} \int_{P}^{Q_i} \omega = \sum_{i=1}^{n} \int_{P}^{R_i} \omega$$

Proof. Just take G in the previous theorem to be the Néron model of the Jacobian of C. □