Étale Fundamental Groups and Cohomology

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Disclaimer

These notes are from a course on étale cohomology by Adrian Iovita, taken by the authors at Concordia University in the winter of 2008. We have tried our best to present a faithful account of these lectures. Be warned that despite our efforts, many errors likely persist. Please do not take this as an indication of the quality of the lectures. Any errors are most likely due to carelessness while typesetting, or the authors' misunderstanding of the material.

Professor Iovita based this course on several courses he has taken, as well as the standard reference books. We will eventually include a full set of references, and give due intellectual credit to the deserving parties.

Ivan Garcia warns us that there is a high overlap between these notes and Lenstra's notes found in:

`http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`, about which we were unaware. You are encouraged to use the material that you find the most suitable for your learning experience. Good luck!

Chapter 1

Étale Coverings and Fundamental Groups

Many classical geometric ideas have analogues in the theory of schemes. The extreme coarseness of the Zariski topology often neccessitates one to reformulate the definitions appropriately, despite the fact that schemes have an honest topological space as their backbone. Two of the first such properties that most students encounter are the Hausdorff and compactness conditions for a topological space. These have the analogues of being separated and proper, respectively, in scheme theory. This chapter will give a useful scheme theoretic definition of a covering space. What happens if one adopts the classical definition? Then one can show, for instance, that $\mathrm{Spec}(\mathbb{C}[x])$ does not admit any nontrivial covering spaces. Thus, in order to describe an interesting theory, we will need to alter our definition of a covering space of a scheme.

1.1 Projective and separable algebras

In what follows, rings are always commutative with 1, and ring homomorphisms map $1 \mapsto 1$. Algebras are also assumed to be commutative and unital. Let A be a ring. In this section we introduce some of the algebraic tools that are necessary to define étale morphisms.

Definition 1.1.1. An A-module P is said to be **projective** if it is a direct summand of a free module. This means that there exists a complementary A-module M, such that $P \oplus M$ is free.

Note that all free modules are projective. There are several equivalent ways to formulate the property of being projective:

Lemma 1.1.2. *Let P be an A-module. The following are equivalent:*

i) P is projective.

ii) The functor $M \mapsto Hom_A(P, M)$ is exact.

iii) For every surjective morphism $g : M \to M'$ of A-modules, and every morphism $f : P \to M'$, there exists a morphism $h : P \to M$ such that $f = g \circ h$.

iv) Every exact sequence $0 \to M' \to M \to P \to 0$ splits. Recall that this means $M \to P$ admits a right inverse, or equivalently that $M' \to M$ admits a left inverse. In this case, $M \simeq M' \oplus P$.

Proof. (i) implies (ii): For all modules P, the functor $M \mapsto \mathrm{Hom}_A(P, M)$ is left exact. It thus suffices to show that it preserves surjections $g : M \to M'$ when P is projective. Let N be such that $P \oplus N$ is free with basis $\{p_i + n_i\}_{i\in I}$. Note that the p_i's generate P, albeit perhaps not freely. Let $f : P \to M'$; we must produce a morphism $h : P \to M$ with $f = g \circ h$. Choose $m_i \in M$ such that $g(m_i) = f(p_i)$. Define a map $h' : P \oplus N \to M \oplus N$ by putting $h'(p_i + n_i) = m_i + n_i$. The composition:

$$P \longrightarrow P \oplus N \xrightarrow{h'} P \oplus M \longrightarrow M$$

is a morphism $h : P \to M$ such that $h(p_i) = m_i$. Since the p_i's generate P, we get $f = g \circ h$.

(ii) implies (iii): This follows immediately since $M \mapsto \mathrm{Hom}_A(P, M)$ preserves surjections.

(iii) implies (iv): Apply (iii) to the given $M \to P$ and $f : P \to P$ the identity map.

(iv) implies (i): Since every module is a quotient of a free module, there is an exact sequence $0 \to M' \to M \to P \to 0$ with M free. A splitting yields an isomorphism $M \simeq M' \oplus P$, so that P is a direct summand of a free module. □

Corollary 1.1.3. *If M is a finitely generated projective A-module, then M is a direct summand of a finite free module.*

Proof. Since M is finitely generated, it is a quotient of a finite free module N. If N' is the kernel of the quotient map, then we can apply property (iv) of the lemma above to the exact sequence:

$$0 \longrightarrow N' \longrightarrow N \longrightarrow M \longrightarrow 0$$

and obtain a decomposition $N \simeq M \oplus N'$. □

Examples 1.1.4.

i) If $A = K$ is a field, then all A-modules are free and hence projective.

ii) Let $A = A_1 \times A_2$ for rings A_1 and A_2. Then A_1 and A_2 are projective A-modules, but they are not free. Let P be an A-module. Multiplication by the idempotents $(1, 0)$ and $(0, 1)$ furnishes an isomorphism $P \simeq P_1 \times P_2$ where P_i is an A_i-module. One can show that P is projective as an A-module if and only if each P_i is projective as an A_i-module.

iii) Let $A = K[G]$ where G is a finite abelian group and K is a field such that $\mathrm{char}(K) \mid |G|$. Then A is semi-simple. Although not every A-module is free, it follows from the previous example that they are all projective.

iv) If A is a principal ideal domain, then projective A-modules are free.

v) Let A be a Dedekind domain and let P be a finitely generated A-module. Then P is projective if and only if P is torsion free. This is equivalent to having a decomposition $P \simeq A^n \oplus I$ where $I \subset A$ is an ideal. Moreover, one can show that two modules $A^n \oplus I$ and $A^m \oplus J$ are isomorphic if and only if $n = m$ and I and J lie in the same ideal class of A.

vi) Let A be an integral domain and let K be the fraction field of A. Then an A-submodule $P \subset K$ is projective if and only if P is invertible.

vii) If $A = K[x_1, \ldots, x_n]$ for K a field, then every projective A-module is free.

viii) If A is a local ring, then every projective A-module is free. In the case of finitely generated projective modules, this is a standard result that can be proven easily via Nakayama's lemma.

One can show that for finitely generated A-modules, projectivity is a local property:

Theorem 1.1.5. *Let P be an A-module. Then the following are equivalent:*

i) P is projective and finite.

ii) P is of finite presentation and for every prime ideal $\mathfrak{p}$ of A, $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

iii) P is of finite presentation and For every maximal ideal $\mathfrak{m}$ of A, $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module.

iv) The coherent sheaf $\tilde{P}$ *on* $\operatorname{Spec}(A)$ *is locally free and locally finite.*

Proof. □

We now proceed to define the rank and trace for *finite* and *projective* modules.

Definition 1.1.6. Let P be a finite, projective A-module. Define the rank function:

$$[P : A]\colon \operatorname{Spec}(A) \to \mathbb{Z}, \quad \mathfrak{p} \mapsto \operatorname{rk}_{A_\mathfrak{p}}(P_\mathfrak{p})$$

This function is locally constant, and hence continuous when $\mathbb{Z}$ is given the discrete topology. This follows easily from the previous theorem. It follows that $[P : A]$ is *constant* on the connected components of $X = \operatorname{Spec}(A)$. In particular, if $\operatorname{Spec}(A)$ is connected then $[P : A]$ is constant. Recall that the connectedness of $\operatorname{Spec}(A)$ is equivalent to the fact that A does not contain any nontrivial idempotents.

Definition 1.1.7. If P is a finite, projective A-module, we say that P is **faithfully-projective** if $[P : A](\mathfrak{p}) \geq 1$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

The rank function characterises properties of the structure morphism of finite projective A-algebras:

Proposition 1.1.8. *Let B be a finite projective A-algebra, and let $f\colon A \to B$ be its structure (ring) morphism. Then:*

i) f is injective $\iff [B\colon A] \geq 1$;

ii) f is surjective $\iff [B\colon A] \leq 1 \iff B \otimes_A B \to B, x \otimes y \mapsto xy$ is an isomorphism; and

iii) f is an isomorphism $\iff [B\colon A] = 1$.

Proof. First note that (3) follows from (1) and (2). Also, recall that injectivity and surjectivity can be determined on the stalks (that is, they are local properties).

(a),$\Rightarrow$

Suppose that $\exists \mathfrak{p} \in \operatorname{Spec}(A)$ such that $[B\colon A](\mathfrak{p}) = 0$, that is $B_\mathfrak{p} = 0$. Then $A_\mathfrak{p} \neq 0$, and hence $f_\mathfrak{p}\colon A_\mathfrak{p} \to B_\mathfrak{p}$ is not injective.

(a),$\Leftarrow$

Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Consider $f_\mathfrak{p}\colon A_\mathfrak{p} \to B_\mathfrak{p}$. Then, the rank of $B_\mathfrak{p}$ is nonzero, and so in particular $B_\mathfrak{p} \neq 0$.

Let $I \overset{\text{def}}{=} \ker f_\mathfrak{p}$. Then $I \cdot B_\mathfrak{p}$, which is by definition $f_\mathfrak{p}(I) \cdot B_\mathfrak{p}$, is zero. As $B_\mathfrak{p}$ is free, it is torsion-free in particular, and so $I = 0$. Hence $f_\mathfrak{p}$ is injective, as wanted.

(b),$[B\colon A] \leq 1 \Rightarrow f$ is surjective.
We may assume that A is already local (after localizing at a prime $\mathfrak{p}$). So $\mathrm{Spec}(A)$ is connected (because local rings do not have idempotents). Hence $[B\colon A]$ is constant. If $[B\colon A] = 0$, then $B = 0$, so f is surely surjective. So assume then that $[B\colon A] = 1$. Then B is a free A-algebra of rank 1.

Consider now $\mathrm{End}_A(B)$, which is free of rank 1 over A, with basis Id_B. The map $\alpha\colon B \to \mathrm{End}_A(B)$, $b \mapsto [x \mapsto bx]$ is A-linear. It is also injective (because $\alpha(b)(1) = b$). We can then take the compositum $\alpha \circ f$, which sends $a \in A$ to the endomorphism "multiplication by a". So $\alpha f(a) = a\mathrm{Id}_B \implies \alpha \circ f$ is an isomorphism, which implies (as α is injective) that f is surjective.

(b),f surjective $\implies B \otimes_A B \to B$ is an isomorphism.
Let $I \stackrel{\text{def}}{=} \ker f$. As we are assuming that f is surjective, we get that $B \simeq A/I$. Then we compute:

$$B \otimes_A B \simeq B \otimes_A (A/I) \simeq B/(IB) = B/(f(i)B) = B/0 = B$$

and this corresponds to the map $x \otimes y \mapsto xy$, so this is an isomorphism. (b),$B \otimes_A B \to B$ iso. $\implies [B\colon A] \leq 1$.
Just note that $[B\colon A] = [B \otimes_A B\colon A] = [B\colon A]^2$ (the last equality follows from a homework exercise), and from that we deduce $[B\colon A] \leq 1$. □

Next we define the trace. Let M and N be A-modules. Let $M^\vee = \mathrm{Hom}_A(M, A)$ denote the dual module of M. There is a natural bilinear map:

$$M^\vee \times N \longrightarrow \mathrm{Hom}_A(M, N),$$

such that $(\phi, n) \mapsto (m \mapsto \phi(m)n)$. This corresponds to a morphism:

$$M^\vee \otimes_A N \longrightarrow \mathrm{Hom}_A(M, N).$$

If M and N are finite and free, it is not difficult to show that this is an isomorphism. One can then use this fact to give a coordinate free definition of the trace of an endomorphism of a finite free module. We will show that this isomorphism holds if M and N are finitely generated and projective, and hence define the trace for endomorphisms of finitely generated projective modules.

Lemma 1.1.9. *Let M and N be projective modules. Then $M^\vee$ and $M \otimes_A N$ are projective.*

Proof. Let M', N' be such that $M \oplus M'$ and $N \oplus N'$ are free. The dual of a free module is free. Thus, $(M \oplus M')^\vee \simeq M^\vee \oplus (M')^\vee$ verifies the first claim. The tensor product of two free modules is free. Thus, $(M \oplus M') \otimes_A (N \oplus N') \simeq (M \otimes_A N) \oplus M''$ verifies the second claim. □

Proposition 1.1.10. *Let M and N be finitely generated projective modules. Then the map discussed above is an isomorphism:*

$$M^\vee \otimes_A N \simeq Hom_A(M, N).$$

Proof. Let M' and N' be such that $M \oplus M'$ and $N \oplus N'$ are finite free. Let:

$$\phi\colon (M \oplus M')^\vee \otimes_A (N \oplus N') \simeq \mathrm{Hom}_A(M \oplus M', N \oplus N')$$

be the isomorphism discussed above. There are canonical isomorphisms:

$$(M \oplus M')^\vee \otimes_A (N \oplus N') \simeq (M^\vee \otimes_A N) \oplus ((M')^\vee \otimes_A N) \oplus (M^\vee \otimes_A N') \oplus ((M')^\vee \oplus N')$$

and:

$$\mathrm{Hom}_A(M \oplus M', N \oplus N') \simeq \mathrm{Hom}_A(M, N) \oplus \mathrm{Hom}_A(M', N) \oplus \mathrm{Hom}_A(M, N') \oplus \mathrm{Hom}_A(M', N').$$

Via these isomorphisms, ϕ decomposes as $\phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \phi_4$. Since ϕ is an isomorphism, each of the component morphisms is as well. In particular, ϕ_1 is an isomorphism. One checks that it is precisely the natural map $M^\vee \otimes_A N \to \mathrm{Hom}_A(M, N)$. □

Suppose that M, N and P are arbitrary modules. The universal property of the tensor product allows one to show that tensoring and homing are adjoint functors:

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \simeq \mathrm{Hom}_A(M \otimes_A N, P).$$

Letting $N = M^\vee$ and $P = A$, this says:

$$\mathrm{Hom}_A(M, M^{\vee\vee}) \simeq \mathrm{Hom}_A(M \otimes_A M^\vee, A).$$

Evaluation of functionals gives an injection $M \to M^{\vee\vee}$; let $e \in \mathrm{Hom}_A(M \otimes_A M^\vee, A)$ be the corresponding element. Then if M is finitely generated and projective, the proposition above yields an A-linear map:

$$\mathrm{Tr}\colon \mathrm{End}_A(M) \to M \otimes_A M^\vee \xrightarrow{e} A.$$

called the **trace**. It is not difficult to verify that this map is the usual trace when M is finite and free. If $M \oplus M'$ is finite and free, then Tr is the same as the natural injection $\mathrm{End}_A(M) \to \mathrm{End}_A(M \oplus M')$ followed by the trace in the free case. The advantage of our definition is that it is coordinate free.

Let B be an A-algebra. We say that B is **finite** if it is finitely generated as an A-module. It is similarly said to be projective. For any $b \in B$, let $m_b\colon B \to B$ be multiplication by b.

Definition 1.1.11. Let B be a finite projective A-algebra. The **trace** from B to A is the A-linear map:

$$\mathrm{Tr}_{B/A}\colon B \longrightarrow A,$$

such that $\mathrm{Tr}_{B/A}(b) = \mathrm{Tr}(m_b)$.

The trace can be used to define a bilinear pairing on B:

$$(b, b') \mapsto \mathrm{Tr}_{B/A}(bb').$$

Definition 1.1.12. A finite projective A-algebra B is said to be **separable** if the bilinear pairing above induces an isomorphism $\phi\colon B \to B^\vee$. We will sometimes drop the adjective projective when describing a finite separable A-algebra.

Remark. In the definition of separability it is important that the dualising isomorphism is given by the trace map. We will see later that this map measures the "ramification" of B/A. The map ϕ being an isomorphism implies that B/A behaves like an unramified covering. One can show that there exist finite projective A-algebras B such that $\phi = 0$ despite the fact that $B \simeq B^\vee$.

Examples 1.1.13.

Let A be a ring and let $B = A^n$, where multiplication is defined componentwise. Then B is an A-algebra via the homomorphism $A \to B$ given by:

$$a \mapsto (a, \ldots, a).$$

It is clear that B is finite and free; it is also separable.

Indeed, let $e_i = (0, \ldots, 1, \ldots, 0)$ have a 1 in the i-th spot and zeros elsewhere. The e_i's form a free A-basis for B. Let $x = (x_1, \ldots, x_n) \in B$. One uses this basis to check that $\mathrm{Tr}(x) = \sum x_i$ (the matrix of m_x in this basis is diagonal with the x_i's as entries on the diagonal). The map $\alpha\colon B^\vee \to B$ given by:

$$\alpha(f) = (f(e_1), \ldots, f(e_n))$$

is an isomorphism. With ϕ as defined above we have:

$$(\alpha \circ \phi)(x) = (\mathrm{Tr}(e_1 x), \ldots, \mathrm{Tr}(e_n x)) = x.$$

Since α is an isomorphism, this shows that ϕ is also an isomorphism.

ii) All finite separable $\mathbb{Z}$-algebras are of the form $\mathbb{Z}^n$.

iii) If $A = k$ is a field, then a k-algebra B is finite separable if and only if B is a product of finite separable field extensions of k. If k is algebraically closed, we again see that all finite separable k-algebras are of the form k^n.

We will see later that pulling back an étale covering via a faithfully flat algebra trivialises the covering. This motivates the following two results.

Lemma 1.1.14. *Let A be a ring, B an A-algebra and C an A-algebra which is faithfully flat. Then B is finite projective if and only if $B \otimes_A C$ is finite projective as a C-algebra.*

Proof. If B is finite projective, then there is an A-module Q such that $Q \oplus P \simeq A^n$ for some integer n. Tensoring this with C yields $(Q \otimes_A C) \oplus (P \otimes_A C) \simeq C^n$ as C-modules.

Now assume that $B \otimes_A C$ is finite projective. We show first that B is a finitely generated A-module. Note that $B \otimes_A C$ is finitely generated over C, and there exists a generating set of the type $x_1 \otimes 1, \ldots, x_n \otimes 1$ with $x_i \in B$ for all i. Now define a map $f \colon A^n \to B$ by mapping the standard generating idempotents to the x_i. Since C is faithfully flat, f is surjective if and only if the map obtained by tensoring with C is surjective. The tensored map is surjective by construction, hence so is f. So B is finitely generated as an A-module.

Now let $Q = \ker f$, so that we have an exact sequence:

$$0 \longrightarrow Q \longrightarrow A^n \longrightarrow B \longrightarrow 0.$$

Tensor this with the flat A-module C to obtain the exact sequence:

$$0 \longrightarrow Q \otimes_A C \longrightarrow C^n \longrightarrow B \otimes_A C \longrightarrow 0.$$

Since $B \otimes_A C$ is assumed to be finite projective, this sequence splits and $C^n \simeq (B \otimes_A C) \oplus (Q \otimes_A C)$. We deduce that $Q \otimes_A C$ is a finite projective C-module. But now, the same proof as given above for B shows that Q is finitely generated as an A-module. We thus obtain an exact sequence:

$$A^m \longrightarrow A^n \longrightarrow B \longrightarrow 0$$

of A-modules, which shows that B is of finite presentation. Now it is a standard fact of commutative algebra that since B is of finite presentation and C is flat as an A-module, there is a natural C-module isomorphism:

$$\mathrm{Hom}_A(B, M) \otimes_A C \to \mathrm{Hom}_C(B \otimes_A C, M \otimes_A C)$$

for any A-module M.

We can now show that B is projective. Let:

$$M \xrightarrow{f} N \longrightarrow 0$$

be exact. We must show that the map $f_*\colon \mathrm{Hom}_A(B, M) \to \mathrm{Hom}_A(B, N)$ induced by composition is surjective. Tensoring this with C and using the canoncial isomorphism mentioned above yields a morphism of C-modules:

$$f_* \otimes \mathrm{Id}_C\colon \mathrm{Hom}_C(B \otimes_A C, M \otimes_A C) \to \mathrm{Hom}_C(B \otimes_A C, N \otimes_A C).$$

Now since $B \otimes_A C$ is projective and $f \otimes \mathrm{Id}_C$ is surjective, this implies that the induced composition map $f_* \otimes \mathrm{Id}_C$ is surjective. But now we apply the faithfull flatness of C to deduce that then f_* must also be surjective. Thus, the functor $\mathrm{Hom}_A(B, -)$ is exact and B is projective. □

Proposition 1.1.15. *Let A be a ring, B an A-algebra and C an A-algebra which is faithfully flat. Then B is separable projective over A if and only if $B \otimes_A C$ is separable projective over C.*

Proof. If B is finite and projective, so is $B \otimes_A C$ by above. Assume that it is furthermore separable, so that $\phi\colon B \to B^\vee$ is an isomorphism. Then so is $\phi \otimes \mathrm{Id}_C$ since C is flat. Since B is finite projective, it is of finite presentation. As C is flat, we have an isomorphism $\psi\colon \mathrm{Hom}_C(B \otimes_A C, C) \to \mathrm{Hom}_A(B, A) \otimes_A C$ of C-modules. One can show that:

$$\begin{array}{ccc} B \otimes_A C & \xrightarrow{\phi_{B \otimes_A C}} & \mathrm{Hom}_C(B \otimes_A C, C) \\ & {\scriptstyle \phi_B \otimes \mathrm{Id}_C} \searrow \quad \nearrow {\scriptstyle \psi} & \\ & \mathrm{Hom}_A(B, A) \otimes_A C & \end{array}$$

commutes and hence $\phi_{B \otimes_A C}$ is an isomorphism. Thus $B \otimes_A C$ is separable projective over C.

For the converse direction, finiteness and projectivity are again covered by the previous lemma. For separability we still have $\phi_{B \otimes_A C} = \psi \circ (\phi_B \otimes \mathrm{Id}_C)$. This time we are given that $\phi_{B \otimes_A C}$ is an isomorphism by separability. We still have ψ being an isomorphism, as we have already argued that B is projective and hence of finite presentation. So we deduce that $\phi_B \otimes \mathrm{Id}_C$ must be an isomorphism. But then, by faithful flatness, ϕ_B is an isomorphism of A-algebras. □

Our next results show that, thinking geometrically, separable projective algebras behave like projections in a very strong sense.

Lemma 1.1.16. *Let A be a ring and $0 \to P_0 \to P_1 \to P_2 \to 0$ an exact sequence of A-modules, such that P_1 and P_2 are finite projective. Let $g \in End_A(P_1)$ be such that $g(P_0) \subset P_0$. Let $h \in End_A(P_2)$ be the induced map. Then P_0 is finite projective, and:*

$$Tr_{P_1/A}(g) = Tr_{P_0/A}(g) + Tr_{P_2/A}(h).$$

Proof. The proof is simple in the case that the P_i's are free modules. One shows that the trace behaves well with respect to localisation, and then reduces to the free case by localising. □

Lemma 1.1.17. *Suppose that B is a separable projective A-algebra, and $f\colon B \to A$ is an A-algebra homomorphism. Then there is an A-algebra C and an A-algebra isomorphism $g\colon B \simeq A \times C$ such that:*

$$\begin{array}{ccc} B & \xrightarrow{g} & A \times C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ A & = & A \end{array}$$

commutes, where p is the natural projection map.

Proof. Since B is separable, $\phi\colon B \to B^\vee$ is an isomorphism. Let $e \in B$ be such that $\phi(e) = f$. We will show that $e \in B$ is an idempotent that gives the correct product decomposition. By the definition of e:

$$\mathrm{Tr}_{B/A}(ex) = f(x)$$

for all $x \in B$. This implies, in particular, that $\mathrm{Tr}(e) = 1$. Furthermore:

$$\mathrm{Tr}_{B/A}(exy) = f(xy) = f(x)f(y) = f(x)\mathrm{Tr}_{B/A}(ey) = \mathrm{Tr}_{B/A}(ef(x)y)$$

for all $x, y \in B$. We deduce that $\phi(ex) = \phi(ef(x))$ for all $x \in B$, and hence $ex = ef(x)$ for all $x \in B$. Note that this shows that $e \ker f = 0$.

If we can show that $f(e) = 1$, then it follows by the computation above that e is an idempotent. The identities above show that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & B & \xrightarrow{f} & A & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f(e)} & & \\ 0 & \longrightarrow & \ker f & \longrightarrow & B & \xrightarrow[f]{} & A & \longrightarrow & 0 \end{array}$$

with exact rows is commutative. The vertical arrows indicate multiplication by the respective elements. Applying the previous lemma to e shows that:

$$1 = \mathrm{Tr}_{B/A}(e) = \mathrm{Tr}_{A/A}(f(e)) + \mathrm{Tr}_{\ker f/A}(0) = f(e).$$

So $e^2 = e$. Note that $1-e \in \ker f$ since $f(e) = 1$. One deduces that $(1-e)B = \ker f$. The exact sequence:

$$0 \longrightarrow \ker f \longrightarrow B \xrightarrow{f} A \longrightarrow 0$$

splits by projectivity. This yields an isomorphism $B \simeq A \times \ker f$ which maps $x \mapsto (f(x), x \pmod{eB})$. A priori this is a map of A-modules, but one easily checks that it is multiplicative as well. The last point is that $C = (1-e)B$ is a ring with identity $(1-e)$, so that this is a decomposition as A-algebras which makes the stated diagram commute. □

Remark. If B is separable projective over A, consider $B \otimes_A B$ as a B-algebra on the second factor. Then $B \otimes_A B$ is separable projective over B and there is a canoncial map $f\colon B \otimes_A B \to B$ mapping $x \otimes y \mapsto xy$. This map corresponds to the *diagonal embedding*. By the lemma above, there exists a B-algebra C such that:

$$\begin{array}{ccc} B \otimes_A B & \xrightarrow{g} & B \times C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & = & B \end{array}$$

commutes. We will see the connection between C and the differentials of B/A later.

Exercises 1. i) Let A be a ring and $0 \to M' \to M \to M'' \to 0$ an exact sequence of A-modules. We say that the exact sequence is split if there is an A-linear isomorphism $\varphi : M \cong M' \oplus M''$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & M' \oplus M'' & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Prove that the following three assertions are equivalent.

(a) The exact sequence $0 \to M' \to M \to M'' \to 0$ is split.

(b) There is an A-linear map $M'' \to M$ such that the composition$M'' \to M \to M''$ is the identity of M''.

(c) There is an A-linear map $M \to M'$ such that the composition $M' \to M \to M'$ is the identity of M'.

ii) Let A be a ring, M an A-module, $\{P_i\}_{i\in I}$ a family of A-modules and $P = \oplus_{i\in I} P_i$. Prove that $\mathrm{Hom}_A(P, M) = \prod_{i\in I} \mathrm{Hom}_A(P_i, M)$ and $P \otimes_A M \cong \oplus_{i\in I} P_i \otimes_A M$.

iii) Let M, N be modules over a ring A with M finitely presented and let $S \subset A$ be a multiplicatively closed subset. Prove that the obvious map

$$S^{-1}\left(\mathrm{Hom}_A(M, N)\right) \to \mathrm{Hom}_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$$

is an isomorphism of $S^{-1}A$-modules.

iv) Let A be a ring, $(f_i)_{i\in I}$ a collection of elements of A such that $\sum_{i\in I} f_i A = A$ and M an A-module.

(a) Suppose that $M_{f_i} = 0$ for all $i \in I$. Prove that $M = 0$.

(b) Suppose that M_{f_i} is a finitely generated A_{f_i}-module for every $i \in I$. Prove that M is a finitely generated A-module.

v) Let A be a ring and P a finitely generated projective A-module.

(a) Suppose that P has constant rank n. Prove that $\mathrm{Tr}_{P/A}(\mathrm{Id}_P) = n \cdot 1_A \in A$.

(b) In the general case prove that $\mathrm{Tr}_{P/A}(\mathrm{Id}_P)$ is the image of $r_{P/A}$ under the natural map $H_0(A) \to \mathcal{O}_X(X) \cong A$, where $X = \mathrm{Spec}(A)$, $\mathcal{O}_X$ is the structure sheaf of the scheme X and $H_0(A)$ denotes the ring of continuous functions $\mathrm{Spec}(A) \to \mathbb{Z}$.

vi) Let A be a ring, B an A-algebra and P a finitely generated projective A-module.

Prove that the natural diagram

$$\begin{array}{ccc} \mathrm{End}_A(P) & \xrightarrow{\mathrm{Id}_B} & \mathrm{End}_A(P \otimes_A B) \\ \big\downarrow{\scriptstyle \mathrm{Tr}_{P/A}} & & \big\downarrow{\scriptstyle \mathrm{Tr}_{(P\otimes B)/B}} \\ A & \longrightarrow & B \end{array}$$

is commutative.

vii) Let A be a ring, $0 \to P_0 \to P_1 \to P_2 \to 0$ an exact sequence of A-modules in which P_1, P_2 are finitely generated projective. Let $g : P_1 \to P_1$ be an A-linear map such that $g(P_0) \subset P_0$. Denote by h the induced map $P_2 \to P_2$. Prove that P_0 is finitely generated projective and that $\mathrm{Tr}_{P_1/A}(g) = \mathrm{Tr}_{P_0/A}(g) + \mathrm{Tr}_{P_2/A}(h)$.

viii) Let P, Q be two finitely generated projective A-modules and $f : P \to Q$ and $g : Q \to P$ two linear maps. Prove that $\mathrm{Tr}_{Q/A}(f \circ g) = \mathrm{Tr}_{P/A}(g \circ f)$.

ix)

(a) Let P be a finitely generated projective A-module. Prove that the map $\psi : \mathrm{End}_A(P) \to \mathrm{End}_A(P^v)$ defined by $\psi(f)(g) = g \circ f$ is an anti-isomorphism of rings (not necessarily commutative) and that $\mathrm{Tr}_{P^v/A}(\psi(f)) = \mathrm{Tr}_{P/A}(f)$.

(b) Let $f : P \to P$ and $g : Q \to Q$ be endomorphisms of the finitely generated projective A-modules P, Q. Prove that $\mathrm{Tr}_{P\otimes Q/A}(f \otimes g) = \mathrm{Tr}_{P/A}(f) \cdot \mathrm{Tr}_{Q/A}(g)$.

x) Let A be a ring, B an A-algebra and P a projective A-module. Prove that $P \otimes_A B$ is a projective B-module and that if P is finitely generated the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

where the vertical maps are the ranks.

xi) Let P be a finitely generated A-module such that for each $\mathfrak{p} \in \mathrm{Spec}(A)$ the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free of rank $r(\mathfrak{p})$, where $r : \mathrm{Spec}(A) \to \mathbb{Z}$ is a continuous function. Prove that P is finitely generated projective.

xii) Let A be a ring and P a finitely generated A-module. Prove that the following properties are equivalent:

(a) P is faithfully projective.

(b) The map $A \to \mathrm{End}_{\mathbb{Z}}(P)$ giving the A-module structure of P is injective.

(c) P is faithful, i.e. if M is an A-module we have: $M \otimes_A P = 0$ if and only if $M = 0$.

(d) P is faithfully flat.

xiii) Let P, Q be finitely generated projective A-modules and $k \geq 0$, $k \in \mathbb{Z}$. Prove that $P \oplus Q$, $P \otimes_A Q$, $\mathrm{Hom}_A(P, Q)$, P^v, $\wedge^k P$, $P^{\otimes k}$ are finitely generated projective A-modules and the ranks, r are given, as functions on $\mathrm{Spec}(A)$ by: $r(P \oplus Q) = r(P) + r(Q)$, $r(P \otimes Q) = r(P) \cdot r(Q)$, $r(\mathrm{Hom}_A(P, Q)) = r(P) \cdot r(Q)$, $r(P^v) = r(P)$, $r(\wedge^k P)$=combinations of $r(P)$ choose k, $r(P^{\otimes k}) = r(P)^k$.

xiv)

(a) Let A be a ring and I, J ideals of A such that $I + J = A$. Prove that there is an isomorphism of A-modules $I \oplus J \cong (I \cdot J) \oplus A$.

(b) Prove that every ideal of a Dedekind domain A is projective and that an A-module is projective if and only if it is a direct sum of ideals of A.

(c) Let M be a finitely generated module over a Dedekind domain A. Prove that M is projective if and only if it is torsion free.

1.2 Finite étale coverings

This section generalises the property of being a separable projective algebra to arbitrary schemes. To this end we define étale morphisms, or coverings. A main theme of these notes will be to examine how well these defintions replace the notion of a topological covering in the category of schemes. We will also look at the connection between separability and ramification.

Definition 1.2.1. Let X and Y be schemes, $f : Y \to X$ a morphism. We say that f is **affine** if, for every open affine $U \subseteq X$, $f^{-1}(U)$ is an affine subset of Y.

Remark. A finite morphism is always affine (this is part of its definition!).

Definition 1.2.2. Let X and Y be schemes, $f : Y \to X$ a morphism. We say that f is a **finite, locally-free morphism of schemes** if for every open affine subset $U \simeq \mathrm{Spec}(A)$ of X, $f^{-1}(U) \simeq \mathrm{Spec}(B)$ is affine in Y, and the A-algebra structure on B induced by f makes B finite and **projective**.

Definition 1.2.3. Let X and Y be schemes, $f : Y \to X$ a morphism. We say that f is a **finite étale covering** if for every open affine subset $U \simeq \mathrm{Spec}(A)$ of X, $f^{-1}(U) \simeq \mathrm{Spec}(B)$ is affine in Y, and the A-algebra structure on B induced by f makes B finite projective and separable.

Proposition 1.2.4. *Let $f : Y \to X$ be a morphism of schemes. Then f is affine if and only if there is a cover $\{U_i\}$ of X by affine opens, say $U_i \simeq \mathrm{Spec}(A_i)$, such that $f^{-1}(U_i) \simeq \mathrm{Spec}(B_i)$ is affine for each i.*

Proof. □

Proposition 1.2.5. *Let $f : Y \to X$ be a morphism of schemes. Then f is a finite and locally-free morphism if and only if there is a cover $\{U_i\}$ of X by affine opens, say $U_i \simeq \operatorname{Spec}(A_i)$, such that $f^{-1}(U_i) \simeq \operatorname{Spec}(B_i)$ is affine for each i, and the A_i-algebra structure on B_i induced by f makes B_i finite and free.*

Proof. □

Proposition 1.2.6. *Let $f : Y \to X$ be a morphism of schemes. Then f is a finite étale covering if and only if there is a cover $\{U_i\}$ of X by affine opens, say $U_i \simeq \operatorname{Spec}(A_i)$, such that $f^{-1}(U_i) \simeq \operatorname{Spec}(B_i)$ is affine for each i, and the A_i-algebra structure on B_i induced by f makes B_i finite projective and separable.*

Proof. If f is a finite étale covering, then f is finite and locally-free. We can reduce to the case of $X = \operatorname{Spec}(A)$ affine. In that case, as f is finite, $Y = \operatorname{Spec}(B)$ is also affine. By hypothesis, B is a finite separable projective A-algebra. So write $A = \sum f_i A$, such that B_{f_i} is a separable free A_{f_i}-algebra (because if B is separable over A, then B_{f_i} is separable over A_{f_i}.

Conversely, let $U = \operatorname{Spec}(A) \subseteq X$ be any affine open subset of X. Then $f^{-1}(U)$ is affine, $f^{-1}(U) = \operatorname{Spec}(B)$, where B is a finite projective A-algebra.

Consider the morphism $\phi_B \colon B \to \operatorname{Hom}_A(B, A)$. We need to show that ϕ_B is an isomorphism. Let $f_1, \dots f_n \in A$ be such that $A = \sum f_i A$ and B_{f_i} is free over A_{f_i}. In that case,

$$\phi_{B_{f_i}} \colon B_{f_i} \xrightarrow{\simeq} \operatorname{Hom}_{A_{f_i}}(B_{f_i}, A_{f_i})$$

is an isomorphism for each i, and hence ϕ_B is a local isomorphism, which implies that it is an isomorphism, as we wanted to show. □

Remark. When we take $\mathfrak{p} \in \operatorname{Spec}(A)$ and consider $B_{\mathfrak{p}}$, note that this is **not** the stalk of $\operatorname{Spec}(B)$ at any point, as $\mathfrak{p}$ is not a prime ideal of B ($B_{\mathfrak{p}} = S^{-1}B$, where $S = A \setminus \mathfrak{p}$).

Let now $f \colon Y \to X$ be a finite, locally-free morphism of schemes. Let $U \subseteq X$ be an affine open, $U = \operatorname{Spec}(A)$. Then we have seen that $f^{-1}(U) = \operatorname{Spec}(B)$, and B is projective over A. Hence we have the rank function defined, $[B : A] \colon U = \operatorname{Spec}(A) \to \mathbb{Z}$.

If $V \subseteq U$ is another affine, $V = \operatorname{Spec}(A')$, then $[B : A]_{|V} = [B' : A'] \colon V \to \mathbb{Z}$, and so one can glue these functions and get the global rank function:

$$[Y : X] \colon X \to \mathbb{Z}$$

which is defined by $[Y : X]_{|U} = [B : A]$, if $U = \operatorname{Spec}(A)$ and $f^{-1}(U) = \operatorname{Spec}(B)$.

Also, note that, if $x \in X$,

$$[Y : X](x) = \mathrm{rk}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{Y,f^{-1}(x)}\right)$$

but note that the ring $\mathcal{O}_{Y,f^{-1}(x)}$ is not a stalk, but a product of stalks.

Definition 1.2.7. A morphism $f\colon Y \to X$ is *surjective* if $f_{\mathrm{top}}\colon |Y| \to |X|$ is a surjective map.

Proposition 1.2.8. *Let $f\colon Y \to X$ be a finite, locally-free morphism of schemes. Then:*

i) $Y = \varnothing \iff [Y : X] = 0$,

ii) $f\colon Y \to X$ *is an isomorphism* $\iff [Y : X] = 1$,

iii) $f\colon Y \to X$ *is surjective* $\iff [Y : X] \geq 1$, $\iff$ *for each* $U \subseteq X$ *affine,* $U = \mathrm{Spec}(A)$*, then* $f^{-1}(U) = \mathrm{Spec}(B)$ *with* B *faithfully-projective (this implies that the structure morphism is injective).*

Proof. We may assume without loss of generality that X is affine, $X = \mathrm{Spec}(A)$ (because all properties are local on X). Then $Y = \mathrm{Spec}(B)$ is affine as well (since f is finite, it is affine). So the hypothesis translates into B being a finite-projective A-algebra. This makes (i) and (ii) trivial. We need to prove (iii).

The map $\varphi\colon A \to B$ (corresponding to $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ makes B a finite-projective A-algebra.

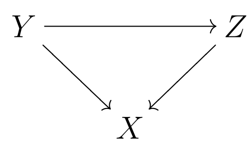
f surjective $\implies [B : A] \geq 1$ ($\iff$ φ injective, by a previous prop.):
If $\mathfrak{q} \in \mathrm{Spec}(B)$, then $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A)$. If f is surjective, let then $\mathfrak{q} \in \mathrm{Spec}(A)$. There exists then $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. We want to prove that $[B : A](\mathfrak{p}) \geq 1$, that is, $B_{\mathfrak{p}}$ is nonzero. But $0 \neq B_{\mathfrak{q}} \simeq (B_{\mathfrak{p}})_{\mathfrak{q}}$, hence $B_{\mathfrak{p}} \neq 0$, as wanted.

$\varphi\colon A \to B$ injective $\implies f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ surjective.
As B is finite over, A, it is a fortiory integral over A. By the going-up theorem, $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, as wanted. □

Examples 1.2.9.
Let $X = \mathrm{Spec}(A)$. Then $Y = X \amalg \cdots \amalg X$ is the geometric analogue of example (i) above. It is the disjoint union of n copies of X, called the trivial finite étale covering of X. Note that this construction works for any scheme X.

ii) Let A be a Dedekind domain, let K be the fraction field of A, L a finite *unramified* extension of K and let B be the integral closure of A in L. The inclusion $A \to B$ corresponds to a map of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. The assumption that L/K is unramified implies that, this map is a finite étale covering. Since $\mathbb{Q}$ does not admit any unramified extensions, this gives an indication of why finite étale coverings of $\mathbb{Z}$ are uninteresting.

Given a scheme X, let $\mathbf{FEt}_X$ denote the category with objects all finite étale coverings $Y \to X$ of X. For two coverings $Y \to X$, $Z \to X$, a morphism of finite étale coverings is a morphism of schemes $Y \to Z$ such that:



commutes.

Definition 1.2.10. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent** (resp. **anti-equivalent**) if there exist covariant (resp. contravariant) functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ and invertible morphisms of functors $\phi\colon FG \to Id_{\mathcal{C}}$, $\psi\colon GF \to Id_{\mathcal{D}}$. In this case we will write $\mathcal{C} \simeq \mathcal{D}$.

The goal of this chapter is to prove the following:

Theorem 1.2.11. *Suppose that X is a connected scheme. Then there is a profinite group π, uniquely determined up to isomorphism, such that there is an equivalence of categories:*

$$\mathbf{FEt}_X \simeq \pi\text{-}\mathbf{Sets}.$$

(In what follows, the category of π-sets has finite sets S as objects, such that π acts continuously on S when S is given the discrete topology and π the profinite topology.)

The profinite group π is called the fundamental group of X. The fundamental group of a path-connected topological space is only defined up to isomorphism. Choosing different basepoints produces isomorphic groups. In the following section we treat the case of the spectrum of a field K. In this case, fixing an algebraic closure of K is the analogue of choosing a basepoint.

1.3 Example: Fundamental group of a field

Our goal in this section is to prove Theorem 1.2.11 above in the case that $X = \mathrm{Spec}(K)$, where K is a field. Note that X is connected since it is a single point.

Let $f\colon Y \to X$ be a finite étale covering. Since X is affine, we deduce that Y must be affine as well, say $Y = \mathrm{Spec}(B)$. The morphism f corresponds to a ring homomorphism $f^*\colon K \to B$ that makes B into a K-algebra. Proposition 1.2.6 above (or the fact that X is a single point) implies that B is a finite separable K-algebra (it is in fact free, not just projective, since K is a field). These remarks show that classifying finite étale coverings of X is equivalent to classifying finite separable K-algebras. More precisely, letting $\mathbf{SAlg}_K$ denote the category of finite separable K-algebras, then we have shown that there is a natural anti-equivalence:

$$\mathbf{FEt}_K \simeq \mathbf{SAlg}_K.$$

It hence suffices to produce a profinite group π and an anti-equivalence $\mathbf{SAlg}_K \simeq \pi - \mathbf{sets}$.

Recall of Galois Theory.

Let K be a field and fix an algebraic closure $\overline{K}$. We assume that the reader is familiar with the definition of normal and separable algebraic extensions (even in the infinite case). An algebraic extension L/K is said to be **Galois** if it is normal and separable. Put:

$$I_L = \{E \mid K \subset E \subset L\ ,\ [E\colon K] < \infty\ ,\ E/K \text{ is Galois}\}.$$

Note that:

$$L = \bigcup_{E \in I_L} E \simeq \varinjlim_{E \in I_L} E.$$

Suppose that L/K is Galois and let:

$$\mathrm{Gal}(L/K) = \{\phi\colon L \to \overline{K} \mid \phi|_K = Id\}.$$

One can show that:

$$\mathrm{Gal}(L/K) \simeq \varprojlim_{E \in I_L} \mathrm{Gal}(E/K).$$

In fact, taking all subgroups $\mathrm{Gal}(L/E)$ for $E \in I_L$ as a basis of neighbourhoods of the identity in $\mathrm{Gal}(L/K)$, and endowing the projective limit with the profinite topology, then one can show that the isomorphism above is a topological isomorphism. Viewed in this way, Galois groups are compact Hausdorff spaces (exercise).

Definition 1.3.1. Let $K_s \subset \overline{K}$ denote the set of elements that are separable over K. This is called the **separable closure** of K in $\overline{K}$. One can show that it is a Galois extension of K. The **absolute Galois group of** K is:

$$G_K \colon = \mathrm{Gal}(K_s/K).$$

Let E/K be Galois. Then every embedding $E \to \overline{K}$ in fact lands in K_s. It follows that:

$$G_K \simeq \varprojlim_{E \in I_{\overline{K}}} \mathrm{Gal}(E/K).$$

As above, this is a toplogical isomorphism. If $K \subset L \subset K_s$, then there is a natural inclusion:

$$\mathrm{Gal}(K_s/L) \to G_K.$$

Since $\mathrm{Gal}(K_s/L)$ is compact and G_K is Hausdorff, we deduce that $\mathrm{Gal}(K_s/L)$ is a closed subgroup of G_K. Note that the fundamental theorem of Galois theory extends to the case that L/K is infinite Galois; however, the statement must be modified so that intermediate fields correspond bijectively to the *closed* subgroups of $\mathrm{Gal}(L/K)$ in the profinite topology.

To prove Theorem 1.2.11 in the case $X = \mathrm{Spec}(K)$, we will take $\pi = G_K$. In light of the first remarks in this section, we will first prove some results concerning finite separable K-algebras.

Lemma 1.3.2. *Let B be a finite K-algebra. Then $B \simeq \prod_{i=1}^{t} B_i$ where B_i is a local K-algebra with a nilpotent maximal ideal.*

Proof. The proof is broken up into two cases. First suppose that B is an integral domain. For any nonzero $b \in B$, the multiplication map m_b is a K-algebra endomorphism of B. Since B is an integral domain, it is actually an isomorphism. This implies that there exists $a \in B$ with $ab = 1$. We deduce that B is a field, hence a finite extension of K.

Now let B be a general finite K-algebra. Let $\mathfrak{p} \in \mathrm{Spec}(B)$. By the case just treated, $B/\mathfrak{p}$ is a field. Hence $\mathfrak{p}$ is maximal. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be distinct primes of B. There is a natural map:

$$B \to \prod_{i=1}^{s} (B/\mathfrak{m}_i).$$

The Chinese remainder theorem implies that this map is surjective (since distinct maximal ideals are relatively prime). We deduce that:

$$\sum_{i=1}^{s} \dim_K(B/\mathfrak{m}_i) \leq \dim_K(B).$$

This implies that B has a finite number of distinct maximal ideals, say $\mathfrak{m}_1, \dots, \mathfrak{m}_t$.

The kernel of

$$B \to \prod_{i=1}^{t} (B/\mathfrak{m}_i).$$

is the nilradical $\mathfrak{N}$ of B. Note that B is Noetherian, and hence $\mathfrak{N}$ is generated by a finite number of elements. This implies that there exists a single integer $N > 0$ such that $\mathfrak{N}^N = 0$. If $i \neq j$ then $(\mathfrak{m}_i, \mathfrak{m}_j) = B$ implies that also $(\mathfrak{m}_i^N, \mathfrak{m}_j^N) = B$. So we also have a natural surjective map:

$$B \to \prod_{i=1}^{t} (B/\mathfrak{m}_i^N).$$

The kernel of this is $\prod_i (\mathfrak{m}_i^N) = \mathfrak{N}^N = 0$. It is hence an isomorphism. Note that each $B/\mathfrak{m}_i^N$ is a local ring with nilpotent maximal ideal. □

Theorem 1.3.3. *Let B be a finite K-algebra, $\overline{K}$ an algebraic closure of K, $\overline{B} = B \otimes_K \overline{K}$. Then the following are equivalent:*

i) B is separable as a K-algebra,

ii) $\overline{B}$ is separable as a $\overline{K}$-algebra,

iii) $\overline{B} \simeq \overline{K}^n$ as $\overline{K}$-algebras,

iv) $B \simeq \prod_{i=1}^{t} B_i$, where the B_i are finite separable field extensions of K.

Proof. i) is equivalent to ii): If B is a separable A-algebra, then ϕ as defined above via the trace is an isomorphism. Let $\{\omega_i\}$ be a free basis for B and let $\{f_i\}$ be the corresponding dual basis for $B^\vee$. The map $\psi \colon B^\vee \to B$ mapping $f_i \mapsto \omega_i$ is an isomorphism. Hence so is $\psi \circ \phi$; let M be the matrix of this morphism in the basis of ω_i's. Then M is an invertible matrix. Since the determinant is a homomorphism, we deduce that $\det(M)$ is invertible. One can easily show that the converse also holds. This result will be applied to both B and $\overline{B}$: one checks that the matrices obtained for both algebras are equal. The equivalence follows.

ii) implies (iii): By the previous exercise we can write:

$$B \simeq \prod_{i=1}^{t} B_i$$

where B_i is a local K-algebra with maximal ideal $\mathfrak{m}_i$ where $\mathfrak{m}_i^N = 0$ (the same N for all i). Then we have:

$$\overline{B} \simeq \prod_{i=1}^{t} \overline{B_i}.$$

Each $\overline{B_i}$ is a local free $\overline{K}$-algebra with nilpotent maximal ideal; we will show that each is a field. Fix an i and let $\mathfrak{m}$ be the maximal ideal. Let $c \in \mathfrak{m}$, so that for all $x \in \overline{B_i}$ we have $(cx)^N = 0$. Then the multiplication by cx map m_{cx} satisfies $m_{cx}^N = 0$. The minimal polynomial of this endomorphism is hence x^m for some m. We deduce that $\mathrm{Tr}(m_{cx}) = \mathrm{Tr}(cx) = \phi(c)(x) = 0$ for all $x \in \overline{B_i}$. But this means $\phi(c) = 0$ if $c \in \mathfrak{m}$. Since $\overline{B}$ is a separable $\overline{K}$-algebra, ϕ is an isomorphism. Hence $c = 0$. So $\mathfrak{m} = 0$ and $\overline{B_i}$ is a field. It is a finite extension of $\overline{K}$ and is hence isomorphic to $\overline{K}$.

(iii) implies (ii) is obvious.

(iii) implies (iv): We still have a decomposition of B as above. By assumption, the nilradical of $\overline{B}$ is zero. It is the product of the nilradical of each $\overline{B_i}$ (which is $\prod(\mathfrak{m}_i \otimes \overline{K})$). One deduces that $\mathfrak{m}_i \otimes \overline{K} = 0$ for each i, hence also $\mathfrak{m}_i = 0$ for each i. We deduce that each B_i is a field. It remains to argue that they are separable.

Let $b \in B_i$, so that $K[b] \subset B_i$. Note that $K[b] \simeq K[x]/(f)$ where f is the minimal polynomial for b over K. Then:

$$K[b] \otimes_K \overline{K} \subset \overline{B_i}$$

since tensoring with $\overline{K}$ preserves injections. But:

$$K[b] \otimes_K \overline{K} \simeq \overline{K}[b] \simeq \overline{K}[x]/(f) \simeq \prod_j \left(\overline{K}[x]/(x - b_j)^{r_j}\right)$$

where the b_j are the distinct roots of f. Since $\overline{B_i}$ does not contain any nilpotent elements, we must have $r_j = 1$ for all j. This implies that f has distinct roots in $\overline{K}$ and is hence separable over K. Since $b \in B_i$ was arbitrary we deduce that B_i is separable.

(iv) implies (iii): It suffices to show that if B is a finite separable field extension of K then $\overline{B} \simeq \overline{K}^n$ for some n. By the primitive element theorem, there exists $\alpha \in B$ such that $B = K[\alpha]$. Let f be the minimal polynomial of α over K, and let $\alpha_1, \ldots, \alpha_n$ be the distinct roots of f. Then:

$$\overline{B} \simeq \overline{K}[x]/(f) \simeq \prod_{i=1}^{n} \left(\overline{K}[x]/(x - \alpha_j)\right) \simeq \overline{K}^n.$$

as we wanted to prove. □

Theorem 1.2.11 will follow from the following theorem and the first remark of this section:

Theorem 1.3.4. *Let K be a field; fix an algebraic closure $\overline{K}$ and separable closure K_s. Let $\pi = \mathrm{Gal}(K_s/K)$ be the absolute group of K. Then there is a natural anti-equivalence of categories:*

$$\mathbf{SAlg}_K \simeq \pi{-}\mathbf{Sets}.$$

Proof. The desired functors $F\colon \mathbf{SAlg}_K \to \pi{-}\mathbf{Sets}$, $G\colon \pi{-}\mathbf{Sets} \to \mathbf{SAlg}_K$ will be almost representable. The minor technicality is that the object that represents them, K_s, is a limit of objects in the respective categories and not necessarily itself an object in them. Let:

$$\begin{aligned} F(-) &\stackrel{\text{def}}{=} \mathrm{Hom}_{K-alg}(-, K_s) \\ G(-) &\stackrel{\text{def}}{=} \mathrm{Hom}_{\pi-sets}(-, K_s). \end{aligned}$$

We break the proof up into several steps.

F is well-defined. Let B be a finite separable K-algebra. It is not clear that $F(B)$ is a π-set. We must show that it is finite and that it is supplied with a π-action. The π-action is easy; since π acts on K_s, π acts on $F(B)$ by composition on the left.

Write $B = \prod B_i$ where the B_i are finite separable field extensions of K (by Theorem 1.3.3). Note that each B_i injects into K_s. Fix an embedding $f_i\colon B_i \to K_s$ for each i. Put $\pi_i = \mathrm{Gal}(K_s/B_i)$. Note that this is an open subgroup of π since B_i/K is finite. We have $B_i = K_s^{\pi_i}$ and

$$\phi\colon \pi/\pi_i \simeq \mathrm{Hom}_{K-alg}(B_i, K_s).$$

If $[\tau] \in \pi/\pi_i$ then $\phi([\tau]) = \tau \circ f_i$. Since π_i is open, π/π_i is finite, and hence so is $\mathrm{Hom}_{K-alg}(B_i, K_s)$. We can let π act on π/π_i via composition on the left, $\sigma \cdot [\tau] \mapsto [\sigma \circ \tau]$. This corresponds to the composition action on $\mathrm{Hom}_{K-alg}(B_i, K_s)$ via ϕ: let $\sigma \in \pi$, $[\tau] \in \pi/\pi_i$ so that:

$$\phi(\sigma \cdot [\tau]) = \phi([\sigma \circ \tau]) = (\sigma \circ \tau) \circ f_i = \sigma \circ (\tau \circ f_i) = \sigma \cdot \phi([\tau]).$$

One checks that:

$$F(B) = \mathrm{Hom}_{K-alg}\left(\prod B_i, K_s\right) \simeq \coprod \mathrm{Hom}_{K-alg}(B_i, K_s).$$

Hence $F(B)$ is a finite π-set.

Now let B, B' be two finite separable K-algebras with $\phi\colon B \to B'$ a map of K-algebras. We must show that $F(\phi)$ is a map of π-sets. Let $\sigma \in \pi$ and $f \in F(B)$. Then:

$$F(\phi)(\sigma \cdot f) = (\sigma \cdot f) \circ \phi = \sigma \circ f \circ \phi = \sigma \circ (F(\phi)(f)) = \sigma \cdot (F(\phi)(f)).$$

So $F(\phi)$ is a morphism of π-sets. The axioms for a functor hold since they are satisfied quite generally by functors defined by forming Hom-sets.

G is well-defined. Let E be a finite π-set. Since K_s is a K-algebra, we can define a K-algebra structure on $G(E) = \mathrm{Hom}_{\pi-sets}(E, K_s)$ pointwise. We must show that $G(E)$ is a finite separable K-algebra. Write E as a disjoint union of orbits:

$$E = \coprod \pi e = \coprod \pi/\mathrm{stab}_\pi(e)$$

Since E has the discrete topology and we have assumed the π-action on E to be continuous, the stabiliser of each orbit representative e is an open subgroup of E. So we have a decomposition:

$$E \simeq \coprod_i \pi/\pi_i$$

with the π_i open subgroups of π. Then:

$$G(E) = \mathrm{Hom}_{\pi-sets}(E, K_s) \simeq \prod_i \mathrm{Hom}_{\pi-sets}(\pi/\pi_i, K_s).$$

It suffices to show that each factor is a finite separable field extension of K.

We claim that there is a natural isomorphism $\alpha\colon \mathrm{Hom}_{\pi-sets}(\pi/\pi_i, K_s) \simeq K_s^{\pi_i}$. Note that since π_i is open, $K_s^{\pi_i}$ is a finite separable field extension of K. Let:

$$\alpha(f) = f([Id]).$$

This certainly maps to K_s and is a map of K-algebras. Furthermore, if $\sigma \in \pi_i$ then $[\sigma] = [Id]$ and:

$$f([Id]) = f([\sigma]) = f(\sigma \cdot [Id]) = \sigma \cdot f([Id]).$$

So $f([Id]) \in K_s^{\pi_i}$.

To see that α is an isomorphism, we will define the inverse K-algebra map $\beta\colon K_s^{\pi_i} \to \mathrm{Hom}_{\pi-sets}(\pi/\pi_i, K_s)$. Given $a \in K_s^{\pi_i}$ define $\beta(a)\colon \pi/\pi_i \to K_s$ by putting:

$$\beta(a)([\sigma]) = \sigma \cdot a.$$

This map is well-defined since a is fixed by π_i. We must next check that $\beta(a)$ is a map of π-sets:

$$\beta(a)(\sigma \cdot [\tau]) = \beta(a)([\sigma \circ \tau]) = (\sigma \circ \tau)(a) = \sigma \cdot (\tau(a)) = \sigma \cdot (\beta(a)([\tau])).$$

We let the reader verify that β is a map of K-algebras, and that $\alpha \circ \beta = Id$, $\beta \circ \alpha = Id$. We thus have:

$$G(E) = \mathrm{Hom}_{\pi-sets}(E, K_s) \simeq \prod K_s^{\pi_i},$$

so it is a finite separable K-algebra. This verifies that G is well-defined on objects.

Now let E, E' be finite π-sets with $\phi\colon E \to E'$ a morphism of π-sets. Since the K-algebra structures on $G(E)$, $G(E')$ are defined pointwise, $G(\phi)$ is a K-algebra map.

It remains to argue that there invertible natural transformations:

$$\begin{aligned} \theta &: Id_{\mathbf{SAlg}_K} \to G \circ F \\ \eta &: Id_{\pi-sets} \to F \circ G \end{aligned}$$

Such natural transformations are called natural ismorphisms. We leave it as an exercise to check that a natural transformation α is a natural isomorphism if and only if α_x is an isomorphism for each object x.

Definition of θ. Given a finite separable K-algebra B, we must define an isomorphism of K-algebras

$$\theta_B\colon B \to GF(B).$$

Note that:

$$GF(B) = G(\mathrm{Hom}_{K-alg}(B, K_s)) = \mathrm{Hom}_{\pi-sets}\left(\mathrm{Hom}_{K-alg}(B, K_s), K_s\right).$$

For $b \in B$ let $\theta_B(b)(f) = f(b)$. There are several things to check. The first is that $\theta_B(b)$ is a map of π-sets:

$$\theta_B(b)(\sigma \cdot f) = (\sigma \cdot f)(b) = (\sigma \circ f)(b) = \sigma \cdot f(b) = \sigma \cdot \theta_B(b)(f).$$

One must next verify that θ_B is a K-algebra homomorphism. We leave this to the reader. Thirdly, one must argue that θ_B is an isomorphism. To do this we will show that θ_B is injective, and that the dimensions of B and $GF(B)$ are the same.

Write $B = \prod B_i$ where the B_i are finite separable field extensions of K. Given $b = (b_1, \ldots, b_n) \in B$ nonzero, there exists an index i so that $b_i \in B_i$ is nonzero. Then any embedding of $f\colon B_i \to K_s$ corresponds to a K-algebra homomorphism $f\colon B \to K_s$ that does not vanish on $b \in B$. Hence $\theta_B(b) \neq 0$ and θ_B is injective.

Let $\pi_i = \mathrm{Gal}(K_s/f(B_i))$. Then $K_s^{\pi_i} \simeq B_i$ and:

$$F(B_i) \simeq \mathrm{Hom}_{K-alg}(K_s^{\pi_i}, K_s) \simeq \pi/\pi_i.$$

Thus:

$$GF(B_i) \simeq G(\pi/\pi_i) = \mathrm{Hom}_{\pi-sets}(\pi/\pi_i, K_s) \simeq K_s^{\pi_i} \simeq B_i.$$

We see that the dimension of $GF(B_i)$ is equal to the dimension of B_i for each i. But we have:

$$GF(B) \simeq \mathrm{Hom}_{\pi-sets}\left(\coprod \mathrm{Hom}_{K-alg}(B_i, K_s), K_s\right) \simeq \prod GF(B_i).$$

This shows that $\dim_K(GF(B)) = \dim_K(B)$, hence θ_B is an isomorphism of K-algebras.

It remains to show that θ is a natural transformation. If B, C are K-algebras, $\phi\colon B \to C$ a morphism of K-algebras, we must check that:

$$\begin{array}{ccc} B & \xrightarrow{\phi} & C \\ {\scriptstyle\theta_B}\downarrow & & \downarrow{\scriptstyle\theta_C} \\ GF(B) & \xrightarrow[GF(\phi)]{} & GF(C) \end{array}$$

commutes. The only tricky part is unravelling what $GF(\phi)$ does to $\theta_B(b)$ for $b \in B$. Quite generally, take $f \in GF(B)$ and $g \in \mathrm{Hom}_{K-alg}(C, K_s)$. Then one checks that:

$$GF(\phi)(f)(g) = f(g \circ \phi).$$

If we take $f = \theta_B(b)$ then:

$$GF(\phi)(\theta_B(b))(g) = \theta_B(b)(g \circ \phi) = g(\phi(b)) = \theta_C(\phi(b))(g).$$

This holds for every $g \in \mathrm{Hom}_{K-alg}(C, K_s)$ and $b \in B$, so that $GF(\phi) \circ \theta_B = \theta_C \circ \phi$.

Definition of η. We will simply define the natural isomorphism η; the verification that it is a natural isomorphism is similar to the work above. Given a finite π-set E we define $\eta_E\colon E \to FG(E)$ by putting

$$\eta_E(e)(h) \colon = h(e).$$

This concludes the proof of our theorem. □

Note that by composing the this anti-equivalence with the anti-equivalence of $\mathbf{FEt}_K$ and $\mathbf{SAlg}_K$ yields Theorem 1.2.11 in the case that $X = \mathrm{Spec}(K)$:

Corollary 1.3.5. *Let K be a field, $X = \mathrm{Spec}(K)$ and let π be the absolute Galois group of K. Then there is a natural equivalence of categories:*

$$\mathbf{FEt}_X \simeq \pi - \mathbf{Sets}.$$

1.4 Totally split morphisms

In order to prove theorem 1.2.11 in full generality, we will need to study properties of étale morphisms more closely. This section will prove a variety of results that will hopefully convince the reader that the notion of an étale morphism is a good algebraic analogue for the topological notion of a covering space.

Definition 1.4.1. A morphism $f\colon Y \to X$ is called *totally split* if $X = \coprod_{n\geq 0} X_n$, such that $f^{-1}(X_n) \simeq X_n \coprod \cdots \coprod X_n$ (n-copies), and such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(X_n) & \xrightarrow{\simeq} & X_n \coprod \cdots \coprod X_n \\ \downarrow f & & \downarrow \\ X_n & = & X_n \end{array}$$

Remarks. If $f\colon Y \to X$ is totally split, then f is finite and étale. This follows since A^n is a finite, separable A-algebra. If X is connected, then $f\colon Y \to X$ being totally split means that $Y \simeq X^n$, for some $n \geq 0$. In the analogy with covering spaces, the totally split morphisms correspond to the trivial covers.

We will prove the following theorem:

Theorem 1.4.2. *Let $f\colon Y \to X$ be a morphism of schemes. Then f is finite étale if and only if there exists a finite, locally-free and surjective morphism $g\colon W \to X$ such that $Y \times_X W \to W$ is totally split.*

To aid us in proving theorem 1.4.2, we will first show that finite étale morphisms behave very nicely with respect to certain base changes.

Proposition 1.4.3. *Let $f\colon Y \to X$ be an affine morphism of schemes. Let $g\colon W \to X$ be surjective finite and locally free. Then f is finite étale if and only if $h\colon Y \times_X W \to W$ is finite étale.*

Proof. First suppose that f is finite étale. Let $U \subset X$ be an affine open, say $U = \operatorname{Spec}(A)$. Since f is affine, we can write $f^{-1}(U) = \operatorname{Spec}(B)$. Finite morphisms are affine as well, and we can hence write $V = g^{-1}(U) = \operatorname{Spec}(C)$. Note that C is a finite A-algebra since g is finite, and that such V's cover W.

Now, $h^{-1}(V)$ is defined via the fibre product as $\operatorname{Spec}(B \otimes_A C)$. The induced map $h\colon \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec}(C)$ corresponds to the ring homomorphism $c \mapsto 1 \otimes c$ in the opposite direction. Since B is finite projective over A (f is finite étale), $B \otimes_A C$ is certainly finite and projective over C. Moreover, one can show that the morphism

$\phi_B\colon B \to B^\vee$ induced by the trace behaves well with respect to tensoring with C. So since B/A is separable, so is $B\otimes_A C/C$. Since the V's cover W, we conclude that h is finite étale.

For the other direction, let U, V, A, B and C be as above. We must show that B/A is finite projective and separable. We are assuming that $B \otimes_A C$ is finite projective and separable over C. The crucial point is that, by a previous proposition, the surjectivity of g implies that C is faithfully projective over A. It is not hard to show that a finitely generated faithfully projective module is faithfully flat. So C is faithfully flat over A. Why is B/A finite projective? For the separability of B/A, note that $\phi_B \otimes id_C$ differs from $\phi_{B\otimes_A C}$ by a natural isomophism, and is hence an isomorphism. Then by faithfull flatness, so is ϕ_B and B/A is separable. □

We now turn to the proof of theorem 1.4.2:

Proof. First suppose that $g\colon W \to X$ is a surjective finite and locally free morphism such that $Y \times_X W \to W$ is totally split. Since totally split morphisms are clearly finite étale, the previous proposition implies that $f\colon Y \to X$ is finite étale. The other direction will require more work.

Let $f\colon Y \to X$ be finite étale. We will first treat the case that $[Y\colon X] = n$ is constant by induction on n. When $n = 0$, $Y = \emptyset$ and the condition of being totally split is vacuously satisfied. So suppose $n \geq 1$. Since $[Y\colon X] \geq 1$, a previous proposition implies that f is surjective. Note that it is finite and locally free, since f is finite étale.

Base change by f and consider the morphism $p\colon Y \times_X Y \to Y$. This is finite étale by the previous proposition. Moreover, one can show that the degree of the map is still n. This essentially follows from the fact that tensoring a finite free A-module with B gives a finite free B-module of the same rank. Let $\Delta\colon Y \to Y \times_X Y$ be the diagonal morphism, so that $p \circ \Delta = id_Y$.

We claim that Δ is an open and closed immersion. This is a local question and, since f is affine, we may as well suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. In this case $Y \times_X Y = \mathrm{Spec}(B \otimes_A B)$ and Δ corresponds to the ring homomorphism $\Delta^\sharp\colon B \otimes_A B \to B$ given by $x \otimes y \mapsto xy$. Since B is a finite projective and separable A-algebra, lemma 1.1.17 yields a commutative diagram:

$$\begin{array}{ccc} B \otimes_A B & \xrightarrow{\sim} & B \times C \\ {\scriptstyle \Delta^\sharp}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & = & B \end{array}$$

for some B-algebra C, where π is the natural projection. This diagram corresponds to scheme maps:

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{\sim} & Y \coprod \mathrm{Spec}(C) \\ {\scriptstyle \Delta}\uparrow & & \uparrow{\scriptstyle inc} \\ Y & = & Y \end{array}$$

We see that Δ identifies Y with a connected component of $Y \times_X Y$, and is hence locally an open and closed immersion. To show that it is globally an open and closed immersion, one covers X by affine opens and uses the affineness of f and the argument above.

One can glue together all of the local decompositions and obtain a commutative diagram:

$$\begin{array}{ccc} Y \times_X Y & \xleftarrow{\sim} & Y \coprod Y' \\ {\scriptstyle p}\downarrow & & \downarrow \\ Y & = & Y \end{array}$$

for some scheme Y'. Since $Y \otimes_X Y \to Y$ is finite étale of degree n, so is $Y \coprod Y' \to Y$. The induced map $Y \to Y \coprod Y' \to Y$ is the identity, and is hence of degree 1. We claim that this implies $Y' \to Y \coprod Y' \to Y$ is of degree $n-1$. We may hence apply the inductive hypothesis to obtain a $g\colon W \to Y$ which is surjective and locally free, such that:

$$\begin{array}{ccc} Y' \times_Y W & \xleftarrow{\sim} & W \amalg \overset{n-1}{\cdots} \amalg W \\ \downarrow & & \downarrow \\ W & = & W \end{array}$$

commutes.

Let $h = f \circ g\colon W \to X$. Since both of f and g are surjective finite and locally free, so is h. It remains to show that $Y \times_X W \to W$ is totally split:

$$\begin{aligned} Y \times_X W &\simeq Y \times_X (Y \times_Y W) \\ &\simeq (Y \times_X Y) \times_Y W \\ &\simeq (Y \amalg Y') \times_Y W \\ &\simeq (Y \times_Y W) \amalg (Y' \times_Y W) \\ &\simeq W \amalg (W \amalg \overset{n-1}{\cdots} \amalg W), \end{aligned}$$

This concludes the proof in the case that $[Y\colon X]$ is constant.

For the general case, write $X = \coprod_{n \geq 0} X_n$ where:

$$X_n = \{x \in X \mid [Y : X](x) = n\}.$$

Note that each X_n is an open and closed subset of X. Each is thus naturally an open (and closed) subscheme of X. Let $Y_n = f^{-1}(X_n)$ for each n. Then $f : Y_n \to X_n$ is finite étale of *constant degree* equal to n. There hence exist surjective finite and locally free morphisms $W_n \to X_n$ for each n, such that $Y_n \times_{X_n} W_n \to W_n$ is totally split. Putting these g_n's together yields the desired morphism:

$$W = \coprod_{n \geq 0} W_n \to X = \coprod_{n \geq 0} X_n.$$

It is surjective finite and locally free since each g_n is. It has the desired splitting property by construction. This concludes the proof of the theorem. □

One might ask how far our analogy between covering spaces and étale morphisms can be pushed. Proceeding naively will lead one to disappointment. For instance, it is not true in general that given a finite étale morphism $f : Y \to X$, there exists a covering $\{U_i\}$ such that $f|_{U_i}$ is totally split for all i. The solution to this dilemma is to alter our notion of "open cover", by introducing a **Grothendieck topology** on X.

For instance, declare:

$$\{f : W \to X \mid f \text{ is surjective finite and locally free}\}$$

to be the open subsets of X. Then the theorem that we just proved says that, in this Grothendieck topology, every finite étale morphism is "locally" totally split. Here locally means that it is totally split after base-changing with one of the open sets above. We will have more to say about Grothendieck topologies in chapter 2.

Proposition 1.4.4. *Suppose that X, Y and Z are schemes, with finite étale morphisms $f : Y \to X$, $g : Z \to Y$. Then $h = g \circ f : Z \to X$ is finite étale.*

Proof. First assume that f is totally split, say of constant rank n:

$$\begin{array}{ccc} Y & \longrightarrow & X_1 \coprod \cdots \coprod X_n \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & = & X \end{array}$$

Here each $X_i \simeq X$. Put $Z_i = g^{-1}(X_i)$, so that $Z \simeq Z_1 \amalg \cdots \amalg Z_n$ and g is the union of the n maps $Z_i \to X_i$. Since g is finite étale, so is each $Z_i \to X_i = X$. But h is just the map obtained by "glueing" these n maps together, and is hence finite étale. If f is totally split, but not of constant rank, then proceed as at the end of the previous theorem.

For the general case, take a surjective finite locally free morphism $\alpha\colon W \to X$ such that $Y \times_X W \to W$ is totally split. Then $Z \times_X W \to Y \times_X W$ is finite étale because it is a base-change, and $Y \times_X W \to W$ is totally split. So by the previous step it follows that the composition $Z \times_X W \to W$ is finite étale. By Proposition 1.4.3, this implies that $h\colon Z \to X$ is finite étale (note that h is affine since it is a composition of two affine morphisms). $\square$

We need to be more systematic when working with totally-split morphism. More concretely, we need a way to label the distinct components of a split cover in a *functorial* way. For this purpose we introduce new terminology.

Definition 1.4.5. Given a scheme X, and a finite set $E = \{e_1, \ldots, e_n\}$, define X^E to be $X^E \overset{\text{def}}{=} X_{e_1} \amalg \cdots \amalg X_{e_n}$, with $X_{e_i} \simeq X$.

Locally, if A is a ring and E is the given finite set, define $A^E \overset{\text{def}}{=} \operatorname{Hom}(E, A)$, the ring of functions from E to A, with pointwise addition and multiplication. Note that the map $f \mapsto (f(e_1), \ldots, f(e_n))$ gives an isomorphism $A^E \simeq A^n$, if $\#E = n$.

Moreover, $\operatorname{Spec}(A^E) = \operatorname{Spec}(A)^E \simeq X \amalg \overset{n}{\cdots} \amalg X$, if $X = \operatorname{Spec}(A)$.

This assignment is functorial. That is, if D and E are finite sets, and $\phi\colon D \to E$ is a map, then it induces a map $\phi^*\colon A^E \to A^D$, sending $f \mapsto \phi^* f = f \circ \phi$.

The natural map $A \to A^E$, sending $a \in A$ to the constant function $e \mapsto a \forall e \in E$ induces a morphism $X^E \to X$.

Also, the map ϕ^* associated to $\phi\colon D \to E$ induces a map $\phi_*\colon X^D \to X^E$.

In general, if X is any scheme and E is a finite set, we write $X = \cup_i U_i$, with $U_i = \operatorname{Spec}(A_i)$ affines, and one can take the amalgamated union $X^E \overset{\text{def}}{=} \amalg_i U_i^E$, which is the gluing of the U_i^E, with the intersections identified.

Remark. One can prove that $X^E \simeq X \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(Z^E)$. If $\phi\colon D \to E$ is any map of finite sets, then $\phi_*\colon X^D \to X^E$ is *finite-étale*.

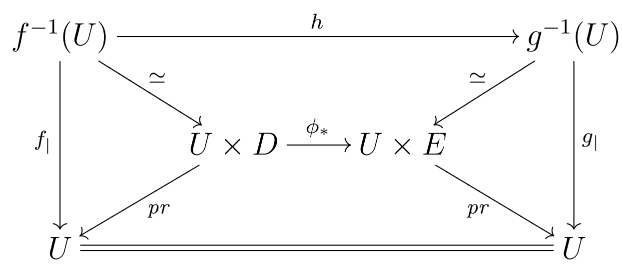
We want to prove an important themorem:

Theorem 1.4.6. *Let $f\colon Y \to X$, $g\colon Z \to X$, $h\colon Y \to Z$ be morphisms of schemes, such that $f = g \circ h$. If f and g are* finite étale*, then so is h.*

$$\begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & Z \\ & {\scriptstyle f.e.}\ \searrow^{f} \quad {}^{g}\swarrow\ {\scriptstyle f.e.} & \\ & X & \end{array}$$

To prove this, we need a technical lemma:

Lemma 1.4.7. *With notation as above, if f and g are totally split, then f,g and h are locally trivial. That is, if $x \in X$, there exists a neighborhood $x \in U \subseteq X$ and a map of finite sets $\phi\colon D \to E$ such that $f^{-1}(U) \simeq U \times D$, $g^{-1}(U) \simeq U \times E$, and the following diagram commutes:*



Proof. Let $x \in X$. We can find an affine open neighborhood V of x such that $[Y : X]|_U$ and $[Z : X]|_U$ are constant.

As f and g are totally-split, $f^{-1}(V) \simeq V^D$ for a finite set D, and similarly with g: $g^{-1}(V) \simeq V^E$ ($\#D$ and $\#E$ are the degrees of Y above X and of Z above X, respectively).

Writing $V = \mathrm{Spec}(A)$, $x \in V$ corresponds to a prime ideal $\mathfrak{p} \subsetneq A$. Then $V \times D = \mathrm{Spec}(A^D)$, and $V \times E = \mathrm{Spec}(A^E)$. We get a map $h\colon V \times D \to V \times E$, which corresponds to a ring homomorphism $\psi\colon A^E \to A^D$.

By localizing at x, we get $\psi_x\colon (A^E)_{\mathfrak{p}} = (A_{\mathfrak{p}})^E \to (A_{\mathfrak{p}})^D = (A^D)_{\mathfrak{p}}$. As $A_{\mathfrak{p}}$ is local, it does not contain any nontrivial idempotents, so we claim that $\psi_x = \phi^*$ for some $\phi\colon D \to E$.

So consider the morphisms $\phi_*, h\colon V \times D \to V \times E$. We know that their localizations are equal: $\psi_x = \phi^*_x$. Think of ψ and ϕ^* as elements of $\mathrm{Hom}_A(A^E, A^D)$ (where we think of them as A-modules, forgetting their ring structure). Localizing at $\mathfrak{p}$, and because A^E is finitely-presented, we get $\big(\mathrm{Hom}_A(A^E, A^D)\big)_{\mathfrak{p}} \simeq \mathrm{Hom}_{A_{\mathfrak{p}}}(A^E_{\mathfrak{p}}, A^D_{\mathfrak{p}})$, and the images of ψ and phi^* are equal in there, so that they are equal in a neighborhood $U \subseteq V$ of x. □

We now proceed to the proof of the theorem we stated:

Proof (of theorem 1.4.6). First, note that if f and g are both totally split, then the lemma implies that h is étale, and we are done.

In the general case, by Theorem 1.4.2 there exists surjective finite and locally free morphisms $W_1 \to X$ and $W_2 \to X$ such that $Y \times_X W_1 \to W_1$ and $Z \times_X W_2 \to W_2$ are totally-split.

Define then $W = W_1 \times_X W_2 \to X$. This is still surjective, finite and locally-free.

Moreover, the maps $Y \times_X W \to W$ and $Z \times_X W \to W$ are totally split, and so by the lemma we deduce that:

$$h \times \mathrm{Id}_W \colon Y \times_X W \to Z \times_X W$$

is finite-étale. We can rewrite: $Y \times_X W = (Y \times_Z Z) \times_X W = Y \times_Z (Z \times_X W)$, and the following diagram commutes:

$$\begin{array}{ccc} Y \times_X W & \xrightarrow{\simeq} & Y \times_Z (Z \times_X W) \\ {\scriptstyle h\times \mathrm{Id}_W}\downarrow & & \downarrow{\scriptstyle h\times \mathrm{Id}_{Z\times_X W}} \\ Z \times_X W & = & Z \times_Z (Z \times_X W) \end{array}$$

From this, one deduces that h is étale by noting that $Z \times_X W \to W$ is surjective, finite and locally-free, and invoking Proposition 1.4.3. □

Exercises 2. In the problems below be very careful about how you reduce proofs to the "affine case".

i) Let $Y_i \to X$ be morphisms of schemes for $1 \le i \le n$ and let $Y = \amalg_{i=1}^n Y_i \to X$ be the induced morphism. Prove that $Y \to X$ is finite and locally free if and only if each $Y_i \to X$ is finite and locally free. Prove that $[Y : X] = \sum_i [Y_i : X]$.

ii) Let $\{X_i\}_{i\in I}$ be a collection of schemes and $Y_i \to X_i$ a finite and locally free morphism for each $i \in I$. Prove that the induced morphism $Y := \amalg_i Y_i \to X := \amalg_i X_i$ is finite and locally free. Prove that $[Y : X]|_{|X_i|} = [Y_i : X_i]$, for all $i \in I$.

iii) Let $Y \to X$ be a finite and locally free morphism of schemes and $W \to X$ be any morphism of schemes.

   (a) Prove that $Y \times_X W \to W$ is finite and locally free.

   (b) Prove that the diagram

$$\begin{array}{ccc} |W| & \longrightarrow & |X| \\ \downarrow{\scriptstyle [Y\times_X W]} & & \downarrow{\scriptstyle [Y:X]} \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

   is commutative.

(c) Suppose that $Y \to X$ is surjective. Prove that $Y \times_X W \to W$ is also surjective.

iv) Suppose that $Z \to Y$ and $Y \to X$ are finite and locally free morphisms. Prove that the composition $Z \to X$ is finite and locally free.

v) Let $Y \to X$ and $Z \to X$ be finite and locally free morphisms of schemes.

(a) Prove that $Y \times_X Z \to X$ is finite and locally free.

(b) Prove that $[Y \times_X Z : X] = [Y : X][Z : X]$.

(c) If both $Y \to X$ and $Z \to X$ are surjective then $Z \to X$ is surjective.

vi) Do exercises (1)-(5) above with everywhere "finite locally free" replaced by "finite étale".

vii) Prove that a morphism $f : Y \to X$ is surjective, finite and locally free if and only if for each affine open $U = \text{Spec}(A)$ of X, the open subscheme of Y, $f^{-1}(U)$ is affine $f^{-1}(U) = \text{Spec}(B)$ where B is a (finite) faithfully projective A-algebra.

viii) If E is a finite set and A a ring we write A^E for the ring of functions $E \to A$, with pointwise addition and multiplication.

(a) For a scheme X and a finite set E prove that $X \times E \cong X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}^E)$, where $X \times E$ was defined in class.

(b) Let X, Y be schemes and E a finite set. Prove that there is a bijection, natural in X, Y, E, from the set of morphisms of schemes $\text{Hom}(X \times E, Y)$ to the set of maps $E \to \text{Hom}(X, Y)$.

(c) Suppose that A is a ring that has no non-trivial idempotents, and let E, D be finite sets. Prove that any A-algebra homomorphism $A^E \to A^D$ is induced by a map $D \to E$.

ix) Let $Y \to Z$ and $Z \to X$ be morphisms of schemes such that: $Z \to X$ and the composed morphism $Y \to X$ are affine. Prove that $Y \to Z$ is affine.

x) Prove that an open immersion is a monomorphism in the category of schemes.

1.5 Galois categories

This section follows almost verbatim the exposition given in ([4], exposé V, section 4).

In order to prove Theorem 1.2.11 in complete generality, we will give a list of categorical axioms and prove an analogue of Theorem 1.2.11 for any category satisfying them. It will then be shown, in the following section, that if X is a connected scheme, $\mathbf{FEt}_X$ satisfies the given axioms.

Definition 1.5.1. Let $\mathcal{C}$ be a category. A **terminal object** in $\mathcal{C}$ is an object X of $\mathcal{C}$ such that every other object Y of $\mathcal{C}$ has a unique morphism $Y \to X$ in $\mathcal{C}$. An **initial object** is the dual notion to a terminal object. Let f be a morphism of $\mathcal{C}$. Then f is a **monomorphism** if whenever $f \circ g = f \circ h$ for morphisms g, h of $\mathcal{C}$, then $g = h$. Dually, f is an **epimorphism** if it can be canceled on the right.

1.5.1 Quotients under group actions

Let Y be an object of $\mathcal{C}$ and $G \subset \mathrm{Aut}_{\mathcal{C}}(Y)$ a finite group of automorphisms of Y. The **quotient of** Y **by** G is an object Y/G along with a morphism $\rho\colon Y \to Y/G$ satisfying $\rho\sigma = \rho$ for all $\sigma \in G$ and the obvious universal property: if Z is an object of $\mathcal{C}$ and $f\colon Y \to Z$ satisfies $f\sigma = f$ for all $\sigma \in G$, then there is a unique morphism $g\colon Y/G \to Z$ such that $f = g \circ \rho$.

Example 1.5.2. If $\mathcal{C} = \textbf{f-Sets}$, then for Y an object of $\mathcal{C}$, the group $G \subset \mathrm{Aut}_{\mathcal{C}}(Y)$ acts on Y, and one can check that Y/G is the set of G-orbits of Y.

1.5.2 Galois categories

Definition 1.5.3. A **Galois category** is a category $\mathcal{C}$ along with a covariant functor $F\colon \mathcal{C} \to \mathbf{FinSets}$ satisfying the following list of axioms:

(G_1) There is a terminal object in $\mathcal{C}$. Fibered products (over arbitary objects) exist in $\mathbf{C}$.

(G_2) Finite coproducts exist in $\mathcal{C}$. In particular, an initial object $\mathcal{O}_{\mathbf{C}}$ exists in $\mathcal{C}$. Quotients by finite groups of automorphisms exist in $\mathcal{C}$.

(G_3) Any morphism $X \xrightarrow{u} Y$ in $\mathcal{C}$ factors as

$$X \xrightarrow{u'} Y' \xrightarrow{u''} Y''$$

where u' is a strict epimorphism and u'' is a monomorphism inducing an isomorphism of Y' with a direct summand of Y.

(G_4) The functor F is left-exact. That is, it preserves right-units and fibered products. (G_5) F preserves with finite direct sums, tranforms strict epimorphisms in epimorphisms, and preserves quotients by finite groups of automorphisms.

(G_6) If u is a morphism in $\mathcal{C}$ such that $F(u)$ is an isomorphism, then u is an isomorphism.

Examples 1.5.4.

i) $\mathcal{C} = \mathbf{FinSets}$ and F is the identity functor.

ii) Let π be a profinite group, $\mathcal{C} = \pi - \mathbf{Sets}$ and let F be the forgetful functor.

iii) Let K be a field, $\mathcal{C} = \mathbf{SAlg}_K$ and $F(-) = \mathrm{Hom}_{K-alg}(-, K_s)$.

If $(\mathcal{C}, F)$ is a Galois category such that $\mathcal{C}$ is **essentially small** [1], then we will define a profinite group π such that $\mathcal{C}$ is equivalent to the category of $\pi - \mathbf{Sets}$. Consider the group $\mathrm{Aut}(F)$ of automorphisms of the functor F. Elements of $\mathrm{Aut}(F)$ are natural isomorphisms of F with itself. Given an object $X \in \mathcal{C}$, $F(X)$ is a finite set. Let $S_{F(X)}$ denote the permutation group of $F(X)$. Then there is a natural injection:

$$\mathrm{Aut}(F) \longrightarrow \prod_{X \in \mathcal{C}} S_{F(X)}$$

given by $\sigma \mapsto (\sigma_X)_X$. Give each $S_{F(X)}$ the discrete topology and endow the product above with the product topology. It is then a compact Hausdorff space.

Given any $\phi \colon F(Y) \to F(Z)$, define a subset:

$$\mathcal{F}_\phi = \{(s_X)_X \in \prod_{X \in \mathcal{C}} S_{F(X)} \mid s_Z \circ \phi = \phi \circ s_Y\}.$$

Since only two coordinates have been restricted above, and the sets $S_{F(Z)}$, $S_{F(Y)}$ have the discrete topology, $\mathcal{F}_\phi$ is a closed subset. But then:

$$\mathrm{Aut}(F) = \bigcap_{g \colon X \to Y} \mathcal{F}_{F(g)}$$

shows that $\mathrm{Aut}(F)$ is closed and hence compact. One can in fact show the following:

Lemma 1.5.5. *The group* $\mathrm{Aut}(F)$ *is profinite.*

[1] A category is said to be **small** if all objects and Hom sets are sets in a given universe. It is **essentially small** if it is equivalent to a small category.

Let $\pi = \mathrm{Aut}(F)$. There is a natural action of π on $F(X)$ for every object $X \in \mathcal{C}$. If $\sigma \in \pi$ and $u \in F(X)$ then let:

$$\sigma \cdot u = \sigma_X(u).$$

Lemma 1.5.6. *If $F(X)$ is given the discrete topology and π the profinite topology, then π acts continuously on $F(X)$.*

Proof. □

Each $F(X)$ is now a π-set. Given a morphism $f\colon X \to Y$ in $\mathcal{C}$, $F(f)$ is compatible with the π-action:

$$F(f)(\sigma \cdot u) = F(f)(\sigma_X(u)) = (F(f) \circ \sigma_X)(u) = (\sigma_Y \circ F(f))(u) = \sigma \cdot (F(f)(u)).$$

We may thus regard F as a functor $F'\colon \mathcal{C} \to \pi - \mathbf{Sets}$. In this view, we have the following fundamental:

Theorem 1.5.7. *Let $(\mathcal{C}, F)$ be a a Galois category. Then the functor $F'\colon \mathcal{C} \to \pi - \mathbf{Sets}$ is an equivalence of categories:*

$$\mathcal{C} \simeq \pi - \mathbf{Sets}$$

where $\pi = \mathrm{Aut}(F)$.

Proof. We proceed in several steps.

Lemma 1.5.8. *Let $X \xrightarrow{u} Y$. Then u is a monomorphism if, and only if, $F(u)$ is a monomorphism.*

Proof. The fact that u is a monomorphism is equivalent to the first projection $X \times_Y X \to X$ being an isomorphism (uses G_1,G_4,G_6). □

Lemma 1.5.9. *Every object X of $\boldsymbol{C}$ is artinian.*

Proof. Let $X' \to X'' \to X$ be monomorphisms, such that $F(X')$ and $F(X''$ have the same image in $F(X)$. Then, by the previous lemma, $F(X') \to F(X'')$ is an isomorphism, and we get $X' \to X$ is an isomorphism, too (by G_6). □

Lemma 1.5.10. *The functor F is strictly pro-representable.*

Proof. This follows formally from the previous lemma together with G_4. □

We can thus find an ordered projective system I which filters P as $P = (P_i)_i \in I$, thought of as a pro-object in $\mathbf{C}$, together with a functorial isomorphism:

$$F(X) = \mathrm{Hom}_{\text{Pro-}\mathbf{C}}(P, X)(\stackrel{\text{def}}{=} \varinjlim_i \mathrm{Hom}_{\mathbf{C}}(P_i, X))$$

We say that an object X in $\mathbf{C}$ is **connected** if it can't be put as $X = X_1 \cup X_2$, with neither of $X_i \simeq \mathcal{O}_{\mathbf{C}}$.

Lemma 1.5.11. *The objects P_i are connected and nontrivial ($\not\simeq \mathcal{O}_{\mathbf{C}}$).*

Lemma 1.5.12. *Any $X \xrightarrow{u} Y$ with $X \not\simeq \mathcal{O}_{\mathbf{C}}$, and Y connected is an epimorphism. If X is connected, then $\mathrm{End}(X)$ is a division ring.*

Lemma 1.5.13. *The following statements are equivalent:*

i) The natural injection $Hom(P_i, P_i) \to Hom(P, P_i) \simeq F(P_i)$ is also surjective.

ii) The group $\mathrm{Aut}(P_i)$ acts transitively on $F(P_i)$.

iii) The group $\mathrm{Aut}(P_i)$ acts simply transitively on $F(P_i)$.

*If these are true, we say that P_i is **Galois object**.*

Lemma 1.5.14. *For each object X in $\mathbf{C}$, thre is a Galois object P_i such that, for each morphism $P \xrightarrow{u} X$, u factors as $u = u' \circ \varphi_i$, where φ_i is the canonical map $P \xrightarrow{\varphi_i} P_i$.*

We conclude then that those P_i's which are Galois form a cofinal system the system of all P_i's.

Lemma 1.5.15. *We have the following chain of equalities:*

$$Hom(P, P) = \mathrm{Aut}(P) = \varprojlim_i F(P_i) = \varprojlim_i \mathrm{Aut}(P_i)$$

where the projective limit is taken over the Galois P_i's.

From the previous lemma, we see that $\mathrm{Aut}(P)$ is the projetive limit of a projective system of finite groups, with surjective transition morphisms. It can be endowed with the limit topology, giving the discrete topology to each of the terms. Let then π, the **fundamental group** (of the pair $(\mathbf{C}, F)$), be defined as the opposite group to $\mathrm{Aut}(P)$. Note that it acts on the right on P, and it's the projective limite of finite groups π_i which also act on the right on the Galois P_i's (π_i is the opposite group to $\mathrm{Aut}(P_i)$).

Now, thanks to the functorial isomorphism $F(X) = \mathrm{Hom}(P, X)$ and from the definition of π, we get an action of π on $F(X)$. This action is continuous, thanks to lemma 1.5.14, because the action factors through π_i. Note now that, given a morphism $u\colon X \to Y$ in $\mathbf{C}$, the resulting morphism $F(u)\colon F(X) \to F(Y)$ is compatible with the actions of π, and in this way we obtain a covariant functor:

$$F\colon \mathbf{C} \to \pi - \mathbf{Sets}$$

Now, we define an inverse functor:

$$G\colon \pi - \mathbf{Sets} \to \mathbf{C}$$

which sends an object E of $\mathbf{C}$ to $P \times_\pi E$, and where this is, by definition, the solution to the universal problem:

$$\mathrm{Hom}_{\mathbf{C}}\left(P \times_\pi E, X\right) \simeq \mathrm{Hom}_\pi\left(E, \mathrm{Hom}(P, X)\right)$$

We need to prove that $P \times_\pi E$ actually exists.

Lemma 1.5.16. *Let Q be an object in $\boldsymbol{C}$, with a right action by a finite group G. Let E be a finite set with a left G-action. Then $Q \times_G E$ exists, and the canoncial map $F(Q) \times_G E \to F(Q \times_G E)$ is an isomorphism.*

Proof. □

Lemma 1.5.17. *Let E be an object of $\pi - \mathbf{Sets}$, and let P_i be a Galois object, such that π acts on E through π_i. Then $P_i \times_{\pi_i} E$ exists, and there is a canonical isomorphism*

$$E \xrightarrow{\simeq} F(P_i \times_{\pi_i} E)$$

If $j \geq i$ is such that P_j is also Galois, then the canonical isomorphism $P_j \times_{\pi_j} E \to P_i \times_{\pi_i} E$ is an isomorphism.

Proof. □

Lemma 1.5.18. *Under the same hypothesis as in the previous lemma, the object $P \times_\pi E$ exists and is canonically isomorphic to $P_i \times_{\pi_i} E$.*

Finally, there is a functorial homomorphism $\alpha\colon \mathrm{Id}_{\pi-\mathbf{Sets}} \to FG$, sending $E \mapsto F(P \times_\pi E)$. It is easy to check that this is an isomorphism, as thus concluding the proof of the theorem. □

1.6 Fundamental group of a connected scheme

In order to prove Theorem 1.2.11, it now suffices to show that $\mathbf{FEt}_X$ is a Galois category whenever X is a connected scheme. We must thus describe the fibre functor for this category, and verify the axioms (G_i).

We begin by characterising the epi and monomorphisms in $\mathbf{FEt}_X$.

Proposition 1.6.1. *Let X be a connected scheme, and $h\colon Y \to Z$ a morphism in $\mathbf{FEt}_X$. Then h is an epimorphism if and only if h is surjective as a morphism of schemes.*

Proof. Suppose that h is an epimorphism. Since h is finite étale, the degree $[Y\colon Z]$ is well-defined. Let Z_0 be the open and closed subscheme of Z consisting of the points of degree 0. Then the complement Z_1 is also an open and closed subscheme. Proposition 1.2.8 implies that $h^{-1}(Z_0) = \emptyset$. Thus, h in fact induces a finite étale morphism:

$$h\colon Y \to Z_1$$

and $[Y\colon Z_1] \geq 1$. This implies that h surjects onto Z_1, again by 1.2.8. We must now show that $Z_0 = \emptyset$.

Let $T = Z_0 \amalg Z_0 \amalg Z_1$. Note that since $Z \to X$ is a finite étale, this map induces finite étale morphisms $Z_0 \to X$ and $Z_1 \to X$. So $T \in \mathbf{FEt}_X$. We would like to define morphisms $a, b\colon Z \to T$ which map Z_0 to the first and second copy of Z_0 in T, respectively. To be precise, the map will be defined locally.

Suppose $X = \mathrm{Spec}(A)$ is affine. Then both of Z_0 and Z_1 are affine, say $Z_1 = \mathrm{Spec}(C_i)$. In this case $Z = \mathrm{Spec}(C_0 \times C_1)$ and $T = \mathrm{Spec}(C_0 \times C_0 \times C_1)$. Y is also affine if X is, say $Y = \mathrm{Spec}(B)$. We showed above that the morphism induced by h:

$$h\colon C_0 \times C_1 \to B$$

factors through C_1:

$$h\colon C_1 \to B.$$

Define maps $a\colon C_0 \times C_0 \times C_1 \to C_0 \times C_1$ by $(x, y, c) \mapsto (x, c)$ and $b\colon C_0 \times C_0 \times C_1 \to C_0 \times C_1$ by $(x, y, c) \mapsto (y, c)$. These induce scheme maps:

$$Y \xrightarrow{h} Z \rightrightarrows T$$

where the maps $Z \to T$ are a and b, respectively. One checks that $a \circ h = b \circ h$. Then since h is an epimorphism, $a = b$. The only way that this can be true is if C_0 is the zero ring. So $Z_0 = 0$, $Z = Z_1$ and h surjects. For the general case, cover X by affines and show that each part of Z_0 is zero by the argument above, hence $Z_0 = 0$ in the general case as well.

Suppose instead that h is surjective. Let $p, q\colon Z \to W$ be finite étale maps over X such that $p \circ h = q \circ h$. Let $\{U_i\}$ be an affine open cover of X. Then the inverse images $\{g^{-1}(U_i)\}$ under $g\colon Z \to X$ give an affine open cover of Z. It suffices to show that $p = q$ over each $g^{-1}(U_i)$. We may as well suppose that X, and hence all schemes appearing, are affine.

In the affine case, surjectivity of the map of schemes h implies that the corresponding map of rings is injective. The relation $p \circ h = q \circ h$ becomes $h \circ p = h \circ q$ in the category of rings, and then the injectivity of h implies $p = q$. So the corresponding maps of schemes must also be equal. This concludes the proof. □

Proposition 1.6.2. *Let $h\colon Y \to Z$ be a morphism in $\mathbf{FEt}_X$. Then h is a monomorphism if and only if h is both an open and closed immersion.*

Proof. One can show quite generally in the category of schemes that an open immersion is a monomorphism. We leave this as a simple exercise. So suppose that h is a monomorphism in $\mathbf{FEt}_X$. Note that $Y \times_Z Y \to Z$ and $Y \times_Z Y \to X$ are finite étale since $Y \to X$ and $Z \to X$ are both finite étale. So $Y \times_Z Y \to Z$ is a morphism in $\mathbf{FEt}_X$. Let u, v be the two projections $Y \times_Z Y \to Y$. Then the commutativity of the pullback square of the fibre product shows that $h \circ u = h \circ v$. But h is a monomorphism, so $u = v$. We claim it follows that u is an isomorphism.

As above, proving this is a local property. We may hence assume that $X = \operatorname{Spec}(A)$ is affine. Then so are $Y = \operatorname{Spec}(B)$ and $Z = \operatorname{Spec}(C)$. The scheme maps u, v correspond to maps:

$$B \to B \otimes_C B$$

given by $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$, respectively. So since $u = v$, we have $b \otimes 1 = 1 \otimes b$ for all $b \in B$. One easily checks that the tensored map:

$$\mathrm{Id} \otimes h\colon B \otimes_C C \to B \otimes_C B$$

is an isomorphism, since $b \otimes 1 = 1 \otimes b$. But $B \otimes_C C \simeq B$, and one checks that the map above is just u under this identification. So u is an isomorphism, $B \simeq B \otimes_C B$. We hence have on the one hand:

$$[B\colon C] = [B \otimes_C B\colon C]$$

due to this isomorphism. On the other hand, one can show generally that:

$$[B \otimes_C B\colon C] = [B\colon C]^2.$$

We deduce that $[B\colon C] \leq 1$. This extends globally, so that we can write:

$$Z = Z_0 \amalg Z_1$$

with notation as in the previous proposition. Again we have $h^{-1}(Z_0) = \emptyset$, so that h induces a morphism $h\colon Y \to Z_1$. Since the degree is 1 above Z_1, proposition 1.2.8 implies that h identifies Y with a component of Z, $h\colon Y \simeq Z_1$. Hence, h is an open and closed immersion. □

Recall that Galois categories come equipped with a fibre functor, and that the group of automorphisms of the functor plays the role of the fundamental group. We turn now to the topic of defining a fibre functor for $\mathbf{FEt}_X$. Let X be a nonempty connected scheme. A **geometric point** of X is a morphism:

$$x\colon \operatorname{Spec}(\Omega) \to X$$

where Ω is an algebraically closed field. We usually identify the morphism x with its image in X. If $a \in X$ is any point, let Ω be an algebraic closure of $k(a)$. Then performing a base-change yields a geometric point corresponding to a. This shows that nonempty schemes have plenty of geometric points.

Definition 1.6.3. Let X be a nonempty connected scheme. Fix a geometric point $x \in X$ over an algebraically closed field Ω. Base-changing yields a functor:

$$H_x\colon \mathbf{FEt}_X \to \mathbf{FEt}_{\operatorname{Spec}(\Omega)}.$$

We have already seen that $\mathbf{FEt}_{\operatorname{Spec}(\Omega)} \simeq \operatorname{Gal}(\boldsymbol{\Omega}) - \mathbf{sets}$. Composing with the forgetful functor $\operatorname{Gal}(\boldsymbol{\Omega}) - \mathbf{sets} \to \mathbf{FinSets}$ yields:

$$F_x\colon \mathbf{FEt}_x \to \mathbf{FinSets}.$$

This is the **fibre functor** of X over x.

Our next goal is to show that quotients by finite groups of automorphisms exist in $\mathbf{FEt}_X$. To do so, we will first study the category of schemes that are *affine over* X. This category properly contains $\mathbf{FEt}_X$, so at the end we will have to show that the construction restricts well.

1.6.1 Characterisation of $\mathbf{Aff}_X$

This section does most of exercise $II.5.17$, in [7].

Let X be a scheme, $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules on X.

Given a ring A and an A-module M, one defines a sheaf of $\mathcal{O}_X$-modules on $X = \mathrm{Spec}(A)$, as follows:

Definition 1.6.4. The **sheaf associated to** M, written $\tilde{M}$, is the sheaf characterized by $\tilde{M}(X_f) = M_f = M \otimes_A A_f$, for any $f \in A$.

Definition 1.6.5. A sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is **quasicoherent** if there exists an open covering $\{U_i\}_{i\in I}$ of X by affines, such that $\mathcal{M}_{|U_i} = \widetilde{\mathcal{M}(U_i)}$, as $\mathcal{O}_{U_i}$-modules, for all $i \in I$. If moreover $\mathcal{M}(U_i)$ is finitely-generated for all $i \in I$, then $\mathcal{M}$ is said to be **coherent**.

Lemma 1.6.6. *Let $f\colon Y \to X$ be an affine morphism of schemes. Then $f_*\mathcal{O}_Y$ is a sheaf of $\mathcal{O}_X$-algebras which, as a sheaf of $\mathcal{O}_X$-modules, is quasicoherent.*

Remark. The converse is also true, but we won't need it for now.

Proof. Consider the corresponding morphism of sheaves $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$, which gives to $f_*\mathcal{O}_Y$ the structure of a $\mathcal{O}_X$-algebra.

Because the property of being quasicoherent is local on X, we may assume that $X = \mathrm{Spec}(A)$ is affine. In this case, $f^{-1}(X) = Y = \mathrm{Spec}(B)$, and $f^\#_X\colon A \to B$ is a ring homomorphism. Let $a \in A$. Then $X_a = \mathrm{Spec}(A_a) \subseteq X$ is an open affine in X, and:

$$(f_*\mathcal{O}_Y)(X_a) = \mathcal{O}_Y(f^{-1}(X_a)) = \mathcal{O}_Y(f^{-1}(\mathrm{Spec}(A_a))) = B_a$$

and hence $f_*\mathcal{O}_Y = \tilde{B}$, as wanted. □

Fix a scheme X, and consider the category $\mathbf{QCoh}_{\mathcal{O}_X}$, whose objects are sheaves $\mathcal{A}$ of $\mathcal{O}_X$-algebras on X, such that they are quasicoherent as $\mathcal{O}_X$-modules.

Define a contravariant functor

$$\Gamma\colon \mathbf{Aff}_X \to \mathbf{QCoh}_{\mathcal{O}_X} \quad (f\colon Y \to X) \mapsto f_*\mathcal{O}_Y$$

Lemma 1.6.7. *The functor Γ is an anti-equivalence of categories.*

Proof. We will describe the inverse functor $\mathbf{Spec}\colon \mathbf{QCoh}_{\mathcal{O}_X} \to \mathbf{Aff}_X$. So let $\mathcal{A}$ be a quasicoherent sheaf of $\mathcal{O}_X$-algebras on X. We want to construct $\mathbf{Spec}(\mathcal{A}) = (f\colon Y \to X)$, an affine morphism to X.

Let $\{U_i\}_{i\in I}$ be an open affine covering of X, $U_i = \mathrm{Spec}(A_i)$. Let $Y_i = \mathrm{Spec}(\mathcal{A}(U_i))$. So we get a ring homomorphism $\mathcal{O}_X(U_i) = A_i \to \mathcal{A}(U_i)$, which induces morphisms of schemes $f_i : Y_i \to U_i$. We will glue this local data together, along the intersections. So let $U_{ij} \stackrel{\text{def}}{=} U_i \cap U_j$, which can be seen as inside U_i and U_j. Let $Y_{ij} \stackrel{\text{def}}{=} f_i^{-1}(U_{ij})$. Note then that Y_{ij} is a subscheme of Y_i, and then Y_{ji} is a subscheme of Y_j. We want to identify these pairs, but they are not affines, so they have to be covered by affines and be identified through them.

Let $V = \operatorname{Spec}(S) \subseteq U_{ij}$ be an open affine. By quasicoherence, $f_i^{-1}(V) = \operatorname{Spec}(\mathcal{A}(V)) \subseteq Y_{ij}$, and at the same time, $f_j^{-1}(V) = \operatorname{Spec}(\mathcal{A}(V)) \subseteq Y_{ji}$, and so $f_i^{-1}(V) \simeq f_j^{-1}(V)$. Varying V along an affine covering of U_{ij}, to get an isomorphism $\varphi_{ij} : Y_{ij} \simeq Y_{ji}$.

By the glueing lemma, one obtains then $Y = \mathbf{Spec}(\mathcal{A})$, together with a map $f : Y \to X$, and it is an easy check to verify that $f : Y \to X$ is affine.

One should then check that the two constructions are inverse to each other, but we omit this detail. □

1.6.2 Quotients under group actions in $\mathbf{Aff}_X$

Fix a scheme X, and let $f : Y \to X$ be an affine morphism.

Let $G \subseteq \operatorname{Aut}_{\mathbf{Aff}_X}(f : Y \to X) = \operatorname{Aut}_X(Y)$ be a finite subgroup of the group of automorphisms of Y that fix X.

Via the equivalence given by the previous lemmas, Y corresponds to a quasicoherent sheaf of $\mathcal{O}_X$-algebras, say $\mathcal{A}$, and G corresponds to a subgroup of $\operatorname{Aut}_{\mathcal{O}_X}(\mathcal{A})$, which will be called G by abuse of notation. This really acts on $\mathcal{A}$, fixing $\mathcal{O}_X$ (e.g. if $X = \operatorname{Spec}(A)$, then $\mathcal{A} = B$ is an A-algebra, and G acts on B over A, like in the setting of Galois theory).

Define then $\mathcal{A}^G$ as a sheaf on X, given by (if $U \subseteq X$ is an open):

$$\mathcal{A}^G(U) \stackrel{\text{def}}{=} \mathcal{A}(U)^G = \{a \in \mathcal{A}(U) \mid \sigma a = a, \forall \sigma \in G\}$$

Note now that the map $\mathcal{O}_X(U) \to \mathcal{A}(U)$ factors through $\mathcal{A}(U)^G \hookrightarrow \mathcal{A}(U)$. Moreover, if $U \subseteq V \subseteq X$ are two opens, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}(V) & \xrightarrow{\rho_{UV}} & \mathcal{A}(U) \\ \uparrow & & \uparrow \\ \mathcal{A}(V)^G & \longrightarrow & \mathcal{A}(V)^G \end{array}$$

because $\sigma \in G$ gives a morphism of sheaves.

This makes $\mathcal{A}^G$ into a presheaf, and it is easy to verify that it is actually a sheaf.

The last thing to do is to verify that it is *quasicoherent*.

Note now that, for each $U \subseteq X$, we have a map

$$\varphi_U \colon \mathcal{A}(U) \to \bigoplus_{\sigma \in G} \mathcal{A}(U)$$

defined by $a \mapsto (\sigma a - a)_{\sigma \in G}$, which is $\mathcal{O}_X(U)$-linear. The key point is to note that $\ker(\varphi_U) = \mathcal{A}^G(U)$.

One checks that the $\{\varphi_U\}$ give a morphism of $\mathcal{O}_X$-modules, $\varphi\colon \mathcal{A} \to \mathcal{A}^{\#G}$, and $\mathcal{A}^G = \ker(\varphi)$.

As both $\mathcal{A}$ and $\mathcal{A}^{\#G}$ are quasicoherent, then it follos that $\ker(\varphi) = \mathcal{A}^G$ is quasicoherent as well (see [7], chapter II, proposition 5.7, or Serre's FAC).

Again under the correspondence described before, we get Y/G, together with an affine morphism $\alpha\colon Y/G \to X$, which corresponds to the embedding $\mathcal{A}^G \hookrightarrow \mathcal{A}$. Remark that $Y/G = \mathbf{Spec}(\mathcal{A}^G)$.

Claim. *The map $\alpha\colon Y/G \to X$ satisfies the universal property.*

Proof. The inclusion $\mathcal{A}^G \hookrightarrow \mathcal{A}$ gives the morphism $Y \to Y/G$, and the universal property of Y/G follows from the corresponging universal property of $\mathcal{A}^G$ in the category $\mathbf{QCoh}_{\mathcal{O}_X}$. □

1.6.3 Quotients under group actions in $\mathbf{FEt}_X$

Let $f\colon YtoX$ be a finite étale morphism, and $G \subset \mathrm{Aut}_X(Y)$ a finite group of automorphisms. Then $Y/G \to X$ exists as an affine scheme. We will now show that in fact $Y/G \in \mathbf{FEt}_X$.

Proposition 1.6.8. *Let $f\colon Y \to X$ be an affine morphism, $G \subset \mathrm{Aut}_X(Y)$ a finite group and $g\colon W \to X$ a finite locally free morphism. Then there is a canonical isomorphism:*

$$(Y \times_X W)/G \simeq (Y/G) \times_X W.$$

Proof. Affine morphisms are stable under basechange, so that $Y \times_X W \to W$ is affine. For every $\sigma \in G$, $f \circ \sigma = f$, and hence:

$$\begin{array}{ccccc} Y \times_X W & & \longrightarrow & & W \\ \downarrow & & & & \downarrow g \\ Y & \xrightarrow{\sigma} & Y & \xrightarrow{f} & X \end{array}$$

commutes. The universal property of the fibre product thence yields a unique endomorphism, which we also call σ, of $Y \times_X W \to W$ such that:

$$\begin{array}{ccccc} Y \times_X W & \xrightarrow{\sigma} & Y \times_X W & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow g \\ Y & \xrightarrow{\sigma} & Y & \xrightarrow{f} & X \end{array}$$

commutes. Applying this to σ^{-1} allows one to show that $\sigma \in \mathrm{Aut}_W(Y \times_X W)$. Similarly, one verifies that since G acts on $Y \to X$, the definition above gives a canonical action of G on $Y \times_X W \to W$. We henceforth regard $G \subset \mathrm{Aut}_W(Y \times_X W)$ in this canonical fashion.

Let $h\colon Y \to Y/G$ be a morphism in $\mathbf{Aff}_X$ that is invariant under G. Then:

$$h \times \mathrm{Id}_W \colon Y \times_X W \to (Y/G) \times_X W$$

is a morphism over W. Moreover, it follows from the definition of the action of G on $Y \times_X W$ that this map is invariant under G since h is. The universal property of quotients thus supplies a unique morphism:

$$\phi\colon (Y \times_X W)/G \to (Y/G) \times_X W$$

over W. We must argue that this is an isomorphism, and such a property can be checked locally on the base.

Assume $X = \mathrm{Spec}(A)$, so that then also $Y = \mathrm{Spec}(C)$ and $W = \mathrm{Spec}(B)$ (f and g are affine morphisms over X). Let $f\colon A \to C$ and $g\colon A \to B$ denote the corresponding ring homomorphisms. We have seen that:

$$\begin{aligned} Y \times_X W &= \mathrm{Spec}(B \otimes_A C), \\ Y/G &= \mathrm{Spec}(C^G), \\ (Y/G) \times_X W &= \mathrm{Spec}(C^G \otimes_A B), \\ (Y \times_X W)/G &= \mathrm{Spec}((C \otimes_A B)^G). \end{aligned}$$

The map $(Y \times_X W)/G \to (Y/G) \times_X W$ is induced by the inclusion of rings:

$$C^G \otimes_A B \hookrightarrow (C \otimes_A B)^G.$$

We would like to show that this map is actually a ring isomorphism, so that the associated morphism if schemes is as well. Consider the following exact sequence of A-modules:

$$0 \longrightarrow C^G \longrightarrow C \longrightarrow \bigoplus_{\sigma \in G} C,$$

where the last map sends $c \mapsto (\sigma c - c)_{\sigma \in G}$. Since $Y \to X$ is finite and locally free, B is finite and projective over A, and hence flat. Tensoring the sequence above over A with B thus gives an exact sequence:

$$0 \longrightarrow C^G \otimes_A B \longrightarrow C \otimes_A B \longrightarrow \bigoplus_{\sigma \in G} (C \otimes_A B),$$

where the last map sends $c \otimes b \mapsto ((\sigma c - c) \otimes b)_{\sigma \in G}$. The kernel of this map is $(C \otimes_A B)^G$ by definition of the action of G on this ring, and thus:

$$C^G \otimes_A B \simeq (C \otimes_A B)^G.$$

□

Proposition 1.6.9. *Let $f\colon Y \to X$ be a finite étale morphism. Let $G \subset \mathrm{Aut}_X(Y)$ be a finite subgroup. Then Y/G exists in $\mathbf{FEt}_X$.*

We have seen that there is an affine morphism $Y/G \to X$. Suppose we can show that if $Y \to X$ is finite étale, then so is $Y/G \to X$. Then it will follow that $Y/G \to Y$ is finite étale by a previous result, since we have a commutative diagram:

$$\begin{array}{ccc} Y/G & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

So it suffices to prove that $Y/G \to X$ is finite étale under the hypotheses above. We prepare the proof of this proposition with a lemma that generalises a previous result:

Lemma 1.6.10. *Let X, Y, Z be schemes with $f\colon Y \to X$ and $g\colon Z \to X$ both totally split. Let $\sigma_1, \dots, \sigma_n\colon Y \to Z$ be morphisms such that $f = g \circ \sigma_i$ for all i. Then for each $x \in X$, there exists an affine open neighbourhood $U \subset X$ of x such that each σ_i is trivial over U. By this we mean that there exist two finite sets D, E with $f^{-1}(U) \simeq U \times D$ and $g^{-1}(U) \simeq U \times E$, and maps $\phi_1 \dots, \phi_n\colon D \to E$ such that:*

$$\begin{array}{ccccccc} f^{-1}(U) & & & \xrightarrow{\quad\sigma_i\quad} & & & g^{-1}(U) \\ & \searrow^{\alpha} & & & & {}^{\beta}\swarrow & \\ \Big\downarrow f & & U \times D & \xrightarrow{\phi_i} & U \times E & & \Big\downarrow g \\ & \swarrow & & & & \searrow & \\ U & & & = & & & U \end{array}$$

commutes for each i.

In the case of a single σ_i, this lemma was proved in the section on totally split morphisms. This generalisation is proved by applying the single case to each σ_i and then intersection the U_i's obtained. An affine open contained in this intersection yields the desired $U \subset X$.

This lemma will be applied when $Y = Z$, $f = g$ and $\{\sigma_1, \ldots, \sigma_n\} = G \subset \mathrm{Aut}_X(Y)$. Specialised to this situation, the lemma says that each $x \in X$ has an affine open neighbourhood U such that there exists a finite G-set D such that $f^{-1}(U) \simeq U \times D$ and the action of G on Y keeps U stable. Moreover, the action above U is induced by the action of G on D. We return now to the proof of the proposition:

Proof. We proceed in three stages. First suppose that $Y = X \times D$ where D is a finite G-set. We have:

$$Y/G = (X \times D)/G = (X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}^D))/G = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}((\mathbb{Z}^D)^G).$$

Now, $\mathbb{Z}^D$ is the ring of functions from $D \to \mathbb{Z}$. The G-invariant functions are uniquely determined by their values on orbits representatives of D/G. One observes that thus $(\mathbb{Z}^D)^G \simeq \mathbb{Z}^{D/G}$ and hence:

$$Y/G = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}((\mathbb{Z}^D)^G) \simeq X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}^{D/G}) = X \times (D/G).$$

Moreover, this isomorphism is a morphism over X. Hence, Y/G is totally split over X. We deduce that $Y/G \to X$ is finite étale.

Next suppose that $Y \to X$ is totally split. The previous lemma yields an affine open cover $\{U_i\}_{i \in I}$ of X such that $f^{-1}(U_i) \simeq U_i \times D_i$ for some finite G-set D_i, and G acts trivially above U_i. It follows by the case just treated that $s\colon Y/G \to X$ is finite étale over each U_i, and hence itself finite étale.

In the general case, we use our old trick of reducing to the totally split case by making a surjective and locally-free base change. Since $Y \to X$ is finite étale, we have seen that there exists a surjective, finite and locally free morphism $W \to X$ so that $Y \times_X W \to W$ is totally split. We showed above how to define an action of G on $Y \times_X W$. The totally split case just treated hence implies that $(Y \times_X W)/G \to W$ is finite étale. However, the first proposition of this section showed that $(Y \times_X W)/G \simeq (Y/G) \times_X W$. We deduce that $(Y/G) \times_X W \to W$ is finite étale. Since $W \to X$ is surjective, finite and locally free, it follows by a previous result that $Y/G \to X$ is finite étale as well. □

This result can be used to weaken the hypotheses of the first proposition proved in this section:

Proposition 1.6.11. *Let $f\colon Y \to X$ be an affine morphism, $G \subset \mathrm{Aut}_X(Y)$ a finite group and $g\colon Z \to X$ any morphism of schemes. Then there is a canonical isomorphism:*

$$(Y \times_X Z)/G \simeq (Y/G) \times_X Z.$$

Proof. As above, the universal property of the quotient yields a morphism over Z:

$$(Y \times_X Z)/G \to (Y/G) \times_X Z.$$

We will show that this is an isomorphism in three steps.

First assume that $Y = X \times D$, where D is a finite G-set, and G acts on Y via D. In this case there is nothing to prove since:

$$Y \times_X Z = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}^D) \times_X Z = (X \times_X Z) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}^D) = Z \times D.$$

Moreover G acts on the fiber product via D in this decomposition, so that:

$$(Y \times_X Z)/G \simeq Z \times (D/G).$$

Similarly $Y/G \simeq X \times (D/G)$ since G acts via D, and one also computes $(Y/G) \times_X Z \simeq Z \times (D/G)$. This concludes the proof in this case.

Next assume that $f\colon Y \to X$ is totally split. Then X can be covered by affines U_i such that $Y_i = f^{-1}(U_i) \to U_i$ is trivial, and the action of G is trivial (permutes the slices of $U_i \times D_i$). By the previous case we obtain isomorphisms for each i:

$$(Y_i \times_X Z)/G \simeq ((Y_i/G) \times_X Z).$$

Glueing these togethers gives the desired isomorphism $(Y \times_X Z)/G \simeq (Y/G) \times_X Z$.

Now for the general (and most interesting) case. Choose a surjective, finite and locally free morphism $W \to X$ such that $Y \times_X W \to W$ is totally split. Given a fibre product $U \times_V W$ we introduce the notation U_W for the scheme $U \times_V W \to W$ over W. So in this notation, $Y_W \to W$ is totally split. By the case just treated:

$$(Y_W \times_W Z_W)/G \simeq (Y_W/G) \times_W Z_W.$$

But note that:

$$Y_W \times_W Z_W = (Y \times_X W) \times_W (Z \times_W W) = (Y \times_X Z) \times_Z W_Z,$$

and $W_Z \to Z$ is surjective, finite and locally free since it is the base change of $W \to X$. Thus, the first proposition of this section (the one we are generalising) implies that:

$$((Y \times_X Z) \times_Z W_Z)/G \simeq ((Y \times_X Z)/G) \times_Z W_Z.$$

Also by that proposition:

$$Y_W/G = (Y \times_X W)/G \simeq (Y/G) \times_X W = (Y/G)_W.$$

Hence:

$$(Y_W/G) \times_W Z_W = (Y/G)_W \times_W Z_W = ((Y/G) \times_X Z) \times_Z W_Z.$$

We thus have an isomorphism:

$$((Y \times_X Z)/G) \times_Z W_Z \simeq ((Y/G) \times_X Z) \times_Z W_Z.$$

We claim that the isomorphism above is the base change to W_Z of the map $(Y \times_X Z)/G \to (Y/G) \times_X Z$. Hence, the map under consideration is an isomorphism after base-changing via a surjective, finite and locally free morphism. It follows by previous work that $(Y \times_X Z)/G \to (Y/G) \times_X Z$ is already an isomorphism. □

We turn now to the long awaited:

Theorem 1.6.12. *Let X be a connected scheme and let x be a geometric point of X. Then $(\mathbf{FEt}_X, F_x)$ is a Galois category.*

Proof. We have been studying $\mathbf{FEt}_X$ for some time, and have alreay shown many of the relevant details. We will prove two more details, and leave the remaining ones to the reader.

Let $h\colon Y \to Z$ be a morphism in $\mathbf{FEt}_X$. We will show that $h = f \circ g$ factors as an epimorphism g and a monomorphism f. For this, we write $Z = Z_0 \amalg Z_1$ as above, where Z_0 is the subscheme of points of Z of degree 0, and Z_1 is the subscheme of points of degree ≥ 1. We have seen that $h^{-1}(Z_0) = \emptyset$, so that h factors: $Y \to Z_1 \to Z$. Since $[Y\colon Z_1] \geq 1$, $Y \to Z_1$ is surjective. By earlier work it is hence an epimorphism. Similarly, $Z_1 \to Z$ is a monomorphism.

We next show that if $h\colon Y \to Z$ in $\mathbf{FEt}_X$ is such that $F_x(h)$ is an isomorphism, then in fact h is an isomorphism. Note first that since X is connected, $[Y\colon X]$ is constant. In fact, one sees that $[Y\colon X] = |F_x(Y)|$.As above, write $Z = Z_0 \amalg Z_1$. Apply the fibre functor to the diagram:

$$Y \to Z_1 \hookrightarrow Z$$

to obtain:

$$F_x(Y) \to F_x(Z_1) \hookrightarrow F_x(Z) = F_x(Z_0) \amalg F_x(Z_1).$$

By assumption the composition is an isomorphism. The only way that this can happen is if $F_x(Z_0) = \emptyset$. But then $[Z_0 \colon X] = 0$ and hence $Z_0 = \emptyset$. Hence $Z = Z_1$, h is surjective and $[Z \colon X] = [Y \colon X]$ (since $|F_X(Z)| = |F_X(Y)|$). We argue that h must be an isomorphism.

Suppose first that $Y = X \times D$ and $Z = X \times E$ and h is induced by a map $\phi \colon D \to E$. Surjectivity of h implies that ϕ is surjective. Also $|D| = [Y \colon X] = [Z \colon X] = |E|$. It follows that ϕ is actually a bijection between sets, and hence h is an isomorphism.

Next assume that Y and Z are both totally split. One can check locally that h is an isomorphism, and it thence follows from the previous case just treated.

In the general case, we can find a surjective finite and locally free morphism $W \to X$ so that both $Y \times_X W \to W$ and $Z \times_X W \to W$ are totally split. By the previous case, the induced map $h \times \mathrm{Id}_W$ is an isomorphism. Then previous work shows that since $W \to X$ is surjective, fintie and locally free, the original map h is also an isomorphism. □

1.7 Étale Morphisms

In this section we introduce *étale morphisms*, and relate them with what we have studied on *finite* étale morphisms.

Let $f \colon Y \to X$ be a morphism of schemes.

Definition 1.7.1. We say that f is **flat** if, for every $y \in Y$, letting $x = f(y)$, the natural map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ makes $\mathcal{O}_{Y,y}$ into a flat $\mathcal{O}_{X,x}$-module (algebra).

Remark. This is a definition which is local on Y, not on X.

Definition 1.7.2. We say that f is **unramified** if, for every $y \in Y$, letting $x = f(y)$, the ring $\mathcal{O}_{Y,y}/\mathfrak{m}_{X,x}\mathcal{O}_{Y,y}$ is a *finite, separable, field extension* of $\mathcal{O}_{X,y}/\mathfrak{m}_{X,x}$.

Equivalently, we require that $\mathfrak{m}_{X,x}\mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$ and that $\kappa(y)$ is a finite separable extension of $\kappa(x)$.

Definition 1.7.3. We say that f is **étale** if it is both *flat* and *unramified.*

Note that we need to work a little bit to relate this new definition to the definition of a morphism f being *finite-étale.* For this we make an ad-hoc and nonstandard definition:

Definition 1.7.4. The morphism $f \colon Y \to X$ is said to be of **finite presentation as modules** (**fpm**) if there exists an open affine covering $\{U_i\}_{i \in I}$ of X, $U_i = \mathrm{Spec}(A_i)$, such that for all $V = \mathrm{Spec}(B) \subseteq f^{-1}(U_i)$, one has that B is an A_i-module of finite presentation.

With this definition, we can relate the different notions of étale maps:

Proposition 1.7.5. *Let $f\colon Y \to X$ be a morphism of schemes. Then f is étale and* fpm *if, and only if, it is finite-étale (as defined before).*

Corollary 1.7.6. *If X is a locally noetherian scheme (that is, X can be covered by the spectra of noetherian rings), and $f\colon Y \to X$ is a morphism, then f is finite and étale if, and only if, f is finite-étale.*

To be able to prove Proposition 1.7.5, we need some more preparation.

Proposition 1.7.7. *Let A, B be rings, and let $f\colon A \to B$ be a ring homomorphism. Then the following statements are equivalent:*

i) f is flat (that is, B is a flat A-algebra).

ii) For each $\mathfrak{q} \in \operatorname{Spec}(B)$, let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, and then $f_\mathfrak{q}\colon A_\mathfrak{p} \to B_\mathfrak{q}$ is flat.

iii) The induced morphism of schemes $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is flat.

iv) For each $\mathfrak{q} \in \operatorname{Spec}(B)$, let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, and then $f_\mathfrak{q}\colon A_\mathfrak{p} \to B_\mathfrak{q}$ is flat.

Proof. $(i) \Rightarrow (ii)$:
Let $\mathfrak{q} \in \operatorname{Spec}(B)$, $\mathfrak{p} = f^{-1}(\mathfrak{q}) \in \operatorname{Spec}(A)$. Let $S = A \setminus \mathfrak{p}$. As localization is exact, the resulting homomorphism $A_\mathfrak{p} \to S^{-1}B = B_\mathfrak{p}$ is flat. Moreover, the map $S^{-1}B \to B_\mathfrak{q}$ is also flat, as it is a further localisation. Hence, the composition of the two is flat as well (flatness is preserved under composition).

$(ii) \Rightarrow (iii)$ is just by definition, and $(iii) \Rightarrow (iv)$ is trivial because, in fact, (ii) is equivalent to (iii) (and (iv) is a weakening of (ii)).

$(iv) \Rightarrow (i)$:
Let M and N be A-modules, and $\varphi\colon M \to N$ an injective A-linear map. We want to show that $\varphi \otimes \mathrm{Id}_B\colon M \otimes_A B \to N \otimes_A B$ is still injective. So let $K \overset{\text{def}}{=} \ker(\varphi \otimes \mathrm{Id}_B)$.

Let $\mathfrak{m}$ be a maximal ideal of B, and $\mathfrak{n} = f^{-1}(\mathfrak{m}) \in \operatorname{Spec}(A)$. As localisation is exact, $\varphi \otimes \mathrm{Id}_{A_\mathfrak{n}}\colon M \otimes_A A_\mathfrak{n} \to N \otimes_A A_\mathfrak{n}$ is injective. Moreover, by hypothesis $A_\mathfrak{n} \to B_\mathfrak{m}$ is flat, so we can apply $\otimes_{A_\mathfrak{n}} B_\mathfrak{m}$ to the previous injection, to get an injective map $(M \otimes_A B) \otimes_B B_\mathfrak{m} \simeq M \otimes_A B_\mathfrak{m} \to N \otimes_A B_\mathfrak{m} \simeq (N \otimes_A B) \otimes_B B_\mathfrak{m}$. This implies that $K \otimes_B B_\mathfrak{m} = 0$ for all $\mathfrak{m}$ maximal ideals of B. As the property of being 0 is local on the maximals, we conclude that $K = 0$, as we wanted. □

We now we restate the geometric version of the previous proposition:

Proposition 1.7.8. *Let $f\colon Y \to X$ be a morphism of schemes. Then the following statements are equivalent:*

i) f is flat.

ii) For any pair $U = \mathrm{Spec}(A) \subseteq X$, $V = \mathrm{Spec}(B) \subseteq Y$ such that $f(V) \subseteq U$, the induced map $A \to B$ is flat.

iii) There is an open affine covering $\{V_i\}_{i \in I}$ of Y such that for every $i \in I$ and every open affine $U = \mathrm{Spec}(A) \subseteq X$ such that $f(V_i) \subseteq U$, the induced map $A \to B_i$ is flat.

iv) For every closed point *$y \in Y$, $x = f(y) \in X$, the induced morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat.*

The following easy result will help us in the proof of the next lemma:

Lemma 1.7.9. *Let A be a ring. Let M be a finitely-presented A-module, and let N be a flat A-module. Then the map $\phi\colon M^\vee \otimes_A N \to Hom_A(M, N)$ which maps $f \otimes n \mapsto [m \mapsto f(m)n]$ is an isomorphism.*

Proof. It is clear if M is free (we actually have proven it for M projective). Start then with a presentation $A^m \to A^n \to M \to 0$, and apply the functors $\mathrm{Hom}(-, N)$ and $\mathrm{Hom}(-, A)$, to get the following diagram, where the second row is the result of applying also the $\otimes_A N$ functor, which is exact because N is flat:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_A(M, N) & \longrightarrow & \mathrm{Hom}_A(A^n, N) & \longrightarrow & \mathrm{Hom}_A(A^m, N) \\
& \phi \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
0 \longrightarrow & \mathrm{Hom}_A(M, A) \otimes_A N & \longrightarrow & \mathrm{Hom}_A(A^n, A) \otimes_A N & \longrightarrow & \mathrm{Hom}_A(A^m A) \otimes_A N
\end{array}$$

By diagram chasing, it follows that ϕ is also an isomorphism. □

We continue with a lemma which relates the notions of *flatness* and *projectiveness*:

Lemma 1.7.10. *Let A be a ring, and let P be an A-module. Then P is* finitely-generated *and* projective *if, and only if, P is* finitely-presented *and* flat.

Proof. We know already that projective modules are always flat, and that finitely-generated projectives are finitely-presented.

Conversely, suppose that P is finitely-presented and flat. We want to prove that P is projective. Apply the previous lemma to $M = N = P$, to get an isomorphism:

$$\phi_P\colon P^\vee \otimes_A P \to \mathrm{Hom}_A(P, P)$$

Consider the element $\mathrm{Id}_P \in \mathrm{Hom}_A(P, P)$. Then, consider elements $f_i \in P^\vee$, $p_i \in P$, such that $\phi_P\left(\sum f_i \otimes p_i\right) = \mathrm{Id}_P$.

This says that, for any $x \in P$, $\sum f_i(x)p_i = x$, so we have a map $f\colon P \to A^n$ sending x to the tuple $(f_1(x), \ldots, f_n(x))$, and also a map $g\colon A^n \to P$ sending $e_i \mapsto p_i$. Then $g \circ f = \mathrm{Id}_P$, and so g is surjective. Moreover, f provides a splitting, and hence P is a direct summand of A^n, hence it is projective. □

The following is the geometric version of the previous lemma:

Proposition 1.7.11. *Let $f\colon Y \to X$ be a morphism of schemes. Then f is* finite *and* locally free *if, and only if, f is* fpm *and* flat*.*

We now restate and prove Proposition 1.7.5:

Proposition. *Let $f\colon Y \to X$ be a morphism of schemes. Then f is étale and* fpm *if, and only if, it is finite-étale (as defined before).*

Proof. After the previous proposition, the only remaining fact to prove is that f is *separable* if, and only if f is *unramified*.

So let $U = \mathrm{Spec}(A) \subseteq X$ be an open affine, $V = \mathrm{Spec}(B) \subseteq Y$ be such that $f(V) \subseteq U$. By taking U small enough, we may assume that the ring homomorphism $f\colon A \to B$ is finite and free (not just projective).

This allows one to reduce to the case of A being a field K.

In this case, $B = \prod_{i=1}^{n} B_i$, with B_i a local K-algebra, and $\mathfrak{m}_i \subseteq B_i$ nilpotent. □

Let now X be a *normal* and *integral* scheme. For instance, if k is an algebraically-closed field and $X \to \mathrm{Spec}(k)$ is smooth and connected, then X is always normal and integral –and for curves, the converse is true–).

Let $\eta \in X$ be its generic point. Let $K = K(X)$ be the function field of X, that is $K = \mathcal{O}_{X,\eta}$. Then one has that, for any open $U \subseteq X$, $\mathcal{O}_X(U) \subseteq K$.

We want to relate the Galois theory of K with the étale covers of X.

Let L be a finite separable field extension of K, and denote by $Y \to X$ the normalisation of X in L. This is done by defining the sheaf $\mathcal{A}$ by, for each $U \subseteq X$, $\mathcal{A}(U) \overset{\text{def}}{=}$ integral closure of $\mathcal{O}_X(U)$ in L, and then $\mathcal{A}$ turns out to be a quasicoherent sheaf. One then defines $Y \overset{\text{def}}{=} \mathbf{Spec}(A)$.

Definition 1.7.12. We say that L/K is **unramified** if $Y \to X$ is unramified.

Theorem 1.7.13.

i) If L/K is finite, separable and unramified, then $Y \to X$ is finite étale.

ii) Every connected, finite étale covering of X is obtained as in (1)*.*

iii) Let $\overline{K}$ be a fixed algebraic closure of K. Let M be the compositum of all the finite, separable, unramified extensions of K in $\overline{K}$. Then:

$$\pi_1(X) \simeq \mathrm{Gal}(M/K)$$

Exercises 3. In all the problems A is a ring and B, C are A-algebras with structure morphisms $f : A \to B$ and $g : A \to C$.

Properties U and E.

composition Suppose we have a ring homomorphism $h : B \to C$ such that $g = h \circ f$.

i) If B has property U over A and C has property U over B then C has property U over A.

ii) If B has property E over A and C has property E over B then C has property E over A.

base-change

i) If B has property U over A then $B_C := B \otimes_A C$ has property U over C.

ii) If B has property E over A then $B_C := B \otimes_A C$ has property E over C.

tensor product

i) If B has property U over A and C has property U over A then $B \otimes_A C$ has property U over A.

ii) If B has property E over A and C has property E over A then $B \otimes_A C$ has property E over A.

faithfully flat descent Suppose that C is a faithfully flat A-algebra.

i) If $B_C := B \otimes_A C$ has property U over C then B has property U over A.

ii) If $B_C := B \otimes_A C$ has property E over C then B has property E over A.

localization Suppose that $f \in B$ is an element.

i) If B has property U over A then B_f has property U over A.

ii) If B has property E over A then B_f has property E over A.

properties U,E are local on Spec(B) Suppose that for all $\mathfrak{q} \in \operatorname{Spec}(B)$ there exists $f \in B \setminus \mathfrak{q}$ such that

i) B_f has property U over A. Then B has property U over A.

ii) B_f has property E over A. Then B has property E over A.

examples

i) If $I \subset A$ is an ideal then $B = A/I$ has property U over A.

ii) Let $f, g \in A[X]$ such that f is a monic polynomial and denote by $f' \in A[X]$ the formal derivative of f. Let $B := \big(A[X]/fA[X]\big)_g$. Show that if $f' \in B^\times$ then B has property E over A. For example one can take $g = f'$.
Remark. An algebra B as in the above problem is called an E-standard A-algebra.

Properties of derivations and differentials

localization Let $S \subset B$ be a multiplicatively closed set. Let M be an $B_S := S^{-1}B$-module and $j : B \to B_S$ the canonical morphism. We may regard M also as a B module via j and denote it M_j.

i) Show that the composition with j given an isomorphism

$$\operatorname{Der}_A(B_S, M) \cong \operatorname{Der}_A(B, M_j).$$

ii) Deduce that we have a canonical isomorphism of B_S-modules $\Omega_{B/A} \otimes_B B_S \cong \Omega_{B_S/A}$.

base-change Let us denote as before $B_C := B \otimes_A C$ and think of it as a C-algebra. Prove that we have a natural isomorphism of B_C-algebras $\Omega_{B_C/C} \cong \Omega_{B/A} \otimes_A C$.

first fundamental exact sequence Let $A \xrightarrow{f} B \xrightarrow{h} C$ be a sequence of rings and ring homorphisms. Let M be a C-module which we regard as a B-module via h.

i) Prove that the natural sequence of B-modules is exact

$$0 \to \operatorname{Der}_B(C, M) \to \operatorname{Der}_A(C, M) \to \operatorname{Der}_A(B, M).$$

ii) Prove that the following sequence of C-modules is exact

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

second fundamental exact sequence Suppose that $J \subset B$ is an ideal and denote $C = B/J$. Let M be a C-module which we may also view as a B-module via the natural projection $B \to C$.

i) Prove that the natural sequence of B-modules is exact

$$0 \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \to \mathrm{Hom}_C(J/J^2, M).$$

ii) Deduce that the sequence of C-modules below is exact

$$J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0.$$

explicit calculations

i) Let $B = A[X_1, ..., X_d]$ be the polynomial algebra. Prove that $\Omega_{B/A}$ is the free B module with basis $d(X_1), ..., d(X_d)$.

ii) If $C = B/J = A[X_1, ..., X_d]/(f_1, ..., f_m)$, with $f_1, ..., f_m \in A[X_1, ..., X_d]$ prove that
$$\Omega_{C/A} \cong \left(\oplus_{i=1}^d C d(X_i)\right) / \left(d(f_1), ..., d(f_m)\right).$$

iii) Let $A = K$ be a field and $B = K[X, Y]/(Y^2 - X^3 + 1)$. Calculate $\Omega_{B/K}$ and prove that it is a projective B-module of rank 1.

1.8 Differentials

Let $f \colon Y \to X$ be a morphism of schemes.

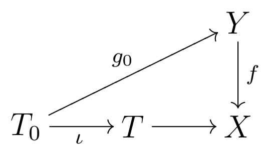
Definition 1.8.1. We say that f is **locally of finite type** (resp. **locally of finite presentation**) if for every point $y \in Y$, $x = f(y) \in X$, there exist open affine neighbourhoods $V = \mathrm{Spec}(B)$ and $U = \mathrm{Spec}(A)$ of y and x, respectively, such that $f(V) \subset U$ and $B \simeq A[x_1, \dots, x_d]/I$ for some ideal I (resp. some finitely generated ideal I).

Note that if Y is locally Noetherian, then these two definitions are equilvalent.

Definition 1.8.2. Let $\iota \colon T_0 \hookrightarrow T$ be a closed immersion corresponding to a quasi-coherent sheaf of ideals $\mathcal{I}$ on T. We say that ι is a **first order thickening** (or that T is a first order thickening of T_0) if $\mathcal{I}^2 = 0$.

If $T = \mathrm{Spec}(C)$ is affine, then first order thickenings correspond to ideals $I \subset C$ such that $I^2 = 0$. For instance, take $C = k[x]/(x^2)$ for some field k. Consider the ideal $I = (x)$ generated by the image of x in C. Then $C/I \simeq k$, and the ring homomorphism $C \to k$ corresponds to a first order thickening $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(C)$ of $\mathrm{Spec}(k)$.

Definition 1.8.3. Let $f\colon Y \to X$ be locally of finite presentation. We say that:
a) f is smooth
b) f has property U
c) f has property E
if for every commutative diagram:



where ι is a first order thickening,
a) locally on T, there exist morphisms $g\colon T \to Y$ "lifting g_0" (N.B. these morphisms need not glue together to give a global morphism).
b) there exists at most one global morphism $g\colon T \to Y$ "lifting g_0".
c) there exists exactly one global morphism $g\colon T \to Y$ "lifting g_0".

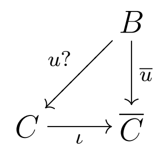
Remark. The terminology of properties U and E is nonstandard. We will see presently that they correspond to the morphism being unramified and étale, respectively. Also, the maps g are really extensions of g_0 to the first order thickening. This terminology comes from the ring theoretic side, where the extensions become liftings after reversing arrows.

The following theorem is an important characterisation of the properties above. It motivates our coming study of differentials; we postpone the proof until after we have covered that material.

Theorem 1.8.4. *Let $f\colon Y \to X$ be locally of finite presentation. Then:*
a) f has property U if and only if $\Omega^1_{Y/X} = 0$, if and only if f is unramified.
b) f has property E if and only if f has property U and is flat, if and only if f is étale.

Our discussion of differentials will begin by considering the affine case. Before delving into this topic, we will hence describe the ring theoretic versions of some the definitions above (as well as a new definition):

Definition 1.8.5. Let $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$, $f\colon A \to B$. We say that:
a) f is smooth (or B is smooth over A)
b) f has property U (or B has property U over A)
c) f has property E (or B has property E over A)
if for every A-algebra C, and ideal $J \subset C$ with $J^2 = 0$ such that:



commutes, where $\overline{C} = C/J$, then:
a) there exists an A-algebra homomorphism u making the diagram commute.
b) there exists at most one u making the diagram commute.
c) there exists exactly one u making the diagram commute.

These conditions can be phrased a different way. There is a natural map:

$$\phi\colon \mathrm{Hom}_{A-alg}(B, C) \to \mathrm{Hom}_{A-alg}(B, \overline{C})$$

where $u \mapsto \overline{u} = u \circ p$ (p is the quotient map $C \to \overline{C}$). With this notation:
a) is equivalent to ϕ being surjective,
b) is equivalent to ϕ being injective,
c) is equivalent to ϕ being an isomorphism.

Definition 1.8.6. Let A be a ring, B an A-algebra and M a B-module. Then $\partial\colon B \to M$ is called an A-**derivation** if it is A-linear and satisfies the Leibniz rule for products:

$$\partial(bb') = b\partial(b') + b'\partial(b)$$

for all $b, b' \in B$. Let:

$$\mathrm{Der}_A(B, M) = \{\partial\colon B \to M \mid \partial \text{ is an } A\text{-derivation}\}.$$

Note that since $1_B^2 = 1_B$, the Leibniz property implies that $\partial(1_B) = 0$, and hence that $\partial(a1_B) = 0$ for all $a \in A$. One can thus think of the elements in A as "scalars" annihilated by differentiation. Since M is a B-module, $\mathrm{Der}_A(B, M)$ carries a natural B-module structure. The operations are defined pointwise.

Suppose that $f\colon M \to N$ is a map of B-modules, and $\partial\colon B \to M$ is an A-derivation. Then one easily checks that $f \circ \partial\colon B \to N$ is also an A-derivation. It follows that $\mathrm{Der}_A(B, -)$ is a covariant functor on the category of B-modules.

Lemma 1.8.7. *The functor $Der_A(B, -)$ is representable. We denote the representing object (which is only determined up to unique isomorphism) $\Omega_{B/A}$. There is hence an isomorphism of functors:*

$$Der_A(B, -) \simeq Hom_B(\Omega_{B/A}, -).$$

We must show that for every B-module M, there are canonical isomorphisms:

$$\mathrm{Der}_A(B, M) \simeq \mathrm{Hom}_B(\Omega_{B/A}, M)$$

which are functorial in M. If we apply this to the B-modular $\Omega_{B/A}$, then we obtain an isomorphism:

$$\mathrm{Der}_A(B, \Omega_{B/A}) \simeq \mathrm{Hom}_B(\Omega_{B/A}, \Omega_{B/A}).$$

There hence exists a natural derivation $d\colon B \to \Omega_{B/A}$, which corresponds to the identity map of $\Omega_{B/A}$. With these observations, one can show that the lemma above is equivalent to proving the more usual characterisation of $\Omega_{B/A}$:

There exists a B-module $\Omega_{B/A}$ and a derivation $d\colon B \to \Omega_{B/A}$, such that the following universal property is satisfied: for every B-module M and derivation $\partial\colon B \to M$, there exists a unique B-linear map $f\colon \Omega_{B/A} \to M$ such that $\partial = f \circ d$.

It is this formulation of the lemma that we will prove.

Proof. The proof boils down to a simple and obvious construction. Let F denote the free B-module on formal symbols db for $b \in B$. Let G denote the B-submodule of F generated by all elements of the form:

a) d(ab + a'b') - adb - a'db',

b) d(bb') - bd(b') - b'db

for all $a, a' \in A$ and $b, b' \in B$. Let $\Omega_{B/A} = F/G$ and define $d\colon B \to \Omega_{B/A}$ by mapping b to the image of db in the quotient. We will often abuse notation and write the image simply as db. This should cause no confusion.

Suppose that $\partial\colon B \to M$ is a derivation. Define a map $\alpha\colon F \to M$ by putting $\alpha(db) = \partial(b)$. Then one checks that $\alpha(G) = 0$, and hence that α factors through $\Omega_{B/A}$. One easily deduces that $\Omega_{B/A}$ satisfies the desired universal property. □

We will give a second construction of the module of differentials $\Omega_{B/A}$. This is particularly nice for scheme theory, as it lends itself to a simple definition of sheaves of differentials on a scheme X. For the time being, however, we continue to focus on the affine case.

Given a ring homomorphism $A \to B$, then there is a corresponding surjective map $m\colon B \otimes_A B \to B$ given by multiplication, $b \otimes b' \mapsto bb'$. This map corresponds to the diagonal embedding of schemes. If we let i_1, i_2 be the canonical maps $B \mapsto$

$B \otimes_A B$, then one has $m \circ i_1 = m \circ i_2 = \mathrm{Id}_B$. Let $I = \ker(m)$. Note that I/I^2 is naturally a $(B \otimes_A B/I)$-module, since I acts trivially on I/I^2. Since m is surjective, $B \otimes_A B/I \simeq B$. We will describe explicitely the B-module structure on I/I^2 obtained under these identifications.

Let $x \in I/I^2$ and $b \in B$. Let $\beta \in B \otimes_A B$ be such that $m(\beta) = b$. Then we define:

$$bx = \beta x,$$

where the action of β is induced by the fact that $I \subset B \otimes_A B$ is an ideal. Note that, in particular, we can take $\beta = 1 \otimes b$ or $b \otimes 1$. So if we regard i_1, i_2 as defining a B-module structure on $B \otimes_A B$, then the induced structure on I/I^2 is the corresponding B-module structure.

Now consider the map $j = i_1 - i_2 \colon B \to B \otimes_A B$, which is a map of A-modules (not B-algebras!). Since $m \circ i_1 = m \circ i_2$, we have $m \circ j = 0$. This implies that the image of j is contained in I, so that we can really consider j as mapping into I. If we compose this with the natural projection map $\pi \colon I \to I/I^2$, then we obtain:

$$d \colon B \to I/I^2.$$

Claim. *d is an A-derivation.*

Proof. The A-linearity is clear. One simply verifies the Leibniz property directly:

$$\begin{aligned} d(bb') &= i_1(bb') - i_2(bb') \pmod{I^2} \\ &= bb' \otimes 1 - 1 \otimes bb' \pmod{I^2} \\ &= bb' \otimes 1 - b \otimes b' + b \otimes b' - 1 \otimes bb' \pmod{I^2} \\ &= (b \otimes 1)(b' \otimes 1 - 1 \otimes b') + (1 \otimes b')(b \otimes 1 - 1 \otimes b) \pmod{I^2} \\ &= bd(b') + b'd(b) \end{aligned}$$

□

Now, the universal property of $\Omega_{B/A}$ supplies a B-linear map $f \colon \Omega_{B/A} \to I/I^2$. We will prove the following:

Theorem 1.8.8. *$f \colon \Omega_{B/A} \to I/I^2$ is a B-linear isomorphism.*

This theorem will be proved by showing that I/I^2 represents $\mathrm{Der}_A(B, -)$. Before we can attack this theorem, we will first provide some more background material to help explain connections between $\Omega_{B/A}$ and other geometric notions introduced in this course.

Relationships between liftings of maps and differentials

Suppose that B, C are A-algebras, with $J \subset C$ an ideal such that $J^2 = 0$. Put $\overline{C} = C/J$, let $p\colon C \to \overline{C}$ be the quotient map, and fix an A-algebra morphism $u\colon B \to C$. Put $\overline{u} = p \circ u\colon B \to \overline{C}$. Since $J^2 = 0$, it follows that J is a $\overline{C}$-module. It is also a B-module via the map $\overline{u}$. We denote this B-module by $J_{\overline{u}}$. Note that this B-module structure only depends on $\overline{u}$; any other lifting of $\overline{u}$ to a map $u'\colon B \to C$ will induce the same B-module structure on J.

Lemma 1.8.9. *Let $\partial \in Der_A(B, J_{\overline{u}})$. Then $v = u + \partial\colon B \to C$ is an A-algebra homomorphism lifting $\overline{u}$ to C. Moreover, the map:*

$$Der_A(B, J_{\overline{u}}) \to \{v\colon B \to C \mid v \text{ lifts } \overline{u}\}$$

given by $\partial \mapsto u + \partial$, is a bijection.

Remark. The map u is itself a lifting of $\overline{u}$. It corresponds to the zero derivation under this association.

Proof. We first show that the map v is a map of A-algebras. The A-linearity is clear. We need only verify the multiplicativity. This is a simple calculation. Note that since $J^2 = 0$, $\partial(b)\partial(b') = 0$ for all $b, b' \in B$.

$$\begin{aligned} v(bb') &= u(bb') + \partial(bb') \\ &= u(b)u(b') + b\partial(b') + b'\partial(b) + \partial(b)\partial(b') \\ &= u(b)u(b') + u(b)\partial(b') + u(b')\partial(b) + \partial(b)\partial(b') \\ &= (u(b) + \partial(b))(u(b') + \partial(b')) \\ &= v(b)v(b') \end{aligned}$$

Now we will verify that v lifts $\overline{u}$. But this is simple:

$$p \circ v = p \circ (u + \partial) = p \circ u + p \circ \partial = \overline{u} + 0 = \overline{u}.$$

Note that $p \circ \partial = 0$ since $J = \ker(p)$.

For the second claim of this lemma, we will construct the inverse map. Suppose that $v\colon B \to C$ lifts $\overline{u}$. We will show that $\partial = v - u$ is a derivation into $J_{\overline{u}}$. A priori it is an A-module map from $B \to C$. Note that:

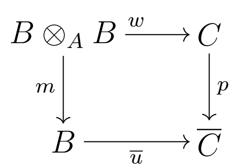
$$p \circ \partial = p \circ v - p \circ u = \overline{u} - \overline{u} = 0,$$

since v lifts $\overline{u}$. We deduce that the image of ∂ is in fact contained in $J = \ker(p)$. It remains to verify the Leibniz property:

$$\begin{aligned}
\partial(bb') &= v(bb') - u(bb') \\
&= v(b)v(b') - u(b)u(b') \\
&= v(b)v(b') - v(b)u(b') + v(b)u(b') - u(b)u(b') \\
&= v(b)(v(b') - u(b')) + u(b')(v(b) - u(b)) \\
&= v(b)\partial(b') + u(b')\partial(b)
\end{aligned}$$

Now, by the remark above, either of u or v can be used to define the B-module structure on J. The last line above is hence $b\partial(b') + b'\partial(b)$, which is the Leibniz property. So ∂ is an A-derivation. Obviously this map is inverse to $\partial \mapsto u + \partial$, which verifies the bijection. □

Fix $u\colon B \to C$, a lift of $\overline{u}$. Let $v\colon B \to C$ be another lift. These maps give a ring homomorphism $w\colon B \otimes_A B \to C$ by $w(b \otimes b') = u(b)v(b')$. This obviously satisfies $w \circ i_1 = u$ and $w \circ i_2 = v$. Moreover, we have commutative diagram:



This implies, in particular, that $w(I) \subset J$. But then:

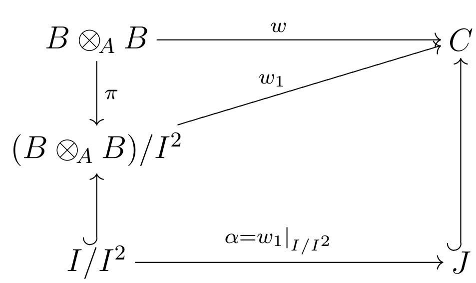
$$w(I^2) \subset w(I)^2 \subset J^2 = 0.$$

Thus, w actually induces a map:

$$\alpha\colon I/I^2 \to J.$$

This is a B-linear map when $J = J_{\overline{u}}$ is given the B-module structure induced by $\overline{u}$.

Here there is a diagram showing all the involved maps:



The following proposition is a step towards showing that I/I^2 represents $\mathrm{Der}_A(B, -)$:

Proposition 1.8.10. *The map:*

$$\{v\colon B \to C \mid v \text{ lifts } \overline{u}\} \to Hom_B(I/I^2, J_{\overline{u}})$$

sending $v \to \alpha$ (in the notation above) is a bijection (of sets).

Proof. We will construct the inverse map. For this, consider the exact sequence of A-modules:

$$0 \xrightarrow{I} B \otimes_A B \underset{\overset{i_1}{\curvearrowleft}}{\xrightarrow{m}} B \longrightarrow 0$$

which is split thanks to i_1. THen, as A-modules, $B \otimes_A B \simeq B \oplus I$, the identification being $(x \otimes y) \mapsto (xy, x \otimes y - xy \otimes 1$. On the right hand side, one can put the ring structure induced by the left hand side, and this isomorphisms $(B \otimes_A B)/I^2 \simeq (B \oplus I)/I^2 \simeq B \oplus (I/I^2)$, multiplication being:

$$(b, i) \cdot (b', i') = (bb', bi' + b'i)$$

Note also that the image of I in $B \oplus I/I^2$ has square 0, as expected.

Define now $w_1\colon B \otimes_A B/I^2 \simeq (B \oplus I)/I^2 \to C$ by

$$w_1(b, i) \overset{\text{def}}{=} u(b) + \alpha(i)$$

One needs to check that w_1 is an A-algebra homomorphism, and that $w \overset{\text{def}}{=} w_1 \circ \pi$ and $v \overset{\text{def}}{=} w \circ i_2$, defined respectively from $B \otimes_A B \to C$ and from $B \to C$ are the desired maps, which is routine calculation.

We can compute that, given $b \in B$,

$$v(b) = u(b) + \alpha(\overline{1 \otimes b - b \otimes 1})$$

Finally, one should check that this assignment from α to v is the inverse to the given in the statement of the proposition. □

From the two bijections we have so far established, we get a third one by composition:

$$\begin{array}{ccc} \{v\colon B \to C \mid v \text{ lifts } \overline{u}\} & \xleftarrow{v(b)=u(b)+\alpha(\overline{1\otimes b - b\otimes 1})} & \mathrm{Hom}_B(I/I^2, J_{\overline{u}}) \\ \big\downarrow{\scriptstyle v \mapsto u - v} & \swarrow{\scriptstyle \phi} & \\ \mathrm{Der}_A(B, J_{\overline{u}}) & & \end{array}$$

where ϕ is given by $(\phi(\alpha))\,(b) = u(b) - v(b) = -\alpha(\overline{1 \otimes b - b \otimes 1}) = \alpha(\overline{b \otimes 1 - 1 \otimes b})$, and it is an isomorphism of B-modules.

Remark. If $J^2 \neq 0$ in C, then one can lift $\overline{u}\colon B \to C/J$ to $u_1\colon B \to C/J^2$, and then to B/J^4, and continue in the same fashion to lift $\overline{u}$ to $\hat{u}\colon B \to \varprojlim C/J^n = \hat{C}$.

So far we have assumed that there is at least one lifting of $\overline{u}$, and we were also working only with a particular B-module, namely $J_{\overline{u}}$. We will see now that this wasn't actually a restriction:

Lemma 1.8.11. *If M is a B-module, then $M = J_{\overline{u}}$ for some A-algebra C, some $J \subseteq C$ of square 0, and some $\overline{u}\colon B \to \overline{C}$, such that there exists at least one lifting of $\overline{u}$ to $u\colon B \to C$.*

Proof. Define $C \stackrel{\text{def}}{=} B \oplus M$ as B-modules. Put a multiplication on C, given by:

$$(b, m) \cdot (b', m') \stackrel{\text{def}}{=} (bb', b'm + bm')$$

Then $J_{\overline{u}} \stackrel{\text{def}}{=} M \simeq \{(0, m) : m \in M\}$, and it is an ideal of square 0. The lifting is the map $b \mapsto (b, 0)$. □

From the previous discussion, we get the following proposition:

Proposition 1.8.12. *The quotient I/I^2 represents the functor $M \mapsto Der_A(B, M)$. In particular, $I/I^2 \simeq \Omega^1_{B/A}$.*

We will next sheafify this constructions, to get a geometric analogue.

Let $f\colon Y \to X$ be a morphism of schemes.

Definition 1.8.13. We say that f is **separated** if $\Delta\colon Y \to Y \times_X Y$ is a closed immersion.

Assume from now on that f is separated (if it weren't, then the image would be a closed subset inside some open, and we still can do it). We will define the **relative sheaf of differentials** of Y over X, written $\Omega^1_{Y/X}$, as a quasi-coherent sheaf of $\mathcal{O}_Y$-modules on Y.

Let $\mathcal{I} \subseteq \mathcal{O}_{Y \times_X Y}$ be the (quasi-coherent) ideal sheaf defining the image of Δ.

Definition 1.8.14. The **relative sheaf of differentials** is the quasi-coherent sheaf of $\mathcal{O}_Y$-modules given by $\Omega^1_{Y/X} \stackrel{\text{def}}{=} \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

Proposition 1.8.15. *For each pair of affine opens $V = \operatorname{Spec} B \subseteq Y$, $U = \operatorname{Spec} A \subseteq C$ such that $f(V) \subseteq U$, we have:*

$$\Omega^1_{Y/X}|_V = \widetilde{\Omega_{B/A}}$$

In particular, $\Omega^1_{Y/X}(V) = \Omega_{B/A}$.

Proof. Just note that $V \times_U V = \operatorname{Spec}(B \otimes_A B)$, and that Δ corresponds to the multiplication map $m\colon B \otimes_A B \to B$ sending $b \otimes b' \mapsto bb'$. Hence $\mathcal{I}|_{V \times_U V} = \ker \Delta = \tilde{I}$, and $\Delta^*(\mathcal{I}/\mathcal{I}^2)|_V = \widetilde{I/I^2} = \widetilde{\Omega_{B/A}}$. □

Chapter 2

Grothendieck Topologies

2.1 Grothendieck topologies

Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. We will restrict ourselves to sheaves of sets and abelian groups, although the definitions here will clearly be seen to be much more widely applicable. Recall that a presheaf can be described very simply in categorical language. Indeed, one considers the category $\mathbf{C}$ with objects the open subsets of X, and morphisms inclusions of sets. Then a presheaf is simply a contravariant functor from $\mathbf{C}$ to the category of abelian groups. More generally, one might call *any* contravariant functor a presheaf.

In order to generalise the notion of a sheaf, one needs to generalise the sheaf axioms. In the case of a topological space X, one can formulate the sheaf property for $\mathcal{F}$ very succinctly. Let $U \subset X$ be any open subset and $\{U_i\}_{i\in I}$ an open cover of U. Consider the maps:

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

Here $\alpha(s) = (s|_{U_i})$, $\beta(s_i) = (s_i|_{U_i\cap U_j})$ and similarly for γ. The sheaf axioms are equivalent to the exactness of:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\beta-\gamma} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

for every open set $U \subset X$ and open cover $\{U_i\}_{i\in I}$ of U.

If $\mathcal{F}$ is a sheaf of sets, then we cannot form the difference $\beta - \gamma$. However, it is still possible to formulate the sheaf property using the formalism above. In this case we will say that:

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is exact if α is injective, and $\mathrm{im}(\alpha) = \mathrm{Eq}(\beta, \gamma)$. Here:

$$\mathrm{Eq}(\beta, \gamma) = \{x \in \prod_{i\in I} \mathcal{F}(U_i) \mid \beta(x) = \gamma(x)\}$$

is the **equaliser** of β and γ.

One can make this definition slightly more abstract by noting that $U_i \cap U_j = U_i \times_U U_j$, where the fibre product is taken with respect to the natural injections $U_i \to U$.

Definition 2.1.1. A **Grothendieck Topology** T consists of the following data: a category, denoted **Cat** T, along with a collection of *coverings*, denoted **Cov** T. By a collection of coverings we mean that **Cov** T contains families of morphisms:

$$\{\phi_i \colon U_i \to U\}_{i\in I},$$

where U, U_i are objects in **Cat** T and ϕ_i is a morphism in **Cat** T. These families must satisfy the following three axioms:

1) If $V \to U$ is an isomorphism in **Cat** T, then $\{V \to U\} \in$ **Cat** T.

2) If $\{\phi_i \colon U_i \to U\}_{i\in I} \in$ **Cov** T is such that for each $i \in I$, there exists $\{\phi_{ij} \colon V_{ij} \to U_i\}_{j\in J_i} \in$ **Cov** T, then:

$$\{\phi_i \circ \phi_{ij} \colon V_{ij} \to U\}_{i\in I,\ j\in J_i} \in \mathbf{Cov}\ T.$$

3) If $\{\phi_i \colon U_i \to U\}_{i\in I} \in$ **Cov** T and $V \to U$ is any morphism in **Cat** T, then $U_i \times_U V$ exists for each $i \in I$ and:

$$\{U_i \times_U V \to V\}_{i\in I} \in \mathbf{Cov}\ T.$$

The first axiom of a covering corrsponds to the fact that U is itself a covering of an open subset $U \subset X$. The second says that if you have an open covering of U by U_i's, and an open covering of each U_i, then putting all of these together gives an open covering of U. The third axiom says that if $\{U_i\}$ is a covering of U, then $\{V \cap U_i\}$ is a covering of $U \cap V$.

Definition 2.1.2. A **presheaf** of sets on T is a contravariant functor $\mathcal{F}$: **Cat** $T \to$ **Sets**. A **sheaf** of sets on T is a presheaf $\mathcal{F}$ of sets such that for every $\{\phi_i \colon U_i \to U\}_{i\in I} \in$ **Cov** T, the sequence:

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact in the sense discussed above. Note that now β and γ are induced by $\mathcal{F}$ applied to the natural maps $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$.

Remark. Although we have only formulated the definitions of sheaves of sets, one can easily generalise this to sheaves of abelian groups, or other objects (one requires arbitrary products to exist in the target category). In the case of abelian groups, one can formulate the sheaf property in terms of an ordinary exact sequence:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\beta-\gamma} \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)\ .$$

Examples 2.1.3.

i) Let X be a topological space. Let **Cat** T be the category with objects the open subsets of X and morphisms inclusion. Let **Cov** T be the collection of open covers of open subsets of X. Then T is a Grothendieck topology.

ii) Let **Cat** T be the category of sets (in some universe, so that the Hom sets are actually sets). Let **Cov** T be the collection of surjective families of maps $\{\phi_i \colon U_i \to U\}$; by this we mean that $U = \bigcup_{i\in I} \phi_i(U_i)$. Then T is a Grothendieck topology. One can show that every functor $\mathrm{Hom}_{\mathbf{Sets}}(-, X)$ is a sheaf of sets on T, and that every sheaf $\mathcal{F}$ of sets on T is representened by a set X; this means that there is a natural isomorphism of functors, $\mathrm{Hom}_{\mathbf{Sets}}(-, X) \simeq \mathcal{F}(-)$.

iii) Let G be a group. Let **Cat** T_G be the category of left G-sets. Let **Cov** T_G denote the collection of surjective families of morphisms in **Cat** T_G. Then T_G is a Grothendieck topology. The set valued sheaves on T are classified up to isomorphism by the functors $\mathrm{Hom}_G(-, X)$ for $X \in$ **Cat** T_G, as above.

iv) Let π be a profinite group. Let **Cat** T_π denote the category of finite π-sets with continuous action (when the set is endowed with the discrete topology). Let **Cov** T_π consist of all surjective families of morphisms that are indexed by a *finite* set. Then T_π is a Grothendieck topology and sheaves of sets on T_π are classified as above. One can show that the category of sheaves of abelian groups on T_π is equivalent to the category of π-modules.

v) Let X be a connected scheme. Let **Cat** $T_X^f = \mathbf{FEt}_X$ (f is for finite) and let **Cov** T_X^f denote the collection of surjective families of morphisms indexed by a finite set. Then T_X^f is a Grothendieck topology. The sheaves of abelian groups on T_X^f are naturally identified with π-modules, where π is the fundamental group of the scheme X.

vi) Let X be a Noetherian connected scheme. Let $\mathbf{Cat}\, T_X^{\text{et}}$ denote the category of schemes Y of finite type and étale over X. Let $\mathbf{Cov}\, T_X^{\text{et}}$ denote the collection of surjective families of morphisms indexed by a finite set. Then T_X^{et} is a Grothendieck topology.

2.2 Inductive and projective limits

In order to define stalks and completions of sheaves, we will need to generalise the notions of inductive and projective limits. Let $I, \mathbf{C}$ be categories. For each object $X \in \mathbf{C}$, define the constant functor $c_X : I \to \mathbf{C}$ by $c_X(i) = X$ for all objects $i \in I$ and $c_X(\phi) = \mathrm{Id}_X$ for every morphism $\phi \in I$. Let $F : I \to \mathbf{C}$ be a functor.

Definition 2.2.1. Consider the functor $\widetilde{\varinjlim} F : \mathbf{C} \to \mathbf{Sets}$ defined on objects by $(\widetilde{\varinjlim} F)(x) = \mathrm{Hom}_{\mathbf{functors}}(F, c_X)$, where $\mathrm{Hom}_{\mathbf{functors}}$ denotes the collection of morphisms (natural transformations) between the functors F and c_X. If this functor is represented by some object $Y \in \mathbf{C}$, meaning that there are natural isomorphisms:

$$\mathrm{Hom}_{\mathbf{functors}}(F, c_X) \simeq \mathrm{Hom}_{\mathbf{C}}(Y, X)$$

for each $X \in \mathbf{C}$, then we say that Y is the **direct limit** of F and write $\varinjlim F \stackrel{\text{def}}{=} Y$.

Analogously, define $\varprojlim F$ as representing the functor $\widetilde{\varprojlim} F : \mathbf{C} \to \mathbf{Sets}$, defined on objects by $(\widetilde{\varprojlim} F)(X) = \mathrm{Hom}_{\mathbf{functors}}(c_X, F)$, and so that there exist natural isomorphisms:

$$\mathrm{Hom}_{\mathbf{functors}}(c_X, F) \simeq \mathrm{Hom}_{\mathbf{C}}(X, \varprojlim F)$$

for each $X \in \mathbf{C}$.

Remark. We stress that the universal property of the direct limit of F is encapsulated in the natural isomorphism:

$$\mathrm{Hom}_{\mathbf{functors}}(F, c_X) \simeq \mathrm{Hom}_{\mathbf{C}}(\varinjlim F, X).$$

Suppose that $\varphi : F \to c_X$ is a morphism of functors. That is, for each i an object of I, we have $\varphi_i : F_i \to X$ (we denote $F_i \stackrel{\text{def}}{=} F(i)$), and also for each $i \to j$ morphism of I, we have a commutative diagram:

$$\begin{array}{ccc} F_i & \xrightarrow{\varphi_i} & X \\ \downarrow & & \| \\ F_j & \xrightarrow{\varphi_j} & X \end{array}$$

In this case, there exists a unique morphism $\psi : \varinjlim F \to X$.

Proposition 2.2.2. *If* $\boldsymbol{C} = \boldsymbol{Sets}$ *or* $\boldsymbol{C} = \boldsymbol{AbGrp}$*, then inductive and projective limits exist, for any functor* $F\colon I \to \boldsymbol{C}$*.*

Proof. Note that inductive and projective limits share the same definition, as long as one is allowed to change the categories in which one works (to the opposite ones). As we are proving a general statement, it is enough to prove it then for inductive limits.

If $\mathbf{C} = \mathbf{Sets}$, denote $F_i = F(i)$ as before, and consider the set:

$$\varinjlim F \stackrel{\text{def}}{=} (\amalg_{i\in I} F_i)/R$$

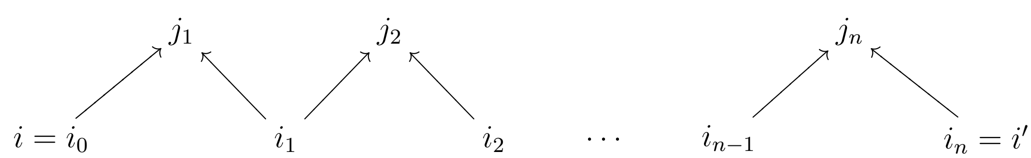
where R is the equivalence relation generated by the pairs $(x, y) \in F_i \times F_j$ such that there exists a morphism $\phi\colon i \to j$ such that $F(\phi)(x) = y$. One then checks that this has the right universal property.

If $\mathbf{C} = \mathbf{AbGrp}$, then one defines

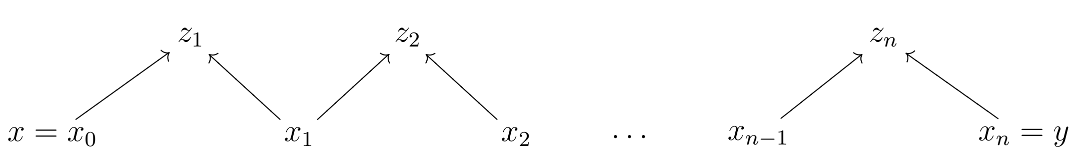
$$\varinjlim F \stackrel{\text{def}}{=} (\bigoplus_{i\in I} F_i)/R$$

where now R is the *subgroup* generated by the $x - y$ where (x, y) is as before. □

We proceed now to explain better what is the relation R defined in the proof of the previous theorem. It is easy to see that $(x, y) \in R$ if, and only if, there exists two chains $i = i_0, i_1, \ldots, i_n = i'$ and $j_1, \ldots, j_n$ of objects in I, such that there exists a diagram:



together with elements $x_0 = x, x_1, \ldots, x_n = y$, with $x_k \in F_{i_k}$, and $z_1, \ldots, z_n$ with $z_k \in F_{j_k}$ related through the following diagram:



These diagrams are called a **connection** between i and i'.

Next, we describe some properties that I may or may not satisfy, but that will allow us to prove stronger statements about limits.

L1 Given a diagram like:

$$\begin{array}{ccc} & & j \\ & \nearrow & \\ i & & \\ & \searrow & \\ & & j' \end{array}$$

then there exists another diagram

$$\begin{array}{ccc} j & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ j' & & \end{array}$$

such that the resulting square commutes:

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & & \searrow & \\ i & & & & k \\ & \searrow & & \nearrow & \\ & & j' & & \end{array}$$

L2 Given $i \rightrightarrows j$ in I, then there exists a morphism $j \to k$ such that the compositions are the same.

L3 (connectedness) For each pair of objects $i, i' \in I$, there is a *connection* between them.

Let now $\mathcal{F}$ be the category of covariant functors $F \colon I \to \mathbf{AbGrp}$, where morphisms are natural transformations of functors. This category is very well behaved, as the following theorems show:

Theorem 2.2.3.

i) The category $\mathcal{F}$ is abelian*,*

ii) The functor $\varinjlim \colon \mathcal{F} \to \boldsymbol{AbGrp}$ which sends F to the object $\varinjlim F$ is right exact*.*

Proof. We first show that $\mathcal{F}$ is an abelian category. So let F, G be two functors $I \to \mathbf{AbGrp}$. We want to give a group structure to $\mathrm{Hom}_{\mathcal{F}}(F, G)$. This is done pointwise, which means that given φ, ψ two natural transformations, one defines another transformation by $(\varphi + \psi)(i) \stackrel{\text{def}}{=} \varphi_i + \psi_i$. It is easy to see that this is a natural transformation. One needs to check as well that composition of natural transformations is a group homomorphism, which is easy as well.

So far, we have sketched why $\mathcal{F}$ is an additive category. Let now $\varphi\colon F \to G$ be a morphism in $\mathcal{F}$. We want to define $\ker\varphi$. Again, we do it pointwise:

$$(\ker\varphi)_i \stackrel{\text{def}}{=} \ker(\varphi\colon F_i \to G_i)$$

If $i \to j$ is a morphism in I, then one defines $(\ker\varphi)(i \to j)$ through the following diagram:

$$\begin{array}{ccccc}
(\ker\varphi)_i & \longrightarrow & F_i & \xrightarrow{\varphi_i} & G_i \\
\downarrow & & \downarrow{\scriptstyle F(i\to j)} & & \downarrow{\scriptstyle G(i\to j)} \\
(\ker\varphi)_j & \longrightarrow & F_j & \xrightarrow{\varphi_j} & G_j
\end{array}$$

The cokernel is defined in the same way. As usual, the image is defined as $\operatorname{im}\varphi \stackrel{\text{def}}{=} \ker(G \to \operatorname{coker}\varphi)$

Given a sequence $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$, one says that it is exact if $\ker\psi = \operatorname{im}\varphi$. Remark that this is equivalent to saying that, for each $i \in I$, $\ker\psi_i = \operatorname{im}\varphi_i$.

Next we show that $\varinjlim$ is actually a functor, and will leave as an exercise to check that is right-exact.

Given $\varphi\colon F \to G$ a morphism in $\mathcal{F}$, we want to show that there exists $\varinjlim\varphi\colon \varinjlim F \to \varinjlim G$. We have that $\operatorname{Hom}_{\mathcal{F}}(F, c_X) \simeq \operatorname{Hom}_{\mathbf{AbGrp}}(\varinjlim F, X)$, for each object X of **AbGrp**. Apply this to $X = \varinjlim G$. To find an element of $\operatorname{Hom}_{\mathbf{AbGrp}}(\varinjlim F, \varinjlim G)$ is then equivalent to finding an element of $\alpha \in \operatorname{Hom}_{\mathcal{F}}(F, c_{\varinjlim G})$. That is, we need to find $\alpha_i\colon F_i \to \varinjlim G$ in a functorial way. By the property satisfied by $\varinjlim G$, we have that $\operatorname{Hom}_{\mathcal{F}}(G, c_{\varinjlim G}) \simeq \operatorname{Hom}_{\mathbf{AbGrp}}(\varinjlim G, \varinjlim G)$, and this last set always contains the element $\operatorname{Id}_{\varinjlim G}$. Also, there is a natural map $\operatorname{Hom}_{\mathcal{F}}(\varinjlim G, c_{\varinjlim G}) \to \operatorname{Hom}_{\mathcal{F}}(F, c_{\varinjlim G})$, which is φ^*, the pullback by φ. Applying then this pullback with the image of $\operatorname{Id}_{\varinjlim G}$ we get an element of $\operatorname{Hom}_{\mathcal{F}}(F, c_{\varinjlim G})$, as wanted. $\square$

Theorem 2.2.4. *If the category I satisfies properties* ***L1,L2*** *and* ***L3****, then the functor $\varinjlim\colon \mathcal{F} \to$* ***AbGrp*** *is exact.*

Proof. It is enough to show, thanks to the previous theorem, that if $0 \to F \to G$ is exact, then $0 \to \varinjlim F \to \varinjlim G$ is exact as well.

Consider $F'\colon I \to \mathbf{Sets}$ to be the forgetful functor $\mathbf{AbGrp} \to \mathbf{Sets}$ applied after F. Denote by $\varinjlim' F \stackrel{\text{def}}{=} \varinjlim F'$. There is an obvious morphism $\varinjlim F' \to \varinjlim F$.

We use without proof the following proposition, whose proof is not trivial:

Proposition. *If I satisfies* ***L1,L2*** *and* ***L3****, then $\varinjlim' F \simeq \varinjlim F$ is bijective (as sets!).*

To prove the theorem is enough then to prove it on the underlying sets, as a group homomorphism is injective if it is so as a map of sets. By the proposition, it is enough then to show that if $F \to G$ is injective, then the map of sets $\varinjlim' F \to \varinjlim' G$ is injective as well.

Recall that $\varinjlim' F = (\amalg F_i')/R$, and because of **L1**, the relation R is simplified to $(x, y) \in R$ if, and only if, $x \in F_i$, $y \in F_j$ and there exists $k \in I$, together with morphisms $i \to k$ and $j \to k$ such that the image of x and of y are the same in F_k.

Consider now the map

$$\begin{aligned}(\amalg F_i)/R_F &\xrightarrow{\overline{\varphi}=\varinjlim' \varphi} (\amalg G_i)/R_G \\ x \in F_i &\longmapsto \overline{\varphi_i(x)}\end{aligned}$$

Suppose now that $x \in F_i$ and $y \in F_j$ are such that $(\varphi_i(x), \varphi_j(y)) \in R_G$. We want to see that $(x, y) \in R_F$. But by hypothesis, there exists $k \in I$ with morphisms $u \colon i \to k$ and $u \colon u \to k$ such that:

$$(G'(u))(\varphi_i(x)) = (G'(v))(\varphi_j(y))$$

But the left hand side in the previous equation is $\varphi_k(F'(u)(x))$, while the right hand side is $\varphi_k(F'(v)(y))$. As by assumption φ_k is injective, we get that $F'(u)(x) = F'(v)(y)$, and so $(x, y) \in R_F$ as wanted. □

Remark. The property **L3** is not necessary for the previous theorem to be true, but it simplifies its proof.

2.3 Pushforwards, pullbacks and adjunctions

Let **C** be a category and let $\mathcal{P}_{\mathbf{C}} = \mathcal{P}$ (we drop the **C** whenever the category is understood) denote the category of contravariant functors $\mathbf{C} \to \mathbf{AbGrp}$. The previous section showed that for any category **C**, $\mathcal{P}$ is an abelian category. It is not hard to show that a sequence:

$$F_1 \to F_2 \to F_3$$

in $\mathcal{P}$ is exact if and only if for all $U \in \mathbf{C}$,

$$F_1(U) \to F_2(U) \to F_3(U)$$

is exact. We will use this fact constantly in what follows.

Example. We will now explain the primary example of presheaves. Throughout this section we will constantly return to this example, which we will refer to as the "canonical example". Let X be a topological space, and let $\mathbf{C}_X$ denote the category with objects the open subsets of X, and morphisms inclusions of sets. The $\mathcal{P}_{\mathbf{C}_x} = \mathcal{P}$ is the usual category of presheaves of abelian groups on X.

Suppose that $\mathbf{C}$ and $\mathbf{C}'$ are categories, and $f\colon \mathbf{C} \to \mathbf{C}'$ a covariant functor. Define a morphism:

$$f_*\colon \mathcal{P}_{\mathbf{C}'} \to \mathcal{P}_{\mathbf{C}}$$

by composition:

$$f_*(F') = F' \circ f.$$

Example. In the canonical example, if X and Y are topological spaces, then any continuous map $f\colon X \to Y$ induces a covariant functor from $\mathbf{C}_Y$ to $\mathbf{C}_X$ by sending $U \mapsto f^{-1}(U)$. Denote this functor by f^{-1}. If F is a presheaf on X, then $f_*^{-1}F$ is a presheaf on Y, such that:

$$f_*^{-1}F(V) = F(f^{-1}(V)).$$

So in this case, f_*^{-1} acts like a pushforward from X to Y.

Lemma 2.3.1. *f_* is an exact functor.*

Proof. We apply the remark at the beginning of this section. Let

$$F_1' \to F_2' \to F_3'$$

be an exact sequence in $\mathcal{P}_{\mathbf{C}'}$. Then:

$$f_*F_1' \to f_*F_2' \to f_*F_3'$$

is exact if and only if for all $U \in \mathbf{C}$,

$$f_*F_1'(U) \to f_*F_2'(U) \to f_*F_3'(U)$$

is exact. But this is just:

$$F_1'(f(U)) \to F_2'(f(U)) \to F_3'(f(U)),$$

which is exact since the first sequence above is exact. The lemma follows. □

Remark. We stress that f_* is exact on *presheaves.*

We recall now the notion of an adjoint pair of functors. Let M, N be (small) categories with covariant functors $\alpha\colon M \to N$ and $\beta\colon N \to M$.

Definition 2.3.2. We say that β is **left adjoint** to α (or that β is **right adjoint** to α) if there is an isomorphism of functors:

$$\mathrm{Hom}_N(-, \alpha(-)) \simeq \mathrm{Hom}_M(\beta(-), -)$$

from $M \times N$ to **Sets**.

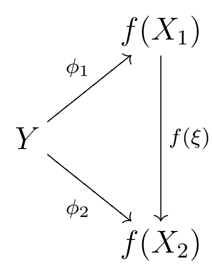
If α or β is given, then the other is determined uniquely up to isomorphism (if it exists at all).

Example. Let $M = N = \mathbf{AbGrp}$. Let $Z \in \mathbf{AbGrp}$ and put $\alpha_Z\colon \mathbf{AbGrp} \to \mathbf{AbGrp}$ by $\alpha_Z(X) = \mathrm{Hom}_{\mathbf{AbGrp}}(Z, X)$, $\beta_Z\colon \mathbf{AbGrp} \to \mathbf{AbGrp}$ by $\beta_Z(Y) = Y \otimes_{\mathbb{Z}} Z$. Then there are natural isomorphisms for all $X, Y \in \mathbf{AbGrp}$:

$$\mathrm{Hom}(Y, \mathrm{Hom}(Z, X)) \simeq \mathrm{Hom}(Y \otimes_{\mathbb{Z}} Z, X).$$

Theorem 2.3.3. *Let $\boldsymbol{C}$ and $\boldsymbol{C'}$ be categories, $f\colon \boldsymbol{C} \to \boldsymbol{C'}$ a covariant functor. There exists a left adjoint functor $f^*\colon \mathcal{P} \to \mathcal{P}'$ to f_*. Moreover, f^* is right exact.*

Proof. We will only sketch the proof. Let $Y \in \mathbf{C}'$ and define I_Y to be the following category: objects are pairs (X, ϕ) where $X \in \mathbf{C}$ and $\phi\colon Y \to f(X)$ is a morphism in $\mathbf{C}'$. So if $(X, \phi) \in I_Y$ then $\mathrm{Hom}_{\mathbf{C}'}(Y, f(X)) \neq \emptyset$. If (X_1, ϕ_1) and (X_2, ϕ_2) are objects in I_Y, a morphism is a morphism $\xi\colon X_1 \to X_2$ in $\mathbf{C}$ such that the diagram:



commutes.

Suppose that $\varepsilon\colon Y \to Z$ is a morphism in $\mathbf{C}'$. Then there is a natural functor $\overline{\varepsilon}\colon I_Z \to I_Y$ defined by:

$$\overline{\varepsilon}(X, \phi) = (X, \phi \circ \varepsilon).$$

If $F \in \mathcal{P}$, then for every $Y \in \mathbf{C}'$, we get a functor $F_Y\colon I_Y \to \mathbf{AbGrp}$ defined by:

$$F_Y(X, \phi) = F(X).$$

Definition 2.3.4. Let $F \in \mathcal{P}$ and define:

$$f^*F\colon \mathbf{C}' \to \mathbf{AbGrp}$$

by putting:

$$(f^*F)(Y) = \varinjlim_{I_Y} F_Y.$$

With this definition, f^*F is a contravariant functor in $\mathcal{P}'$. In other words, if $\varepsilon\colon Y \to Z$ is a morphism in $\mathbf{C}'$, then we get a functor $\overline{\varepsilon}\colon I_Z \to I_Y$ which induces maps between the injective limits. This is used to define $(f * F)(\varepsilon)$. A simple formal argument shows that f^*F is right exact: let

$$F_1 \to F_2 \to F_3 \to 0$$

be exact in $\mathcal{P}$. Consider:

$$f^*F_1 \to f^*F_2 \to f^*F_3 \to 0.$$

This is exact if and only if for every $Y \in \mathbf{C}'$, the sequence:

$$f^*F_1(Y) \to f^*F_2(Y) \to f^*F_3(Y) \to 0$$

is exact. By definition, this is just:

$$\varinjlim_{I_Y} F_{1,Y} \to \varinjlim_{I_Y} F_{2,Y} \to \varinjlim_{I_Y} F_{3,Y} \to 0.$$

One checks that since the first sequence is exact, so is:

$$F_{1,Y} \to F_{2,Y} \to F_{3,Y} \to 0.$$

Taking injective limits, which is an exact functor, concludes the proof that f^*F is right exact.

We will omit most of the remaining details, save for giving an indication of why f^* is left adjoint to f_*. Let $F \in \mathcal{P}$ and $G \in \mathcal{P}'$. We want to show that:

$$\mathrm{Hom}_{\mathcal{P}}(F, f_*G) \simeq \mathrm{Hom}_{\mathcal{P}'}(f^*F, G).$$

Suppose that $\phi\colon F \to f_*G$ is a natural transformation. This means that for every $X \in \mathbf{C}$ we have:

$$\phi_X\colon F(X) \to (f_*G)(X) = G(f(X)).$$

We will define map $\phi \to \psi$ where ψ is a natural transformation $f^*F \to G$. Let $Y \in \mathbf{C}'$. To define ψ, we must give maps:

$$\psi_Y \colon (f^*G)(Y) \to G(Y).$$

Take $(X, \phi) \in I_Y$, so that we have a commutative diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & G(f(X)) \\ & \searrow & \downarrow {\scriptstyle G(\phi)} \\ & & G(Y) \end{array}$$

Since we have such maps $F(X) \to G(Y)$ for every object in I_Y, the universal property of the injective limit implies that there exists a corresponding map $(f^*F)(Y) \to G(Y)$. We call this ψ_Y. One checks that this actually defines a natural transformation $\psi\colon f^*F \to G$.

Now we explain how to define the inverse map. Take $\psi\colon f^*F \to G \in \mathcal{P}'$. This time we want to define maps:

$$\phi_X \colon F(X) \to (f_*G)(X) = G(f(X)),$$

for each $X \in \mathbf{C}$. Fix $X \in \mathbf{C}$ and put $Y = f(X) \in \mathbf{C}'$. Then:

$$(X, \mathrm{Id}_Y) \in I_Y,$$

so that $I_Y \neq \emptyset$. We thus certainly have a commutative diagram:

$$\begin{array}{ccc} F(X) = F_Y(X, \mathrm{Id}_Y) & \longrightarrow & (f^*F)(Y) = \varinjlim_{I_Y} F_Y \\ & \searrow & \downarrow {\scriptstyle \psi_Y} \\ & & G(Y) = G(f(X)) \end{array}$$

Let ϕ_X be the diagonal arrow. We claim that this gives a natural transformation ϕ, and sets up a natural bijection:

$$\mathrm{Hom}_{\mathcal{P}}(F, f_*G) \simeq \mathrm{Hom}_{\mathcal{P}'}(f^*F, G).$$

The remaining details are left to the reader. $\square$

Example. Return to the standard example, so that $f\colon X \to Y$ is a continuous map of topological spaces. We will describe $(f^{-1})^*$. Let F be a presheaf on Y. Let U be an open subset of X. Then objects of I_U are pairs (V, ϕ) where $\phi\colon U \to f^{-1}(V)$ is a morphism in $\mathbf{C}_X$; in other words, $U \subset f^{-1}(V)$. We see that objects of I_U are open subsets V in Y such that $f(U) \subset V$. In this case:

$$((f^{-1})^*F)(U) = \varinjlim_{f(U)\subset V//V\in\mathbf{C}_Y} F(V).$$

This agrees with the standard definition (see Hartshorne, for instance).

Recall that in an abelian category, an object Z is said to be **injective** if the functor $X \mapsto \mathrm{Hom}(X, Z)$ is exact. More will be said about injective objects in the following section on cohomology. In preparation for this discussion, we prove now a useful corollary:

Corollary 2.3.5. *Let $\boldsymbol{C}$ and $\boldsymbol{C'}$ be categories, with $f\colon \boldsymbol{C} \to \boldsymbol{C'}$ a covariant functor. Suppose that f^* is exact. Then f_* preserves injectives.*

Proof. Let Z be an injective object in $\mathcal{P}'$. We want to show that f_*Z is injective in $\mathcal{P}$. Let:

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

be exact in $\mathcal{P}$. We want to show that:

$$0 \to \mathrm{Hom}_{\mathcal{P}}(F_3, f_*Z) \to \mathrm{Hom}_{\mathcal{P}}(F_2, f_*Z) \to \mathrm{Hom}_{\mathcal{P}}(F_1, f_*Z) \to 0$$

is exact. By adjunction this is the same as:

$$0 \to \mathrm{Hom}_{\mathcal{P}}(f^*F_3, Z) \to \mathrm{Hom}_{\mathcal{P}}(f^*F_2, Z) \to \mathrm{Hom}_{\mathcal{P}}(f^*F_1, Z) \to 0$$

Apply the exact functor f^* to the first exact sequence above and use the fact that Z is injective to conclude the proof. □

Let $\mathbf{C}$ be a category and let $X \in \mathbf{C}$. Let $\{X\}$ be the category with a single object, X, and a single morphism, Id_X. Then there is an obvious inclusion functor $i_X\colon \{X\} \to \mathbf{C}$. If we note that presheaves on $\{X\}$ are equivalent to the category **AbGrp**, then we obtain a functor:

$$i_{X,*}\colon \mathcal{P}_{\mathbf{C}} \to \mathbf{AbGrp}.$$

It is simply evaluation at X, $F \mapsto F(X)$. By the work above, there is also a functor $i_X^*\colon \mathbf{AbGrp} \to \mathcal{P}_{\mathbf{C}}$.

Claim. i_X^* *is exact.*

Proof. Let $Y \in \mathbf{C}$. Then in this case, since $\mathcal{P}_{\{X\}} \simeq \mathbf{AbGrp}$ is such a simple category, one easily verifies that:

$$I_Y = \mathrm{Hom}_{\mathbf{C}}(Y, X).$$

By this we mean that the objects are the morphisms, and the morphisms in I_Y are just the identity morphisms for each object. Such a category is said to be *discrete*. It follows from the discreteness that we don't need to divide by any relations in the definition of the following direct limit:

$$(i_X^* A)(Y) = \varinjlim_{I_Y} F_Y = \bigoplus_{\phi \in \mathrm{Hom}(Y,X)} A.$$

(recall the explicit description of the direct limit given in the previous section). With this description of i_X^*, it is not hard to verify that it is an exact functor. □

If one applies the previous corollary to i_X, one hence obtains:

Corollary 2.3.6. *If F is an injective object in $\mathcal{P}_{\boldsymbol{C}}$, then for all $X \in \boldsymbol{C}$, $F(X)$ is injective in* ***AbGrp****.*

Given an object $X \in \mathbf{C}$, let $\zeta_X = i_X^*(\mathbb{Z})$. Then for every presheaf $F \in \mathcal{P}_{\mathbf{C}}$, the adjunction property implies that there is a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{P}}(\zeta_X, F) \simeq \mathrm{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}, i_{X,*}F).$$

But note that $i_{X,*}F = F(X)$, and that $\mathrm{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}, F(X)) \simeq F(X)$. We thus see that:

$$F(X) \simeq \mathrm{Hom}_{\mathcal{P}}(\zeta_X, F).$$

For a fixed object X, we see that the functor from $\mathcal{P} \to \mathbf{AbGrp}$ defined by $F \mapsto F(X)$ is represented by $\zeta_X \in \mathcal{P}$. We will see below (and explain what these words mean) that the collection of $\zeta_X \in \mathcal{P}$ is a *family of generators*, and that $\mathcal{P}$ has *enough injectives*.

2.4 Cohomological δ-functors

We first recall the notion of a complex for an arbitrary abelian category. So let $\mathbf{A}$ be an abelian category (e.g. presheaves, modules over any ring, ...).

Definition 2.4.1. A **complex** $E^\bullet$ is a sequence $E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \longrightarrow \cdots$ of morphisms in **A** such that $d_i \circ d_{i-1} = 0$, for each $i \geq 1$.

Definition 2.4.2. A **morphism of complexes** $f^\bullet \colon E^\bullet \to E'^\bullet$ is a sequence of morphisms $f^i \colon E^i \to E'^i$, such that the following diagram commutes:

$$\begin{array}{ccccccc} E^0 & \xrightarrow{d_0} & E^1 & \xrightarrow{d_1} & E^2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\ E'^0 & \xrightarrow{d'_0} & E'^1 & \xrightarrow{d'_1} & E'^2 & \longrightarrow & \cdots \end{array}$$

Definition 2.4.3. The **cohomology objects** (or also called cohomology groups), written $H^n(E^\bullet)$, are defined as $\ker d_n / \operatorname{im} d_{n-1}$ (note that $d_n \circ d_{n-1} = 0$ means that $\operatorname{im} d_{n-1}$ embeds in $\ker d_n$, and $H^n(E^\bullet)$ is the cokernel of this map).

The cohomology groups are functorial. That is, if $f^\bullet \colon E^\bullet \to E'^\bullet$ is a morphism of complexes, then for each $n \geq 0$ we have morphisms:

$$H^n(f^\bullet) \colon H^n(E^\bullet) \to H^n(E'^\bullet)$$

Moreover,if $0 \to E'^\bullet \to E^\bullet \to E''^\bullet \to 0$ is an exact sequence of complexes, then we get a long exact sequence in cohomology:

$$\begin{array}{l} 0 \longrightarrow H^0(E'^\bullet) \longrightarrow H^0(E^\bullet) \longrightarrow H^0(E''^\bullet) \xrightarrow{\delta_0} \\ \longrightarrow H^1(E'^\bullet) \longrightarrow H^1(E^\bullet) \longrightarrow H^1(E''^\bullet) \xrightarrow{\delta_1} \\ \longrightarrow H^2(E'^\bullet) \longrightarrow H^2(E^\bullet) \longrightarrow H^2(E''^\bullet) \longrightarrow \cdots \end{array}$$

The morphisms δ_i are called connecting homomorphisms, and their existence is proved using the snake lemma.

Consider now a pair of morphism of complexes $f^\bullet, g^\bullet \colon E^\bullet \to E'^\bullet$.

Definition 2.4.4. We say that $f^\bullet$ and $g^\bullet$ are **homotopic** if there exists a sequence of morphisms $h_n \colon E^n \to E'^{n-1}$, for each $n \geq 1$, such that

$$f_n - g_n = d'_{n-1} \circ h_n + h_{n+1} \circ d_n$$

.

The importance of this definition is seen in the following easy result, whose proof we leave as an easy exercise in tracking definitions.

Lemma 2.4.5. *If $f^\bullet$ and $g^\bullet$ are homotopic, then $H^n(f^\bullet) = H^n(g^\bullet)$, for each $n \geq 0$.*

We finally define the objects of study of this section. Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories. Recall that a functor $F\colon \mathbf{A} \to \mathbf{B}$ is called **additive** if the natural map $\mathrm{Hom}_{\mathbf{A}}(M, N) \to \mathrm{Hom}_{\mathbf{B}}(FM, FN)$ is a group homomorphism.

Definition 2.4.6. A **cohomological δ-functor** is a sequence of additive functors $\{F^n\}_{n\geq 0}$, with $F^n\colon \mathbf{A} \to \mathbf{B}$ such that:

i) For every exact sequence $- \to M' \to M \to M'' \to 0$ of objects in $\mathbf{A}$ we have a sequence of morphisms $\delta_n\colon F^n(M'') \to F^{n+1}(M')$, for $n \geq 0$ such that the resulting sequence is exact:

$$\begin{array}{l} 0 \longrightarrow F^0(M') \longrightarrow F^0(M) \longrightarrow F^0(M'') \xrightarrow{\delta_0} \\ \quad\; F^1(M') \longrightarrow F^1(M) \longrightarrow F^1(M'') \xrightarrow{\delta_1} \\ \quad\; F^2(M') \longrightarrow F^2(M) \longrightarrow F^2(M'') \longrightarrow \cdots \end{array}$$

ii) For any given commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

then the following squares commute:

$$\begin{array}{ccc} F^n(M'') & \xrightarrow{\delta_n} & F^{n+1}(M') \\ \downarrow{\scriptstyle F^n(\gamma)} & & \downarrow{\scriptstyle F^{n+1}(\alpha)} \\ F^n(N'') & \xrightarrow{\delta_n} & F^{n+1}(N') \end{array}$$

Of course, this definition has its motivation in the example of the cohomology functors H^n.

Definition 2.4.7. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor. We say that F is **effaçable** if for every object M of $\mathbf{A}$ there exists an exact sequence $0 \to M \xrightarrow{u} A$ such that $F(u) = 0$.

Definition 2.4.8. A cohomological δ-functor is called **effaçable** if F^n is so, for each $n \geq 1$ (note that we don't require F^0 to be effaçable).

Lemma 2.4.9. *Let $\{F^n\}_{n\geq 0}$ and $\{G^n\}_{n\geq 0}$ be cohomological δ-functors such that $\{F^n\}_{n\geq 0}$ is effaçable. Suppose also that there exists a morphism of functors $f_0\colon F^0 \to G^0$. Then there exists a* unique *sequence of morphisms $f_n\colon F^n \to G^n$ which commutes with the corresponding δ morphisms.*

Proof. Let M be an object of $\mathbf{A}$. We want to construct a sequence $f_{n,M}\colon F^n(M) \to G^n(M)$ in a natural way. Consider the exact sequence $0 \to M \xrightarrow{u} A \to P \to 0$, where P is the cokernel of u, and such that $F(u) = 0$.

We get then a long exact sequence:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & F^0(A) & \xrightarrow{\alpha} & F^0(P) & \longrightarrow & F^1(M) & \xrightarrow{F^1(u)=0} & F^1(A) & \longrightarrow \cdots \\
& \downarrow f^0_A & & \downarrow f^0_P & & \downarrow f^1_M & & & \\
\cdots \longrightarrow & G^0(A) & \xrightarrow{\beta} & G^0(P) & \longrightarrow & G^1(M) & \longrightarrow & G^1(A) & \longrightarrow \cdots
\end{array}$$

To construct f^1_M, we observe that $F^1(M) \simeq \operatorname{coker}(\alpha)$, which maps to $\operatorname{coker}(\beta)$, and this last object maps canonically into $G^1(M)$. Composing, we get the desired map.

The remaining morphisms are constructed inductively in the same way. See [8] for more details. □

The following is an immediate corollary of the previous lemma.

Corollary 2.4.10. *If $\{F^n\}_{n\geq 0}$ and $\{G^n\}_{n\geq 0}$ are effaçable cohomological δ-functors and $F^0 \simeq G^0$ (as functors), then there are unique isomorphisms $F^n \simeq G^n$, which commute with the δ morphisms.*

2.4.1 Right derived Functors

We will see now a method to construct effaçable δ-functors, given F^0. Recall first that we call an object I of the category $\mathbf{A}$ *injective* if the functor $\operatorname{Hom}_{\mathbf{A}}(-, I)$ is exact.

Proposition 2.4.11. *Let I be an object of $\mathbf{A}$. The following statements are equivalent:*

i) *I is injective.*

ii) *For every exact sequence $0 \to M' \xrightarrow{u} M$ of objects in $\boldsymbol{A}$ and $f\colon M' \to I$, there exists a morphism $g\colon M \to I$ such that $g \circ u = f$.*

iii) *Every short exact sequence $0 \to I \to M \to M'' \to 0$ is split.*

Proof. Exercise. □

Recall that we say that **A has enough injectives** if every object M of $\mathbf{A}$ can be embedded in an injective object.

Theorem 2.4.12. *Let $F\colon \boldsymbol{A} \to \boldsymbol{B}$ be an* additive left-exact functor. *Then there exists an effaçable cohomological δ-functor $\{R^n F\}_{n\geq 0}$ such that $R^0 F = F$. The functor $R^n F$ is called the n^{th} **right derived functor of** F.*

Remark. This construction is unique, by the previous lemma.

Proof. Let M be an object of $\mathbf{A}$. Consider an injective resolution of M, that is an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots$ in which I^n is injective for each $n \geq 0$. This is done using the property of having enough injectives: $0 \to M \xrightarrow{u} I^0$ is given by the property, and then one applies it again to $\operatorname{coker} u$, which embeds in I^1, and one keeps going indefinitely.

From this, one considers the "deleted complex" $I^\bullet$:

$$I^\bullet = 0 \to I^0 \to I^1 \to I^2 \to \cdots$$

Finally, define $(R^n F)(M) \stackrel{\text{def}}{=} H^n\left(F(I^\bullet)\right)$. One should see that this is well defined (independent of the resolution one starts with), and that it is functorial.

We use the following lemma, which can be easily proved:

Lemma 2.4.13. *Let M and M' be two objects in $\boldsymbol{A}$, and let $\varphi\colon M \to M'$ be a morphism. Let $0 \to M \to I^0 \to I^1 \to \cdots$ and $0 \to M' \to I'^0 \to I'^1 \to \cdots$. Then φ extends to a morphism of the two resolutions, and any two of them is homotopy equivalent.*

Applying the previous lemma to $\mathrm{Id}_M\colon M \to M$, one gets well-definedness of $R^n F$. Applying it to a morphism $\varphi\colon M \to M'$, one gets functoriality.

Next, one needs to see that $R^0 F = F$, which is a direct consequence of the left-exactness of F.

To see that $R^n F$ is effaçable for each $n \geq 1$, we will prove the stronger fact that, if I is injective, $(R^n F)(I) = 0$. For this, one can take the trivial injective resolution for I, which is given by $I^\bullet = 0 \to I \overset{\mathrm{Id}_I}{\to} I \to 0 \to 0 \to \cdots$. The resulting complex $F(I^\bullet)$ is still exact, so that it has 0 cohomology on positive degrees, as we wanted.

Finally, we should see that $R^n F$ is a δ-functor. So consider $0 \to M' \to M \to M'' \to 0$ be an exact sequence. We need to construct injective resolutions for each of the terms, such that they are good enough for us to work with them.

Lemma 2.4.14. *If $I'^\bullet$ is an injective resolution for M' and $I''^\bullet$ is one for M'', then $I^\bullet \overset{def}{=} I'^\bullet \oplus I''^\bullet$ is an injective resolution for M.*

This lemma can be easily proven using that F is left exact, together with the fact that short exact sequences with an injective as the first term always split.

Moreover, F will preserve exactness of split exact sequences. Therefore, the exact sequence $0 \to F(I'^\bullet) \to F(I^\bullet) \to F(I''^\bullet) \to 0$ gies a long exact sequence in cohomology, which in turn gives the δ_n's. □

Example 2.4.15. Let G be a group. Let $\mathbf{Mod}_G$ be the category of G-modules. Consider the functor $H^0(G, -)\colon \mathbf{Mod}_G \to \mathbf{AbGrp}$ defined on objects by $H^0(G, M) \overset{\text{def}}{=} M^G$ (the G-invariant elements).

Then $\mathbf{Mod}_G$ is an abelian category with enough injectives, and $H^0(G, -)$ is additive and left exact. In this case, the right derived functors $R^n H^0(G, -) \overset{\text{def}}{=} H^n(G, -)$ are the group cohomology.

If G was a topological group, then the category of discrete groups with continuous G-action might not lead to a δ-functor.

Example 2.4.16. Let X be a topological space, and let $\mathbf{A} \overset{\text{def}}{=} \mathbf{Sh}(X)$, the category of sheaves of abelian groups on X. Let $H^0(X, -)\colon \mathbf{A} \to \mathbf{AbGrp}$ be defined on objects by $H^0(X, \mathcal{F}) \overset{\text{def}}{=} \mathcal{F}(X)$. Then $\mathbf{A}$ is again an abelian category, and we get $H^n(X, \mathcal{F})$, which is the sheaf cohomology of X.

2.5 Čech Cohomology

Let T be a Grothendieck topology, $\mathcal{C} = \mathbf{Cat}\ T$, and let $\mathcal{P} = \mathcal{P}_\mathcal{C}$ be the category of presheaves on $\mathcal{C}$. Let V be an object of $\mathcal{C}$, and let $\{U_\alpha \to V\}_{\alpha \in I}$ be in $\mathbf{Cov}\ T$.

We have a diagram of maps:

$$V \longleftarrow \{U_\alpha\} \;\overset{\hat{1}}{\underset{\hat{0}}{\longleftarrow\!\longleftarrow}}\; \{U_\alpha \times_V U_\beta\} \;\overset{\hat{2},\,\hat{1}}{\underset{\hat{0}}{\longleftarrow\!\longleftarrow\!\longleftarrow}}\; \{U_\alpha \times_V U_\beta \times_V U_\gamma\} \qquad \cdots$$

Applying to it a presheaf $F\colon \mathcal{C} \to \mathbf{AbGrp}$, we get:

$$\prod_\alpha F(U_\alpha) \longrightarrow \prod_{\alpha,\beta} F(U_\alpha \times_V U_\beta) \underset{F(\hat{1})}{\overset{F(\hat{0})}{\rightrightarrows}} \prod_{\alpha,\beta,\gamma} F(U_\alpha \times_V U_\beta \times_V U_\gamma) \overset{F(\hat{0}),\,F(\hat{1}),\,F(\hat{2})}{\rightrightarrows} \cdots$$

We get a complex $C^\bullet$, which is called the **Čech complex** attached to the pair $(\{U_\alpha \to V\}, F)$, by defining $d_n \overset{\text{def}}{=} \sum_{i=0}^{n+1} (-1)^i F(\hat{i})$:

$$d_n \colon \prod_{\alpha_0,\ldots,\alpha_n} F(U_{\alpha_0} \times \cdots \times U_{\alpha_n}) \to \prod_{\alpha_0,\ldots,\alpha_{n+1}} F(U_{\alpha_0} \times \cdots \times U_{\alpha_{n+1}})$$

$$C^\bullet : \qquad \prod_\alpha F(U_\alpha) \xrightarrow{d_0} \prod_{\alpha,\beta} F(U_\alpha \times_V U_\beta) \xrightarrow{d_1} \prod_{\alpha,\beta,\gamma} F(U_\alpha \times_V U_\beta \times_V U_\gamma) \xrightarrow{d_2} \cdots$$

It is a routine check to see that $d_{n+1} \circ d_n = 0$. From this, one defines the **Čech cohomology groups**:

$$\check{H}^i(\{U_\alpha \to V\}, F) \overset{\text{def}}{=} H^i(C^\bullet) = \ker d_i / \operatorname{im} d_{i-1}$$

Theorem 2.5.1. *The functor $\check{H}^i(\{U_\alpha \to V\}, -)$ is effaçable, for $i \geq 1$. This implies, as we have seen, that $\check{H}^0$ determines everything, by using its right-derived functors.*

Proof. Let F be a presheaf. Then, there exists an injective presheaf I and $F \hookrightarrow I$. We will actually show a stronger statement than needed. Namely, we will see that $\check{H}^i(\{U_\alpha \to V\}, I) = 0$ for all $i \geq 1$.

Consider the complex:

$$\prod_\alpha I(U_\alpha) \overset{d_0}{\to} \prod_{\alpha,\beta} I(U_\alpha \times U_\beta) \overset{d_1}{\to} \prod_{\alpha,\beta,\gamma} I(U_\alpha \times U_\beta \times U_\gamma) \to \cdots \tag{2.1}$$

Recall that, for X an object in the category $\mathcal{C}$, we had defined a functor $\{X\} \overset{i}{\hookrightarrow} \mathcal{C}$ giving a push-forward $i_* \colon \mathcal{P} \to \mathbf{AbGrp}$, and that i_* has a left-adfoint $i^* \colon \mathbf{AbGrp} \to \mathcal{P}$. Define, as we did before, $\zeta_X \overset{\text{def}}{=} i^*(\mathbb{Z})$, an object in $\mathcal{P}$. Then one has:

$$\operatorname{Hom}_{\mathcal{P}}(\zeta_X, I) \simeq \operatorname{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}, I(X)) \simeq I(X)$$

Moreover, $\zeta_X(Y) = \bigoplus_{\operatorname{Hom}(Y,X)} \mathbb{Z}$. Then the sequence (2.1) becomes:

$$\prod_\alpha \operatorname{Hom}_{\mathcal{P}}(\zeta_{U_\alpha}, I) \overset{d_0}{\to} \prod_{\alpha,\beta} \operatorname{Hom}_{\mathcal{P}}(\zeta_{U_\alpha \times U_\beta}, I) \overset{d_1}{\to} \prod_{\alpha,\beta,\gamma} \operatorname{Hom}_{\mathcal{P}}(\zeta_{U_\alpha \times U_\beta \times U_\gamma}, I) \to \cdots$$

which in turn is isomorphic to:

$$\mathrm{Hom}_{\mathcal{P}}\left(\bigoplus_{\alpha}\zeta_{U_\alpha}, I\right) \to \mathrm{Hom}_{\mathcal{P}}\left(\bigoplus_{\alpha,\beta}\zeta_{U_\alpha\times U_\beta}, I\right) \to \mathrm{Hom}_{\mathcal{P}}\left(\bigoplus_{\alpha,\beta,\gamma}\zeta_{U_\alpha\times U_\beta\times U_\gamma}, I\right) \to \cdots$$

So to prove that the first complex (2.1) is exact, it is enough to show that the following is exact (as I is injective):

$$\bigoplus_\alpha \zeta_{U_\alpha} \longleftarrow \bigoplus_{\alpha,\beta}\zeta_{U_\alpha\times U_\beta} \longleftarrow \bigoplus_{\alpha,\beta,\gamma}\zeta_{U_\alpha\times U_\beta\times U_\gamma} \longleftarrow \cdots$$

So let Y be an object in $\mathcal{C}$. Consider the resulting sequence:

$$\bigoplus_\alpha \zeta_{U_\alpha}(Y) \longleftarrow \bigoplus_{\alpha,\beta}\zeta_{U_\alpha\times U_\beta}(Y) \longleftarrow \bigoplus_{\alpha,\beta,\gamma}\zeta_{U_\alpha\times U_\beta\times U_\gamma}(Y) \longleftarrow \cdots$$

which comes from the sequence:

$$\coprod_\alpha \mathrm{Hom}(Y, U_\alpha) \leftleftarrows \coprod_{\alpha,\beta}\mathrm{Hom}(Y, U_\alpha\times U_\beta) \leftleftarrows\!\!\!\leftarrow \coprod_{\alpha,\beta,\gamma}\mathrm{Hom}(Y, U_\alpha\times U_\beta\times U_\gamma) \leftleftarrows\!\!\!\leftleftarrows \cdots$$

Fix now $\phi\colon Y \to V$, and let $S(\phi) \stackrel{\text{def}}{=} \{f_\alpha\colon Y \to U_\alpha \text{ such that } \phi_\alpha \circ f_\alpha = \phi\}$.

Then this diagram becomes:

$$\mathbb{Z}^{S(\phi)} \longleftarrow \mathbb{Z}^{S(\phi)\times S(\phi)} \longleftarrow \mathbb{Z}^{S(\phi)\times S(\phi)\times S(\phi)} \longleftarrow \cdots$$

which is exact, as we wanted. □

Fix a Grothendieck topology T, with $\mathcal{C} = \mathbf{Cat}\ T$ and P a presheaf of abelian groups on T. Fix $U \in \mathcal{C}$ and two coverings $\{U_\alpha \to U\}_{\alpha\in I}$, $\{V_\nu \to U\}_{\nu\in J}$. A **morphism of coverings**:

$$f\colon \{U_\alpha \to U\}_{\alpha\in I} \to \{V_\nu \to U\}_{\nu\in J}$$

consists of the data $f = (\varepsilon, (f_\alpha)_{\alpha\in I})$ where $\varepsilon\colon I \to J$ is a map of sets, and $f_\alpha\colon U_\alpha \to V_{\varepsilon(\alpha)}$ is a morphism such that:

$$\begin{array}{ccc} U_\alpha & \longrightarrow & V_{\varepsilon(\alpha)} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

commutes.

If f is a morphism of coverings, then one obtains maps:

$$\begin{array}{ccccccc}
\{U_\alpha\} & \leftleftarrows & \{U_\alpha \times_U U_\beta\} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & \{U_\alpha \times_U U_\beta \times_U U_\gamma\} & & \cdots \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f\times f\times f} & & \\
\{V_\nu\} & \leftleftarrows & \{V_\nu \times_U V_{\nu'}\} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & \{V_\nu \times_U V_{\nu'} \times_U V_{\nu''}\} & & \cdots
\end{array}$$

Applying P yields a morphism of Čech complexes. If $f^* = P(f)$, then one obtains maps:

$$f^i\colon \check{H}^i(\{U_\alpha \to U\}, P) \to \check{H}^i(\{V_\nu \to U\}, P).$$

We say that these maps are "induced by the refinement" f. One can prove the following:

Proposition 2.5.2. *With notation as above, if $f, g\colon \{U_\alpha \to U\}_{\alpha\in I} \to \{V_\nu \to U\}_{\nu\in J}$ are two morphisms of coverings, then f^* and g^* are homotopic maps of Čech complexes. This implies, in particular, that $f^i = g^i$ for all i.*

2.6 Sheafification

Fix a Grothendieck topology T with $\mathcal{C} = \mathbf{Cat}\ T$. Let $\mathcal{P}$ denote the category of presheaves on T and let $\mathcal{S}$ denote the category of sheaves. Let $\iota\colon \mathcal{S} \to \mathcal{P}$ denote the natural fully-faithful inclusion functor (so morphisms of sheaves are just morphisms of presheaves). This section is devoted to proving the following crucial:

Theorem 2.6.1. *There exists a left adjoint $\sharp\colon \mathcal{P} \to \mathcal{S}$ to ι. This is called the **sheafification functor**.*

Remarks.

i) If P is a presheaf and S a sheaf, then:

$$\mathrm{Hom}_{\mathcal{S}}(P^\sharp, S) \simeq \mathrm{Hom}_{\mathcal{P}}(P, \iota(S)).$$

If we take $S = P^\sharp$, then we see that there exists a canonical morphism $\alpha\colon P \to \iota(P^\sharp)$ corresponding to the identity $P^\sharp \to P^\sharp$.

ii) $P^\sharp$ has the following universal property: for every sheaf S and morphism $f\colon P \to \iota(S)$, there exists a unique morphism $g\colon P^\sharp \to S$ such that $f = g \circ \alpha$.

Proof. Our proof begins by defining a functor $+\colon \mathcal{P} \to \mathcal{P}$. Fix $U \in \mathcal{C}$ and let J_U be the category of coverings of U with morphisms as defined in the previous section. If $P \in \mathcal{P}$, then we obtain a functor:

$$P_U\colon J_U \to \mathbf{AbGrp}$$

by putting:

$$P_U(\{U_\alpha \to U\}) = \check{H}^0(\{U_\alpha \to U\}, P).$$

We define $+$ using the limit of this functor:

$$P^+(U) = \varinjlim_{J_U} P_U.$$

Claim. *P^+ is a presheaf.*

We must describe how P^+ acts on morphism $\phi\colon V \to U$ in $\mathcal{C}$. We begin by describing a functor $J(\phi)\colon J_U \to J_V$. Given a covering $\{U_\alpha \to U\}$, $J(\phi)$ maps it to:

$$\{U_\alpha \times_U V \to V\}$$

This can be extended to morphisms of coverings. We thus have a diagram of functors:

$$\begin{array}{ccc} J_U & \xrightarrow{J(\phi)} & J_V \\ {\scriptstyle P_U}\downarrow & & \downarrow{\scriptstyle P_V} \\ \mathbf{AbGrp} & = & \mathbf{AbGrp} \end{array}$$

and maps:

$$\check{H}^0(\{U_\alpha \to U\}, P) \to \check{H}^0(\{U_\alpha \times_U V \to V\}, P).$$

These induce map between the direct limits, which we define to be:

$$\phi^+\colon P^+(U) \to P^+(V).$$

One checks that this actually defines a contravariant functor $P \to P^+$, and hence that $+$ is a presheaf.

If $P \in \mathcal{P}$ and $U \in \mathcal{C}$ then there is a map $P(U) \to P^+(U)$. Indeed, given a covering $\{U_\alpha \to U\}$, one obtains a sequence:

$$P(U) \to \prod_\alpha P(U_\alpha) \to \prod_{\alpha,\beta} P(U_\alpha \times_U U_\beta)$$

These two maps always compose to the zero map, as is seen by applying P to the appropriate fibre product diagrams. Since $\check{H}^0$ is defined as the kernel of the second map, we hence obtain maps:

$$P(U) \to \check{H}^0(\{U_\alpha \to U\}, P)$$

for every covering. Taking the direct limit of these maps gives $P(U) \to P^+(U)$. One can check that this actually defines a morphism of presheaves $P \to P^+$. If $P = S$ is actually a sheaf, then the sequence:

$$0 \to S(U) \to \prod_\alpha S(U_\alpha) \to \prod_{\alpha,\beta} S(U_\alpha \times_U U_\beta)$$

is actually exact. In this case $S(U) \to \check{H}^0(\{U_\alpha \to U\}, S)$ is an isomorphism for each covering, and hence $S \simeq S^+$.

Technical remark: The category J_U is not a nice category for taking inductive limits. One can replace J_U by J'_U, defined in the following way: objects of J'_U are still coverings of U in T. We restrict morphisms in the following way:

$$\mathrm{Hom}_{J'_U}(\{U_\alpha \to U\}, \{V_\nu \to U\}) = \begin{cases} \text{one morphism if there is a map between the coverings,} \\ \emptyset \text{ otherwise.} \end{cases}$$

The reason that we can restrict morphisms like this is essentially because any two morphisms of coverings (or refinements) induce the same maps on cohomology (see the remarks at the end of the previous section). Now J'_U is just a partially ordered set via the morphisms and, moreover, it satisfies axiom L_3 from the section on inductive limits above. For if $U_\alpha \to U$ and $V_\nu \to U$ are coverings, then:

$$\{U_\alpha \times_U V_\nu \to U\}$$

is a covering that is larger than both.

Returning to the proof, if P is any presheaf and S a sheaf such that $f\colon P \to S$, then we have a commutative diagram:

$$\begin{array}{ccc} P & \longrightarrow & P^+ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^+} \\ S & \longrightarrow & S^+ \end{array}$$

We have seen that the bottom arrow is an isomorphism. We thus see that f factors through $P \to P^+$. If P^+ were a sheaf, then this would be one half of the desired universal property for $\sharp$ (we have not shown uniqueness of the map $P^+ \to S$). Although it need not be true that P^+ is a sheaf, we claim that:

Claim. *For every presheaf P, P^{++} is a sheaf.*

With this claim, we can hence put $P^\sharp = P^{++}$. One can then show, using the diagram above, that $P^\sharp$ satisfies the desired universal property. We will prove the claim in a somewhat roundabout manner. First, say that a presheaf *has property* (+) if for every $U \in \mathcal{C}$ and covering $\{U_\alpha \to U\}$, the map:

$$P(U) \to \prod_\alpha P(U_\alpha)$$

is injective. With this terminology, we can prove the claim in two steps:

(i) If P is a presheaf, then P^+ has property (+).

(ii) If P has property (+), then P^+ is a sheaf.

For (i), let $U \in \mathcal{C}$ and let $\{U_\alpha \to U\}$ be a covering. Take $\overline{\xi}_1\overline{\xi}_2 \in P^+(U)$, where we put overlines to emphasize that $P^+(U)$ is a direct limit. Suppose that the restrictions $\overline{\xi}_1|_{U_\alpha} = \overline{\xi}_2|_{U_\alpha}$ are equal for all α. We want to show that $\overline{\xi}_1 = \overline{\xi}_2$.

Let ξ_1, ξ_2 represent $\overline{\xi}_1, \overline{\xi}_2$; by this we mean, find a covering $\{V_\nu \to U\}$ and take:

$$\xi_1, \xi_2 \in \ker\left(\prod_\nu P(V_\nu) \to \prod_{\nu,\nu'} P(V_\nu \times_U V_{\nu'}\right)$$

mapping to $\overline{\xi}_1$ and $\overline{\xi}_2$ in the direct limit. Then the images of ξ_1, ξ_2 in $P^+(U_\alpha)$ are represented by:

$$\xi_{1,\alpha}, \xi_{2,\alpha} \in \ker\left(\prod_\nu P(V_\nu \times_U U_\alpha) \to \prod_{\nu,\nu'} P(V_\nu \times_U V_{\nu'} \times_U U_\alpha\right)$$

There exists a finer covering of U_α, $\{W_{\alpha,\mu} \to U_\alpha\}$, such that the images of $\xi_{1,\alpha}$ and $\xi_{2,\alpha}$ in $\prod_\mu P(W_{\alpha,\mu})$ are the same. Then:

$$\{W_{\alpha,\mu} \to U_\alpha \to U\}_{\alpha,\mu}$$

is a covering of U. Now the families $\{\xi_{1,\alpha,\mu}\}$ and $\{\xi_{2,\alpha,\mu}\}$ represent $\overline{\xi}_1$ and $\overline{\xi}_2$ in $\prod_{\alpha,mu} P(W_{\alpha,\mu})$. But we have chosen this covering so that the two families are equal in this product. This implies that $\overline{\xi}_1$ and $\overline{\xi}_2$ are equal in the direct limit $P^+(U)$. This concludes the proof of (i).

We now turn to proving (ii). For this, we first show that if P has property (+) and $f\colon \{V_\nu \to U\} \to \{U_\alpha \to U\}$ is a morphism of coverings, then $f^*\colon \check{H}^0(\{U_\alpha \to$

$U\}, P) \to \check{H}^0(\{V_\nu \to U\}, P)$ is injective. For each α, $\{V_\nu \times_U U_\alpha \to U_\alpha\}$ is a covering. We deduce that:

$$P(U_\alpha) \to \prod_\nu P(V_\nu \times_U U_\alpha)$$

is injective, since P has property (+). But then it follows that the product of these maps is injective:

$$0 \to \prod_\alpha P(U_\alpha) \to \prod_{\alpha,\nu} P(V_\nu \times_U U_\alpha).$$

One argues similarly that the natural diagram:

$$0 \to \prod_{\alpha,\beta} P(U_\alpha \times_U U_\beta) \to \prod_{\alpha,\beta,\nu,\nu'} P(V_\nu \times_U V_{\nu'} \times_U U_\alpha \times_U U_\beta)$$

is exact. We thus have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \check{H}^0(\{U_\alpha \to U\}, P) & \longrightarrow & \prod_\alpha P(U_\alpha) & \longrightarrow & \prod_{\alpha,\nu} P(V_\nu \times_U U_\alpha) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \check{H}^0(\{U_\alpha \times_U V_\nu \to U\}, P) & \longrightarrow & \prod_{\alpha,\beta} P(U_\alpha \times_U U_\beta) & \longrightarrow & \prod_{\alpha,\beta,\nu,\nu'} P(V_\nu \times_U V_{\nu'} \times_U U_\alpha \times_U U_\beta) \end{array}$$

such that the rightmost two vertical arrows are injective. One deduces, for example by the snake lemma, that the first vertical map is also injective. But, by functoriality, this map is just $p^* \circ f^*$, where p^* is induced by the projections:

$$V_\nu \times_U U_\alpha \to V_\nu.$$

So f^* must itself be injective, as was claimed above.

We now want to show that P^+ is a sheaf whenever P has property (+). This will follow if we can show that for each covering $\{U_\alpha \to U\}$ and collection of $\overline{\xi}_\alpha \in P^+(U_\alpha)$ satisfying:

$$\overline{\xi}_\alpha|_{U_\alpha \times_U U_\beta} = \overline{\xi}_\beta|_{U_\alpha \times_U U_\beta},$$

then there exists $\overline{\zeta} \in P^+(U)$ such that $\overline{\zeta}_\alpha = \overline{\xi}|_{U_\alpha}$.

We will only give a proof in the case of a covering by two sets $\{U_1, U_2 \to U\}$. We thus have $\overline{\xi}_i \in P^+(U_i)$ for $i = 1, 2$ such that $\overline{\xi}_1|_{U_1 \times_U U_2} = \overline{\xi}_2|_{U_1 \times_U U_2}$. We want to show that there exists $\alpha \in P^+(U)$ such that $\alpha|_{U_i} = \overline{\xi}_i$.

These $\overline{\xi}_i$ are represented by

$$\xi_i \in \check{H}^0(\{V_{\alpha,i} \to U_i\}, P) \in \prod_\alpha P(V_{\alpha,i}),$$

for two coverings $\{V_{\alpha,i} \to U_i\}$, $i = 1, 2$. The restrictions $\xi_i|_{U_1 \times_U U_2}$ are the image of ξ_i in:

$$\check{H}^0(\{V_{\alpha,i} \times_U U_j \to U_1 \times_U U_2\}, P).$$

where $j \in \{1, 2\}$ is taken $j \neq i$. We must take a refinement of these two covers so that we can compare the restrictions of the ξ_i.

The two $\xi_i|_{U_1 \times_U U_2}$ are represented by the same element in $P^+(U_1 \times_U U_2)$. This implies that there exists a covering $\{W_\gamma \to U_1 \times_U U_2\}$ which is a common refinement of:

$$\{V_{\alpha,1} \times_U U_2 \to U_1 \times_U U_2\}$$

and

$$\{U_1 \times_U V_{\alpha,2} \to U_1 \times_U U_2\},$$

such that the images of the ξ_i in:

$$\check{H}^0(\{W_\gamma \to U_1 \times_U U_2\}, P)$$

are equal.

However, P has (+) and so, by what was proved above, ξ_1 and ξ_2 have the same image in:

$$\check{H}^0(\{W'_{\gamma'} \to U_1 \times_U U_2\}, P)$$

where $\{W'_{\gamma'} \to U_1 \times_U U_2\}$ is any common refinement of

$$\{V_{\alpha,1} \times_U U_2 \to U_1 \times_U U_2\}$$

and

$$\{U_1 \times_U V_{\alpha,2} \to U_1 \times_U U_2\}.$$

This is true, in particular, for:

$$\{V_{\alpha,1} \times_U V_{\beta,2} \to U_1 \times_U U_2\}.$$

Thus, ξ_1 and ξ_2 have the same image in:

$$\check{H}^0(\{V_{\alpha,1} \times_U V_{\beta,2} \to U_1 \times_U U_2\}), P),$$

and hence also in:

$$\prod_{\alpha,\beta} P(V_{\alpha,1} \times_U V_{\beta,2}).$$

This shows that (ξ_1, ξ_2) are in the kernel of:

$$\prod_\alpha P(V_{\alpha,1}) \times \prod_\beta P(V_{\beta,2}) \to \prod_{\alpha,\beta} P(V_{\alpha,1} \times_U V_{\beta,2}).$$

So by definition, $(\xi_1, \xi_2) \in \check{H}^0(\{V_{\alpha,i} \to U\}_{\alpha,i}, P)$ and this maps into the direct limit $P^+(U)$ of the cohomology groups. Call the image α. Then α satisfies $\alpha|_{U_i} = \overline{\xi}_i$. This concludes the proof of our claim in the case of a covering by two sets. The general case is similar, but requires much more burdensome notation. The general case implies (ii) above, and hence completes the proof. □

Given a presheaf P on T, we call $P^\sharp$ the **associated sheaf**, or the **sheafification** of P. We end this section with a discussion of another crucial result:

Theorem 2.6.2. *a) $\mathcal{S}$ is an abelian category.*
b) $\mathcal{S}$ has enough injectives.
c) ι is left exact and $\sharp$ is exact.

We will not prove this theorem fully, but will explain how to define kernels, cokernels and images in $\mathcal{S}$. Let F, G be sheaves. A morphism between them is a morphism as presheaves, and so $\mathrm{Hom}_{\mathcal{S}}(F, G)$ has a natural structure of abelian group. Let $f\colon F \to G$ be a morphism of sheaves. We let $\ker f$ be the presheaf kernel of f defined previously. A priori this is just a presheaf, but we claim that it is in fact a sheaf. Let $K = \ker f$, let $U \in \mathcal{C}$ and let $\{U_\alpha \to U\}_{\alpha\in I}$ be a covering of U. Then one has the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & K(U) & \longrightarrow & \prod_{\alpha\in I} K(U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta} K(U_\alpha \times_U U_\beta) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & F(U) & \longrightarrow & \prod_{\alpha\in I} F(U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & G(U) & \longrightarrow & \prod_{\alpha\in I} G(U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta} G(U_\alpha \times_U U_\beta)
\end{array}$$

The bottom two rows are actually exact, since F and G are sheaves. One concludes that the top row is exact by the snake lemma. This shows that the presheaf kernel is actually a sheaf.

It remains to argue that K is actually a kernel for f in $\mathcal{S}$. This amounts to showing that the sequence:

$$0 \to \mathrm{Hom}_{\mathcal{S}}(X, K) \to \mathrm{Hom}_{\mathcal{S}}(X, F) \to \mathrm{Hom}_{\mathcal{S}}(X, G)$$

induced by the maps:

$$K \to F \xrightarrow{f} G$$

is actually exact, for every sheaf X. Note that since X is a sheaf, $X \simeq \iota(X)^\sharp$. We can thus employ the adjointness property of ι and $\sharp$ to obtain an equivalent sequence:

$$0 \to \mathrm{Hom}_{\mathcal{P}}(\iota(X), \iota(K)) \to \mathrm{Hom}_{\mathcal{P}}(\iota(X), \iota(F)) \to \mathrm{Hom}_{\mathcal{P}}(\iota(X), \iota(G))$$

Now since $\iota(K)$ is the kernel of $\iota(f)$ in $\mathcal{P}$, this last sequence is exact. It follows that the first one from this paragraph is as well, and hence that K is a kernel for f in $\mathcal{S}$.

We turn to the subject of defining $\operatorname{coker} f$. In general, the presheaf cokernel $C = \operatorname{coker} \iota(f)$ given by:

$$C(U) = G(U)/\mathrm{im}(f_U)$$

is not a sheaf. We thus sheafify:

$$\operatorname{coker} f = C^\sharp = (\operatorname{coker} \iota(f))^\sharp .$$

There is a map of $\operatorname{coker} f$ obtained from the natural map $G \to C$ followed by the canonical map $C \to C^\sharp$.

We will now argue that $G \to C^\sharp$ as defined is a cokernel for f. This amounts to showing that for each sheaf $X \in \mathcal{S}$, the sequence:

$$0 \to \mathrm{Hom}_{\mathcal{S}}(C^\sharp, X) \to \mathrm{Hom}_{\mathcal{S}}(G, X) \to \mathrm{Hom}_{\mathcal{S}}(F, X)$$

is exact. But again, $G = (\iota G)^\sharp$ since G is a sheaf, and similarly for F. Applying the adjunction property yields the equivalent sequence:

$$0 \to \mathrm{Hom}_{\mathcal{P}}(C, \iota(X)) \to \mathrm{Hom}_{\mathcal{P}}(\iota(G), \iota(X)) \to \mathrm{Hom}_{\mathcal{P}}(\iota(F), \iota(X))$$

This is exact since $C = \operatorname{coker} \iota(f)$. So $C^\sharp$ is a cokernel for f, as claimed. We leave the remaining details of the theorem above to the reader. We end this section by recalling some definitions pertaining to abelian categories:

Definition 2.6.3. Given a morphism of sheaves $f\colon F \to G$, define the **image** of f:

$$\mathrm{im}(f) = \ker (G \to \operatorname{coker} f) .$$

A sequence of sheaves:

$$F \xrightarrow{f} G \xrightarrow{g} H$$

is said to be **exact** if $\ker g = \mathrm{im}\, f$.

Remark. Given a presheaf F and an object $U \in \mathcal{C}$, how should one think about sections $s \in F^+(U)$? Such a section is represented by pairs:

$$(\{U_\alpha \to U\}_{\alpha \in I}, (s_\alpha)_{\alpha \in I})$$

consisting of a covering and sections $s_\alpha \in F(U_\alpha)$, compatible on overlaps:

$$s_\alpha|_{U_\alpha \times_U U_\beta} = s_\beta|_{U_\alpha \times_U U_\beta}$$

for all $\alpha, \beta \in I$. Two representatives:

$$(\{U_\alpha \to U\}_{\alpha \in I}, (s_\alpha)_{\alpha \in I}), \quad (\{V_\eta \to U\}_{\eta \in J}, (t_\eta)_{\eta \in J})$$

are equal in $F^+(U)$ if and only if there exists a common refinement:

$$\{W_\nu \to U\}_{\nu \in A}$$

of the two coverings, corresponding to index set maps:

$$\begin{array}{ccc} & A & \\ {\scriptstyle \varepsilon} \swarrow & & \searrow {\scriptstyle \delta} \\ I & & J \end{array}$$

such that for every $\nu \in A$,

$$s_{\varepsilon(\nu)}|_{W_\nu} = t_{\delta(\nu)}|_{W_\nu}.$$

2.7 Sheaves on X^{et}

There are several different Grothendieck topologies on a scheme relating to the étale condition that are commonly in use. We will use one of the more common ones. Given a scheme X, we define a Grothendieck topology X^{et} in the following way: **Cat** (X^{et}) is the category of étale morphisms $f\colon U \to X$. The admissible coverings **Cov** (X^{et}) are given by the collection of finite surjective families of morphisms:

$$\left\{U_i \xrightarrow{f_i} U \mid U \to X \text{ and } U_i \to U \text{ are étale, } U = \bigcup f_i(U_i)\right\}_{i=1}^{n} \in \mathbf{Cov}\ (X^{et}).$$

Remark. Given a covering as above, we can always consider:

$$U' = \coprod_{i=1}^{n} U_i$$

with the induced surjective map $U' \to U$. This map is étale since each $U_i \to U$ is, and hence $\{U' \to U\} \in \mathbf{Cov}\ (X^{et})$. These two coverings are isomorphic, under a natural notion of isomorphism in $\mathbf{Cov}\ (X^{et})$. One can thus work solely with coverings consisting of a single morphism.

We turn to proving that an abelian group scheme (to be defined below) defines a sheaf of abelian groups on X^{et}. First let $G \to X$ be a scheme over X. Note well that we do not suppose G is étale over X. Define a set valued contravariant functor:

$$G\colon X^{et} \to \mathbf{Sets}$$

by:

$$G(U) = \mathrm{Hom}_X(U, G),$$

where we mean only to consider scheme morphisms over X. This defines a presheaf on X^{et}. In fact, one has the following:

Theorem 2.7.1. *With notation as above, G is a set-valued sheaf on X^{et}.*

Proof. The remark above implies that we need only verify the sheaf axiom for coverings of the form $\{U' \to U\}$. We will thus be concerned with the sequence:

$$0 \longrightarrow G(U) \longrightarrow G(U') \rightrightarrows G(U' \times_U U')$$

We treat the case that all schemes appearing are affine, and leave the various reductions to the reader. Let $X = \mathrm{Spec}(R), U = \mathrm{Spec}(A), U' = \mathrm{Spec}(B), G = \mathrm{Spec}(S)$. By assumption, $A \to B$ is affine and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. This implies that $f\colon A \to B$ is faithfully flat. We hence have the *descent exact sequence*:

$$0 \longrightarrow A \xrightarrow{f} B \underset{i_2}{\overset{i_1}{\rightrightarrows}} B \otimes_A B$$

where $i_1(b) = b \otimes 1$ and $i_2(b) = 1 \otimes b$.

By definition of G, we must show the exactness of:

$$0 \longrightarrow \mathrm{Hom}_{R-alg}(S, A) \longrightarrow \mathrm{Hom}_{R-alg}(S, B) \rightrightarrows \mathrm{Hom}_{R-alg}(S, B \otimes_A B).$$

Suppose that $\phi, \phi'\colon S \to A$ satisfy $f \circ \phi = f \circ \phi'$. Since f is faithfully flat, it follows that $\phi_1 = \phi_2$. This proves the injectivity of the first map above. For exactness of the last map, suppose that $\phi\colon S \to B$ satisfies $i_1 \circ \phi = i_2 \circ \phi$. Then from the descent exact sequence, for each $s \in S$ there exists a unique $a \in A$ so that $\phi(s) = f(a)$. Define a map $\psi\colon S \to A$ by putting $\psi(s) = a$. One can argue that this actually gives an R-algebra map $\psi\colon S \to A$. By construction, ψ maps to ϕ in the sequence above. This shows that the equaliser of the last map is contained in the image of second map in our sequence. The other inclusion is trivial. □

We say that G is an **abelian group scheme** if the associated functor is such that:

$$G(U) = \mathrm{Hom}_X(U, G)$$

has the structure of an abelian group for every $U \in$ **Cat** (X^{et}), and for every $f\colon U \to V$ with $U, V \in$ **Cat** (X^{et}), $G(f)$ is a group homomorphism. If one begins with a representable set-valued covariant functor $G\colon$ **Cat** $(X^{et}) \to$ **Sets** satisfying this same property, then the representing object is an abelian group scheme. With this definition, one can prove the following corollary to the theorem above:

Corollary 2.7.2. *If $G \to X$ is an abelian group scheme, then G defines a sheaf of abelian groups on X^{et}.*

We will illustrate the definition of an abelian group scheme via a series of examples.

Examples 2.7.3.

Additive group over X: $\mathbb{G}_{a,X}$

Let $\mathcal{A} = \mathcal{O}_X[T]$ denote the quasi-coherent sheaf of $\mathcal{O}_X$-algebras on X. Recall that this is defined as:

$$\mathcal{O}_X[T](U) = \mathcal{O}_X(U)[T].$$

The sheaf of $\mathcal{O}_X$-algebras defines a natural scheme:

$$Y = \mathrm{Spec}(\mathcal{A}) \to X.$$

This is defined by taking an affine cover of X and glueing together the $\mathcal{O}_X[T](U)$ for the opens U in the cover. We put:

$$\mathbb{G}_{a,X}(U) = \mathrm{Hom}_X(U, Y),$$

for $U \to X$ étale.

We claim that $\mathrm{Hom}_A(U, Y) = (\mathcal{O}_U(U), +)$ is the additive group of $\mathcal{O}_U(U)$. In the special case that $X = \mathrm{Spec}(K)$ for a field K, then $Y = \mathrm{Spec}(K[T])$. Take $U = \mathrm{Spec}(A) \to X$ étale. Then:

$$\begin{aligned} \mathbb{G}_{a,K} &= \mathrm{Hom}_X(U, Y) \\ &\simeq \mathrm{Hom}_{K-alg}(K[T], A) \\ &\simeq A, \end{aligned}$$

since K-algebra homomorphisms as above are determined by where they map T.

Multiplicative group over X: $\mathbb{G}_{m,X}$

In this case one considers the quasi-coherent sheaf $\mathcal{A}_m = \mathcal{O}_X[T, T^{-1}]$ of $\mathcal{O}_X$-algebras and puts:

$$Y_m = \operatorname{Spec}(\mathcal{O}_X[T, T^{-1}]) \to X.$$

Then:

$$\mathbb{G}_{m,X}(U) = \operatorname{Hom}_X(U, Y_m) \simeq ((\mathcal{O}_U(U))^\times, \cdot).$$

n-th roots of unity: μ_n

Given an integer $n > 0$ put:

$$\mathcal{A}_n = \mathcal{O}_X[T]/(T^n - 1)$$

and

$$Y_n = \operatorname{Spec}(\mathcal{A}_n) \to X.$$

Then put $\mu_n(U) = \operatorname{Hom}_X(U, Y_n) = (\mu_n(\mathcal{O}_U(U)), \cdot)$.

Note that this group scheme can be defined in a second, perhaps more conceptual, way. We define a map of sheaves $[n] \colon \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ in the following way: for each $U \to X$ étale, let:

$$[n] = [n]_U \colon \mathbb{G}_{m,X}(U) \to \mathbb{G}_{m,X}(U)$$

be given by $x \mapsto x^n$. Then define $\mu_n = \ker[n]$. This definition of μ_n is only applicable to étale maps $U \to X$.

Constant group scheme.

Given an abelian group M, define $Y_M = X \times M \to X$ (recall that $X \times M = \coprod_{m \in M} X$). Then define the constant sheaf $\underline{M}$ on X^{et} by:

$$\underline{M}(U) = \operatorname{Hom}_X(U, Y_M) \simeq M^{\sharp \text{ of components of } U},$$

for $U \to X$ étale.

2.8 Sheaf Cohomology

Let X be a scheme. Consider on it the Grothendieck topology $X^{\text{ét}}$, and let F be a sheaf (that is, an object of $\mathbf{Sh}(X^{\text{ét}})$).

Define first $H^0(X^{\text{ét}}, F) \stackrel{\text{def}}{=} F(X)$ (note that the one-set covering $\{X \xrightarrow{\text{Id}} X\}$ is étale).

Lemma 2.8.1. *The functor* $H^0(X^{ét}, -)\colon \boldsymbol{Sh}(X^{ét}) \to \boldsymbol{AbGrp}$ *is* left exact.

Proof. Let $0 \to F' \to F \to F''$ be an exact sequence of sheaves on $X^{\text{ét}}$. This means that $F' \simeq \ker(F \to F'')$ as presheaves as well. Hence the presheaf sequence $0 \to \iota(F') \to \iota(F) \to \iota(F'')$ is exact, and in particular, the sections sequence $0 \to \iota(F')(X) \to \iota(F)(X) \to \iota(F'')(X)$ is exact as well. But this last sequence is $0 \to F'(X) \to F(X) \to F''(X)$, and proves the lemma. □

Recall now that the category of sheaves on $X^{\text{ét}}$, which we call $\mathbf{Sh}(X^{\text{ét}})$, has enough injectives, so we can define its right derived functors:

$$H^n(X^{\text{ét}}, -) \overset{\text{def}}{=} R^n(H^0(X^{\text{ét}}, -)) = n^{\text{th}} \text{ right derived functor of } H^0(X^{\text{ét}}, -)$$

We know then that the sequence $\{H^n(X^{\text{ét}}, -)\}_{n\geq 0}$ is an effaçable cohomological δ-functor.

2.8.1 Calculating cohomology: direct and inverse images

Let $f\colon X' \to X$ be a morphism of schemes. We want to generalize the constructions of f_* and f^*, to the étale topology.

For this, consider first the functor:

$$f_t\colon (X^{\text{ét}}) \to (X'^{\text{ét}})$$

whose action on objects is: if $U \to X$ is étale, $f_t(U) \overset{\text{def}}{=} X' \times_X U \to X'$ (which is étale as well!).

Proposition 2.8.2 (Properties of f_t). *i) The functor f_t commutes with fiber products: if $U \to X$ and $V \to X$ are étale, then $f_t(U \times_X V) \simeq f_t(U) \times_{X'} f_t(V)$.*

ii) The functor f_t takes coverings to coverings: if $\{U_\alpha \to U\}_{\alpha\in I}$ is a covering in $X^{ét}$, then the family $\{f_t(U_\alpha) \to f_t(U)\}_{\alpha\in I}$ is a covering in $X'^{ét}$.

These two properties actually generalise the property of a map being continuous, but to Grothendieck topologies:

Definition 2.8.3. Given T and T' two Grothendieck topologies, and a functor $f\colon \mathcal{T} \to \mathcal{T}'$, we say that f is **continuous** if f commutes with fiber products, and if f takes coverings to coverings.

From the (continuous) functor f_t we obtain a new functor:

$$f_*\colon \mathbf{Sh}(X'^{\text{ét}}) \to \mathbf{Sh}(X^{\text{ét}})$$

by sending a sheaf F on $X'^{\text{ét}}$ to $(f_*F)(U) \stackrel{\text{def}}{=} F(f_t(U))$.

Claim. *The presheaf f_*F is a sheaf.*

Proof. As f_t and F are both functors, the composition is as well, and so f_*F is a presheaf. We need to show that it verifies the sheaf property. So let $\{U_\alpha \to U\}_{\alpha\in I} \in$ **Cov** $(X^{\text{ét}})$ be a covering, and let F be a sheaf on $X'^{\text{ét}}$. The sheaf property sequence is:

$$0 \longrightarrow F\left(f_t(U)\right) \longrightarrow \prod_{\alpha\in I} F\left(f_t(U_\alpha)\right) \rightrightarrows \prod_{\alpha,\beta\in I} F\left(f_t(U_\alpha \times_X U_\beta)\right)$$

but note that $f_t(U_\alpha \times_X U_\beta) = f_t(U_\alpha) \times_{X'} f_t(U_\beta)$, and so the result follows from observing that the family $\{f_t(U_\alpha) \to f_t(U)\}_{\alpha\in I}$ is an element of **Cov** $(X'^{\text{ét}})$, and that F is a sheaf on $X'^{\text{ét}}$. □

The functor f_* is left-exact, and has a left-adjoint, which we will denote by $f^*\colon \mathbf{Sh}(X^{\text{ét}}) \to \mathbf{Sh}(X'^{\text{ét}})$.

Remark. Consider the composition of functors:

$$\mathbf{Sh}(X'^{\text{ét}}) \xrightarrow{\ f_*\ } \mathbf{Sh}(X^{\text{ét}}) \xrightarrow{H^0(X^{\text{ét}},-)} \mathbf{AbGrp}$$

If $F \in \mathbf{Sh}(X'^{\text{ét}})$, we compute:

$$H^0(X^{\text{ét}}, f_*F) = (f_*)F(X) = F(X') = H^0(X^{\text{ét}}, F)$$

so $H^0(X^{\text{ét}}, -) \circ R^0 f_* = H^0(X^{\text{ét}}, -)$. We have then the *Leray spectral sequence*:

$$E_2^{i,j} = H^i(X^{\text{ét}}, R^j f_*F) \Longrightarrow H^{i+j}(X'^{\text{ét}}, F)$$

We should think of this as a way to obtain the cohomology of X' in terms of the cohomology of the base X and the cohomology of the fibers.

Example 2.8.4. For $i + j = 0$ this gives:

$$H^0(X'^{\text{ét}}, F) \simeq H^0(X^{\text{ét}}, R^0 f_*F)$$

which we already knew. For $i + j = 1$, we get the following exact sequence:

$$0 \to H^1(X^{\text{ét}}, R^0 f_*F) \to H^1(X'^{\text{ét}}, F) \to H^0(X^{\text{ét}}, R^1 f_*F) \to H^2(X^{\text{ét}}, R^0 f_*F) \to \cdots$$

Corollary 2.8.5 (to the remark). *If $R^i f_*F = 0$ for all $j \geq 1$, then:*

$$H^n(X'^{ét}, F) \simeq H^n(X^{ét}, f_*F)$$

2.9 Stalks on $X^{\text{ét}}$

For general Grothendieck topologies we don't have a good notion of stalks. However, for the étale topology we do have them, and they are as useful as when working with the usual Zariski topology. So let X be a scheme, and consider the étale topology on it $X^{\text{ét}}$, and the sheaves on $X^{\text{ét}}$, which we write $\mathbf{Sh}(X^{\text{ét}})$.

Definition 2.9.1. A **geometric point** $\overline{x}$ of X is a morphism of schemes:

$$\overline{x}\colon \operatorname{Spec}(K) \to X$$

where K is a separably closed field ($K = K^{\text{sep}}$).

This is the same to fixing a point $x \in X$ and an embedding of $\kappa(x)$ into some separable closure K.

Equivalently, it is the same to having a ring homomorphism $\mathcal{O}_X(X) \to K$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_X(X) & \longrightarrow & K \\ \downarrow & & \uparrow \\ \mathcal{O}_{X,x} & \twoheadrightarrow & \kappa(x) \end{array}$$

We call $x \in X$ the **support** of $\overline{x}$.

Example 2.9.2. Let $X = \operatorname{Spec} \mathbb{Z}$, $x = (p) \in \operatorname{Spec} \mathbb{Z}$, $\kappa(x) = \mathbb{F}_p \hookrightarrow \overline{\mathbb{F}}_p$.

But also one can take $x = (0) \in X$, and $\kappa(x) = \mathbb{Q}$. Then a geometric point with support x would correspond to fixing an algebraic closure of $\mathbb{Q}$.

Definition 2.9.3. Let $\overline{x} \to X$ be a geometric point of X. An **étale neighbourhood** of $\overline{x}$ is a pair $(U, \overline{u})$ where $U \to X$ is étale, and $\overline{u}\colon \overline{x} \to U$ is such that it makes the following diagram commutative:

$$\begin{array}{ccc} \overline{x} & \xrightarrow{\overline{u}} & U \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Remark. Let $\overline{x} \xrightarrow{\iota} X$ be a geometric point, and let $f\colon U \to X$ be an étale map. Let $x \in X$ be the support of $\overline{x}$, and suppose that $x \in f(U)$ (note that f being étale means that it's open, so $f(U)$ is an open in X).

Then there one can construct an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ in the following way:

As $x \in f(U)$, there is some $u \in U$ such that $f(u) = x$. Write then $\overline{x} = \operatorname{Spec} K$, and so ι is given by an inclusion $\kappa(x) \hookrightarrow K$.

As U is étale over X, then $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ is unramified, and the extension $\kappa(x) \hookrightarrow \kappa(y)$ is finite and separable. Hence there is an extension $\overline{u}\colon \kappa(u) \to K$ giving the desired geometric point in U. In fact, there are as many possible $\overline{u}$ as the degree of $\kappa(u)$ over $\kappa(x)$.

Let now F be a sheaf on $X^{\text{ét}}$, and fix $\overline{x} \to X$ a geometric point of X. We can then consider the category of étale neighbourhoods of $\overline{x}$, where a morphism $\varphi\colon (U, \overline{u}) \to (V, \overline{v})$ is defined to be a morphism $\varphi\colon U \to V$ over X such that $\varphi \circ \overline{u} = \overline{v}$.

Moreover, we can construct fibered products:

$$(U, \overline{u}) \times_{(X,\overline{x})} (V, \overline{v}) \stackrel{\text{def}}{=} (U \times_X V, \overline{u} \times \overline{v})$$

Note that the category of étale neighbourhoods of $\overline{x}$ satisfies the $(L3)$ property (it is a filtered category).

Definition 2.9.4. The **stalk** of a presheaf on $X^{\text{ét}}$, F, at $\overline{x}$ is defined as:

$$F_{\overline{x}} \stackrel{\text{def}}{=} \varinjlim_{(U,\overline{u})} F$$

Proposition 2.9.5 (properties of stalks).

i) Fix a geometrix point $\overline{x} \to X$, and let $F \in \mathcal{P}(X^{\acute{e}t})$ be a presheaf. Then the elements of $F_{\overline{x}}$ are classes of triples $(U, \overline{u}, s)$, where $(U, \overline{u})$ is an étale neighbourhood of $\overline{x}$, and $s \in F(U)$. Two triples $(U, \overline{u}, s) \equiv (V, \overline{v}, t)$ are equivalent if there exists an étale neighbourhood $(W, \overline{w})$ of $\overline{x}$ and morphisms

$$\begin{array}{ccc} (W, \overline{w}) & \longrightarrow & (U, \overline{u}) \\ \downarrow & & \\ (V, \overline{v}) & & \end{array}$$

such that $s|_W = t|_W$.

ii) If F is a presheaf on $X^{\acute{e}t}$, and $\overline{x} \to X$ is a geometric point, then:

$$P_{\overline{x}} = (P^{\#})_{\overline{x}}$$

iii) One can characterise the exacntess of sequences by looking at the stalks at all geometric points: a sequence of sheaves $F' \to F \to F''$ is exact if, and only if, the sequence of stalks $F'_{\overline{x}} \to F_{\overline{x}} \to F''_{\overline{x}}$ is exact, for all geometric points $\overline{x} \to X$.

Proof. The first assertion follows from the fact that the category over which we are taking the inductive limit satisfies property (L3).

To prove the second assertion, recall that $P^{\#} = P^{++}$, and so we only need to prove it for $+$. Consider the canonical map $l\colon P \to P^{+}$. We will see that it induces isomorphisms on the stalks: $l_{\overline{x}}\colon P_{\overline{x}} \to (P^{+})_{\overline{x}}$.

For injectivity, let $\alpha = [(U, \overline{u}, s)] \in P_{\overline{x}}$ be such that $l_{\overline{x}}(\alpha) = 0$. This means that there exists $(U', \overline{u'}) \to (U, \overline{u})$ such that $s|_{U'} = 0$ (in $P^{+}(U')$). So that there is a covering $\{U_\beta \to U'\}_{\beta \in I}$ and sections $g_\beta \in P(U_\beta)$ satisfying the glueing property, and such that $0 = s|_{U_\beta} = g_\beta$. We need to lift $\overline{u'}$ to U_{β_0}, for some β_0. So take U_{β_0} such that there exists some y_{β_0} with the property that y_{β_0} maps to x under the obvious maps. Then, by a previous remark, there will exist $\overline{u_{\beta_0}}\colon \overline{x} \to U_{\beta_0}$ making all diagrams commutative. This gives a morphism $(U_{\beta_0}, \overline{u_{\beta_0}}) \to (U, \overline{u})$ such that $s|_{U_{\beta_0}} = g_{\beta_0} = 0$, as wanted. Surjectivity is done with a similar argument, and we leave it to the reader.

Lastly, we prove the last statement of the proposition. First, if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves, note that from both the section functor and the inductive limit functor being left-exact, we obtain already exactness on the left. So we just need to prove surjectivity, and the argument is very similar to what has been done before in this proof.

We will show, to ilustrate how to do the converse, that injectivity can be detected at the stalks. Suppose that $\phi\colon F' \to F$ is such that, for all geometric points $\overline{x} \to X$, $\phi_{\overline{x}}\colon F'_{\overline{x}} \to F_{\overline{x}}$ is exact. Note that it is enough to prove that ϕ_U is injective for all U in a basis for the Zariski topology, and so we may take U to be affine (we just need that U is quasicompact). Let $x \in U$, and take $\overline{x} \overset{\text{def}}{=} \operatorname{Spec}(\kappa(x)^{\text{sep}}) \xrightarrow{\overline{u}} U \to X$, so that $\overline{x}$ becomes in this way a geometric point of X. By hypothesis then, $0 \to F'_{\overline{x}} \to F_{\overline{x}}$ is injective, and $\beta \overset{\text{def}}{=} [(U, \overline{u}, s)] \in F'_{\overline{x}}$. Also, $\phi_{\overline{x}}(\beta)$ is zero in $F_{\overline{x}}$, and so, as $\phi_{\overline{x}}$ is injective, we deduce that $\beta = 0$ in $F'_{\overline{x}}$. Hence there exists a morphism $(U_x, \overline{u_x}) \to (U, \overline{u})$ such that $s|_{U_x} = 0$. This can be done for each $x \in U$, and so we get a surjective family $\{U_x \to U\}_{x \in U}$. As U is quasicompact, we can extract a finite covering, under which $s|_{U_{x_i}} = 0$. As F' is a sheaf, we deduce that $s = 0$, as we wanted to show. □

Example 2.9.6. Consider the sheaf $\mathcal{O}_{X^{\text{ét}}}$ on $X^{\text{ét}}$ given by, if $U \to X$ is étale, then $\mathcal{O}_{X^{\text{ét}}}(U) \overset{\text{def}}{=} \mathcal{O}_U(U)$. Given $x \in X$, fix a separable closure K of $\kappa(x)$, and then we obtain a geometric point $\overline{x} = \operatorname{Spec} K \to X$. Consider the stalk at $\overline{x}$:

$$B \overset{\text{def}}{=} \mathcal{O}_{X^{\text{ét}},\overline{x}} = \varinjlim (U, \overline{u})\mathcal{O}_U(U)$$

Note that we have a map $A \to B$, since $A = \varinjlim_{U'} \mathcal{O}_X(U')$, where $U' \subseteq X$ are Zariski opens (because $U' \subseteq X$ are étale). This makes B an A-algebra. It is actually the *stric henselization* of A (note that A is a local ring) which we will see below. As we will see next $A \simeq B$ precisely when the residue field of x is separably closed.

Definition 2.9.7. A local ring is called **henselian** if Hensel's lemma hold in it. It is called **strict henselian** if it is henselian and its residue field is separably closed.

Example 2.9.8. A strict Henselization of the ring of p-adic integers is given by the maximal unramified extension, generated by all roots of unity of order prime to p. It is not "universal" as it has non-trivial automorphisms.

Proposition 2.9.9 (properties of the strict henselisation). *Let A be a local ring, and let B be a strict henselisation of A. Then:*

i) B is a local ring, and the structure morphism $A \to B$ is a local homomorphism ($\mathfrak{m}_A B \subseteq \mathfrak{m}_B$).

ii) B is unramified over A ($\mathfrak{m}_A B = \mathfrak{m}_B$) and the residue field of B is a separable closure of that of A.

iii) B can be written as $B = \varinjlim_\alpha B_\alpha$, where B_α are étale A-algebras, for each α.

iv) If $B \to B'$ is étale and $B'/(\mathfrak{m}_B B') \neq 0$ (so that $\varphi : \operatorname{Spec} B' \twoheadrightarrow \operatorname{Spec} B$ is surjective), then there is a section $s\colon \operatorname{Spec} B \to \operatorname{Spec} B'$ to φ (that is, $\varphi \circ s = Id_{\operatorname{Spec} B}$). In particular, B is a direct factor of B'.

Proof. We just sketch certain parts of the proof. First, note that one can write:

$$B = \varinjlim_{(U,\overline{u})} \mathcal{O}_U(U) = \varinjlim_{(U,\overline{u})} \varinjlim_{U' \subseteq U} \mathcal{O}_{U'}(U') = \varinjlim_{(U,\overline{u})} \mathcal{O}_{U,u}$$

where the open subsets $U' \subseteq U$ considered are those containing the image of $\overline{u}$. This shows that B is a direct limit of local rings. After checking that the transition maps are also local homomorphisms, one deduces that B is a local ring. Moreover, as $A \to \mathcal{O}_{U,u}$ is a local homomorphism, one deduces that $A \to B$ is local as well.

We don't prove unramifiedness, but note that at least $A \to \mathcal{O}_{U,u}$ is unramified. One needs to see what happens when taking the direct limits.

For the second part, let $V = \operatorname{Spec}(R) \subseteq X$, and $x \in V$. We may restrict the inductive limit only to affine subsets $U = \operatorname{Spec}(R)$. Write then:

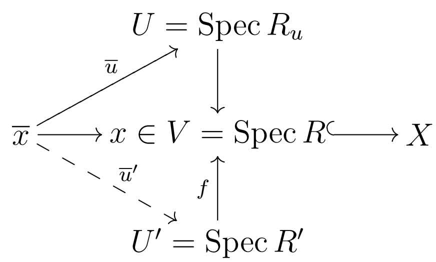
$$B = \varinjlim_{(\operatorname{Spec} R_u, \overline{u})} R_u$$

As B is an A-algebra, we have:

$$B \simeq B \otimes_R A = \left(\varinjlim_{(\operatorname{Spec} R_u, \overline{u})} R_u \right) \otimes_R A \simeq \varinjlim_{(\operatorname{Spec} R_u, \overline{u})} R_u \otimes_R A$$

As $R \to R_u$ is étale, we get that $R_u \otimes_R A$ is also étale over A (by base change) as wanted.

Lastly, given $B \to B'$ an étale B-algebra with $B'/(\mathfrak{m}_B B') \neq 0$, from what we have observed so far we deduce that there exists an étale neighbourhood of $\overline{x}$, say $(U, \overline{u})$, with $U = \operatorname{Spec}(R_u)$, such that $B' = R_u \otimes_R R'$, and with $R \to R'$ étale. We just need to find then a map $R' \to B$. For this, let $U' \stackrel{\text{def}}{=} \operatorname{Spec}(R')$.



Note that $x \in f(U')$, because $B'/\mathfrak{m}_B B' \neq 0$. Hence there exists a lift $\overline{u}' \colon \overline{x} \to U'$ making the diagram commutative. This means that $U', \overline{u}')$ is an étale neighbourhood of $\overline{x}$, and so there is a map $R' \to B$.

This allows us to define a section:

$$\begin{aligned} R' \otimes_R R_u &\to B \\ r' \otimes a &\mapsto r'a \end{aligned}$$

□

Corollary 2.9.10. *Let X be a scheme, and let $\overline{x} \to X$ be a geometric point with support $x \in X$. Let $B \stackrel{def}{=} \mathcal{O}_{X^{\acute{e}t}, \overline{x}}$ and $Y \stackrel{def}{=} \operatorname{Spec}(B)$. Then, for any sheaf F on $Y^{\acute{e}t}$,*

$$H^i(Y^{\acute{e}t}, F) = 0 \quad \textit{for all } i \geq 1$$

Theorem 2.9.11 (pull-backs and stalks). *Let $f \colon Z \to X$ be a morphism of schemes. Let $\overline{z} \to Z$ be a geometric point, and let $\overline{x} \stackrel{def}{=} f(\overline{z}) \to X$ be the corresponding geometric point on X. Let F be a sheaf on $X^{\acute{e}t}$. Then:*

$$((f^* F)_{\overline{z}} \simeq F_{f(\overline{z})}$$

Proof. Exercise. □

Corollary 2.9.12. *The functor* $f^*\colon \textbf{\textit{Sh}}(X^{ét}) \to \textbf{\textit{Sh}}(Z^{ét})$ *is exact.*

Theorem 2.9.13 (push-forwards and stalks). *Let* $f\colon Z \to X$ *be a morphism of schemes, and let* $\overline{x} \to X$ *be a geometric point, with support* $x \in X$. *Let* $\overline{X} \stackrel{def}{=} \operatorname{Spec}(\mathcal{O}_{X^{ét},\overline{x}})$, *and define* $\overline{Z} \stackrel{def}{=} \overline{X} \times_X Z$. *Denote by* $\alpha\colon \overline{Z} \to Z$ *the second projection. Let* F *be a sheaf on* $Z^{ét}$. *Then:*

$$(R^q f_* F)_{\overline{x}} = H^q(\overline{Z}^{ét}, \alpha^* F)$$

Remark. We think of $R^q f_* F$ as the "cohomology on the fibers". Actually, if f is proper, then the previous quotes can be removed (see [6]).

Corollary 2.9.14. *Suppose that* $f\colon Z \to X$ *is finite. Let* F *be a sheaf on* Z. *Then:*

i) $R^q f_* F = 0$ *for all* $q \geq 1$.

ii) $H^n(Z^{ét}, F) \simeq H^n(X, f_* F)$ *for all* $n \geq 0$

Proof. Note first that the second statement is a consequence of the first, because of the Leray spectral sequence. So we prove the first statement in the remaining of the proof.

Let $x \in X$, and let $\overline{x} = \operatorname{Spec}(\kappa(x)^{\text{sep}}) \to X$ be a geometric point with support x. As $Z \to X$ is finite, then:

$$\overline{Z} \stackrel{\text{def}}{=} \operatorname{Spec}\left(\mathcal{O}_{X^{\text{ét}},\overline{x}}\right) \times_X Z \simeq \coprod_{i=1}^{t} \operatorname{Spec}(B_i)$$

is a disjoint union of affine rings, where B_i are strict henselian rings (some argument is needed to complete this claim).

Now, we have, for all $q \geq 1$:

$$(R^q f_* F)_{\overline{x}} = H^q(\overline{Z}, \alpha^* F) = \prod_{i=1}^{s} H^q(\operatorname{Spec}(B_i)^{\text{ét}}, \alpha_i^* F) = 0$$

It follows that $R^q f_* F = 0$, as it vanishes at all of its stalks. □

2.10 Application: The Cohomology of Curves

Fix $k = \overline{k}$ an algebraically-closed field, and let X be a smooth, projective, irreducible algebraic variety of dimension 1 (a curve).

We want to calculate the cohomology groups of $X^{\text{ét}}$, wich values on the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}$, when $\text{char}(k) \nmid n$. That is, we are interested in computing $H^n(X^{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}})$.

There is a (non-canonical) isomorphism:

$$H^n(X^{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}}) \simeq H^n(X^{\text{ét}}, \mu_{n,X})$$

(where $\mu_{n,X}$ is the sheaf of n^{th} roots of unity over X). We will actually compute the latter groups. For this, we will compute the cohomology with values on the sheaf $\mathbb{G}_{m,X}$, and then show that the following sequence is exact:

$$0 \longrightarrow \mu_{n,X} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{[m]} \mathbb{G}_{m,X} \longrightarrow 0$$

2.10.1 The Cohomology of $\mathbb{G}_{m,X}$

We first recall some notions from classical algebraic geometry. We write $\text{Div}(X)$ for the free abelian group generated by the set of closed points on X. There is a group homomorphism, the degree map:

$$\deg\colon \text{Div}(X) \to \mathbb{Z}, \quad \sum_{x \in |X|^{\text{cl}}} n_x x \mapsto \sum_x n_x$$

and we define $\text{Div}^0(X)$ to be the kernel of the degree map.

Let $K(X)$ be the function field of X. We also have a map:

$$\text{div}\colon K(X) \to \text{Div}(X), \quad f \mapsto \sum_{x \in |X|^{\text{cl}}} \text{ord}_x(f) \cdot x$$

As X is projective, we have that $\deg(\text{div}(f)) = 0$ for all $f \in K(X)$. Denote by $P(X) \subseteq \text{Div}^0(X)$ the image of div, and write $\text{Pic}(X) \stackrel{\text{def}}{=} \text{Div}(X)/P(X)$. for the Picard group of X. Also, define $\text{Pic}^0(X) \stackrel{\text{def}}{=} \text{Div}^0(X)/P(X) \subseteq \text{Pic}(X)$.

Theorem 2.10.1. *There exists an abelian variety $J(X)$, called the* ***Jacobian of*** *X such that $J(X)(k) = \text{Pic}^0(X)$.*

Theorem 2.10.2. *Let X be a (smooth, proper) curve over an algebraically closed field k. Then:*

$$H^0(X^{ét}, \mathbb{G}_{m,X}) = k^\times, \quad H^1(X^{ét}, \mathbb{G}_{m,X}) \simeq \mathrm{Pic}(X)$$

and the higher cohomology vanishes.

Proof. Let $k(X)$ be the function field of X and let:

$$j\colon \eta \to X$$

be the generic point. Let $G = \mathrm{Gal}(k(X)^{sep}/k(X))$ be the Galois group of a separable closure of $k(X)$. Then since $\eta = \mathrm{Spec}(k(X))$, we have observed that:

$$\eta^{et} \simeq G - \mathbf{Sets}.$$

Furthermore, we have seen that sheaves on η^{et} correspond to G-modules under this identification. Under this correspondence:

$$\mathbb{G}_{m,\eta} \leftrightarrow (k(X)^{sep})^\times.$$

Let $x \in X$ be a closed point. Then $k(x)$ is an algebraic extension of k; since k is algebraically closed, $k(x) = k$. This shows that every closed point is a geometric point. Let $\mathbb{Z}_x$ be the constant sheaf $\mathbb{Z}$ on x^{et}. Put $i_x\colon x \to X$. We first claim that:

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,\eta} \to \bigoplus_{x \in |X|^{cl}} i_{x,*}(\mathbb{Z}_x) \to 0$$

is exact.

To prove this, we must first explain precisely how this sequence is defined. Let $f\colon U \to X$ be étale with U connected. It follows that since X is an irreducible curve and $U \to X$ is étale, U is an irreducible curve. Let $\xi = \mathrm{Spec}(k(U))$ be the generic point of U. We have $\mathbb{G}_{m,X}(U) = \mathcal{O}_U(U)^\times$ and:

$$j_*(\mathbb{G}_{m,\eta}(U) = \mathbb{G}_{m,\eta}(U \times_X \eta) = \mathbb{G}_{m,\eta}(f^{-1}(\eta)) = \mathbb{G}_{m,\eta}(\xi) = k(U)^\times.$$

Let $\phi_U\colon \mathcal{O}_U(U)^\times \to k(U)^\times$ denote the natural embedding. Then ϕ defines the first mapping.

We must next analyze:

$$i_{x,*}(\mathbb{Z}_x)(U) = \mathbb{Z}_x(U \times_X x) = \mathbb{Z}_x(\{x \in U \mid f(u) = x\}) = \bigoplus_{f(u)=x} \mathbb{Z}u.$$

This shows that:

$$\bigoplus_{x\in|X|^{cl}} i_{x,*}(\mathbb{Z}_x)(U) = \bigoplus_{x\in|X|^{cl}} \bigoplus_{f(u)=x} \mathbb{Z}u = \mathrm{Div}(U).$$

Let $\psi_U : k(U)^\times \to \mathrm{Div}(U)$ be given by $g \mapsto \mathrm{div}(g)$. Then ψ defines the second map. We must now argue that the sequence is exact. Note that:

$$0 \to \mathcal{O}_U(U)^\times \to k(U)^\times \to \mathrm{Div}(U)$$

is exact. The first map is naturally injective, and a function in $k(U)^\times$ without poles is a unit in $\mathcal{O}_U(U)$. It remains to show that the last map is surjective.

Let $D = \sum_{i=1}^m n_i u_i$ with $n_i \in \mathbb{Z}$ and $u_i \in U$. For each i, we can find Zariski open neighbourhoods $U_i \in U$ of u_i, and parameters $t_i \in k(U_i)^\times$ on U_i, such that:

$$\mathrm{div}_{U_i}(t_i) = U_i.$$

Put $U_0 = U - (U_1 \cup \cdots \cup U_m)$. Note that $D|_{U_0} = 0$, which is the image of $1 \in k(U)^\times$. For $i \neq 0$ we have $D|_{U_i} = n_i u_i$. This is the image of $t_i^{n_i} \in k(U_i)^\times$. So for each i, $D|_{U_i} \in \mathrm{Im}(\psi_{U_i})$. It follows by the sheaf property that ψ is surjective. This concludes the proof of our first claim.

We will now consider the associated long exact cohomology sequence:

$$0 \longrightarrow H^0(X^{et}, \mathbb{G}_{m,X}) \longrightarrow H^0(X^{et}, j_*(\mathbb{G}_{m,\eta})) \longrightarrow \bigoplus_{x\in|X|^{cl}} H^0(X^{et}, i_{x,*}(\mathbb{Z}_x)) \longrightarrow$$

$$\longrightarrow H^1(X^{et}, \mathbb{G}_{m,X}) \longrightarrow H^1(X^{et}, j_*(\mathbb{G}_{m,\eta})) \longrightarrow \bigoplus_{x\in|X|^{cl}} H^1(X^{et}, i_{x,*}(\mathbb{Z}_x)) \longrightarrow$$

$$\vdots$$

The theorem will follow by applying the following two lemmas to this sequence.

Lemma 2.10.3. *For $n \geq 0$:*

$$H^n(X^{et}, j_*(\mathbb{G}_{m,\eta})) \simeq H^n(\eta^{et}, \mathbb{G}_{m,k}) \simeq \begin{cases} k(x)^\times & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We will use the Leray spectral sequence. Hence, we begin by computing:

$$R^q_{j_*}(\mathbb{G}_{m,\eta})$$

for $q \geq 1$.

Let $y \to X$ be a geometric point. We will show that all the stalks $(R^q_{j_*}(\mathbb{G}_{m,\eta}))_y$ vanish, treating two cases.

For the first case, suppose $y = \overline{\eta} = \operatorname{Spec}(k(X)^{sep}) \to X$. Then $\mathcal{O}_{X^{et},\overline{\eta}}$ is the strict Henselization of $\mathcal{O}_{X,\eta} = k(X)$. But the strict Henselization of a field is just the separable closure, and thus:

$$\mathcal{O}_{X^{et},\overline{\eta}} = k(X)^{sep}.$$

Therefore:

$$(R^q_{j_*}(\mathbb{G}_{m,\eta}))_{\overline{\eta}} = H^q(\operatorname{Spec}(k(X)^{sep}) \times_X \overline{\eta}, \alpha^*(\mathbb{G}_{m,\eta})) = H^q(\operatorname{Spec}(k(X)^{sep})^{et}, \alpha^*(\mathbb{G}_{m,\eta})).$$

But again, this can be computed in terms of Galois cohomology by previous work. Since $k(X)^{sep}$ is separably closed, the Galois group of this field is trivial. Hence the Galois cohomology groups are trivial, and so the group above is zero. This concludes the first case.

In the second case, suppose that $y = x$ is a closed point of X with $i\colon x \to X$ the natural embedding. Then $\mathcal{O}_{X^{et},x}$ is the strict Henselization of $\mathcal{O}_{X,x}$. It is hence an inductive limit of local rings of curves U étale over X. Put $\operatorname{Frac}(\mathcal{O}_{X^{et},x}) = K$ and note that $\operatorname{Frac}(\mathcal{O}_{X,x}) = k(X)$. Then $K \subset k(X)$. Note that this extension is separable, since K is an inductive limit of fraction fields which are each separable over $k(X)$. After one shows that $\alpha^*(\mathbb{G}_{m,\eta}) \simeq \mathbb{G}_{m,K}$, it follows that:

$$\begin{aligned} (R^q j_*(G_{m,\eta}))_x &= H^q(\operatorname{Spec}(\mathcal{O}_{X^{et},x}) \times_X \eta, \alpha^*(\mathbb{G}_{m,\eta})) \\ &= H^q(\operatorname{Spec}(K), \mathbb{G}_{m,K}) \\ &\simeq H^q(G_K, (k(X)^{\mathrm{sep}})^\times), \end{aligned}$$

where the last group is a Galois cohomology group with $G_K = \operatorname{Gal}(K^{sep}, K)$. When $q = 0$, take Galois invariants to obtain $K^\times$. When $q = 0$ the cohomology vanishes by Hilbert theorem 90. When $q \geq 2$ it also vanishes, this time by Tsen's theorem. This concludes the second case. Thus:

$$R^q_{j_*}(\mathbb{G}_{m,\eta}) = 0$$

for $q \geq 1$, since all the stalks vanish. The Leray spectral sequence then gives:

$$\begin{aligned} H^n(X^{et}, j_*(\mathbb{G}_{m,\eta}) &\simeq H^n(\eta^{et}, \mathbb{G}_{m,\eta}) \\ &\simeq H^n(G, (k(X)^{sep})^\times) \\ &= \begin{cases} k(X)^\times & n = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This concludes the proof of the lemma. □

Lemma 2.10.4. *For all $n \geq 0$,*

$$H^n(X^{et}, i_{x,*}(\mathbb{Z}_x)) \simeq H^n(x^{et}, \mathbb{Z}_x) = \begin{cases} \mathbb{Z}_x & n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The second equality of the lemma follows similarly to above, using Galois cohomology. For the first, we will again make use of the Leray spectral sequence. Since k is algebraically closed, x^{et} is equivalent to the category of finite sets. Moreover, sheaves on x^{et} are the same thing as abelian groups. If M is an abelian group (corresponding to a constant sheaf on x) then $(i_{x,*}(M))_y = 0$ if $x \neq y$ and M if $x = y$. One deduces that the functor $i_{x,*}$ is exact. Thence $R^q i_{x,*} = 0$ for $q \geq 1$. The lemma then follows by the Leray spectral sequence. □

This concludes the proof of the theorem. □

Theorem 2.10.5. *Let $n > 0$ be an integer. Suppose that char(k) does not divide n. Then:*

$$\begin{aligned} H^0(X^{et}, \underline{\mu}_n) &= \mu_n(k) \\ H^1(X^{et}, \underline{\mu}_n) &= Pic^0(X)[n] \simeq Jac(X)[n] \\ H^2(X^{et}, \underline{\mu}_n) &= \mathbb{Z}/n\mathbb{Z} \\ H^q(X^{et}, \underline{\mu}_n) &= 0 \end{aligned}$$

for $q \geq 3$.

Proof. We first claim that:

$$0 \to \overline{\mu}_n \to \mathbb{G}_{m,X} \xrightarrow{[n]} \mathbb{G}_{m,X} \to 0$$

is an exact sequence of sheaves on X^{et}. Let $U \to X$ be étale. Then:

$$0 \to \mu_n(\mathcal{O}_U(U)) \to \mathcal{O}_U(U)^\times \to \mathcal{O}_U(U)^\times$$

where the last map is $x \mapsto x^n$, is exact. It thus remains to show that $[n]$ is a surjective map of sheaves.

Let $g \in \mathbb{G}_{m,X}(U) = \mathcal{O}_U(U)^\times$. We must construct an étale cover of U containing an n-th root of g. Define a quasi-coherent sheaf of $\mathcal{O}_U$-algebras by putting:

$$\mathcal{T} = \mathcal{O}_U[T]/(T^n - g)$$

and $Y = \mathrm{Spec}(\mathcal{T})$. Then $Y \to U$ is a surjective affine map. We will show that it is étale. Let $V = \mathrm{Spec}(A) \subset U$ be an affine open subset. Then:

$$W = f^{-1}(V) = \mathrm{Spec}(A[T]/(T^n - g|_V)).$$

Put $B = \mathrm{Spec}(A[T]/(T^n - g|_V)$. Note that $(d/dT)(T^n - g|_V) = nT^{n-1}$. Since n is relatively prime to the characteristic of k, it follows that n is a unit in B. Moreover, $T^n = g \in A^\times \subset B^\times$ and hence T is also a unit in B. A homework problem thus shows that B/A is étale. One can show that $W \to V$ is surjective by the going-up theorem of commutative algebra (since B/A is integral). It follows that $Y \to U$ is étale and surjective, and hence is an étale cover.

Let $t \in \mathcal{O}_U(U)$ be the image of $T \in \mathcal{O}_Y(Y)$. Then it follows that $[n]$ is surjective and:

$$0 \to \underline{\mu}_n \to \mathbb{G}_{m,X} \xrightarrow{[n]} \mathbb{G}_{m,X} \to 0$$

is exact. Consider the corresponding long exact sequence of cohomology:

$$0 \longrightarrow H^0(X^{et}, \underline{\mu}_n) \longrightarrow H^0(X^{et}, \mathbb{G}_{m,X}) \longrightarrow H^0(X^{et}, \mathbb{G}_{m,X}) \longrightarrow$$

$$\longrightarrow H^1(X^{et}, \underline{\mu}_n) \longrightarrow H^1(X^{et}, \mathbb{G}_{m,X}) \longrightarrow H^1(X^{et}, \mathbb{G}_{m,X}) \longrightarrow$$

$$\longrightarrow H^2(X^{et}, \underline{\mu}_n) \longrightarrow H^2(X^{et}, \mathbb{G}_{m,X}) \longrightarrow H^2(X^{et}, \mathbb{G}_{m,X}) \longrightarrow$$

$$\vdots$$

For $q \geq 3$, we deduce that $H^q(X^{et}, \underline{\mu}_n) = 0$ by the previous theorem. Since k is algebraically closed, the map $k^\times \to k^\times$ given by $x \mapsto x^n$ is surjective. Since H^0 is just the global sections, it follows that we must analyze the sequence:

$$0 \to H^1(X^{et}, \underline{\mu}_n) \to \mathrm{Pic}(X) \xrightarrow{[n]} \mathrm{Pic}(X) \to H^2(X^{et}, \underline{\mu}_n) \to 0.$$

Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{deg} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [n]} & & \downarrow{\scriptstyle [n]} & & \downarrow{\scriptstyle [n]} & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{deg} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Apply the snake lemma to this diagram. The kernel of the last map is zero, since $\mathbb{Z}$ has no torsion. The cokernel of the first map is zero since $\mathrm{Pic}^0(X)$ is a divisible group. We hence obtain two exact sequences, one for the kernels:

$$0 \to \mathrm{Pic}^0(X)[n] \to H^1(X^{et}, \underline{\mu}_n) \to 0$$

and one for the cokernels:

$$0 \to H^2(X^{et}, \underline{\mu}_n) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

The theorem follows. □

Theorem 2.10.6. *Let $n > 0$ be an integer and suppose that n and char(k) are relatively prime. Let g be the genus of X. Then there are (noncanoncial) isomorphisms:*

$$\begin{aligned} H^0(X^{et}, \underline{\mathbb{Z}/n\mathbb{Z}}) &= \mathbb{Z}/n\mathbb{Z} \\ H^1(X^{et}, \underline{\mathbb{Z}/n\mathbb{Z}}) &= (\mathbb{Z}/n\mathbb{Z})^{2g} \\ H^2(X^{et}, \underline{\mathbb{Z}/n\mathbb{Z}}) &= \mathbb{Z}/n\mathbb{Z} \\ H^q(X^{et}, \underline{\mathbb{Z}/n\mathbb{Z}}) &= 0 \end{aligned}$$

for $q \geq 3$.

Proof. Choose a primitive n-th root of unity $\zeta \in \mu_n(k)$. Then we claim that there is an isomorphism of sheaves on X^{et}:

$$\underline{\mathbb{Z}/n\mathbb{Z}} \simeq \underline{\mu}_n.$$

It is given in the following way. If $U \to X$ is étale, let s be the number of connected components of U. Then:

$$\underline{\mathbb{Z}/n\mathbb{Z}}(U) = (\mathbb{Z}/n\mathbb{Z})^s$$

and:

$$\underline{\mu}_n(U) \simeq \mu_n(k)^s.$$

Define a map $\phi_U\colon (\mathbb{Z}/n\mathbb{Z})^s \to \mu_n(k)^s$ by putting $\phi_U(u_1, \ldots, u_s) = (\zeta^{u_1}, \ldots, \zeta^{u_n})$. Then the ϕ_U's define an isomorphism of sheaves $\phi\colon \underline{\mathbb{Z}/n\mathbb{Z}} \to \underline{\mu}_n$. The theorem now follows from the previous one, using the fact that:

$$\mathrm{Pic}^0(X)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

For a proof of this fact, see Mumfords book on abelian varieties. □

This concludes the course!!

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