mod p- Langlands (R. Olivier)

Recall: Our goal is - given a p-adic field F, look at the theory of smooth representation theory of $GL_n(F)$ where the coefficients are either $\mathbb{C}$ or $\boxed{\overline{\mathbb{F}}_p}$.

Chapter 1: Representations of $GL_n(k)$ $\quad k = \mathbb{F}_q, \; q = p^f$.

(I) G a finite group, R = field (field of coefficients)

(ρ, V) a representation: V an R-vectorspace, $\rho: G \to \mathrm{Aut}(V)$.

- Intertwining operators: given (ρ, V), and (σ, W), then this is $\mathrm{Hom}_G(V, W)$.
- Direct sums, contragredient (dual): $(\rho^\vee, V^\vee)$: $V^\vee = \mathrm{Hom}_R(V, R)$ with action $g\varphi^\vee = \varphi \circ g^{-1}$.

Seeing a rep as an $R[G]$-module, we get the concept of irred. rep.

Eg: Trivial character: $\leftarrow$ a one-dim'l rep.

$$GL_n(k) \to R^\times.$$

eg: Regular representation: $V = R[G]$. (dim = $\#G$).

(II) Induction / Restriction.

$H < G$, and let (σ, W) a rep of H. Then:

$$\mathrm{Ind}_H^G \sigma = R[G] \otimes_{R[H]} \sigma.$$

Frobenius reciprocity: given (σ, W) rep of H and (ρ, V) rep of G,

$$\mathrm{Hom}_G\cancel{(W}(\mathrm{Ind}_H^G \sigma, V) \cong \mathrm{Hom}_H(W, V|_H).$$

eg: Take (σ, W) ~~to~~ be the trivial character. Then:

$$\mathrm{Hom}(\mathrm{Ind}_H^G W, V) \cong V^H = \{v \in V \mid hv = v \ \forall h \in H\}.$$

(H-invariants).

Also we can think of $\mathrm{Ind}_H^G \sigma$ to be:

$$\{ f : G \to W \mid f(hg) = \sigma(h)\cdot f(g) \}.$$

(~~and~~ say that G acts by right translation!):

$$(g \cdot f)(x) = f(xg).$$

<u>Mackey formula</u>: Given $H < G$, and $K < G$,

$$(\mathrm{Ind}_H^G \sigma)|_K = \bigoplus_{HwK} \mathrm{Ind}_H^{HwK} \sigma = \bigoplus_{w} \mathrm{Ind}_{w(H)\cap K}^{K} w(\sigma)$$

where
$$\begin{cases} w(H) = wHw^{-1} \\ w(\sigma) \text{ rep of } w(H) \\ w(\sigma)(whw^{-1}) = w(h) \end{cases}$$

(III) Irreducible reps.

1) If $\#G \in R^\times$, then Mashke's theory gives that the category of fin-dim reps of G is semi-simple.

2) Schur's lemma: given V, W reps of G which are irreducible, then

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \not\cong W \\ \cong R & \text{if } R \text{ is algebraically closed.} \end{cases}$$

Ref: Serre, "Representations of Finite Groups".

3) Spz $R=\mathbb{C}$. Then the number of irred. reps of G = the number of conjugacy classes in G.

Spt $R=\overline{\mathbb{F}}_p$. Then the number of irred reps of G = the number of conjugacy classes whose elements have order coprime to p (called p-regular classes).

Eg: $GL_2(\mathbb{F}_q)$. (Bushnell-Henniart's book LL corresp. for $GL_2(F)$).

Get the conjugacy classes:

$\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}$	$\lambda \in k^\times$	$(q-1)$
$\begin{pmatrix}\lambda & \\ & \mu\end{pmatrix}$	$\lambda \neq \mu \in k^\times$	$\frac{(q-1)(q-2)}{2}$
$\begin{pmatrix}\lambda & 1 \\ 0 & \lambda\end{pmatrix}$	$\lambda \neq 0$	$(q-1)$
$\begin{pmatrix}0 & -b \\ 1 & -a\end{pmatrix}$	Corresp to X^2+aX+b irred	$\frac{q^2-q}{2}$

total: q^2-1

II Reps of $GL_n(k)$

1) Vocabulary:
- Torus
- maximal split torus ~ $\begin{pmatrix}* & & 0 \\ & \ddots & \\ 0 & & *\end{pmatrix}$.
- Borel subgroups: conjugate to $B=\begin{pmatrix}* & & * \\ & \ddots & \\ & & *\end{pmatrix}$

Parabolic subgroups: choose a partition $(n_1,\dots,n_r)$ of n. This corresponds to a Levi subgroup $\begin{pmatrix}GL_{n_1}(k) & & 0 \\ & \ddots & \\ 0 & & GL_{n_r}\end{pmatrix}=M$, and a unipotent subgroup $\begin{pmatrix}\mathrm{id}_{n_1} & & * \\ & \ddots & \\ 0 & & \mathrm{id}_{n_r}\end{pmatrix}=N$.

Then a standard parabolic is:

$$P = M \rtimes N \simeq MN \supset B$$

A parabolic subgroup is a conjugate of a standard parabolic subgroup.

• Bruhat decomposition

Def: The (finite) Weyl group W_0 is defined as $N_{GL_n(k)}(T)/T$.

It can be seen as a copy of $\mathfrak{S}_n$ inside $GL_n(k)$.

$$GL_n(k) = \coprod_{w \in W_0} BwB$$

Writing U for the unipotent sgp $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, then also

$$GL_n(k) = \coprod_{w \in W_0} BwU.$$

And: $W_0 = \mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$, $s_i^2 = 1$, $s_i = (i, i+1)$

A standard parabolic P is associated to a set $I \subset \{1, \dots, n-1\}$:

$$I = \{ i \in \{1, \dots, n-1\} : s_i \in P \}.$$

Then can consider $W_0^I = \langle s_i : i \in I \rangle$, and obtain a new refined decomposition:

$$GL_n(k) = \coprod_{w \in W_0^I \backslash W_0} PwB.$$

(finite) Root datum for $GL_n(\kappa)$

$W_0 = \mathfrak{S}_n$, $S_0 = \{s_1, \dots, s_{n-1}\}$, $\hat{T} = \mathrm{Hom}(T, \kappa^\times)$

$\Phi^\vee$ = Roots for $GL_n(\kappa)$: $\{\alpha^\vee_{ij} : T \to \kappa\}_{i \neq j}$, $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i t_j^{-1}$.

Since associated to the Borel $B = \begin{pmatrix} * & * \\ 0 & \ddots \end{pmatrix}$, the positive roots are

$\Phi^{\vee +} = \{\alpha_{ij},\ i<j\}$.

$\cup$

$\Pi^\vee$ = Simple roots = $\{\alpha_{i,i+1},\ i \in \{1, \dots, n-1\}\}$.

Ref, Bourbaki, Chapter 4, 5, 6 (Lie Groups) or Humphries.

Length on W_0: W_0 acts on $\hat{T}$ and also on $\Phi^\vee$.

$\ell(w) = \#\{\alpha \in \Phi^{\vee +} \mid w\alpha \in \Phi^{\vee -}\}$ $\quad (\Phi^{\vee -} = -\Phi^{\vee +} = \{\alpha_{ij}\ i>j\})$

The maximal possible length is $\#\Phi^{\vee +} = \frac{n(n-1)}{2}$.

Also, $\ell(w) = \ell(w^{-1})$

$\ell(s_i) = 1$ (the only positive roots α s.t. $s_i \alpha \in \Phi^{\vee -}$ is $\alpha = \alpha_{i,i+1}$)

$$\ell(ws_i) = \begin{cases} \ell(w) + 1 & \text{if } w(\alpha_{i,i+1}) \in \Phi^{\vee +} \\ \ell(w) - 1 & \text{otherwise.} \end{cases}$$

From all this, one can show that $\ell(w)$ is the length of a minimal expression of w in terms of s_i's.

It can also be checked that

$$\ell(w) = \# \, U \cap w^{-1} \bar{U} w \qquad \left(\bar{U} = \text{opposite unipotent } \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right).$$

Moreover,

$$q^{\ell(w)} = \# \, {}_{B}\backslash^{BwB} = \# \, BwB/B$$

and $\ell(w) + \ell(w') = \ell(w \cdot w') \iff BwBw'B = Bww'B$.

I.2) Parabolic induction/restriction

Let $P = MN$ be a Levi decomp. of P, a parabolic.

Start with a rep of M (direct sum of reps of smaller gps), and inflate it to $P \twoheadrightarrow M$. Finally, do

$$\mathrm{Ind}_P^G \rho.$$

To do parabolic restriction, or the Jacquet functor, we associate to a rep V of G:

$$J_P(V) = V/V(N), \quad V(N) = \langle nv - v : n \in N, v \in V \rangle.$$

This is right-exact, and note that $M \circlearrowright J_P(V)$.

Adjunction property:

$$\mathrm{Hom}_M(J_P(V), W) = \mathrm{Hom}_G(V, \mathrm{Ind}_P^G W).$$

3: Hecke Algebras $G = GL_n(\kappa)$

$\mathcal{H}_R(G) = \mathrm{Fun}(GL_n(\kappa), R)$ ← Global Hecke algebra of $GL_n(\kappa)$

$(f*f')(x) := \sum_{g\in G} f(g)\, f'(g^{-1}x)$ (unit is $\mathbb{1}_1(x)$, the characteristic function of 1).

Let $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. If p is invertible in R, have a function:

$$e_U = \frac{1}{\#U}\mathbb{1}_U .$$

Then $e_U * \mathbb{1}_{\{g\}} = \frac{1}{\#U}\mathbb{1}_{Ug}$, ... and can prove that e_U is an idempotent of $\mathcal{H}_R(G)$.

So we get $\mathcal{H}_R(G,U) := e_U * \mathcal{H}_R(G) * e_U \cong R[U\backslash G/U]$.

(functions that are left and right bi-invariants).

with convolution: $(f*f')(x) = \sum_{g\in G/U} f(g)\, f'(g^{-1}x)$.

<u>Rmk</u>: even if $p \notin R^\times$, still can define $(\mathcal{H}_R(G,U), *)$

<u>Link with representations</u>:

$$\mathrm{End}_G(\mathrm{Ind}_U^G 1) \cong (\mathrm{Ind}_U^G 1)^U \simeq R[U\backslash G/U]$$

$$\circ \longleftrightarrow *$$

So let R be arbitrary (any char). Take a rep V of G.

$$V^U = \mathrm{Hom}(\mathrm{Ind}_U^G 1, V)$$

So we get

$$\begin{tikzcd} \mathrm{Rep}\, G \arrow[r] & \text{Right } \mathcal{H}_R(G,U)\text{-modules} \\ V \arrow[r, mapsto] & V^U \end{tikzcd}$$

Rmk: 1) Can construct $\mathcal{H}_R(G,H)$ for any H (as B).

2) Given $H < G$, and (ρ, W) a rep then

$$\mathcal{H}_R(G,\rho) := \mathrm{End}_G(\mathrm{Ind}_H^G \rho).$$

The functor $V \overset{F}{\longmapsto} V^U$

A) Complex case.

$$\begin{aligned} \pi : V &\to V^U \\ v &\mapsto \frac{1}{\# U} \sum uv \end{aligned}$$ another projection.

F is left-exact in general, but this π implies that F is also right-exact.

$\Rightarrow \mathrm{Ind}_U^G 1$ is projective in $\mathrm{Rep}\, G$.

Adapting Morita theory, get that F is an equivalence of categories:

$$\left\{ \begin{array}{l} \text{reps } V \text{ generated by} \\ \text{their } U\text{-invariants} \end{array} \right\} \hookrightarrow \{\text{Hecke-modules}\}.$$

Rmk: this won't work mod p !!

1/27/11.

Recall: Hecke algebras.

1) $\mathcal{H}_R(G,B) = (R[B\backslash G/B], *)$ (with convolution).

A basis: $1\!\!1_{BwB}$

Setting $S_i := 1\!\!1_{Bs_iB}$, then this is a basis of $\mathcal{H}_R(G,B)$ as an algebra.

They satisfy: $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$

$$S_i^2 = (q-1)S_i + q$$

Remark: $n=2$, then

$$\mathcal{H}_R(G,B) = \langle S \rangle, \quad (S+1)(S-q) = 0.$$

The simple modules correspond to $S \to -1$

$S \to q$.

2) $\mathcal{H}_R(G,U) = \mathrm{End}(\mathrm{Ind}_U^G 1\!\!1)$, $\quad \# B/U = (q-1)^n$.

(remember that $\mathrm{Ind}_U^G 1\!\!1 = \bigoplus_{\chi \in \hat{T}} \mathrm{Ind}_B^G \chi$).

So we will introduce $\varepsilon_\chi \in \mathcal{H}_R(G,U)$, $\varepsilon_\chi : \mathrm{Ind}_U^G 1\!\!1 \twoheadrightarrow \mathrm{Ind}_B^G 1\!\!1$, which is an idempotent. (for each $\chi \in \hat{T}$).

For $n=2$:

$$\begin{tikzcd} & \#\gamma = 1 \quad \gamma = \{\chi\},\ s\chi = \chi \quad \left(s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right). \\ \gamma \in \hat{T}/W_0 = S_2 \arrow[ur] \arrow[dr] & \\ & \gamma = \{\chi, s\chi\} \end{tikzcd}$$

Setting $\varepsilon_\gamma = \sum_{\chi \in \gamma} \varepsilon_\chi$, this is a central idempotent.

So $\mathcal{H}_R(G,U) = \bigoplus_\gamma \varepsilon_\gamma \mathcal{H}_R(G,U)$

In this case, $\mathcal{H}_R(G,U) = \langle \tau_{s_i}, \tau_t$ for $t \in T \rangle$, where $\tau_{s_i} = \mathbb{1}_{U s_i U}$

$$= \langle \tau_{s_i}, \varepsilon_\chi \;\; \forall i, \forall \chi \rangle.$$

The relations are: (at least for GL_2, although it can be done in general)

$$\begin{cases} \varepsilon_\chi T_{s_\alpha} = T_{s_\alpha} \varepsilon_{s_\alpha \chi} \\ \varepsilon_\chi T_{s_\alpha}^2 = \varepsilon_\chi \left((q-1) T_{s_\alpha} + q \right) \quad \text{if } s_\alpha \chi = \chi \\ \varepsilon_\chi T_{s_\alpha}^2 = q \, \varepsilon_\chi \quad \text{otherwise} \end{cases}$$

The simple $\mathcal{H}$-modules are:

→ if $\chi = s\chi$, ($\# \sigma = 1$), then they are all $\varepsilon_\sigma(\mathcal{H}(G,U))$.

→ if $\chi \neq s\chi$ ↗ char $R \neq 0$: 2 dim simple modules
↘ char $R = 0$: only characters.

The simple modules for $\mathcal{H}_{\overline{\mathbb{F}}_p}(GL_n(k), B)$ have dimension one.
(to find a stable line, take the span of $w_0 \in W_0$, an element of maximal length such that $\tau_{w_0} M \neq 0$).

4) The functor $V \to V^U$.

A) Complex case: it is exact: if $f \in \mathcal{H}(G)$, then $f(\rho) = \sum_g f(g) \rho(g)$

Ex: $e_U = \frac{1}{\# U} \mathbb{1}_U \Rightarrow f(e_U) = V \to V^U$ projection. ⇒ right-exact ⇒ exact
left-exact is obvious.

<u>Rk</u>: Can also do $V \to V_U$, where

$$V / V(U) \simeq V^U, \quad V(U) = \mathrm{Vect}(uv - v, \; u \in U, v \in V).$$

<u>Corollary</u>: $V \xrightarrow{\mathcal{F}} V^U$ induces $\mathrm{Rep}^U G \longrightarrow \mathcal{H}_R(G,U)\text{-mod}$

<u>Pf</u>: . $\mathrm{Ind}_U^G 1$ is projective in $\mathrm{Rep}_R(G)$. (Morita theory)

Or: The functor $M \xrightarrow{\mathcal{T}} M \otimes_{\mathcal{H}_R(G,U)} \mathrm{Ind}_U^G 1_R$ is left-adjoint

to $\mathcal{F}$: $\mathcal{F}\,\mathcal{T}(M) \cong f(e_U)\left(M \otimes_H \mathrm{Ind}_U^G 1\right) = M \otimes_{\mathcal{H}_R(G,U)} \left(\mathrm{Ind}_U^G 1\right)^U \cong M$.

(...)

B) <u>R of characteristic p</u>: Let $H = \mathcal{H}_R(G,U)$, $\overset{H \circlearrowright}{\underset{G \circlearrowleft}{\mathcal{C}}} = \mathrm{Ind}_U^G(1)$ $(G = GL_n(k))$

<u>Lemma</u>: H is a Frobenius algebra: there is a linear form

$$\lambda : H \longrightarrow R \text{ s.t } \forall h \neq 0, \begin{cases} \lambda(hH) \neq 0 \\ \lambda(Hh) \neq 0 \end{cases}$$

($\Rightarrow$ H is self-injective, $\Rightarrow$ fin. gen projectives = fin-gen injectives).

Pf (Cabanes, Enguehard "Modular Repr of Finite Groups")

(for $H_1 = \mathcal{H}(G,B)$, this is very easy: take $\lambda : H_1 \to R$

$T_{w_0} \mapsto 1$
others $\mapsto 0$ ($w_0 \in W_0$ of maximal length).)

Restricting to U-invariants we get a functor:

$$\mathrm{Rep}^U G \xrightarrow{\mathcal{F}} H\text{-mod}$$

which has a left-adjoint $\mathcal{T}$: ~~Hmod~~ $\to \mathrm{Rep}^U G$ defined as before.

$\mathrm{Rep}^U G = \{(\rho, V) : \text{s.t } \exists d \text{ w/ } \mathcal{C}^d \twoheadrightarrow V\}$. $(\mathcal{C} = \mathrm{Ind}_U^G 1)$.

Note that $\mathcal{C}^\vee \simeq \mathcal{C}$ (as reps)

$\mathrm{ev}(1) \leftarrow\!\shortmid 1_U$

Consider a subcategory $\mathcal{B} \subseteq \mathrm{Rep}^U G$:

$$\mathcal{B} = \left\{ \begin{array}{l} V \text{ is generated by } V^U \text{ and} \\ V^\vee \text{ is generated by } (V^\vee)^U \end{array} \right\} = \left\{ \begin{array}{l} V \text{ is the image of a} \\ G\text{-endomorphism } \mathcal{C}^n \to \mathcal{C}^m \end{array} \right\}.$$

Theorem (Morita, Gabriel-Popescu):

$\mathcal{B} \xrightarrow{\mathcal{F}} H$-modules is an equivalence of categories.

Note that every irreducible representation is in $\mathcal{B}$: first, V irred $\Rightarrow$ V generated by V^U!

(recall that a p-group acting on a vectorspace w/ characteristic p has a nontrivial fixed vector) $\Longrightarrow$ $V^U \neq 0$.

c) p-adic reps:

$\mathcal{F}: V \to V^U$ is essentially surjective, and also faithful.

(b/c $f: V \to W$, $f|_{V^U}$ inj $\Leftrightarrow$ f inj)

Corollary: TFAE:

1) $\mathcal{F}$ is full.

2) $\mathcal{C}$ is exact $\quad \left(\mathcal{C}: H\text{-mod} \to \mathrm{Rep}^U G, \quad M \mapsto M \otimes_H \mathcal{C} \right)$

3) $\mathcal{C}$ is flat over H

4) $\mathcal{C}$ is projective over H.

5) $\mathcal{B} = \mathrm{Rep}^U G$ coincide.

Q: When are these true?

For $n=2$, then the previous conditions are satisfied $\iff q=p$.

(But if $\mathcal{C}_1 = \mathrm{Ind}_B^G 1$ $\Rightarrow$ C_1 flat over H_1 for any q.) (U is cyclic)

$H_1 = \mathcal{H}_R(G,B)$

For $n=3$:

- $q=p$: $\mathcal{C}_1$ is flat over $\mathcal{H}_1$
- $\mathcal{C}$ is never flat over $\mathcal{H}$.

For $n \geq 4$: $\mathcal{C}$ is never flat over $\mathcal{H}$.

Open question: is $\mathcal{C}_1$ flat over $\mathcal{H}_1$ (when $q=p$) ?

Let M be ~~the~~ standard Levi subgroups in $G = GL_n(k)$.

$\underset{\text{upper}}{P} = MN$; then $C_M = \mathrm{Ind}_{M\cap U}^{M} 1$, and $\left\{ \begin{array}{l} (C_M)^{U_M} \hookrightarrow C^U \cong H \\ H_M \hookrightarrow H \end{array} \right.$

(because $U = NU_M = U_M N$).

If $\mathcal{C}$ is $\mathcal{H}$-flat, then C_M is H_M-flat:

Pf/ ~~Let~~ $\mathfrak{m}$ be an ideal of H_M: and let P the standard parabolic.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, squiggly, "J_P"'] & K \arrow[r] & \mathfrak{m} \otimes_{H_M} C_M \arrow[r] & C_M \arrow[r] & 0 \\
0 \arrow[r] & J_P(K) \arrow[r] \arrow[d, equal] & J_P(\mathfrak{m} \otimes_{H_M} C_M) \arrow[r] \arrow[d, phantom, "\cong"] & I_P(C_M) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
0 \arrow[r] & J_P(K) \arrow[r] & (\mathfrak{m} \otimes_{H_M} H) \otimes_H \mathcal{C} \arrow[r] & \mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

Flatness of $\mathcal{C}$ $\Rightarrow$ $J_P(K) = 0$.

∎

- Classification of the irreducible mod p reps of $GL_2(k)$

(Carter-Lusztig). All as subreps of $\mathcal{C}$. $\circlearrowleft GL_2(k)$ (on the left)

Consider $Sym^r(R) = Vect_R(X^{r-i}Y^i, i=0..r)$. $\begin{pmatrix}a&b\\c&d\end{pmatrix} X^{r-i}Y^i = (aX+cY)^{r-i}(bX+dY)^i$

Let χ be a character ~~on~~ $k^\times \to R^\times$. Then

$$(\chi\circ\det)\otimes Sym^{\vec{r}} \text{ is irreducible,}$$

and these are all of the irreds.

Here, $q=p^f$, $\vec{r}=(r_1,\dots,r_f)$, $Sym^{\vec{r}}(g) = \bigotimes_{i=1}^{f} Sym^{r_i}\circ Fr^{i-1}(g)$,

$(Fr : k\to k \;;\; x\mapsto x^p)$.

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Remark: The theorem stated in the previous lecture implies, in particular, that if C is not flat over H, then ~~$\mathcal{B}\not\supseteq Rep^U\mathcal{G}$~~

$\mathcal{B}\not\supseteq Rep^U G$.

Let's show this, at least for $G=GL_2(\mathbb{F}_q)$.

Write $H=\bigoplus H\varepsilon_\chi$, $\varepsilon_\chi : C \to C_\chi = Ind_B^G \chi$

$\tau_s = 1_{UsU}$, $s=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. $\tau_s^2\varepsilon\chi \equiv 0$ mod p if $\chi\neq\chi_s$.

So have ~~an~~ s.e.s:

$$0\to \varepsilon_\chi\tau_s H\to \varepsilon_\chi H\to \tau_s\varepsilon_\chi H\to 0$$

If C was flat, we would obtain an exact seq:

$$0\to \tau_s C_{s\chi}\to C_\chi\to \tau_s C_\chi\to 0$$

ez to see that these are irreducible (if $\chi\neq\chi_s$) ↯

(cont)

But then if $q \neq p$, then $\dim(\tau_s C_{sx} + \tau_s C_x) < q+1 = \dim(C_x)$.

Therefore if $q \neq p$, there is a kernel:

$$0 \to K \to C_x \to \tau_s C_x \to 0$$

$$\tau_s C_x \subsetneq K$$

However, taking U-invariants get $\overset{\dim = 1}{(\tau_s C_x)^U} \subseteq K^U \subsetneq \overset{\dim = 2}{(C_x)^U}$

So $K^U = (\tau_s C_x)^U$, and therefore K is not generated by K^U.

Since $K \hookrightarrow C_x$, ~~(C_x)~~ $C^\vee \twoheadrightarrow K^\vee$. But $C^\vee = C$,

so $K^\vee \in \mathrm{Rep}^U G$, but it is not in $\mathcal{B}$.

Chapter III: $GL_n(F)$ and its smooth representation.

Let F be a p-adic field (local, nonarch field with residue char p). Let $\mathcal{O}$ be its valuation ring, and let π be its uniformizer. Write $\mathfrak{p} = \pi \mathcal{O}$ for its maximal, $k = \mathcal{O}/\mathfrak{p}$, and set the valuation so that $val(\pi) = 1$.

$$GL_n(F) \hookrightarrow Mat_n(F)$$

$\uparrow$ locally profinite. (recall profinite means:

1) $\varprojlim$ (finite gps w/disc. top)

or

2) Compact totally disconnected

or

3) Separated, and the set of open normals is a fundamental system of nbhds of 1.)

$GL_n(\mathcal{O})$ is profinite, but $GL_n(F)$ is only locally profinite:

1) $\mapsto$ locally compact + totally disconnected.

or

2) separated and the set $\Omega(G)$ of $\underset{\swarrow \text{ call it COS}}{}$ compact open subgroups is a fundamental system of nbhds of 1.

Let $K = GL_n(\mathcal{O}) \subset G = GL_n(F)$. Inside K there are

$$K_m = 1 + \pi^m \mathrm{Mat}_n(\mathcal{O}) \ : \ \text{prop groups.}$$

II. Geometrical elements

Let T be the diagonal torus (the finite torus will be written $\mathbb{T}$)

- The stabilizers of flags are the parabolic subgroups (in particular, the maximal standard flag built on the canonical basis gives the Borel sgp $\left\{\begin{pmatrix} * & * & * \\ & \ddots & * \\ 0 & & * \end{pmatrix}\right\}$).
- The stabilizers of a certain $F^n = V_1 \oplus \cdots \oplus V_m$ give Levi subgroups

Standard ones: $\begin{pmatrix} GL_{n_1} & & 0 \\ & GL_{n_2} & \\ 0 & & \ddots \end{pmatrix}$

- We have also the unipotents N, for each parabolic P and Levi L.

$$\begin{pmatrix} id & & * & * \\ & id & & \\ & & id & \\ 0 & & & id \end{pmatrix}$$

There is also the Bruhat decomposition:

$$G = \bigsqcup_{w \in W_0 \simeq S_n} B w B$$

and the Iwasawa decomposition:

$$G = P \cdot K = B K \quad , \text{for any parabolic } P.$$

G/B is compact, and K is the unique maximal compact subgroup (up to conjugacy).

<u>Ref</u>: Fiona Murnaghan, "p-adic $\left(\begin{smallmatrix}\text{reductive gps}\\ GL_n\end{smallmatrix}\right)$" (available online).

III. <u>Parabolic subgroups</u>:

$$\begin{tikzcd} K \arrow[r, "\text{mod unif.}"] & \mathbb{G} = GL_n(k) \\ P \arrow[u, dashed] \arrow[r, dashed] & \mathbb{P} \arrow[u, phantom, "\vee"] \end{tikzcd}$$

$\mathbb{P}$ ← parabolic

↑ inverse image of $\mathbb{P}$ is called a <u>parahoric subgroup</u>.

<u>eg</u>: $\mathbb{P} = \mathbb{B} \rightsquigarrow P =$ Iwahori subgroup:

$$\mathcal{P} \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix} \quad , \quad I = T(\mathcal{O}) \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \to \sqcup .$$

<u>eg</u>: $\mathbb{P} = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} < GL_3(k) \rightsquigarrow P = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{GL_2(\mathcal{O})} & \mathcal{O} \\ & & \mathcal{O} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathcal{O}^\times \end{array}\right) > \underset{\overset{\shortparallel}{M}}{\left(\begin{array}{cc|c} \multicolumn{2}{c|}{GL_2(\mathcal{O})} & 0 \\ & & 0 \\ \hline 0 & 0 & \mathcal{O}^\times \end{array}\right)}$

Let N = maximal pro-p gp contained in P.

$$N = \begin{pmatrix} 1+\mathfrak{p} & 0 & \mathcal{O} \\ 0 & 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & 1 \end{pmatrix}$$

Then $P = M \cdot N$.

Decompositions:

Cartan: $G = \coprod K A^{+} K$, $A \simeq \mathbb{Z}^n = \begin{pmatrix} \varpi^{\mathbb{Z}} & & \\ & \ddots & \\ & & \varpi^{\mathbb{Z}} \end{pmatrix}$

(ϖ = uniformizer).

$$A > A^{+} = \left\{ \begin{pmatrix} \varpi^{r_1} & & \\ & \ddots & \\ & & \varpi^{r_n} \end{pmatrix} \quad r_i \geq r_{i+1} \right\}.$$

Proof: Let $\Lambda_0 = \mathcal{O} e_1 \oplus \cdots \oplus \mathcal{O} e_n$ (e_n : canonical basis for F^n).

Then for $g \in G$, $g\Lambda_0$ is another lattice, and

$\exists m \geq 0$ s.t. $\varpi^m g \Lambda_0 \subset \Lambda_0$.

By the invariant factors theorem, $\exists$ basis adapted to both: $\{b_1, \ldots, b_n\}$ $(r_1, \ldots, r_n)$ decreasing s.t. $\Lambda_0 = \mathcal{O} b_1 \oplus \cdots \oplus \mathcal{O} b_n$ and

$\varpi^n g \Lambda_0 = \mathcal{O} \varpi^{r_1} b_1 \oplus \cdots \oplus \mathcal{O} r_n b_n$.

write $(b_n, \ldots, b_n) = x \cdot (e_1, \ldots, e_n)$, $x \in K$ (change of basis).

Then: $\varpi^n g \Lambda_0 = \mathcal{O} x \varpi^{r_1} e_1 \oplus \cdots \oplus \mathcal{O} x \varpi^{r_n} e_n =$

$$= x \begin{pmatrix} \varpi^{r_1} & & \\ & \ddots & \\ & & \varpi^{r_n} \end{pmatrix} \Lambda_0 \text{ , and :}$$

$$\begin{pmatrix} & \end{pmatrix} x^{-1} \varpi^n g \in K.$$

∎

Corollary: G/K is countable.

IV Representations of G:

Smooth representations:

$\rho : G \to GL(V)$ is smooth if the stabilizer of every $v \in V$ is open.

Examples: characters: $\chi : G \to R^\times$, for R an arbitrary algebraically closed field.

χ smooth $\Leftrightarrow$ ker χ is open.

Induction: Let $H < G$ be a closed subgroup.

Let (σ, W) be a smooth rep of H.

$$\mathrm{Ind}_H^G \sigma = \overline{\left\{ f : G \to W \;:\; f(hg) = \sigma(h) f(g) \right\}}^{\,\text{smooth}}$$

To take the smooth part means: to take the functions f such that satisfy $\exists$ open sgp $U = U(f)$ s.t $f(gU) = f(g)$ $\forall g \in G$.

This contains in particular the functions with compact support in G/H. If we only take this part, this is called compact induction. and written $c\text{-}\mathrm{Ind}_H^G \sigma$.

Rmk: if G/H is compact, then the two inductions coincide.

Parabolic Induction:

Let M be a Levi sgp, and (σ, W) be a smooth rep of M. Can inflate it to a rep of P, $P = MN$, and then induce to G. (G/P is compact, so $c\mathrm{Ind} = \mathrm{Ind}$).

This process is called Parabolic induction.

The parabolic induction has a left adjoint

$$J_P : \mathrm{Rep}\, G \to \mathrm{Rep}\, M$$
$$V \longmapsto V/_{V(N)}$$

where $V(N) = \mathrm{Vect}\{(nv - v) : n \in N, v \in V\}$.

Hecke Algebra:

Spherical Hecke Algebra: $\mathcal{H}(G, KZ) = \mathrm{End}\left(c\mathrm{Ind}_{KZ}^{G} 1\right) \simeq R\left[{}_{KZ}\backslash G/K\right]$

$$Z \simeq F^{\times}, \text{ center of } GL_n(F) \qquad \begin{matrix} \wr\wr \\ R[T_1, \dots, T_{n-1}] \end{matrix}$$

The isomorphism $\mathcal{H}(G, KZ) \cong R[T_1, \dots, T_{n-1}]$ is related to the Satake isomorphism.

The Iwahori Hecke algebra is

$$\mathcal{H}(G, IZ) = \mathrm{End}_G\left(c\mathrm{Ind}_{IZ}^{G} 1\right) \simeq R\left[{}_{IZ}\backslash G/I\right]$$

(conv. alg)

Moreover,

$$\mathcal{H}(G, KZ) \overset{\simeq}{\hookrightarrow} \text{center of } \mathcal{H}(G, IZ)$$

Affine building of (P)$GL_n(F)$. (for $n=2$ it's a tree)

- Set of homothety classes of $\mathcal{O}$-lattices in F^n. $\mathcal{X}_0 = \{[\Lambda] : \Lambda \text{ lattice}\}$
- $\lambda, \lambda' \in \mathcal{X}_0$. Say $\lambda \sim \lambda'$ if $\exists$ reps of λ, λ' s.t.

$$\varpi\Lambda' \subset \Lambda \subset \Lambda'.$$

(This is an equivalence relation).

Let $\mathcal{X}$ be the simplicial complex associated to $(\mathcal{X}_0, \sim)$.

Ref: Brown-Abramenko.

Look up: "spherical building at ∞ for affine building".

Feb 3th, 2011:

Chapter V: p adic Hecke algebras.

1) Root datum for $GL_n(F)$.

$$\left(X, X^{\vee}, \phi \cancel{x} \phi^{\vee}, \Pi, \Pi^{\vee}\right)$$

← roots

↑ coweights ↑ weights ↑ coroots ↑ basis for $\phi^{\vee}$

(free ab. $\cong \mathbb{Z}^n$)

Let T be the torus, $G = GL_n(F)$, $K = GL_n(O)$.

$X := T/T\cap K \simeq \mathbb{Z}^n$, $X = \begin{pmatrix} \varpi^{\mathbb{Z}} & & 0 \\ & \ddots & \\ 0 & & \varpi^{\mathbb{Z}} \end{pmatrix}$

Have a perfect pairing btw X and $X^{\vee}$, $\langle \cdot, \cdot \rangle$.

$\Pi^{\vee} = \{\alpha_1^{\vee}, \dots, \alpha_{n-1}^{\vee}\}$, $\alpha_i^{\vee} : \begin{pmatrix} \varpi^{x_1} & & \\ & \ddots & \\ & & \varpi^{x_n} \end{pmatrix} \longmapsto \varpi^{x_{i+1} - x_i}$

↑ this chooses an orientation !

Have a partial order on $\phi^{\vee}$:

$\beta^{\vee} \geq \alpha^{\vee} \iff \beta^{\vee} - \alpha^{\vee}$ is a combination of elts of $\Pi^{\vee}$ with positive coords. (integers)

$\phi^{+\vee} := \{\alpha^{\vee} \geq 0\}$, $\phi^{-\vee} := \{\alpha^{\vee} \leq 0\}$.

b/c we are in GL_n ! ↙ a singleton !

Let $\Pi_m = \{$minimal elements (w.r.t $\leq$) in $\phi\}. = \{-\alpha_1^{\vee} - \alpha_2^{\vee} - \dots - \alpha_{n-1}^{\vee}\}$

To any $\alpha^{\vee} \in \Pi^{\vee}$, there corresponds a reflection $s_{\alpha}^{\vee} : X \longrightarrow X$

$x \longmapsto x - \langle x, \alpha^{\vee} \rangle \alpha$

The finite Weyl group is $W_0 = \langle s_{\alpha}^{\vee} : \alpha \in \Pi^{\vee} \rangle$

(for $G = GL_n$, $W_0 \simeq S_n$).

Note that W_0 acts on X and $X^\vee$, and it preserves $\phi^\vee, \phi$.
Also, X acts on itself by translations, $x \mapsto e^x$.

Df: The <u>extended Weyl group</u>: $W = W_0 \ltimes X$

Classically, the <u>affine Weyl group</u> is $W = W_0 \ltimes \phi$

<u>Affine roots</u>:

Let $\Phi := \Phi^{+\vee} \sqcup \Phi^{-\vee} \subset \phi \times \mathbb{Z}$, where

$$\Phi^{+\vee} = \{(\alpha^\vee, 0) : \alpha^\vee \in \phi^{\vee +}\} \cup \{(\alpha^\vee, k) : \alpha^\vee \in \phi,\ k > 0\}$$
$$\Phi^{-\vee} = \{(\alpha^\vee, 0) : \alpha^\vee \in \phi^{\vee -}\} \cup \{(\alpha^\vee, k) : \alpha^\vee \in \phi,\ k < 0\}$$
$$\overline{\Pi}^\vee = \{(\alpha^\vee, 0) : \alpha^\vee \in \Pi^\vee\} \cup \{(\alpha^\vee, 1) : \alpha^\vee \in \Pi_m\}.$$

For each $\alpha^\vee \in \overline{\Pi}^\vee$, $S_{\alpha^\vee m}$: if in the second type,

$$S_{(\alpha^\vee, 1)} = S_{\alpha^\vee} e^{-\alpha_0}, \quad \alpha_0^\vee = -\alpha_1^\vee - \cdots - \alpha_{n-1}^\vee.$$

W acts on $\Phi^\vee$ by

$$(w = w_0 e^x, (\alpha^\vee, k)) \longmapsto (w_0 \alpha^\vee, k - \langle \alpha^\vee, x \rangle).$$

Ref: Lusztig, "Hecke Algebras and their graded version".

<u>Length on W</u>:

$$\ell(w) = \# \{A \in \Phi^{+\vee} : wA \in \Phi^{-\vee}\}.$$

1) Extends the length on W_0.

2) $\ell(s_i) = 1$ for $i = 1 \ldots n-1$

If $s_0 = S_{(\alpha_0^\vee, 1)}$, $\ell(s_0) = 1$.

3) $\ell(ws) = \begin{cases} \ell(w)+1 & \text{if } w\alpha_s^\vee \in \Phi^+ \\ \ell(w)-1 & \text{o/w} \end{cases}$ $\quad \forall s \in \{s_0, s_1, \ldots, s_{n-1}\}$.

⑫

The affine Weyl group (W_{aff}) is a Coxeter group ind

$$\left(W_{aff}\ S = \{s_0, s_1, \dots, s_{n-1}\}\right).$$

(with relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (with i defined mod n).

Prop: If $w, w' \in W$, then $\ell(w) + \ell(w') = \ell(ww') \Leftrightarrow Iww'I = IwIw'I$.

Proof: ~~Some~~ calculation.

Also, as in the finite case:

Prop: 1) $|IwI/I| = |I\backslash IwI| = q^{\ell(w)}$ (res. field is $\mathbb{F}_q$) $\forall w \in W$.

2) $\ell(\omega) = 0$ because ω normalizes I

$$\left(\omega = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ \varpi & 0 & & 1 \end{pmatrix}\right). \qquad \left(W = \langle\omega\rangle W_{Aff}\right)\ S$$

b/c any $w \in W$ can be written as $w = \omega^k \overbrace{s_{i_1} \cdots s_{i_\ell}}$

and $\ell(w) = \ell$.

Prop: W is a system of representatives of $I\backslash G/I$.

Ref: I. Macdonald, "Spherical functions for p-adic groups";
or
Lecture notes by Casselman (an account of).

Rmk: $I(1)\backslash G/I(1) = W^{(1)} := W \ltimes T(\mathbb{F}_q)$ lifted by Teichmüller.

Hecke algebras:

Let Ω be a COS of G, and let ρ be a smooth rep of Ω.

$$\mathcal{H}(G,\rho) := \mathrm{End}_G(c\text{-}\mathrm{Ind}_{\Omega}^{G}\rho).$$

(eg $\Omega = I$, or $\Omega = K$, with $\rho = 1\!\!1_I$, $\rho = 1\!\!1_K$).

$$\mathrm{Rep}\, G \longrightarrow \mathcal{H}(G,\rho)\text{-modules}$$
$$V \longmapsto V^{\rho} = \{ v \in V : x \cdot v = \rho(x) v \ \forall x \in \Omega \}.$$

This functor is exact if char 0 ($\mathbb{C}$-coeffs), ~~and~~ if $\rho = 1\!\!1_{\Omega}$.

This splits via $p : V \to V^{\Omega}$
$$v \longmapsto \int_{x \in \Omega} \pi(x)\, d\mu(x), \quad \mu \text{ Haar measure on } \Omega.$$

Recall that μ is nonzero, and it is not hard to see that it can't be defined taking values only on $\overline{\mathbb{F}_p}$!

- Equivalences between reps and Hecke modules

Def: A type for $GL_n(F)$ is a pair (Ω, ρ) $\begin{cases} \Omega \text{ cos} \\ \rho \text{ irr. rep of } \Omega \end{cases}$

s.t $\begin{pmatrix}\mathrm{Rep}^{\text{gen}} \\ \text{by } V^{\rho}\end{pmatrix} \to \mathcal{H}(G,\Omega)$

is an equivalence of categories.

Bernstein proved: Rep G decomposes as a product of categories called blocks (full subcategories).

A block is of the form $\{ V \in \mathrm{Rep}\, G$ / generated by ~~the~~ $V^{\rho} \}$.

(closed but not compact)

irred + supercuspidal (ie σ not a constituent of a parabolic induction)

Moreover, if (Ω, ρ) is a type, then $\exists$ Levi subgroup L of $G = GL_n(F)$, $\sigma \in \mathrm{Rep}\, L$, and $P = LN$ an associated parabolic s.t $\forall V \in \mathrm{Rep}\, G$,

$V \in$ block corresp. to $(\Omega, \rho) \iff$ its irred. subquotients are subquotients of $\mathrm{Ind}_P^G \sigma$

An example of a block:

The pair $(I, 1_I)$ is a type:

$$\mathrm{Rep}^I G \longrightarrow \mathcal{H}(G,I)\text{-mod}$$

(Borel, (Inventiones)). + Matsumoto.

The corresponding Levi is T, $P = B$, $\sigma = 1_T$. trivial principal series.

So any irred. rep in $\mathrm{Rep}^I G$ has constituents in $\mathrm{Ind}_B^G 1$

If $V_{irr} \in \mathrm{Rep}^I G$, then V is a constituent of $\mathrm{Ind}_B^G 1$, so V is <u>not</u> supercuspidal.

$\mathbb{F}_p$-coefficients:

$$I(1) = \begin{pmatrix} 1+P & & 0 \\ & \ddots & \\ P & & 1+P \end{pmatrix} \quad \text{pro-}p \text{ subgroup.}$$

V a smooth rep, and let $\underset{\neq 0}{v} \in V$. $\exists m$ s.t. K_m fixes v.

$$\langle I(1) \cdot v \rangle = \langle g \in I(1)/K_m, \ gv \rangle_{\overline{\mathbb{F}}_p} = \text{finite-dimensional.}$$

$\Rightarrow$ can be seen as a rep of $I(1)/K_m$, which is a p-group!

We already saw that $\exists\, w \in \langle I(1) \cdot v \rangle_{\overline{\mathbb{F}}_p}$ which is fixed by $I(1)$.

Feb 8th

Chapter 3: Representation of $GL_2(F)$.

R of char 0 or p.

F a p-adic field. $\mathcal{O}, \mathcal{P}, \varpi$, $k = \mathcal{O}/\mathcal{P}$ as usual.

$B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ $B = TU$, $\bar{B} = T\bar{U}$ ($\bar{U}$ opposite of U).

$I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P} & \mathcal{O}^\times \end{pmatrix}$ & $I(1) = \begin{pmatrix} 1+\mathcal{P} & \mathcal{O} \\ \mathcal{P} & 1+\mathcal{P} \end{pmatrix}$

Iwahori Decomposition: Let H be any c.o.s. of G. Then it has a decomposition:

$$H = H^+ H_0 H^- = (H \cap U)(H \cap T)(H \cap \bar{U}).$$

In particular, I and $I(1)$ have a Iwahori decomposition.

Notation: $U = \{u(x) \mid x \in F\}$, $u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$\bar{U} = \{\bar{u}(x) \mid x \in F\}$, $\bar{u}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$

So for all $a, c \in \mathcal{O}^\times$, $b, d \in \mathcal{O}$,

$$\begin{pmatrix} a & b \\ \varpi d & c \end{pmatrix} = u(b/c) \begin{pmatrix} a - \frac{\varpi d b}{c} & 0 \\ 0 & c \end{pmatrix} \bar{u}\left(\frac{\varpi d}{c}\right)$$

All reps are smooth with coefficients in R. If char $R = p$: no Haar measure on G with values in R

Also, recall that any nonzero mod p rep. of G has a nonzero $I(1)$-invariant vector.

(see Serre, Prop 26).

Prop: Let (π, V) ~~be~~ a rep of G; Suppose $\mathbf{R}$ is of char p.

If V is generated by $\langle V^{I(1)} \rangle_G$ and $V^{I(1)}$ is irreducible as a $\mathcal{H}(G, I(1))$ module, then V is irreducible.

Pf/ If $V' \subset V$ is a nonzero subrep, then:

$(V')^{I(1)} \neq 0$, so $(V')^{I(1)} = V^{I(1)}$. So $V' = V$ since $V^{I(1)}$ generates V.

Barthel-Livne (94-5) used this to classify the irreducible subquotients of $\underbrace{\mathrm{Ind}_B^G \chi}_{\text{principal series}}$, where χ is a character of T.

In the classical theory, an irred representation is called cuspidal ~~(resp supercuspidal~~) if it's not a subrep of a parabolic induction

Barthel-Livné proved also that there are mod p irreducible reps of $GL_2(F)$ which don't appear in principal series. They call them <u>supersingular</u>.

In the classical case, the irreducible cuspidal reps of G are in 1-1 correspondence w/ 2-dim'l irreducible $\mathbb{C}$-reps of $W(\overline{F}/F)$

$\uparrow$ Weil group, dense in $Gal(\overline{F}/F)$

I) <u>Galois representations</u>: (<u>Ref</u>: Bushnell-Henniart)

Let $\mathbf{R}$ be of char $= p$. Let $\overline{F}$ be a separable algebraic closure of F.

Want to study $\mathrm{Rep}_{Gal(\overline{F}/F)}$.

$$\begin{tikzcd}
\overline{F} \arrow[d, dash, "\text{Wild ramif. gp } P \text{ (a prop group).}"] \arrow[dd, dash, bend right=60, "\text{Inertia}"'] \\
F^{tr} \arrow[d, dash] \\
F^{ur} \arrow[d, dash, "Gal(\overline{k}/k)"'] \\
F
\end{tikzcd}$$

By Clifford's theory, any irred. rep of G_F is such that its restriction to $P \triangleleft G_F$ is semisimple. (Ref: Benson, or Curtis-Reiner).

We say that "the irreducible reps are tamely ramified".

Let ρ be a smooth irred. rep of G_F, of dimension n.

Ker ρ is both closed (b/c it's a kernel) and open, so it's a C.O.S.
b/c smooth + fin dim

So $(\overline{F})^{\ker\rho}$ is a finite Galois ext of F which is tamely ramified.

Let $L = (\overline{F})^{\ker\rho}$.

So we get a rep. of a finite group.

L/F has residue field $\mathbb{F}_{q^f}$ ($\mathbb{F}_q$ = residue field of F).

Let F_f/F be the unramified ext of degree F.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Gal}(L/F_f) \arrow[r] & \mathrm{Gal}(L/F) \arrow[r] & \mathrm{Gal}(F_f/F) \arrow[r] & 0 \\
& \text{inertia} \arrow[u] & & \text{cyclic} \arrow[u] & \\
& \text{cyclic order prime to } p \arrow[from=2-4, to=2-2, bend left, "\text{acts by Frobenius } x \mapsto x^q"] & & &
\end{tikzcd}$$

Clifford theory $\Rightarrow 0 \to H \to G \to G/H \to 0$ where H, G/H are both finite (G any finite gp).

Def: A character χ of H is G-regular if $\chi \neq \chi(g\,g^{-1})$ for any $g \in G \setminus H$.

Lemma: Let χ be a character of H.

1) $\mathrm{Ind}_H^G \chi$ is irreducible $\Leftrightarrow$ χ is G-regular.

2) $\mathrm{Ind}_H^G \chi \cong \mathrm{Ind}_H^G \chi'$ with χ, χ' characters $\Leftrightarrow$ χ' and χ are conjugate.

3) An irred rep. ρ of G is always $\mathrm{Ind}_{H'}^G \chi$ for some $H' \triangleright H$, χ some character.

From the lemma, our n-dimensional irreducible rep ρ of G_F is induced from a character of a group $H' \supset \mathrm{Gal}(L/F_\ell)$.

Therefore H' has index n, and

$$(\bar{F})^{H'} = F_n, \text{ the unramified extension of } F \text{ of deg } n.$$

and

$$\rho \simeq \mathrm{Ind}_{G_{F_n}}^{G_F} \chi$$

Since ρ is irreducible, χ is G_F-regular. We get:

<u>Theorem</u>: Up to isomorphism, the n-dim irr. mod p reps of G_F are parametrized by the conjugacy classes of regular characters of G_{F_n}

Moreover:

$$\text{Characters of } G_{F_n} \leftrightarrow \text{Characters of } F_n^\times.$$

$$F_n^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}_n^\times \underset{\substack{\uparrow \\ \text{Teichmüller lift}}}{\cong} \varpi^{\mathbb{Z}} \times \mathbb{F}_{q^n}^\times \simeq \varpi^{\mathbb{Z}} \times (1+\varpi\mathcal{O}_n).$$

So $\chi = \mu_\chi \cdot \lambda$, $\begin{cases} \mu_\chi \text{ trivial on } \mathcal{O}_n^\times \text{ (unramified char).} \quad \overset{\substack{\text{defining} \\ \downarrow \text{ property}}}{(\mu_\chi(\varpi) = x)} \\ \lambda : \mathcal{O}_n^\times \to R^\times \text{ is trivial on } 1+\varpi\mathcal{O}_n, \end{cases}$

So λ can be seen as a character $\mathbb{F}_{q^n}^\times \to R^\times$.

χ regular $\Leftrightarrow$ λ is Frobenius-regular: it can't be factorized through a smaller $\mathbb{F}_{q^n}$, $n < m$.

Also, if $\mathbb{F}_{q^n} = \mathbb{F}_q[\zeta]$, it means also that

$$\lambda(\zeta), \lambda(\zeta)^q, \ldots, \lambda(\zeta)^{q^{n-1}} \text{ are pairwise distinct.}$$

The conjugacy classes of regular characters of G_{F_n} (with a fixed action of ϖ) are parametrized by the irreducible polynomials with coefficients in $\mathbb{F}_q$ of degree n.

<u>Example</u>: For $n=2$, get $\frac{q^2-q}{2}$.

<u>II</u>. Characters of $GL_2(F)$.

<u>Prop</u>: A smooth, finite-dimensional irred rep of $GL_2(F)$ is a character.

Pf The kernel is open, so it contains $U(\mathfrak{p}^m)$ for $m \gg 0$. $= \{u(x) : x \in \mathfrak{p}^m\}$

It also contains $\bar{U}(\mathfrak{p}^m)$.

Conjugating by the torus, get that kernel contains $U(F) = U$, $\bar{U}(F) = \bar{U}$.

<u>Lemma</u>: $\{U, \bar{U}\}$ generate $SL_2(F)$.

Since $GL_2(F)/SL_2(F) \cong F^\times$, we get ρ is an irred rep of $F^\times$.

It's smooth, so ρ is trivial on $1+\varpi^m \mathcal{O}$, $m \geq 0$, so

ρ is a rep of $F^\times/1+\varpi^m\mathcal{O}$ $\cong \pi^{\mathbb{Z}} \times (\text{fin. ab})$

So ρ is a character.

<u>Def</u>: $\lambda : F^\times \to R^\times$ is <u>unramified</u> if it is trivial on $\mathcal{O}^\times$.

Any character $\lambda = \mu_x \cdot \chi$ $\begin{cases} \mu_x \text{ unramified}, \ \mu_x(\varpi) = x \\ \chi : \mathbb{F}_q^\times \to R^\times. \end{cases}$

Remark: The character of $GL_2(F)$ are of the form $\lambda \circ \det$, with $\lambda : F^\times \to R^\times$ smooth.

III. Principal series for $GL_2(F)$. (Char $R = 0$ or p.)

Def: A principal series rep is a rep. $\mathrm{Ind}_B^{GL_2(F)} \lambda_1 \otimes \lambda_2$, $\lambda_1 \otimes \lambda_2 : T \to R^\times$.

Prop: 1) For any V rep of G, $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \lambda) \simeq \mathrm{Hom}_B(V, \lambda)$

2) If $\lambda_1 = \lambda_2$, then $\mathrm{Ind}_B^{GL_2(F)} \lambda_1 \otimes \lambda_1 = (\lambda_1 \circ \det)\, \mathrm{Ind}_B^G \mathbb{1}$.

Rmk: $\mathrm{Ind}_B^{GL_2(F)} \mathbb{1} \cancel{= \mathrm{Ind}_B^B \mathbb{1} \oplus \mathrm{Ind}_B^{BsB} \mathbb{1}}$

b/c $GL_2(F)/B \simeq \mathbb{P}^1(F)$ $\to$ $\wr$ loc. constant

$\mathcal{C}_c^\infty(\mathbb{P}^1(F)) \ni \mathbb{1}_{GL_2(F)}$ ⇝ trivial character of $GL_2(F)$.

Define then $St = \mathrm{Ind}_B^G \mathbb{1} / \langle \text{constants} \rangle$

(for GL_n, $St = \mathrm{Ind}_B^G \mathbb{1} / \sum \mathrm{Ind}_P^{GL_n(F)} \mathbb{1}$, $P \supset B$).

Prop: The irreducible subquotients of the principal series when R has char p are:

1) Characters: $\mathbb{1} \hookrightarrow \mathrm{Ind}_B^G \mathbb{1}$

$\chi \circ \det \hookrightarrow \mathrm{Ind}_B^G \lambda \otimes \lambda$

2) Steinberg: St, $(\lambda \circ \det) \otimes St$

3) Irreducible principal series $\mathrm{Ind}_B^G \lambda_1 \otimes \lambda_2$, $\lambda_1 \neq \lambda_2$.

Rmk
Unlike in the complex case: 1) in char 0 one works with normalized principal series.
$i(\lambda_1 \otimes \lambda_2) = \mathrm{Ind}_B^G (\lambda_1 \otimes \lambda_2) \times \begin{pmatrix}\text{modulus} \\ \text{character}\end{pmatrix}$.

2) For GL_n + char 0, cuspidal $\equiv$ supercuspidal.

In general:

cuspidal : not a <u>subrep</u> of a principal series.

Supercuspidal: not a <u>subquotient</u> of a principal series.

If Char $= 0$, these two conditions are equivalent.

In char p, we call <u>Supersingular</u> an irred. rep which is <u>not</u> a subquotient of a principal series rep.

<u>Goal</u>: prove St is irreducible.

1) Any principal series $\mathrm{Ind}_B^{GL_2(F)} \lambda$ is generated by $(\mathrm{Ind}_B^G \lambda)^{I(1)}$.

Pf $(\mathrm{Ind}_B^G \lambda)^{I(1)} = \langle \{ f_{BwI(1),1} \} \rangle_R$ $w \in B\backslash G / I(1) \cong W_0 \cong S_2$ where we denote by

$f_{BwI(1),1}$ the ~~funct~~ $I(1)$-invariant function with support in $BwI(1)$ and value 1 at w.

So for GL_2, $(\mathrm{Ind}_B^G \lambda)^{I(1)}$ has dimension 2 and basis $\{ f_{BI(1),1} \; f_{BsI(1),1} \}$.

Any $f \in \mathrm{Ind}_B^G \lambda$ has compact open support in $B\backslash G$

Let $t = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}$ (for GL_n we would do $\begin{pmatrix} 1 & & \\ & \varpi & \\ & & \ddots \; \varpi^{n-1} \end{pmatrix}$).

Note that $t I^+ t^{-1} \subset I^+$ and $t I^- t^{-1} \subset I^-$ $\left(\begin{matrix} I^+ = I \cap U \\ I^- = I \cap \bar{U} \end{matrix} \right)$

(Say that t is positive).

The subgroups $\{ t^{-n} I^- t^n \}_n$ are ...

$\begin{pmatrix} 1 & 0 \\ p^{-n-1} & 1 \end{pmatrix}$

We can see that the map

$$\begin{matrix} \mathcal{C}_c^\infty(F,R) & \hookrightarrow & \text{"}\mathrm{Ind}_B^{BsB} 1\text{"} \ (=K) \\ \circlearrowright_B \quad \overset{\cup}{\phi} & & \overset{\cup}{f} \end{matrix}$$

$$\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\phi\right)(x) = \phi\left(\frac{dx+b}{a}\right) \quad ; \quad \phi = \left[x \mapsto f\left(s\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right)\right] \leftarrow\!\shortmid f$$

this turns out to be a isomorphism of reps of B.

$$0 \to \mathcal{C}_c^\infty(F,R) \to \mathrm{Ind}_B^G(1) \to 1\!\!1 \to 0$$

$J_U(V) := V/V(U)$, $B = TU$ is right-exact.

$$J_U(\mathcal{C}_c^\infty(F,R)) \to J_U(\mathrm{Ind}_B^G 1\!\!1) \to J_U(1\!\!1) \to 0$$

But if $\mathrm{char}(R) = p$, then

$J_U(\mathcal{C}_c^\infty(F,R)) = 0$ (otherwise $(\mathcal{C}_c^\infty(F,R))^U \neq 0$, so there is a nonzero linear form on $\mathcal{C}_c^\infty(F,R)$ which is invariant under translation $\Rightarrow$!! (b/c we said there is no Haar measure on F with values in R).

Therefore $\boxed{J_U(\mathrm{Ind}_B^G 1\!\!1) = 1\!\!1.}$

Rmk: in char 0, $J_U(\mathcal{C}_c^\infty(F,R)) = \delta_B^{-1}$ (1-dimensional).

This can be used to prove that St is irreducible.

(See Vigneras "mod p principal series").

As a result:

Cor: $\mathrm{Ind}_B^G \mathbb{1}$ has length 2 as a representation of B.

Cor': $\mathrm{Ind}_B^G 1$ has length 2 as a representation of G. So St is irred.

- Jacquet functor of St:

$$\begin{tikzcd}
0 \arrow[r, dashed] & J_U(1) \arrow[r] & J_U(\mathrm{Ind}_B^G 1) \arrow[r] & J_U(St) \arrow[r] & 0 \\
& \mathrm{Dim}\ 1 \arrow[u] & \mathrm{Dim}\ 1 \arrow[u] & &
\end{tikzcd}$$

because $\bar{f} = 0 \Rightarrow f \in (\mathrm{Ind}_B^G)[U]$, so $f(1) = 0 \Rightarrow f = 0$.

So $\boxed{J_U(St) = 0}$

Therefore St can't be embedded into a principal series because:

Lemma: if V is irreducible smooth, then

$$V_U \neq 0 \Leftrightarrow V \hookrightarrow \text{some } \mathrm{Ind}_B^G \chi.$$

Pf/ $\Leftarrow$] $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) = \mathrm{Hom}_B(V_U, \chi)$

So if $V \hookrightarrow \mathrm{Ind}_B^G \chi$, then $V_U \neq 0$.

$\Rightarrow$] Let $\overset{0}{\overset{\neq}{v}} \in V$. $K = GL_2(\mathcal{O})$. $K \cdot v$ is finite, so $\langle K \cdot v \rangle_{\mathbb{R}}$ generate V as a B-rep.

So V_U is finitely generated as a rep of B.

By Zorn's lemma, there is an irred quotient $V_U \twoheadrightarrow W$, that is a character of the torus, say χ.

Another way of proving the irreducibility of St.

i) $\mathrm{Ind}_B^G \chi$ is generated by $\left(\mathrm{Ind}_B^G \chi\right)^{I(1)}$ (for any χ).

$$\left(\mathrm{Ind}_B^G \chi\right)^{I(1)} = \langle \underset{1 \mapsto 1}{f_{BI(1),1}} , \underset{s \mapsto 1}{f_{BsI(1),1}} \rangle.$$

If $t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$, then

$$t^{-n} I^- t^n = \begin{pmatrix} 1 & 0 \\ p^{n+1} & 1 \end{pmatrix} \subset I^-, \text{ and}$$

$\{t^{-n} I^- t^n\}_n$ are a set of fundamental ~~domains~~ neighborhoods of $\bar{U}$ in U^-.

For $B\backslash G$, $\quad B\backslash B t^{-n} I^- t^n = B\backslash B I(1) t^n$, so for any $k \in K$, $B\backslash B I(1) t^n k$ are a fundamental set of nbhds of the class ~~of~~ Bk.

Let $f \in \mathrm{Ind}_B^G \chi$. Since it is loc. const + compact support,

$f = \sum$ charal functions of $B\backslash B I(1) t^n k$.

$$\text{"} 1_{BI(1)t^n k} \text{"} = (t^n k)^{-1} \underbrace{f_{BI(1),1}}_{I(1)\text{-invariant}}$$

(Schneider-Stuhler, "Cohomology of p-adic symmetric spaces".)

2) $\left(\mathrm{Ind}_B^G \chi\right)^{I(1)}$ as a Hecke module.

Let $\xi = \left(\chi|_{T(\mathbb{F}_q)}\right)^{-1}$.

The Hecke algebra $\mathcal{H}(G, I(1))$ is generated by:

$$\langle \tau_s, \tau_\omega, \tau_t \rangle$$

s reflection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$w = t\omega^k s_{i_1} \cdots s_{i_j} \in W.T(\mathbb{F}_q)$

$\omega = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$

Let V be a rep of G.

$$\begin{array}{ccc} V^{I(1)} & \circlearrowleft & \mathcal{H}(G, I(1)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & & \tau_h \end{array} \qquad h \in W.T(\mathbb{F}_q)$$

$$\phi_v : c\text{-}\mathrm{Ind}_{I(1)}^G 1 \to V$$
$$1_{I(1)} \mapsto v$$

Then $v.\tau_h = (\phi_v \circ \tau_h)(1_{I(1)})$.

Eg: If $h \in T(\mathbb{F}_q)$, $\tau_t(1_{I(1)}) = 1_{I(1)tI(1)} = 1_{I(1)t} = t^{-1} 1_{I(1)}$

So $v\tau_t = t^{-1} \cdot v$

Eg: $V = \mathrm{Ind}_B^G \chi$

$v = f_{BI(1),1}$, $v.\tau_t = t^{-1} \cdot f_{BI(1),1} = \chi(t^{-1}) f_{BI(1),1} = \xi(t) f_{BI(1),1}$.

$\Rightarrow \tau_t$ acts on $\begin{cases} f_{BsI(1)} & \text{by } s\xi(t) \\ f_{BI(1)} & \text{by } \xi(t) \end{cases}$

Further,

- ω normalizes $I(1)$; and

$$f_{BI(1)} \cdot \tau_s = f_{BsI(1)}$$

$$\mathcal{H}(G,I) \circlearrowright$$

Case when χ is unramified, $\xi = 1$.

The torus acts trivially on $(\mathrm{Ind}_B^G 1)^{I(1)} = (\mathrm{Ind}_B^G 1)^{I}$

Set $S = 1_{IsI}$

$\Omega = 1_{I\omega I}$

For $v = f_{BI(1)}$, $S = \begin{array}{cc} v & v\Omega \\ \left[\begin{matrix} 0 & 0 \\ \beta & -1 \end{matrix}\right] \end{array}$ $\Omega = \begin{array}{cc} v & v\Omega \\ \left[\begin{matrix} 0 & \alpha^{-1}\beta^{-1} \\ 1 & 0 \end{matrix}\right] \end{array}$

<u>Rmk</u>: this can be done for GL_n.

<u>Rmk</u>: 1) $(S+1)\Omega + \Omega S$ acts by $\alpha^{-1} \neq 0$.

2) This module is irreducible $\Leftrightarrow \alpha \neq \beta$.

$$\Big(\underbrace{(\mathrm{Ind}_B^G \chi)^{I(1)}}_{\chi \text{ unram}} \Big) \qquad \chi\begin{pmatrix}\pi & \\ & 1\end{pmatrix} = \alpha$$

$$\chi\begin{pmatrix}1 & \\ & \pi\end{pmatrix} = \beta$$

Take $\alpha = \beta = 1$:

$$0 \to \begin{pmatrix} S \mapsto 0 \\ \Omega \mapsto 1 \end{pmatrix} \to (\mathrm{Ind}_B^G 1)^{I(1)} \to \begin{pmatrix} S \mapsto -1 \\ \Omega \mapsto -1 \end{pmatrix} \to 0$$

is exact, but to see St is irred it remains to show:

$(St)^{I(1)}$ is 1-dim: $(\mathrm{Ind}_B^G)^{I(1)} \longrightarrow (St)^{I(1)}$ is surjective.

This will work because we are in GL_2.

Feb 15, 2011

Prop: $(St)^{I(1)} \twoheadleftarrow (\mathrm{Ind}_B^G 1)^{I(1)}$ is surjective.

Pf Let $f \in (\mathrm{Ind}_B^G 1)$ be s.t. the image is $I(1)$-invariant. That is, f is $I(1)$-invariant modulo constants:

$$\forall a \in I(1), \quad af - f = i(a) \in R.$$

This means that $\forall a \in I(1), \forall g \in G, \quad f(ga) - f(g) = i(a)$.

Want to show that f is $I(1)$-invariant: $i(a) = 0 \ \forall a$.

If there is $g \in G$ s.t. $gag^{-1} \in B$, then

$$f(ga) = f(gag^{-1}g) = f(g), \text{ so } i(a) = 0.$$

Therefore i is 0 on
- $I(1)^+ = I(1) \cap U$.
- $I(1)_0 = I(1) \cap T$
- $I(1)^-$ b/c $S I(1)^- S \subset B$.

But $i(ab) = i(a) + i(b)$, so i is identically 0. ∎

Remark: In char $R = 0$, still get $(St)^{I(1)}$ 1-dim, hence irred as a Hecke module. However, one needs to work more to see it's irreducible as a representation. Need to use Borel's result:

$$\left(\text{Borel: } \mathrm{Rep}^{I}(G) \xrightarrow{\sim} \mathcal{H}(G, I)\text{-mod}\right) \leftarrow \text{a \underline{deep} result!}$$

$$V \longmapsto V^I$$

$$M \otimes_{\mathcal{H}} \mathrm{c\text{-}ind}_I^G 1 \longleftarrow M$$

Next we will look at the <u>supersingular reps.</u>

Chapter IV The supersingular reps of $GL_2(F)$.

Def: V smooth is *admissible* (any char) if V^H is finite-dimensional for any c.o.s. H.

(in char 0, irred $\Rightarrow$ admissible, but not known for char p).

Rk: V. Paskunas "Coeff. syst. and supersingular reps": If R has char p, then V admissible $\Leftrightarrow$ $\exists$ a pro-p c.o.s H s.t. V^H is fin. dim.

Recall (Barthel-Livné) (for char $R = p$), if V is irred. admissible, then V^K is fin. dim and $K_1 = 1 + \varpi M_2(\mathcal{O}) \trianglelefteq K$

K/K_1 is finite, so V^{K_1} is a fin. dim rep of a finite group.

$\Rightarrow V^{K_1} \supset \rho$ an irred rep of $GL_2(\mathbb{F}_q)$.

By Frob. reciprocity, $c\text{-ind}_K^G \rho \twoheadrightarrow V$.

Need: study irred quotients of $c\text{-ind}_K^G \rho$.

Start with $\rho = 1_{GL_2(\mathbb{F}_q)}$. Ask also for Z to act trivially.

$c\text{-ind}_{KZ}^G 1$ = functions w/ finite support on $KZ\backslash G$.

Recall that $KZ\backslash G$ may be thought of as the set of vertices of the B-T tree of ~~$GL_2(\mathbb{F}_q)$~~. $GL_2(F)$.

Since $\mathcal{H}(G, KZ) = R[KZ\backslash G/K]$, a basis for $\mathcal{H}(G, KZ)$ is

$$\left\{ T_n = 1_{KZ\begin{pmatrix}1 & \\ & \varpi^n\end{pmatrix}K} \right\}_{n \geq 0}$$

Note now that: (~~$[\sigma_0]$~~ $\sigma_0 = [\mathcal{O}e_1 \oplus \mathcal{O}e_2]$)

$\left\{ kz \begin{pmatrix} 1 & \\ & \varpi^n \end{pmatrix} \cdot \sigma_0 , k \in K, z \in Z \right\}$ is the set of vertices at distance n.

Therefore T_n (seen as an element of $\text{c-ind}_{KZ}^{G} \mathbb{1}$) is the sum of all the vertices at distance n.

Lemma: For $n \geq 2$, ~~$(T_1)^n = T_n$~~. $T_1 T_n = T_{n+1}$

$(T_1^2(\sigma_0) = T_2(\sigma_0) + \sigma_0) \Rightarrow T_1^2 = T_2 + id$

$(\Rightarrow T_2 = T_1^2 - id.)$

Pf/ EZ.

Therefore $\mathcal{H}(G, KZ) = R[T_1]$ (in particular, it's commutative).

Prop: if $\text{c-ind}_{KZ}^{G} \mathbb{1} \twoheadrightarrow V$, then V contains an eigenvector for T_1.

Cor: any irred. quotient of $\text{c-ind}_{KZ}^{G} \mathbb{1}$ is a quotient of

$\pi(\mathbb{1}, \lambda) = \text{c-ind}_{KZ}^{G} \mathbb{1} / T_1 - \lambda$ for some λ.

Thm: if $\lambda \neq 0$, $\pi(\mathbb{1}, \lambda)$ is a principal series (?)

Therefore we don't get anything new.

Thm: If the irreducible quotients of $\pi(\mathbb{1}, 0)$ are not subquotients of principal series.

These are called $\underset{\sim}{\text{supersingular}}$

$\rightarrow$ b/c $\lambda = 0$.

Now (B-L, second paper): work with any ρ.

They prove that $\forall \rho$ irred rep of K/K_1 trivial on Z,

$$\mathcal{H}(G,\rho) = R[T_\rho]$$

Again, consider $\text{c-ind}_{KZ}^{G} \rho \big/ T_\rho - \lambda$ ← generically ppal series ($\lambda \neq 0$)

In 2000, Breuil; $\underline{\lambda = 0}$

$\text{c-ind}_{KZ}^{G} \rho \big/ T_\ell$ are irreducible if $\mathbb{F} = \mathbb{Q}_p$,

and have infinite length otherwise.

⇝ mod p p-adic Langlands conjecture (Breuil-Mézard).

<u>Hecke modules</u>: $V \neq 0 \Rightarrow V^{I(1)} \neq 0$,

↑ action of $\mathcal{H}(G, I(1))$.

- $\text{c-Ind}_{I(1)}^{I} 1 \simeq \bigoplus_{\chi \in \hat{T}(\mathbb{F}_q)} \chi$
- $\forall \chi$ $\varepsilon_\chi : \text{ind}_{I(1)}^{G} 1 \twoheadrightarrow \text{ind}_{I}^{G} \chi \in \mathcal{H}(G, I(1))$ (b/c $\text{ind}_{I(1)}^{G} 1 = \bigoplus_{\chi} \text{ind}_{I}^{G} \chi$)

(rmk ε_χ are orthogonal idempotents).

Mackey: $\text{Hom}_G(\text{c-ind}_I^G \chi, \text{c-ind}_I^G \chi) = 0$ if $\chi \neq \chi'$ and $\chi \neq_s \chi'$.

So if $\varepsilon_\gamma = \sum_{\chi \in \gamma} \varepsilon_\chi$, $\gamma \in \hat{T}(\mathbb{F}_q)/S_2$, then ε_γ are central orthogonal idempotents.

$$\Rightarrow \mathcal{H}(G, I(1)) = \bigoplus_{\gamma} \mathcal{H}(G, I(1)) \cdot \varepsilon_\gamma$$

If $\#\gamma = 1$, then up to twist $\mathcal{H}(G, I(1))\, \varepsilon_\gamma \simeq \mathcal{H}(G, I)$.

This is why sometimes we can change $I(1)$ by I.
Working with I is better, because I is the stabilizer of an edge: if $\omega = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} s$.

$$\mathrm{Stab}(\sigma_1) = IZ.$$

So $c\text{-}\mathrm{Ind}_{IZ}^{G}$ = functions on the oriented edges.

Rmk: G acts transitively on the oriented edges. This is true for GL_2 and GL_3, but not for GL_3!

(and ω ~~reverses orientation~~).
sends σ_1 to $\overline{\sigma_1}$

The element s, seen as an elt. of $c\text{-}\mathrm{Ind}_{IZ}^{G} \mathbb{1}$, is:

$$s(\sigma_1) = \sum_a \begin{pmatrix} 1 & -[a] \\ & 1 \end{pmatrix} s \, \sigma_1$$

(b/c $IZsI = \bigsqcup_{a \in \mathbb{F}_q} IZs \begin{pmatrix} 1 & [a] \\ & 1 \end{pmatrix}$).

Therefore: $s^2(\sigma_1) = (\text{look at the tree}) = q\sigma_1 + (q-1)s(\sigma_1)$
That is: $s^2 = (q-1)s + q$.

As for Ω: $\Omega = 1_{IZ\omega I}$, and ω normalizes I, so $\Omega = 1_{IZ\omega} = \omega^{-1} 1_{IZ}$
$= \omega^{-1}\sigma_1 = \omega\sigma_1 = \overline{\sigma_1} \leftarrow$ opposite edge.

(Therefore Ω sends every edge to its opposite).

$$\Rightarrow \mathcal{H}(G, IZ) = \langle s, \Omega \mid \Omega^2 = 1, s^2 = -s \rangle$$

$\mathcal{H}(G, IZ)$ - simple modules in char p (right-modules)

$S(S+1)=0$

1) S acts by zero $\rightsquigarrow$ $S \to 0$
$\Omega \to \pm 1$

2) S acts by -1 : $S \to -1$
$\Omega \to \pm 1$

3) Otherwise: Ker $S+1$ is stable by ΩS.

So $\exists v \neq 0$ s.t $\begin{cases} v S = -1 \\ v \Omega S = v \lambda \end{cases}$

$\Rightarrow \langle v, v\Omega \rangle$ is stable under S and Ω.

So $\begin{pmatrix} -1 & \mu \\ 0 & 0 \end{pmatrix} \mapsto S$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \Omega$

- If $\mu \neq 0$, nothing new.

Call the module w/ $\mu = 0$ <u>supersingular</u>.

<u>Rmk</u>: The central element $\Omega S + (S+1)\Omega$ acts by μ in general, in particular acts by 0 on the supersingulars.

<u>Rmk</u>: if $\#\gamma = 2$, this can also be done. The simple $\mathcal{H}(G, I(1))\,\varepsilon_\chi$-modules are all 2-dimensional

$\nearrow$ Z_γ doesn't act by 0 : $(\mathrm{Ind}_B^G \lambda)^{I(1)}$

$\searrow$ 1 such that Z_γ acts by 0.

<u>Ref</u>: Vigneras, "Mod p reps of GL_2".

Feb 17

Consider now $\mathcal{F} : \mathrm{Rep}^{I(1)} G \longrightarrow \mathcal{H}(G, I(1)) = \mathcal{H}$.

$$V \longmapsto V^{I(1)}$$

Let $\mathcal{C} = \mathrm{cind}_{I(1)}^{G}$, and consider $\mathcal{G} : \mathcal{H}(G, I(1)) \to \mathrm{Rep}^{I(1)} G$

$$M \longmapsto M \otimes_{\mathcal{H}} \mathcal{C}$$

Rmk: what happens with GL_n?

Rep-side: • principal series: ✓, b/c $(\mathrm{Ind}_B^G \lambda)^{I(1)}$ is easy to compute, and still generates all reps

Thm: $(\mathrm{Ind}_B^G \lambda)$ is irreducible $\iff$ $\lambda_i \neq \lambda_{i+1}$ $i \in \{1, \dots, n-1\}$
$(\lambda = \lambda_1 \otimes \cdots \otimes \lambda_n)$.

Consider $GL_3(\mathbb{Q}_p)$.

$$\left(\mathrm{Ind}_{\begin{smallmatrix}\boxed{2} & \\ & \boxed{1}\end{smallmatrix}}^{GL_3} \rho \otimes \lambda_3\right)^{I(1)}$$ irreducible (Herzig)

For GL_n: $\mathrm{Ind}_{\begin{pmatrix}\boxed{n_1} & * & * \\ & \boxed{n_2} & * \\ & & \boxed{n_3} \\ & & & \boxed{n_k}\end{pmatrix}}^{GL_n} (\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_k)$

irreducible except if $n_i = n_{i+1} = 1$, $\rho_i = \rho_{i+1}$.

Fact: The n-dimensional simple supersingular modules for $\mathcal{H}(GL_n, I(1))$ $\overset{1:1}{\longleftrightarrow}$ n-dimensional Galois reps (fixed action of $\det(\mathrm{Frob})$).

Chapter 5

The functor $\mathcal{F}$, $\mathcal{C}$, Is $\mathcal{C}$ exact? At least, $\mathcal{C}$ is flat over $\mathcal{H}$.

1) Link between the Hecke algebras

$$C = \mathrm{ind}_U^{GL_n(\mathbb{F}_q)} 1 \longrightarrow \mathcal{C}$$

$$1_U \longmapsto 1_{I(1)}$$

(also, $C = \mathrm{ind}_{I(1)}^{K} 1$)

C ↻ K

(Z acts trivially)

Taking $I(1)$ invariants, get

$$\begin{tikzcd} C^{I(1)} \arrow[r] \arrow[d, phantom, "\cong"] & \mathcal{C}^{I(1)} \\ H = \langle T_w, w \in W_0, T_t, t \in T(\mathbb{F}_q) \rangle \arrow[r] & \mathcal{H} \end{tikzcd}$$

called "universal module".

Think of C as functions on $\mathcal{C}$ with support in K.

Prop. $\mathcal{H}$ is a free right module over H.

Proof: There is a system of reps $\mathcal{D}$ of W/W_0 with minimal length.

$$\forall d \in \mathcal{D}, \ \ell(dw_0) = \ell(d) + \ell(w_0).$$

Actually, $\mathcal{D} = \{ d \in W : d\Phi^+ \subset \Phi^+ \}$ + use induction.

Eg: For GL_2, $\mathcal{D} = \{ (sw)^k, w(sw)^k \}$, ; $W_0 = \langle s, t_t, t \in \mathbb{F}_q \rangle$

(and check $\ell((sw)^k s) = \ell((sw)^k) + 1$

$\ell(w(sw)^k s) = \ell(w(sw)^k) + 1$.

→ To finish proof: if $w \in W$, $w = dw_0$, $d \in \mathcal{D}$, $w_0 \in W_0$,

$T_w = T_{dw_0} + T_d T_{w_0}$ ⟹ $\mathcal{H}$ is free over H with basis $\{T_d\}_{d \in \mathcal{D}}$.

Corollary: H is a direct factor of $\mathcal{H}$ as an H-module.

Pf. $\mathcal{H} = H \oplus \overbrace{\bigoplus_{\substack{d\in\mathcal{D}\\ d\neq 1}} \tau_d H}^{\text{stable under } H}$. ∎

Corollary: Let $\mathcal{M}$ be an H-module. (left)

$\mathcal{M}$ is H-flat $\Leftrightarrow$ $\mathcal{H}\otimes_H \mathcal{M}$ is $\mathcal{H}$-flat. (and then both are projective)

$\Rightarrow]$ $\mathcal{M}$ H-flat $\Rightarrow$ $\mathcal{M}$ H-projective (H artinian)!

So $\mathcal{H}\otimes_H \mathcal{M}$ projective.

$\Leftarrow]$ ~~Pf~~ Let A be an ideal (right) of H. Need to prove that

$A \otimes_H \mathcal{M} \to \mathcal{M}$ is injective.

Let $\mathcal{A} = A\mathcal{H} = A\otimes_H \mathcal{H}$.

Know that
$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{H}\otimes_H \mathcal{M} \arrow[r, hook] \arrow[d, equal] & \mathcal{H}\otimes_H \mathcal{M} \\
\mathcal{A}\otimes_H \mathcal{M} \arrow[ur] &
\end{tikzcd}$$
← factors through $\mathcal{A}\otimes\mathcal{M} \hookrightarrow H\otimes_H \mathcal{M} = \mathcal{M}$. ∎

Application: C is H-flat for GL_2 ~~(even projective)~~

So $\mathcal{H}\otimes_H C$ is flat (even projective) over $\mathcal{H}$.

$$\begin{array}{c} \| \\ \bigoplus_{d\in\mathcal{D}} \tau_d C \longrightarrow \mathcal{C}^{K_1} \end{array} \qquad (K_1 = 1+\varpi M_2(\mathcal{O}))$$

Prop: $\mathcal{H}\otimes_H C \xrightarrow{\sim} \mathcal{C}^{K_1}$ is an isomorphism.

Rmk: this is also true for GL_n.

~~Proof~~ Proof:

Injectivity: i) $\bigoplus \tau_d C \longrightarrow \mathcal{C}$

τ_d ~~C~~ C has support in $I(1)dK$, and $\mathcal{D} \cong W/W_0 \simeq {}_{I(1)}\backslash G/K$.

So all the double cosets are disjoint.

$\therefore$ just need to prove that $\tau_d : C \longrightarrow \mathcal{C}$ is injective.

But it's K-invariant, and $\tau_d|_{C^{I(1)}}$ ~~$I(1)$~~ $\simeq H \hookrightarrow \mathcal{H} \subset \mathcal{C}$. is injective b/c

$$\tau_d\left(\sum \lambda_{w_0} \tau_{w_0}\right) = \sum \lambda_{w_0} \underbrace{\tau_{dw_0}}$$

~~T~~these have disjoint support. ✓.

Surjectivity:

Follows from an argument using the BT tree. However, the proof also works for GL_n:

Let $f = 1_{I(1)gK_1}$. We will find an "edge" $e = 1_{I(1)x}$ such that $f \in \mathcal{H} \cdot e = \mathcal{C}^{x^{-1}I(1)x}$, and such that $K_1 \subset x^{-1}I(1)x$.

That is, need: $\begin{cases} K_1 \subset x^{-1} I(1) x \\ I(1) g K_1 x^{-1} I x = I(1) g K_1 \end{cases}$

$g \in G$ belongs to some $I(1) w I($~~4~~$)$, $w \in \tilde{W}$.

We have a nice system of reps of $\tilde{W}/W_0$, namely $\mathcal{D}$.

So $w = dw_0$, $d \in \mathcal{D}$, $w_0 \in W_0$. $\Rightarrow g = I(1) d w_0 \underset{I(1)}{\underset{\uparrow}{\alpha}}$, $w_0 \alpha \in K$.

$\Rightarrow f = 1_{I(1) d w_0 \alpha K_1} = 1_{I(1) d K_1 w_0 \alpha}$.

We suppose that $g = d$ (i.e. $w_0 \alpha = 1$) (if we solve it for $\underset{\mathcal{H}_{I(1)g}}{\underset{\uparrow}{1_{I(1)dK_1}}}$, then act by $(w_0 \alpha)^{-1}$).

The condition translates then into: (use I to mean $I(1)$)

$$\begin{cases} K_1 \subset x^{-1} I(1) x \\ I(1)\, d\, x^{-1} \cancel{I x} = I d K_1 \end{cases} \iff \boxed{K_1 \subset x^{-1} I x \subset d^{-1} I d K_1}.$$

For $d \in \mathcal{D}$, know that $d\, \phi^{+v} \subset \Phi^{+v}$, so $d I^{+} d^{-1} \subset I$.

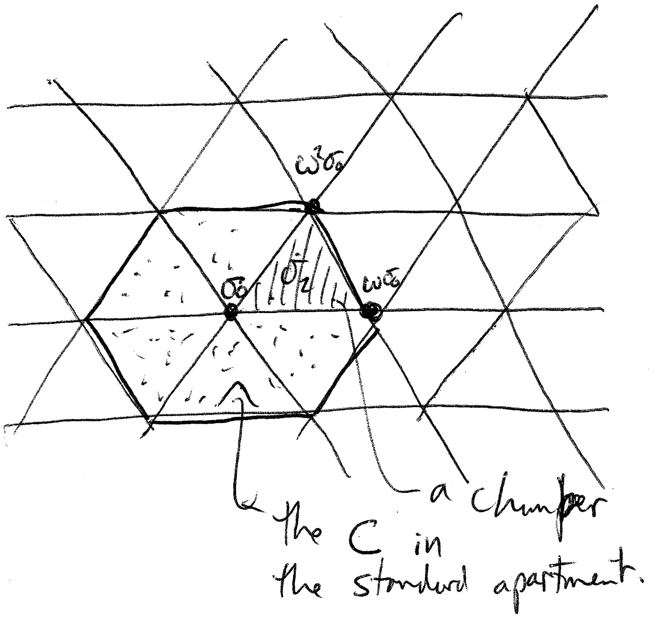
$\Rightarrow I^{+} \subset d^{-1} I d$. So $x = 1$ works !!

$$K_1 \subset I = I \bar{I}_0 I$$

$\blacksquare$

<u>Rmk</u>: For GL_n, the "picture proof" doesn't work anymore. Here's the picture for GL_3:

<u>Standard apartment for $GL_3(\mathbb{F}_q)$:</u>



$$\omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi & 0 & 0 \end{pmatrix}$$

$$\omega\sigma_0 = [\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \varpi\mathcal{O}e_3]$$

$$\omega^2\sigma_0 = [\mathcal{O}e_1 \oplus \omega\mathcal{O}e_2 \oplus \omega\mathcal{O}e_3]$$

<u>Open problem</u>: adapt the proof above to make it geometric on GL_3!

Recall that $\mathcal{C} = \varinjlim \mathcal{C}^{K_m}$. But we need to get flatness of $\mathcal{C}$, and we will need to work harder.

3) Coefficient systems on the tree

For σ a vertex or edge, let $K(\sigma) \subset G$ be its stabilizer.

$K(\sigma_0) = KZ$

$K(\sigma_1) = \langle IZ, \omega \rangle$

Def: A coefficient system on the tree is given by:

- R-vectorspaces $(V_\sigma)_\sigma$ σ a simplex (vertex or edge).
- restriction maps: if $\tau \subset \sigma$, $r^\sigma_\tau : V_\sigma \to V_\tau$, $r^\sigma_\sigma = id_{V_\sigma}$.

Def: An action of G on the coeff. system $(V_\sigma)_\sigma$ is

$g_\sigma : V_\sigma \to V_{g\sigma}$ R-linear $(\forall g \in G)$.

s.t.

$$\begin{tikzcd} V_\sigma \arrow[r] \arrow[d, "r^\sigma_\tau"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{g\sigma} \arrow[d, "r^{g\sigma}_{g\tau}"] \\ V_\tau \arrow[r] & V_{g\tau} \end{tikzcd}$$

$+ \; g_{h\sigma} \circ h_\sigma = (gh)_\sigma$

Homology: (Ref: Schneider-Stuhler, "Resolutions, ...").

Let $\mathcal{V} = (V_\sigma)_\sigma$ be a c.s. Let $i \in \{0,1\}$. $\mathcal{F}_i(\mathcal{V}) = \bigoplus_{\substack{\sigma \\ \dim\sigma = i}} V_\sigma$.

$\partial : \mathcal{F}_1(\mathcal{V}) \to \mathcal{F}_0(\mathcal{V})$

$$f \mapsto \Big(\tau \mapsto \sum_{\substack{\sigma \supset \tau \\ \dim\sigma = 1}} [\sigma : \tau] r^\sigma_\tau(f(\sigma))\Big),$$

$[\sigma:\tau]$ = incidence number $\in \{\pm 1\}$ (so that this becomes a complex).

We obtain an exact sequence:

$$0 \to H_1(\mathcal{V}) \to \mathcal{F}_1(\mathcal{V}) \xrightarrow{\partial} \mathcal{F}_0(\mathcal{V}) \to H^0(V) \to 0.$$

Rmk: if $\tau \in \sigma$, then r_τ^σ is conjugate to $r_{\sigma_0}^{\sigma_1}$,

$\quad \; \|$

$g\sigma_0 \subset g\sigma_1$

so a G-equivariant coefficient system is determined by what happens at the origin.

This is used by Paskunas, to produce actual coefficient systems.

Lemma: If $r_{\sigma_0}^{\sigma_1}$ is injective, then $H_1(\mathcal{V})=0$.

Pf Let $f \in \mathcal{F}_1(\mathcal{V})$. Suppose that $\partial(f)=0$.

Take m minimal s.t $\operatorname{supp}(f) \subset B(o,m)$ ← ball in the tree of radius m

So there is a vertex v at distance m of σ_0 s.t v is contained in only one edge e of the support of f.

$$\underset{\underset{0}{\|}}{(\partial f)(v)} = r_v^e(f(e)) \Rightarrow f(e) = 0 \Rightarrow !! \text{ b/c } e \text{ is in the support of } f.$$

∎

Example: To a simplex σ, associate U_σ a pro-p-group, the unipotent radical of $\kappa(\sigma)$ (eg $\sigma = g\sigma_0 \Rightarrow U_{\sigma_0} = gK_1g^{-1}$
$\sigma = g\sigma_1 \Rightarrow U_{\sigma_1} = gI(1)g^{-1}$)

Let (π, V) be a smooth representation of $GL_2(F)$.

Define $V_\sigma = V^{U_\sigma} \circlearrowleft \kappa(\sigma)$ (note $U_\sigma \triangleleft k(\sigma)$)

In this example,

$r^{\sigma_1}_{\sigma_0} : \pi^{I(1)} \hookrightarrow \pi^{K_1}$, so we can apply the lemma, to get:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1(\mathcal{V}_\pi) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{F}_0(\mathcal{V}_\pi) \arrow[r] \arrow[d, phantom, "\parallel"] & H_0(\mathcal{V}_\pi) \arrow[r] & 0 \\
 & \bigoplus\limits_{\sigma \text{ edges}} \pi^{U_\sigma} & \bigoplus\limits_{\sigma \text{ vertex}} \pi^{U_\sigma} \arrow[r] \arrow[dr, two heads] & H_0(\mathcal{V}_\pi) \arrow[d, "\varepsilon"] & \\
 & & & \pi &
\end{tikzcd}$$

(Rmk: if $R = \mathbb{C}$, then $\pi \simeq H_0(\mathcal{V}_\pi)$ (if π has central character).)

In particular, let $\pi = \mathcal{X}$. We get then:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{\sigma \text{ edge}} \mathcal{X}^{U_\sigma} \arrow[r] & \bigoplus\limits_{\sigma \text{ vertex}} \mathcal{X}^{U_\sigma} \arrow[r] \arrow[dr, two heads] & H_0(V_{\mathcal{X}}) \arrow[r] \arrow[d, "\varepsilon"] & 0 & (!) \\
 & \mathcal{H} \arrow[u, bend left] & \mathcal{H} \arrow[u, bend left] & \mathcal{X} & &
\end{tikzcd}$$

⇝ a complex of left $\mathcal{H}$-modules.

The middle term is (as $\mathcal{H}$-module): $\bigoplus\limits_{g \in G/KZ} g\, \mathcal{X}^{K_1}$. So flat $\Leftrightarrow q = p$.

The left term is flat (as $\mathcal{H}$-module) always, b/c it's $\bigoplus\limits_{g \in G/\langle I, \omega\rangle} g\, \mathcal{X}^{I(1)}$.

<u>Proposition</u>: $\mathcal{X}$ is $\mathcal{H}$-flat $\Leftrightarrow q = p$, in which case it is projective.

Pf/ Based on ~~considering~~ proving first that $H_0(V_{\mathcal{X}}) \simeq \mathcal{X}$ as an $\mathcal{H}$-module. (See Schneider-Stuhler

↓

(cont pf)

For every $m \geq 0$, set $\lambda_m = \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix}$.

The vertices at distance m are precisely $\{ k \lambda_m \cdot \sigma_0 : k \in K \}$.

For any vertex σ at distance m, set e_σ the only edge ~~at distance m cont~~ in the ball $B(\sigma_0, m)$ containing σ. For σ_0, choose $e_{\sigma_0} = \{ \sigma_0, \lambda_{-1} \sigma_0 \} = \sigma_1$ (stabilized by $\langle \varpi, I \rangle$).

<u>Lemma</u>: For $g \in G$, $g\sigma_0$ is at distance $\leq m$

$\Leftrightarrow K_m \subset gKg^{-1}$

$\Leftrightarrow K_{m+1} \subset gK_1 g^{-1}$

Also, $g e_{\sigma_0} = g\sigma_1 \in B(\sigma_0, m)$

$\Leftrightarrow K_m \subset g I(1) g^{-1}$

Granting the lemma, let now σ be a vertex at distance m.

$$\mathscr{C}^{U_\sigma} / \mathscr{C}^{U_{e_\sigma}} \hookrightarrow \mathscr{C}^{K_{m+1}} / \mathscr{C}^{K_m} \quad , \left(\mathscr{C}^{U_{e_\sigma}} = \mathscr{C}^{K_m} \cap \mathscr{C}^{U_\sigma} \right).$$

Pf/ write $\sigma = \lambda_m \sigma_0$, $e_\sigma = \lambda_m \sigma_1$, so $\lambda_m \mathscr{C}^{I(1)} = \mathscr{C}^{K_m} \cap \lambda_m \mathscr{C}^{K_1}$ ...

Now, consider $$\bigoplus_{\substack{\sigma \text{ vertex} \\ \text{at dist. } m}} \mathscr{C}^{U_\sigma} / \mathscr{C}^{U_{e_\sigma}} \xrightarrow{(*)} \mathscr{C}^{K_{m+1}} / \mathscr{C}^{K_m}.$$

Prop: the map $(*)$ is injective $\overset{(\forall m)}{\Leftrightarrow}$ the complex (!) is exact.

Pf Spse that $(*)$ is injective, and let f be a 0-chain s.t $\varepsilon(f) = 0$.

$\varepsilon(f) = \sum_\sigma f(\sigma) = 0$. If $f \neq 0$, let m be smallest s.t

Supp $f \subset B(0, m)$.

↓

If $m = 1$, get

$$\underbrace{\sum_{\sigma \text{ at dist } 1} f(\sigma)}_{\in \mathcal{C}^{U_\sigma}} + \overset{\in \mathcal{C}^{K_1}}{f(\sigma_0)} = 0$$

But $$\bigoplus_{\sigma \text{ dist } 1} \mathcal{C}^{U_\sigma} \big/ \mathcal{C}^{U_{e_\sigma}} \hookrightarrow \mathcal{C}^{K_2} \big/ \mathcal{C}^{K_1}$$

$\Rightarrow f(\sigma) \in \mathcal{C}^{U_{e_\sigma}} \ \forall \sigma$ at dist 1.

Let v be the 1-chain supported in $B(0,1)$ s.t. $e_\sigma \mapsto f(\sigma)$.

Then done.

For $m > 1$, do induction.

Conversely, suppose that the complex is exact.

Want to show that $\forall m \geq 1$,

$$\bigoplus_{\sigma \text{ at dist } m} \mathcal{C}^{U_\sigma} / \mathcal{C}^{U_{e_\sigma}} \hookrightarrow \mathcal{C}^{K_{m+1}} \big/ \mathcal{C}^{K_m}$$

Let $m \geq k$, and let $(v_\sigma)_{\sigma \text{ vertices at dist } m}$ s.t. $v_\sigma \in \mathcal{C}^{U_\sigma}$, and such that $\sum v_\sigma \in \mathcal{C}^{K_m}$.

<u>Lemma</u>: $\forall m$, $\mathcal{C}^{K_m} = \sum_{\substack{\sigma \text{ vertices} \\ \text{at dist } \leq m-1}} \mathcal{C}^{U_\sigma}$ ← does this work in general (for GL_n?)

~~Pf~~ after a couple paragraphs.

So $\exists (v_\sigma)_{\substack{\sigma \text{ vert} \\ \text{at dist } \leq m-1}}$ s.t. $\sum_{\text{at dist } m} v_\sigma + \sum_{\substack{\sigma \text{ dist} \\ \leq m-1}} v_\sigma = 0$

Let f be the 0-chain $f(\sigma) = v_\sigma \ \forall \sigma$ at dist $\leq m$.

Then $f \in \operatorname{Ker} \varepsilon$, so $f \in \operatorname{Im} \partial \Rightarrow \checkmark$.

Corollary: $\mathcal{C}$ is flat $\Leftrightarrow q = p$, in which case it's also projective.

Pf/ $\mathcal{C} = \varinjlim \mathcal{C}^{K_m}$, $\quad \mathcal{C}^{K_{m+1}} / \mathcal{C}^{K_m} \underset{\substack{\uparrow \\ \text{as left } \mathcal{H}\text{-modules.}}}{\cong} \bigoplus \mathcal{C}^{K_1} / \mathcal{C}^{I(1)}$

$$\text{Also, } \mathcal{C}^{K_1} = \mathcal{H} \otimes_H \overset{\substack{\text{flat/proj} \Leftrightarrow q=p \\ \downarrow}}{C}$$

$$\begin{tikzcd} \mathcal{C}^{K_1} = \mathcal{H} \otimes_H C \\ \mathcal{C}^{I(1)} = \mathcal{H} \otimes_H H \arrow[u, hook] \end{tikzcd}$$

So $\mathcal{C}^{K_1} / \mathcal{C}^{I(1)} \cong \mathcal{H} \otimes (C/H)$

H is itself injective $\Rightarrow H \hookrightarrow C \Rightarrow C/H$ is a direct factor.

$\Rightarrow$ If $q = p$, then $\mathcal{C}^{K_{m+1}} / \mathcal{C}^{K_m}$ is projective. So $\mathcal{C}$ is projective $\boxtimes$

We still need ~~to~~ prove the Lemma that we stated above:

Lemma: ~~the complex cont~~ $\mathcal{C}^{K_m} = \sum\limits_{\substack{\sigma \text{ vert} \\ \text{at dist} \\ \leq m-1}} \mathcal{C}^{U_\sigma}$

Pf/ Let $f \in \mathcal{C}^{K_{m+1}}$, $f = 1_{I(1) g K_{m+1}}$.

First to simplify, let's assume $f = 1_{Ig} = g^{-1} \cdot \overset{\substack{e \text{ an oriented edge} \\ \downarrow}}{e}$ is K_{m+1}-inv.

$\to$ if f is K_m-inv, done by induction.

$\to$ if not, $\exists$ vertex $x\sigma_0$ at distance m s.t. $x^{-1} f \in \mathcal{C}^{K_1}$ (i.e. $f \in \mathcal{C}^{U_x}$ ✓).

In the general case, $f = 1_{I g K_{m+1}}$ not K_m-inv.

Want x s.t. $f \in x \cdot \mathcal{C}^{K_1}$. Since K normalizes K_{m+1}, may suppose that $g = d \in D$.

March 1st, 2011

Rmk: From what we have seen, the functors

$$\begin{array}{rcl} & \mathrm{Rep}^{I(1)} GL_2(F) \longrightarrow \mathcal{H}\text{-mod} & \\ \mathcal{F}: & V \longmapsto V^{I(1)} & \\ & M \otimes_{\mathcal{H}} \mathcal{C} \longleftarrow M & : \mathcal{T} \end{array}$$

are an equivalence of categories $\iff q = p$. (b/c $\mathcal{T}$ is <u>not</u> exact when $q \neq p$)

Thm: if $F = \mathbb{Q}_p$, then $\mathcal{F}$ induces an equivalence of categories between reps and $\mathcal{H}$-modules with central character.

Cor: For $GL_2(\mathbb{Q}_p)$ we can construct $\frac{p(p+1)}{2}$ supersingular reps (corresp. to the supersingular modules) (numerical Langlands correspondence).

Idea (of pf of Thm):

If we check that $(M \otimes_{\mathcal{H}} \mathcal{C})^{I(1)} \cong M$, then $\mathcal{F}$ & $\mathcal{T}$ are quasi-inverses: if the above is true, then take V st $\langle V^{I(1)} \rangle = V$. Then:

$$V^{I(1)} \otimes_{\mathcal{H}} \mathcal{C} \xrightarrow{\varphi} V \quad \text{and} \quad \varphi\big|_{(V^{I(1)} \otimes_{\mathcal{H}} \mathcal{C})^{I(1)}} = \varphi\big|_{V^{I(1)}} \text{ is}$$

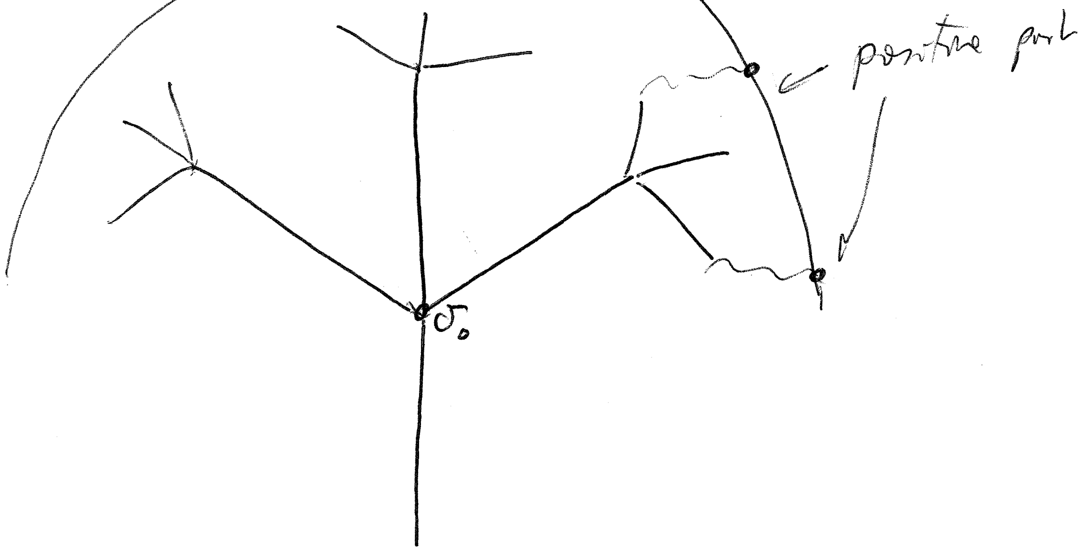
injective. Therefore φ is injective.

Since it's always surjective, it's an isomorphism. ✓.

We need to do computations on the tree to prove that for M an $\mathcal{H}$-module,

$$(M \otimes_{\mathcal{H}} \mathcal{C})^{I(1)} = M. \quad (*)$$

$\downarrow$



Let ρ be an admissible rep of $GL_2(\mathbb{Q}_p)$: ρ^{K_1} is fin. dim, with action of K.

Since $K/K_1 \cong GL_2(\mathbb{F}_q)$, there is an irred rep of $GL_2(\mathbb{F}_q)$ s.t $\sigma \subset \rho^{K_1}$.

$$\text{c-ind}_{KZ}^{G}\,\sigma \twoheadrightarrow \rho$$

eg: if $\sigma = 1\!\!1$, then $\mathrm{End}_G(\text{c-ind}_{KZ}^{G} 1) \cong \mathcal{H}(G, KZ) \cong R[T]$.

and $\exists\, \lambda \in R$ s.t $\text{c-ind}_{KZ}^{G} 1\!\!1 / T-\lambda \twoheadrightarrow \rho$.

This is actually true for any σ (Bartel-Livné).

When $\lambda = 0$, Breuil proved that $\text{c-Ind}_{KZ}^{G} 1\!\!1/T$ is irreducible, b/c its $I(1)$-invariants is an irreducible Hecke module.

The vertices at dist m:

$$K\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\sigma_0 \quad , \quad K = I \cup IsI$$

$$= I\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\sigma_0 \cup Is\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\sigma_0 .$$

$$= \underbrace{I(1)\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\sigma_0}_{\text{positive part}} \cup \underbrace{I(1)s\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\sigma_0}_{\text{negative part}}.$$

We can write down an explicit set of reps for $I(1)\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\sigma_0 =$

$= \left\{ x\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\sigma_0 ,\ x \in I(1) \right\}$.

Need $x \in \dfrac{I(1)}{I(1) \cap \left(\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix} KZ \begin{pmatrix}\varpi^{m} & \\ & 1\end{pmatrix}^{-1}\right)} = \dfrac{I(1)}{I(1) \cap \begin{pmatrix}\mathcal{O} & \varpi^m\mathcal{O} \\ \varpi^{-m}\mathcal{O} & \mathcal{O}\end{pmatrix}}$

$= \dfrac{I(1)}{\begin{pmatrix}1+\varpi\mathcal{O} & \varpi^m\mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O}\end{pmatrix}}$

We take $x = \begin{pmatrix}1 & a(v) \\ 0 & 1\end{pmatrix}$, $u \in \mathbb{F}_q^m$, $a(u) = [u_0] + [u_1]\varpi + \dots + ~~[u_{m-1}]\varpi^{m-1}~~$ $+ [u_{m-1}]\varpi^{m-1}$

$\lrcorner$

...

$\sqcup$

VI: What about $GL_2(F)$, with $F \neq \mathbb{Q}_p$? (Paskunas, Breuil-Paskunas)

Start with an $\mathcal{H}$-module $\mathcal{M}$.

Can associate a) ρ, a semisimple rep of $GL_2(\mathbb{F}_p)$, $= K/K_1 \to$ rep of KZ.

b) an action of $\langle I, w\rangle$

$\uparrow$ trivial action $\nwarrow$ $w.x = x \cdot \Omega^{-1}$. $(\Omega \in \mathcal{H})$

Note that $KZ = k(\sigma_0)$ and $\langle I, w\rangle = k(\sigma_1)$

$$\begin{tikzcd} \rho^{I(1)} \arrow[loop above, "\langle I,w\rangle"] \arrow[r, hook] \arrow[d, phantom, "\wr"] & \rho \arrow[loop above, "KZ"] \\ \mathcal{M} & \end{tikzcd}$$

$\swarrow$ a "diagram", equivalent to a coefficient system on the tree.

6.1: Diagrams & Coefficient systems.

We will see that the category of diagrams is equivalent to the category of coefficient systems (G-equiv).

IZ-equiv

$$\left\{\begin{tikzcd} D_1 \arrow[r, "r_0^1"] \arrow[loop below, "\langle I, w\rangle"] & D_0 \arrow[loop below, "KZ"] \end{tikzcd}\right\}.$$

We will consider oriented coefficient systems: $(V_\tau)_{\tau \text{ simplex}}$.

- $r_\tau^\sigma : V_\sigma \to V_\tau$, $\tau \subset \sigma$; $\tau_\sigma^\sigma = id_{V_\sigma}$.
- G-action: $\forall g \in G$, $g_\sigma : V_\sigma \to V_{g\sigma}$ st

$$\begin{tikzcd} V_\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V_{g\sigma} \arrow[d] \\ V_\tau \arrow[r] & V_{g\tau} \end{tikzcd} \quad + \quad 1_\sigma = id_{V_\sigma} \quad + \quad \cancel{g_{h_\sigma} h_\sigma =} \\ g_{h_\sigma} \circ h_\sigma = (gh)_\sigma.$$

This gives a $K(\sigma)$ action on V_σ $(K(\sigma) = \mathrm{stab}_G(\sigma))$.

Recall that $(V_\tau)_\tau$ is said to be G-equivariant if the actions of $K(\sigma)$ are smooth (on V_σ).

Rk: if $(\tau \subset \sigma) = g(\sigma_0 \subset \sigma_1)$, then $r_\tau^\sigma = g_{\sigma_0} \circ r_{\sigma_0}^{\sigma_1} \circ (g^{-1})_\sigma$.

Denote by $\mathcal{X}$ the tree, and
$$\begin{cases} \mathcal{X}_{(0)} = \text{set of vertices} \\ \mathcal{X}_1 = \text{set of edges} \\ \mathcal{X}_{(1)} = \text{set of oriented edges.} \end{cases}$$

0-chains: $Ch(\mathcal{X}_{(0)}, \mathcal{V}) = \{ \mathcal{X}_{(0)} \to \bigoplus_{\sigma \in \mathcal{X}_0} V_\sigma$ with finite support $\}$.

1-chains: $Ch(\mathcal{X}_{(1)}, \mathcal{V}) = \{ \mathcal{X}_{(1)} \to \bigoplus_{\sigma \in \mathcal{X}_1} V_\sigma$ with finite support st $f((\sigma, \sigma')) = -f(\sigma', \sigma) \}$.

Then G acts on the 0-chains:

$$\cancel{(g\cdot f)(g\sigma)} \qquad (g\cdot f)(g\sigma) = g_\sigma f(\sigma).$$

and on 1-chains:

$$(g\cdot f)(g(\sigma,\sigma')) = g_{\{\sigma,\sigma'\}} \cdot f((\sigma,\sigma')).$$

Define $\partial : Ch(\mathcal{X}_{(1)}, \mathcal{V}) \to Ch(\mathcal{X}_{(0)}, \mathcal{V})$

$$f \longmapsto \sigma \mapsto \sum_{\sigma'} r_{\{\sigma\}}^{\{\sigma,\sigma'\}} f((\sigma,\sigma'))$$

<u>Lemma</u>: ∂ is G-equivariant.

<u>Lemma</u>: if $r_{\sigma_0}^{\sigma_1}$ is injective, then $H_1(\mathcal{X},\mathcal{V}) = 0$.

EX Generalize to GL_n!

<u>Lemma 1</u>: ~~$r_{\sigma_0}^{\sigma}$ is injective~~ Suppose that $r_{\sigma_0}^{\sigma_1}$ is <u>injective</u>.
If f is a 0-chain supported on a single vertex, then $f \notin \mathrm{Im}\,\partial$.

<u>Lemma 2</u>: Suppose that $r_{\sigma_0}^{\sigma_1}$ is <u>surjective</u>.
Let f be a 0-chain. Then there is f_0, a 0-chain supported on (at most) one vertex, and s.t. $f + \mathrm{Im}\,\partial = f_0 + \mathrm{Im}\,\partial$.

Lemmas 1 & 2 allow us to prove:

<u>Prop</u>: if $r_{\sigma_0}^{\sigma_1}$ is bijective, then

$$H_0(\mathcal{X},\mathcal{V})\big|_{k(\sigma_0)} \simeq V_{\sigma_0}$$
$$\wr$$
$$H_0(\mathcal{X},\mathcal{V})\big|_{n(\sigma_1)} \simeq V_{\sigma_1}$$

~~Pf Not too hard~~. (See next page)

We will apply this to $V_\tau := \mathrm{inj}_{\mathcal{J}K}^{\sigma^\omega}\, \rho$.

we'll see what this is.

Pf of prop:

Since $r^{\sigma_1}_{\sigma_0}$ is injective,

$$\begin{tikzcd}
\mathcal{F}(\sigma_0, V_{\sigma_0}) \arrow[r, hook, "\iota"] \arrow[d, "ev_0"', "\wr"] & H_0(\mathcal{X}, \mathcal{V}) \\
V_{\sigma_0} \arrow[ur, hook, "KZ\text{-equiv.}"'] &
\end{tikzcd}$$

Since $r^{\sigma_1}_{\sigma_0}$ is surjective, the map $V_{\sigma_0} \to H_0(\mathcal{X}, \mathcal{V})$ is also surjective.

We also need to check that the map $\psi = \iota \circ ev_0$ is

$\langle I, \omega \rangle$ equivariant.

Let $v \in V_{\sigma_0}$, Then $\psi(\omega v) = f_{\sigma_0, \omega v} + \mathrm{Im}\, \partial$

$$\omega \psi(v) = f_{\omega\sigma_0, \omega v} + \mathrm{Im}\, \partial$$

But $f_{\sigma_0, \omega v} - f_{\omega \sigma_0, \omega v} = \partial(f_{\sigma_1, x})$, where $x \in V_{\sigma_1}$ is s.t. $r^{\sigma_1}_{\sigma_0}(x) = \omega v$. ✓.

$\square$

Rmk: if $\mathcal{V} = (V_\tau)_\tau$ is a G-equiv. coeff system, we obtain a diagram

$$V_{\sigma_1} \xrightarrow{r^{\sigma_1}_{\sigma_0}} V_{\sigma_0} \quad \text{is a diagram.}$$

Start now from a diagram

$$D_1 \xrightarrow{r} D_0 .$$

Consider $\text{c-ind}^G_{KZ} D_0 = \bigoplus_{\sigma \text{ vertices}} D_0$

Let σ be a vertex, $\sigma = g\sigma_0$. Set $V_\sigma := \{ f \in \text{c-ind}^G_{KZ} D_0 : \mathrm{Supp}(f) \subset KZg^{-1} \}$.

σ an edge, $\sigma = ge_1$; $V_\sigma := \{ f \in \text{cind}^G_{K(\sigma_1)} D_1 : \mathrm{supp}(f) \subset K(\sigma_1) g^{-1} \}$.

We can now check that $V_\sigma = g \cdot V_{\sigma_i}$ if $\sigma = g\sigma_i$ $(i \in \{0,1\})$.

Restriction maps:

$$\begin{tikzcd}
V_{\sigma_1} \arrow[r] \arrow[d, "ev_1"'] & V_{\sigma_0} \\
D_1 \arrow[r, "r"] & D_0 \arrow[u, "ev_0^{-1}"']
\end{tikzcd}$$

3) Injective envelopes

(Let ρ be a semi-simple representation of $K(\sigma_0) = KZ$, with trivial action of ϖ
we want to have a $k(\sigma_1)$-rep.)

• Inj. envelopes in the category $\overline{\mathbb{F}}_p[G]$-modules.

Def: Let V be an R-rep of G.

A representation I (an $R[G]$-module) is an injective envelope if:

1) I is an injective object in $\mathbb{F}_p[G]$-mod, with $V \hookrightarrow I$

(i.e.
$$\begin{tikzcd} 0 \arrow[r] & \pi \arrow[r] \arrow[d] & \pi' \arrow[dl, dashed] \\ & I & \end{tikzcd}$$
)

2) $V \hookrightarrow I$ is essential: $(\forall X \hookrightarrow I,\ X \cap V \neq \{0\}.)$

largest semisimple subrep.

Rmk: If V is irreducible, then $\mathrm{Socle}\, \mathrm{inj}_G V = V$.

If J is another injective ~~envelope~~, object containing V,

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d] & \mathrm{inj}_G V \arrow[dl, dashed, "\psi"] \\ & J & \end{tikzcd}$$

ψ injective.

Also, for any V, $\mathrm{inj}_G V = \mathrm{inj}_G(\mathrm{Socle}\, V)$.

Rmk: $R[G]$ is self-dual (via $\begin{aligned} R[G]^\vee &\to R[G] \\ f &\mapsto \sum_g f(g) g \end{aligned}$).

So injective objects $\Leftrightarrow$ projective.

Therefore any injective rep. of G is a direct sum of indecomposable inj objects $\Rightarrow$ parametrized by irreducible reps. (= proj)

So $R[G] \cong \bigoplus (\dim \sigma)\, \mathrm{inj}\, \sigma$.

Pf: $R[G]$ is proj + inj, so it decomposes into a sum of indecomposables $\bigoplus m_\sigma\, \mathrm{inj}\, \sigma$.

$$m_\sigma = \cancel{\#}\dim \mathrm{Hom}_G(\mathrm{inj}\,\sigma, R[G]) = \cancel{\#}\dim \mathrm{Hom}_G(\sigma, R[G]) \overset{\text{Frob rec.}}{=} \dim \mathrm{Hom}_{\{1\}}(\sigma, 1) \cong \dim \sigma$$

∎

<u>Lemma</u>: 1) $H < G$. Then $\mathrm{inj}_G\, \sigma|_H$ is an injective object. (for any rep σ of G).

2) $H \trianglelefteq G$. Then $(\mathrm{inj}_G\, \sigma)^H = \mathrm{inj}_{G/H}\, \sigma^H$.

3) If U is a p-Sylow in G (R of char p is important), then ρ is an injective object (as a rep of G) $\Leftrightarrow$ $\rho|_U$ is injective as a rep of U.

Pf/ See Serre, "Linear reps of finite gps". §14.4 Lemma 20.

Note that in $\overline{\mathbb{F}}_p(U)$-modules, there is only one indecomposable injective: $\mathrm{inj}_U \mathbb{1}$.

and $\mathbb{F}_p[U] \cong \mathrm{inj}_U \mathbb{1}$.

<u>Example</u>: Consider the Steinberg rep of $GL_n(\mathbb{F}_q) = G$.

$$St = \frac{\mathrm{Ind}_B^G 1}{\sum_P \mathrm{Ind}_P^G 1}$$, where P runs thru Parabolics containing B.

We can show that $St|_U \simeq \overline{\mathbb{F}}_p[U]$. So $St|_U$ is injective and hence St is an injective object in $Rep_{\overline{\mathbb{F}}_p}(G)$.

We now do the proof of Paskunas' result.

Let M be supersingular, and s.t ϖ acts trivially : ie $T_{\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix}} = T_{W^2} = T_W^2 = id.$

So $M|_{\underset{\text{finite}}{H}}$ is semisimple, direct sum of 2 characters for H. $\left(w = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right)$.

For χ any character of H, get ρ_χ irred rep of $\overset{K/K_1}{\overset{\shortparallel}{GL_2}}(\mathbb{F}_q)$ s.t $\rho_\chi^{I(1)} \underset{\overset{\uparrow}{H}}{\simeq} \chi$.

Write $M_H = \chi \oplus \chi'$, and let $\rho = \rho_\chi \oplus \rho_{\chi'}$, which is a semisimple rep of K (actually coming from one of K/K_1).

Obtain a diagram:

$$\begin{array}{c} \rho^{I(1)} \hookrightarrow \rho \\ \circlearrowleft \\ \langle I, w \rangle \\ \shortparallel \\ K(\sigma_1) \end{array}$$

Recall:

Prop: Let G be a finite group. Let $H < G$, and let V be an injective in Rep G.

write $V|_H = \oplus n_\sigma \operatorname{Inj}_H \sigma$, σ irred. rep of H.

Rmk: $\oplus n_\sigma \sigma = \operatorname{Socle}_H V$.

Injective envelopes in Rep_K

For any rep ρ of K, define $\operatorname{Inj}_K \rho = \begin{cases} \nearrow \text{an essential extension of } \rho \\ \searrow \text{injective object.} \end{cases}$

If ρ is an irred. rep of K/K_1, what is the link between

$\operatorname{Inj}_K \rho$ and $\operatorname{Inj}_{K/K_1} \rho$?

First, note that K is profinite: $K = \varprojlim K/K_n$, $K_n = 1 + \varpi^n M_2(\mathcal{O})$.

(the argument works, for any profinite gp, in particular for $I = \varprojlim I/I_n$, with $I_n = \begin{pmatrix} 1+\varpi^n\mathcal{O} & \varpi^n\mathcal{O} \\ \varpi^n\mathcal{O} & 1+\varpi^n\mathcal{O} \end{pmatrix}$.)

∂

As K/K_{n+1} rep,

$$\begin{tikzcd}\rho \arrow[r, hook] \arrow[d] & \mathrm{Inj}_{K/K_n}\rho \arrow[dl, dashed] \\ \mathrm{Inj}_{K/K_{n+1}}\rho & \end{tikzcd}$$

← this map is injective. because $\mathrm{Socle}_{K/K_{n+1}} \mathrm{Inj}_{K/K_n} \rho = \rho$.

Set $I := \varinjlim \mathrm{Inj}_{K/K_n} \rho$. (a priori this depends on a choice).

Prop: I is an injective envelope of ρ in Rep_K.

Pf 1) I is an essential ext. of ρ:

Clear b/c $\mathrm{Socle}_K I = \rho$.

2) I is injective:

For $n, m \geq 0$, $(\mathrm{Inj}_{K/K_{n+m}} \rho)^{K_n} \simeq \mathrm{Inj}_{K/K_n} \rho$:

Since K/K_{n+m} is finite,

$$(\mathrm{Inj}_{K/K_{n+m}} \rho)^{K_n} \simeq (\mathrm{Inj}_{K/K_{n+m}} \rho)^{K_n/K_{n+m}} \simeq \mathrm{Inj}_{K/K_n} \overset{\rho}{\overset{\shortparallel}{\left(\rho^{K_n/K_{n+m}}\right)}}$$

Therefore, $\boxed{I^{K_n} = \mathrm{Inj}_{K/K_n} \rho}$ (b/c $I = \varinjlim_m (\mathrm{Inj}_{K/K_{n+m}} \rho)^{K_n}$)

To see I injective, s'pose

$$\begin{tikzcd}0 \arrow[r] & \pi \arrow[r] \arrow[d] & \pi' \arrow[dl, dashed, "?"] \\ & I & \end{tikzcd}$$

Taking K_1-invariants,

$$\begin{tikzcd}0 \arrow[r] & (\pi)^{K_1} \arrow[r] \arrow[d, "\alpha"'] & (\pi')^{K_1} \arrow[dl, dashed, "\psi_1"] \\ & I^{K_1} & \end{tikzcd}$$

b/c I^{K_1} is injective.

Consider the pushout $\pi^{K_2} \oplus_{\pi^{K_1}} (\pi')^{K_1} = P_2$

Since $\pi^{K_2} \hookrightarrow (\pi')^{K_2}$ and $(\pi')^{K_1} \hookrightarrow (\pi')^{K_2}$, get $P_2 \hookrightarrow (\pi')^{K_2}$.

h

But then

$$\begin{tikzcd} & & I \\ \pi^{K_2} \arrow[r] \arrow[rru, bend left] \arrow[d] & P_2 \arrow[ru, dotted] & \\ \pi^{K_1} \arrow[r] & (\pi')^{K_1} \arrow[u] \arrow[ruu, bend right] & \end{tikzcd}$$

Get

$$\begin{tikzcd} 0 \arrow[r] & P_2 \arrow[r, hook] \arrow[d] & (\pi')^{K_2} \arrow[ld, dashed, "\Psi_2"] \\ & I & \end{tikzcd}$$

and check that $\Psi_2|_{(\pi')^{K_1}} = \Psi_1$. Continue by induction. Since π' is smooth, get $\Psi: \pi' \to I$.

∎

Corollary: Let ρ be an irred (or semisimple) rep of K. Let I be the Iwahori.

$$\mathrm{Inj}_K \rho|_I = \bigoplus_{\substack{\chi \text{ char of } I \\ (\text{or } I/I(1))}} n_\chi \mathrm{Inj}\,\chi$$

and

$$\bigoplus n_\chi \chi = \mathrm{Soc}_I(\mathrm{Inj}_K \rho) \underset{=}{\hookrightarrow} \underbrace{\overset{I \curvearrowright}{(\mathrm{Inj}_K \rho)^{I(1)}}}_{\text{semisimple}}$$

In particular,

$$\mathrm{Inj}_K \rho|_I = \mathrm{Inj}_I\left((\mathrm{Inj}_K \rho)^{I(1)}\right).$$

Construction of a supersingular rep associated to m.

We will extend the action of $IF^\times$ on $\mathrm{Inj}_K \rho$ into an action of $\kappa(\sigma_1)$. Once we do this, we get a diagram

$$\mathrm{Inj}_K \rho \to \mathrm{Inj}_K \rho$$

extending $\rho^{I(1)} \Rightarrow \rho$ (i.e. a morphism of diagrams). This induces a map of coefficient systems, and in turn to the homology groups:

$$F: H_0(\mathcal{X}, \mathcal{D}) \to H_0(\mathcal{X}, \mathcal{Y}) \cong \mathrm{Inj}_K \rho|_{K(\sigma_1)}$$

Letting $\pi_m = \mathrm{Img}\, F$, we will see that π_m is irreducible + supersingular.

First, $H_0(\mathcal{X}, \mathcal{V})$ is generated by $\{\sigma_0 \mapsto v \in \rho\}$, and then:

$$\mathrm{Soc}_K(H_0(\mathcal{X}, \mathcal{Y})) = \mathrm{Soc}_K(\mathrm{Ind}_K \rho) = \mathrm{Soc}_{K/K_1}((\mathrm{Ind}_K \rho)^{K_1})$$

$$= \mathrm{Soc}_{K/K_1}(\mathrm{Ind}_{K/K_1} \rho) = ~~\mathrm{So}~~\ \rho \quad \text{b/c } \rho \text{ is semisimple.}$$

Therefore π_m is generated (as a rep of $GL_2(F)$) by $\mathrm{Soc}_K(H_0(\mathcal{X}, \mathcal{Y}))$.

If $0 \neq \pi' \hookrightarrow \pi_m$, then $(\pi')^{I(1)} \hookrightarrow \pi_m^{I(1)}$, and since

$$\pi' \cap \underbrace{\mathrm{Soc}_K(H_0(\mathcal{X}, \mathcal{Y}))}_{= \rho} \neq 0, \quad \text{then}$$

$$\underset{\circlearrowleft \mathcal{H}}{(\pi')^{I(1)}} \cap \rho^{I(1)} \neq 0 \quad \Rightarrow \rho^{I(1)} \hookrightarrow (\pi')^{I(1)} \Rightarrow \rho = \langle K \cdot \rho^{I(1)} \rangle \hookrightarrow \pi'!$$

$m \supseteq \mathcal{H}$
irred module

So that $\pi_m \hookrightarrow \pi' \Rightarrow !!.$

Since the non-supersingular reps are classified, it's easy to see that π_m is supersingular.

Now, need to prove how to extend the action of I on $\mathrm{Ind}_K \rho$ into an action of $\kappa(\sigma_1)$, st $\rho^{I(1)} \hookrightarrow \mathrm{Ind}_K \rho$ is $\kappa(\sigma_1)$-equivariant.

Recall that

$$\mathrm{Ind}_K \rho \big|_I = \mathrm{Ind}_I \Big(\overbrace{(\mathrm{Ind}_K \rho)^{I(1)}}^{\text{sum of characters of } I, \ = \oplus m_\chi \chi} \Big)$$

As an I-rep, $(\mathrm{Ind}_K \rho)^{I(1)} \simeq \rho^{I(1)} \oplus \underset{\nwarrow \text{ sum of characters}}{X}$

$$\Rightarrow \mathrm{Ind}_K \rho \big|_I \simeq \mathrm{Ind}_I \rho^{I(1)} \oplus \mathrm{Ind}_I X$$

Extending the action of w on $\mathrm{Inj}_I\, \rho^{I(1)} \circlearrowleft \rho^{I(1)}$.

(assume for simplicity $p \neq 2$).

Lemma: Let H be finite, $D \triangleleft H$ a p-group w/ $\# H/D$ is coprime to p.
Let σ be a finite dim rep of H. Consider $\mathrm{Inj}_D(\sigma|_D)$.
Then: the action of D on $\mathrm{Inj}_D(\sigma|_D)$ can be extended into an action of H s.t. $\mathrm{Inj}_D(\sigma|_D) \simeq \mathrm{Inj}_H \sigma$. in a unique way
(do not prove uniqueness here)

Pf $(\mathrm{Inj}_H \sigma)^H \cong \mathrm{Inj}_{D/H}\, \sigma^D$. Any rep of D/H in mod-p coeffs is semi-simple,

so $\mathrm{Inj}_{D/H}\, \sigma^D \cong \sigma^D$

Also, $\sigma|_D \hookrightarrow (\mathrm{Inj}_H \sigma)|_D$ is essential: otherwise, there is a subrep $\tau \subset \mathrm{Inj}_H \sigma|_D$ s.t. $\tau \cap \sigma|_D = \{0\}$. But then
$\tau \oplus \sigma|_D \hookrightarrow \mathrm{Inj}_H(\sigma|_D)$. Taking D-inv, get:
$\tau^D \oplus \sigma^D \hookrightarrow \sigma^D \Rightarrow \tau^D = 0 \Rightarrow !!$ b/c D is a p-group + mod-p coeffs.

Therefore $\mathrm{Inj}_H \sigma|_D \cong \underset{\overset{\circlearrowright}{H}}{\mathrm{Inj}_D(\sigma|_D)}$ (b/c it's inj + essential ext).

∎

For each n,

$\mathrm{Inj}_{I/I_n}\, \rho^{I(1)}$ + consider $H = k(\sigma_1)/\varpi^{\mathbb{Z}} I_n \leftarrow$ finite! ; $D = I/I_n$

Lemma $\Rightarrow$ the action of I/I_n on $\mathrm{Inj}_I\, \rho^{I(1)}$ can be extended into an action of $k(\sigma_1)/\varpi^{\mathbb{Z}} I_n$ ~~s.t. $\mathrm{Inj}_D(\sigma|_D) \simeq \mathrm{Inj}_H \sigma$~~ compatible w/ action of $k(\sigma_1)$.

We have decomposed

$$\mathrm{Inj}_K \rho|_I = \mathrm{Inj}_I(\rho^{I(1)}) \oplus \mathrm{Inj}_I(X) \quad \text{s.t. Socle } \mathrm{Inj}_K \rho = \rho^{I(1)} \oplus X.$$

If we prove that $\forall \chi : I/_{I(1)} \to \overline{\mathbb{F}}_p$ characters we have

$W_\chi \cong V_{\chi^s}$ (as vector spaces)

then choose $\phi_{\chi\chi^s} : W_\chi \to W_{\chi^s}$ a vs isomorphism, and set $\phi_{\chi^s\chi} = (\phi_{\chi\chi^s})^{-1}$.

Also, $\phi_{\chi\chi} = \mathrm{id}_{W_\chi}$. Then define $\phi : W_\chi \to W_{\chi^s}$, $w \mapsto \phi_{\chi\chi^s}(w)$ $(W = \oplus W_\chi)$

and this works.

So we are left to prove $W_\chi \overset{\alpha}{\cong} W_{\chi^s}$.

Since w acts on $\mathrm{Inj}_I(\rho^{I(1)})^{=V}$, we have $V_\chi = V_{\chi^s}$. So it's enough that

$$\mathrm{Inj}_K \rho|_\chi \cong \mathrm{Inj}_K \rho|_{\chi^s}.$$

Lemma: $\dim \mathrm{Hom}_K(\mathrm{Ind}_I^K \chi, \mathrm{Inj}_K \sigma) = \dim \mathrm{Hom}_K(\mathrm{Ind}_I^K \chi^s, \mathrm{Inj}_K \sigma)$

(either 0 or 1)

Pf

Claim: $\mathrm{Ind}_I^K \chi$ and $\mathrm{Ind}_I^K \chi^s$ have the same irred. constituents with multiplicity 1.

Rmk: Open: prove for GL_3: given σ an irred rep of $GL_2(\mathbb{F}_q)$, ψ a char of $GL_1(\mathbb{F}_q)$, show that

$\mathrm{Ind}_{\boxed{2\,1}}^{GL_3} \psi \otimes \sigma$ and $\mathrm{Ind}_{\boxed{1\,2}}^{GL_3} \psi \otimes \sigma$ have the same constituents

- **Breuil-Paskunas**: F a p-adic field. w/ residue field $\mathbb{F}_q$, $q=p^f$, F ~~unram~~/$\mathbb{Q}_p$.

Let ρ be a 2-dim irr rep of $\mathrm{Gal}(\bar{F}/F)$.

$$\rho\big|_{I_F} = \begin{pmatrix} \omega_{2f}^{\overbrace{(r_0+1)+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)}^{*}} & \\ & \omega_{2f}^{p^f *} \end{pmatrix} \qquad \left(\rho = \mathrm{Ind}_{\mathrm{Gal}(F_2)}^{\mathrm{Gal}(F)} \omega_{2f}^{*}\right)$$

For $\mathbb{Q}_p$ $(f=1)$:

$$\rho\big|_{I_{\mathbb{Q}_p}} \begin{pmatrix} \omega_2^{r+1} & \\ & \omega_2^{p(r+1)} \end{pmatrix} \rightsquigarrow \mathrm{Ind}_{K_Z}^{GL_2(\mathbb{Q}_p)} \mathrm{Sym}^r R/T.$$

Serre's conjecture

Let $\rho: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ which is continuous, $\overset{\det(\overset{\text{complex conj}}{c})=-1}{\text{odd}}$ and irreducible.

Thm (-Wintenberger): ρ is modular: it is the reduction of a p-adic Galois representation on the space of cusp forms with weight $\underset{\sim}{k}$ and level $\underset{\sim}{N}$.

Now, let K be a totally real field s.t. p is unramified in K.

Let $S_K = \{\tau: K \hookrightarrow \mathbb{R}\}$. its set of real embeddings. ~~$\vec{x}$~~

Let $\vec{k} \in \mathbb{Z}^{\# S_K}$, $k_\tau \geq 1$, k_τ have same parity.

Let $\mathfrak{n}$ be a nonzero ideal of $\mathcal{O}_K$.

Let's consider the space of Hilbert modular forms of wt $\vec{k}$, with action of

$T_{\mathfrak{m}}$ (Hecke-operator, for each $\mathfrak{m} \overset{\neq 0}{\subseteq} \mathcal{O}_K$).

Thm (Carayol, Ohta, Rogawski, Taylor, Tunnel): To a HMF f which is Eigenform, associate a Galois rep $\rho_f : ~~\mathrm{Gal}~~\ G_K \to GL_2(\bar{\mathbb{Q}}_p)$.

They associate $\bar{\rho}_f$, ~~reduction~~ mod = Semisimplification of $(\rho_f \bmod p)$.

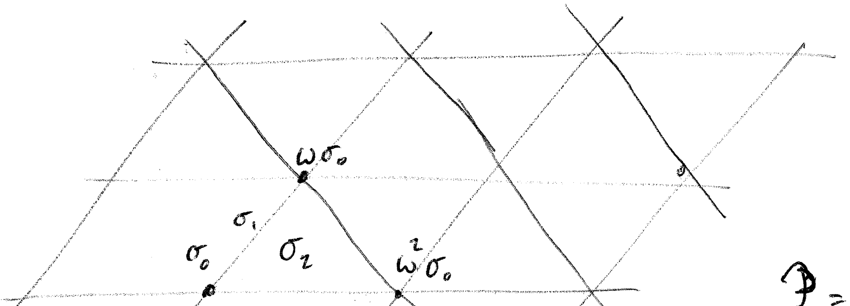
Then Serre's conjecture can be generalized.

To say ρ is modular of weight V = Irred rep of $GL_2(\mathcal{O}_K/p\mathcal{O}_K)$ is to say that there is a quaternion algebra B/K, split at the primes over p, and a small open compact subgroup $U \subset (D \times_K \mathbb{A}_K^f)^\times$ s.t.

$$\rho \text{ is a subquotient of } \left(\mathrm{Pic}^\circ(X_U)[p](\bar{K}) \otimes V\right)^{GL_2(\mathcal{O}/p\mathcal{O})}.$$

Some remarks about GL_3: $G = GL_3$, I = Iwahori, $N_G(I) = \langle I, \omega \rangle$, $\omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & 0 & 0 \end{pmatrix}$

Recall the standard apartment:



$\sigma_0 = [\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}e_3]$

$\omega\sigma_0 = (e_1, e_2, \pi e_3)$

$\omega^2\sigma_0 = (e_1, \pi e_2, \pi e_3)$

$$\mathcal{P} = \left(\begin{array}{c|c} GL_2(\mathcal{O}) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline P \quad P & \mathcal{O}^\times \end{array}\right)$$

$k(\sigma_0) = KZ \triangleright K_1$

$k(\sigma_1) = \mathcal{P}Z \triangleright \mathcal{D} = \begin{pmatrix} 1+P & P & \mathcal{O} \\ P & 1+P & \mathcal{O} \\ P & P & 1+P \end{pmatrix}$

$k(\sigma_2) = \langle I, \omega \rangle \triangleright I(1)$

The coefficient system associated to $\mathcal{V} = \mathrm{ind}_{I(1)}^{G} 1$ (on the building of GL_3) gives an exact complex (by the result of Schneider–Stuhler).

$\perp$

Get:

$$0 \to \overset{\bigoplus g\, \mathscr{C}^{I(1)}}{\mathcal{F}_2(\mathscr{X},\mathscr{C})} \to \overset{\bigoplus g\, \mathscr{C}^{N}}{\mathcal{F}_1(\mathscr{X},\mathscr{C})} \to \overset{\bigoplus g\, \mathscr{C}^{K_1}}{\mathcal{F}_0(\mathscr{X},\mathscr{C})} \to \mathscr{C} \to 0.$$

where $\hat{\mathcal{F}}_i(\mathscr{X}, V)$ = functions with finite support on the set of i-simplices.

s.t. $f(\sigma) \in \mathscr{C}^{U_\sigma}$ $\quad \left(U_{\sigma_2} = I(1),\ U_{\sigma_1} = N,\ U_{\sigma_0} = K_1 \right).$

The boundary maps are:

$$\partial_i : \mathcal{F}_i(\mathscr{X},\mathscr{C}) \to \mathcal{F}_{i-1}(\mathscr{X},\mathscr{C})$$
$$f \longmapsto \Big(\underset{i-1}{\underset{\uparrow}{\tau}} \longmapsto \sum_{\underset{i}{\underset{\uparrow}{\sigma}} \supset \tau} \underbrace{[\sigma:\tau]}_{\underset{\pm 1}{\uparrow}} f(\sigma) \Big)$$

Now:

1) $\mathscr{C}^{I(1)} \cong \mathcal{H}$ free

2) $\mathscr{C}^{K_1} = \mathcal{H} \otimes_H C$ ← not flat, but $\mathscr{C}_0^{K_1}$ is flat b/c C_0 is flat. ($\mathcal{H}$ is flat over H).

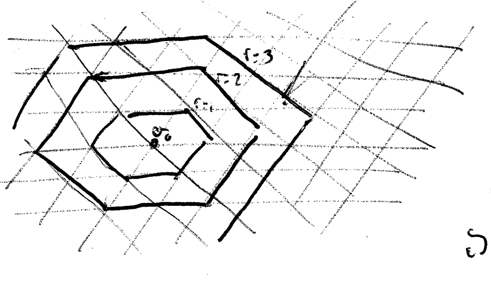
3) $\mathscr{C}^N = \mathcal{H} \otimes_H C^N$, $N = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} = N \pmod{\pi}$

Writing $P = MN$, M Levi subgroup, then $C^N = H \otimes_{H_M} C_M$

where $H_M = \mathrm{End}_M(C_M)$, $C_M = \mathrm{ind}_{U \cap M}^{M} \mathbb{1}$.

So $\mathscr{C}^N$ is flat (actually projective) if $q = p$

And also $\mathscr{C}_0^N$ is flat and projective always.



Balls of radius 1, 2, 3 about σ_0.

<u>Lemma</u> (Bruhat-Tits, Belluche-Otsumoso?).
If σ is a vertex at distance m, then any neighbor of σ at distance $m-1$ is contained in any apartment containing σ and σ_0.

A chamber in the border of the ball is of type

(a) if it has two vertices on the border (i.e at distance m),

(b) if it has only one vertex on the border.

Lemma: Let σ be a chamber at distance m. Let $x \in \sigma$ be a vertex at distance m, $z \in \sigma$ at distance $m-1$.

Let $y \in \sigma$ be the other vertex $\left(\begin{array}{l}\text{if dist } m \Rightarrow \text{type a} \\ \text{if dist } m-1 \Rightarrow \text{type b}\end{array}\right)$.

Then the set of ~~edges~~ chambers containing the edge $\{x, y\}$ which are at distance $\leq m$ is:

- $\rightsquigarrow$ only $\{\sigma\}$ if σ is of type (a),
- $\rightsquigarrow$ $\sigma \cup \{$a set of chambers of type (a)$\}$ if σ is of type (b).

Exercise: Show using above lemma that:

Let $\mathcal{V}_\sigma = (V_\sigma)$ be a coeff. system on the building of GL_3.

Consider $\mathcal{H}_i(\mathcal{X}, \mathcal{V})$ $i \geq 1$. Then:

- If the restriction maps $r_{\tau_1}^{\tau_2}$ are injective, then $\mathcal{H}_2(\mathcal{X}, \mathcal{V}) = 0$.
- If the restriction maps $r_{\tau_1}^{\tau_2}$ are surjective and $r_{\tau_0}^{\tau_1}$ are injective, then $\mathcal{H}_1(\mathcal{X}, \mathcal{V}) = 0$.
- If $r_{\tau_1}^{\tau_2}$ and $r_{\tau_0}^{\tau_1}$ are bijective then $\overset{\rightsquigarrow \text{KZ-reps}}{H_0(\mathcal{X}, \mathcal{V})} = \overset{\text{PZ-rep}}{V_{\sigma_0}} \simeq \overset{\langle I, w\rangle\text{-rep}}{V_{\sigma_1}} \simeq V_{\sigma_2}$.

(Try to do this in general !!)

• Non-flatness of $\mathcal{C}$ (if $q=p$).

$$\mathcal{C} = \underset{\substack{\uparrow \\ \text{not flat b/c } C \text{ is not flat.}}}{\mathcal{F}_0(\mathcal{X},V)} \Big/ \operatorname{Im}\partial_1$$

If $\mathcal{C}$ were flat, then $\operatorname{Im}\partial_1$ flat $\Leftrightarrow$ $\mathcal{F}_0(\mathcal{X},V)$ flat. So it's enough to show that $\operatorname{Im}\partial_1$ is flat.

Have an exact sequence:

$$0 \to \operatorname{Im}\partial_2 \to \underset{\substack{\uparrow \\ \text{flat}}}{\mathcal{F}_1(\mathcal{X},V)} \xrightarrow{\partial_1} \operatorname{Im}\partial_1 \to 0$$

flat ~~(actually free)~~ b/c $\mathcal{C}^N$ is flat since C_M^N is flat ($p=q$).

[**Lemma**: $0 \to E' \to E \to E'' \to 0$ s.e.s of A-modules, with E flat.
Then E'' is flat $\Leftrightarrow$ $\forall\, \mathfrak{a}$ ideal, $\mathfrak{a}E' = \mathfrak{a}E \cap E'$].

So let $\mathfrak{a}$ be an ideal of $\mathcal{H}$. Need to prove that

$$\mathfrak{a}\operatorname{Im}\partial_2 = \mathfrak{a}F(\mathcal{X},V) \cap \operatorname{Im}\partial_2.$$

$\subseteq$ obvious

$\supseteq$: Given $f \in \mathcal{F}_2(\mathcal{X},V)$. Sps $f \neq 0$, and $\partial_2(f) \in \mathfrak{a}F_1(\mathcal{X},V)$.

Let m be the smallest integer s.t $\operatorname{Supp}(f) \subset B(o,m)$.

If $\sigma \in \operatorname{Supp}(f)$ is a chamber & $f(\sigma) \in \mathfrak{a}\,\mathcal{C}^{U_\sigma}$, consider the ~~$\mathbb{Z}$~~-chain
$f' : \sigma \mapsto f(\sigma)$.

Then $\partial_2 f' \in \mathfrak{a}\operatorname{Im}\partial_2$, so if $f = f'$ we are done.

So after some iteration, may suppose that $f(\sigma) \notin \mathfrak{a}\,\mathcal{C}^{U_\sigma}$ for all $\sigma \in \operatorname{Supp}(f)$.

(cont)
Let σ be a chamber ~~at~~ distance m in the support of f.

1) If σ is of type (a). Then $\exists \{x,y\} \subset \sigma$ s.t. x, y are at distance m.

But $\partial_2(f)[\{x,y\}] = \pm f(\sigma)$ b/c σ is the only chamber containing $\{x,y\}$. But this implies that $f(\sigma) \in \mathcal{B}^{U_\sigma}$, which we assume is not the case.

2) If σ is of type (b) (and all other chambers in $\mathrm{Supp}(f)$ at dist m are also of type (b)).

$\sigma \supset \{x,y\}$, x at dist m - y at dist $m-1$.

So chambers in $\mathrm{Supp}(f)$ containing $\{x,y\}$ are all of type (a) except σ, so done again!

Similarly one can prove that if $p = q$, $\mathcal{C}_0$ is $\mathcal{H}$-flat.

More is known for GL_3!

1) Right number of supersingular $\mathcal{H}$-modules $\left(= \frac{(q-1)q(q+1)}{3} \right)$

2) $\operatorname{Ind}_B^{GL_3(F)} \chi$; $\operatorname{Ind}^{GL_3(F)}_{\boxed{2,1}} \underset{\substack{\uparrow \\ \text{supersingular}}}{\rho} \otimes \psi \leftarrow$ character is irreducible. (Florian Herzig)

3) $\mathcal{C}$ is not flat on $\mathcal{H}$ (bad news)

4) $\mathcal{C}$ is flat for $q = p$ on $\mathcal{H}_0$ (good news!) (maybe we have an equivalence $\{\mathcal{H}_0\text{-mod}\} \leftrightarrow \{\text{Reps } ~~by~~ \text{ generated by } I\text{-invariant subspace}\}$

• Colmez's Montreal Functor:

Ref: Colmez, "Représentations de $GL_2(\mathbb{Q}_p)$ et (φ,Γ)-modules".

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times$ (b/c of cyclotomic character).

$$(\zeta_{p^n} \mapsto \zeta_{p^n}^x) \longleftarrow\!\shortmid\ x$$

Iwasawa algebras & completed gp algebras.

Let E be a finite field with char. p. Write $\mathcal{O}=\mathbb{Z}_p$, $P=p\mathbb{Z}_p$.

If H is a profinite group $H = \varprojlim_{\Omega} \underset{\text{finite}}{H/\Omega}$ define:

$$E[[H]] := \varprojlim_{\Omega} E[H/\Omega].$$

If H_0 is a p-group, $E[H_0]$ is a local ring, with maximal ideal

$$\mathfrak{m} = \mathrm{Ker}\deg, \quad \deg: E[H_0] \to E$$
$$f \longmapsto \sum_{h_0\in H_0} f(h_0).$$

For $H=\mathbb{Z}_p = \varprojlim_{m} \mathcal{O}/p^m$, we get $E[[H]] = \varprojlim E[\mathcal{O}/p^m]$, a local ring.

It is easy to see that

$$E[[\mathbb{Z}_p]] \overset{\text{as algebras!}}{\cong} \left(\mathcal{C}^\infty(\mathbb{Z}_p, E)\right)'$$

with convolution product.

$$(\mu * \mu')(f) = \int_{\mathcal{O}\times\mathcal{O}} f(x+y)\, d\mu(x)\, d\mu'(y)$$

The maximal ideal is the set of measures of total measure 0.

Prop: $E[[\mathbb{Z}_p]] \simeq E[[X]] = \varprojlim_{m} E[X]/(X^m)$

$$1 \longmapsto X+1$$

Pf: If h has order p^m, let $P \in E[X]$. $P(h-1)=0 \iff X^{p^m}-1 \mid P(X-1) \iff (X+1)^{p^m}-1 \mid P(X)$
$\iff X^{p^m} \mid P(X)$.

Moreover, the monoid $p^{\mathbb{N}}\mathbb{Z}_p^\times$ acts continuously on $\mathbb{Z}_p$, so also on $E[[\mathbb{Z}_p]]$.

Denote by φ the action of p on $E[[\mathbb{Z}_p]] \cong E[[X]]$.

And Γ acts on $E[[X]]$ via the cyclotomic character.

eg: $\varphi(X) = \varphi((X+1)-1) = (X+1)^p - 1 \underset{\substack{\uparrow \\ \text{char } p.}}{=} X^p$

(if $x \in p^{\mathbb{N}}\mathbb{Z}_p^\times$, $x\cdot(X+1) = (X+1)^x = \lim_m \boxed{(X+1)^{\sum_{i=0}^m a_i p^i}}$ convergent in $E[[X]]$).

The action φ and Γ extends to an action on $E((X)) := \mathrm{Quot}(E[[X]])$.

Def: A (φ,Γ)-module over $E((X))$ is a finite-dimensional vectorspace ~~D~~ on $E((X))$. with a Frobenius φ and ~~[illegible]~~ an action of Γ satisfying:

a) φ, Γ commute: $\varphi(\gamma v) = \gamma\cdot\varphi(v)$ $\forall \gamma \in \Gamma$

b) Semilinear:

$\varphi(fv) = \varphi(f)\varphi(v)$ $\forall f \in E((X))$.

$\gamma\cdot(fv) = (\gamma\cdot f)\cdot\gamma(v)$ $\forall \gamma \in E((X))$.

c) The action of φ is continuous on D, where the topology on D is given by: let $L \subset D$ an $E[[X]]$-lattice, and let $(X^m L)_{m\in\mathbb{N}}$ be a fundamental system of nbhds of 0.

Moreover, D is said to be étale if φ is injective $D \to D$, and $D = \varphi(D) \otimes_{E[[T]]} E((T))$ (matrix of ~~D~~ φ on any basis is nonsingular).

Rmk: D étale $\iff$ $D = \bigoplus_{i=0}^{p-1} (1+X)^i \varphi(D)$

Def: Given D, D' étale (φ, Γ)-module, a morphism $D \to D'$ is an $E((T))$-linear map which is (φ, Γ), (φ', Γ')-equivariant.

Thm (Fontaine '90): There is an equivalence of categories

$$\begin{matrix}\text{Étale } (\varphi, \Gamma)\text{-modules} \\ \text{over } E((T))\end{matrix} \longleftrightarrow \begin{matrix}\text{Smooth fin-dim } E\text{-representations} \\ \text{of } Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p).\end{matrix}$$

Example:

1-dim'l (φ, Γ)-modules $/\sim$ $\Leftrightarrow$ $(\#E-1)(p-1)$ characters of $\mathbb{Q}_p^\times$

Given $\lambda : \mathbb{Q}_p^\times \to E^\times$ a character, associate to it $D_\lambda = E((X)).e$ with:

$$\varphi(e) = \lambda(p)\, e$$

cyclotomic character

$$\gamma(e) = \lambda(\chi(\gamma))\, e \qquad \forall \gamma \in \Gamma.$$

Conversely, given a 1-dim'l (φ, Γ)-module D, $D = E((X))e$.

Let $h \in E((X))$ s.t. $\varphi(e) = h(X)e$ ($h(X) \neq 0$ b/c étale).

Write $h(x) = h_0 X^a f(x)$, with $\begin{cases} a \in \mathbb{Z} \\ f(x) \in 1 + XE((X)) \end{cases}$.

Changing $e \mapsto u(x)e$ for some $u \in E((T))^\times$,

$$\varphi(u(x)e) = \varphi(u(x))\,\varphi(e) = u(x^p)\,\varphi(e) = \frac{u(x^p)}{u(X)} h(x) \cdot (u(x)e)$$

Find $u(x)$ s.t. $\frac{u(x^p)}{u(x)} = \frac{1}{f(x)}$ (just choose $u(x) = \prod_{i \in \mathbb{Z}} (f(X^{p^i}))$, which converges)

So WLOG assume $f(x) = 1$, so $h(x) = h_0 X^a$.

Write $a = b(p-1) + r$ for $a \in \mathbb{Z}$, so:

$$\varphi(X^{-b}e) = X^{-pb+a} h_0 e = X^{a-bp+b} h_0 (X^{-b}e) = X^r h_0 (X^{-b}e).$$

So WLOG assume $\varphi(e) = X^a h_0 e$, $a \in [0, p-2]$. ✓

Let γ be a generator of Γ. Imposing φ, γ commute ~~*~~ and checking degrees, we see that $a = 0$, and $g(\gamma) \in E$. ....

Goal: to a rep of $GL_2(\mathbb{Q}_p)$, associate a rep of $\overset{\text{mirabolic subgroup.}}{P} = \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$

To this, we associate a rep of P^+:

$$t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in P; \quad P_0 = P \cap K = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}; \quad N_0 = U(\mathbb{Z}_p) = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \cong \mathbb{Z}_p$$

and set $P^+ = P_0 t^{\mathbb{N}}$ (just a monoid).

$$P^+ = \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

Def: A smooth rep V of P^+ with coefficients in $\bar{E}$ is said to be étale if the action "φ" of t on V is injective, and

$$V = \bigoplus_{i=0}^{p-1} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \varphi(V).$$

Colmez: an étale (φ, Γ)-module on $E((X))$ gives a smooth representation of P^+ which is étale:

on E (not on $E((X))$)

D a (φ, Γ)-module. Define the action by:

- $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} v = (1+X)^a v$, $a \in \mathbb{Z}_p$
- $t \cdot v = \varphi(t) \cdot v$
- $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v = \underbrace{\chi^{-1}(a)}_{\Gamma} v$ (recall $\chi : \Gamma \to \mathbb{Z}_p^\times$).

Start now from Π, an irreducible E-rep of $\overset{G}{\overset{\shortparallel}{GL_2(F)}}$. with ~~scalar~~ trivial action of the center Z

Denote by $\mathcal{W}(\Pi) = \{ W \subseteq \Pi$ s.t W is E-finite-dimensional + $\underset{\text{max'l compact}}{\underset{\shortparallel}{K}}$-stable + W generates Π as a rep of $G \}$.

If $W \in \mathcal{W}(\Pi)$, then get a surjection

$$\mathrm{ind}_{KZ}^{G} W \twoheadrightarrow \Pi$$

We say that Π has a <u>finite presentation</u> if the kernel is finitely generated.

Also, for $W \in \mathcal{W}(\Pi)$, also $\varpi W \hookrightarrow \Pi$.

We have an action of $\langle \varpi, I \rangle$ on $W \cap \varpi W$, so we get a diagram:

$$\circlearrowleft\, W \cap \varpi W \hookrightarrow W \quad (*)$$

We say that W gives a <u>"standard" presentation</u> for Π if

$$\Pi \cong H_0(\text{Diagram } (*)).$$

In this case, we get an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \text{1-chains} \arrow[r] & \mathrm{ind}_{KZ}^{G} W \arrow[r, two heads] \arrow[d, phantom, "\cong"] & \Pi \arrow[r] & 0 \\ & & \text{0-chains of the coeff system to the diagram.} & & \end{tikzcd}$$

Colmez used that the rep had a standard presentation, but this is not needed.

<u>Rmk</u>: For $GL_2(\mathbb{Q}_p)$, any irred rep has a finite presentation:

$$\Pi = \mathrm{ind}_{KZ}^{G}(\underset{\text{irr. of } GL_2(\mathbb{F}_p)}{\sigma}) / \underset{\text{some Hecke-operator}}{(T}-\lambda)(\mathrm{ind})$$

Prop: Any irreducible rep of $GL_2(\mathbb{Q}_p)$ has a <u>standard</u> presentation.

pf · Hecke mod $\longrightarrow$ (Rep of $GL_2(\mathbb{Q}_p)$ $^{\text{gen by } I_1\text{-inv}}$) (recall that this is an exact equivalence of categories)

$$M \longmapsto M \otimes_{\mathcal{H}} \mathcal{C}$$

So $$\Pi = \Pi^{I(1)} \otimes_{\mathcal{H}} \mathcal{C}.$$

and $\mathcal{C}$ appears in:

$$0 \to \bigoplus_{g\sigma_0 \in \mathcal{X}_1} g\,\mathcal{C}^{I(1)} \longrightarrow \bigoplus_{g\sigma_1 \in \mathcal{X}_0} g\,\mathcal{C}^{K_1} \to \mathcal{C} \to 0$$

↙ exact, G-equivariant + $\mathcal{H}$-left equivariant

Tensoring by $\Pi^{I(1)}$ (and since $\mathcal{C}$ is $\mathcal{H}$-flat we get still exact):

$$0 \to \Pi^{I(1)} \otimes \bigoplus(\quad) \hookrightarrow \Pi^{I(1)} \otimes \bigoplus(\quad) \to \underbrace{\Pi^{I(1)} \otimes_{\mathcal{H}} \mathcal{C}}_{\substack{\cong \\ \Pi}} \to 0$$

So Π is the 0^{th} homology of this diagram. ∎

Def: Let $W^{\circ}(\Pi) \subseteq W(\Pi)$ be the set of $W \in W(\Pi)$ s.t. yield a standard presentation.

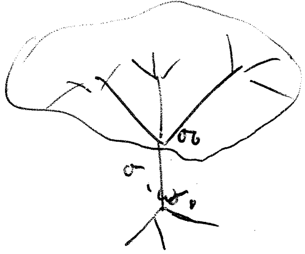
Prop (Colmez, Corollary 2.12): If Π has a standard presentation, then any $W \in W(\Pi)$ is contained in an element of $W^{\circ}(\Pi)$.

Prop: if $\Pi_1, \Pi_2, \Pi_3 \in \mathrm{Rep}(G)$ s.t. Π_1, Π_3 have a standard presentation, and $0 \to \Pi_1 \to \Pi_2 \to \Pi_3 \to 0$, then Π_2 has a standard presentation.

pf (See Colmez 2.13)

Let $G = GL_2(\mathbb{Q}_p)$ now. Let $\Pi \in \mathrm{Rep}_E(G)$ of finite length (or for simplicity, irreducible), with trivial action of Z.

Let $B = \{p\cdot\sigma_0,\ p \in P^+\}$. (= "positive" vertices):



Let
$$0 \to R \to \mathrm{ind}_{KZ}^G W \to \Pi \to 0$$
be a standard pres of Π.

If $Y \subset \mathfrak{X}_0$ is a set of vertices, let $I_Y(W)$ = image in Π of $\bigoplus_{g\sigma_0 \in Y} gW$.

Let $D^+_W(\Pi) := \langle \mu \in \overset{\text{algebraic dual}}{\Pi^\vee} : \mu \text{ is zero on } I_{B^c}(W) \rangle$

Note that P^+ acts on $I_B(W)$, so $(P^+)^{-1}$ acts on $I_{B^c}(W)$, so $D^+_W(\Pi)$ gets an action of P^+.

Def: Let $D(\Pi) := E((X)) \otimes_{E[[X]]} D^+_W(\Pi)$.

We need to check that:

- It is independent of the choice of W.
- It is étale
- Finite-dim'l.

Lemma: if $W_1 \subset W_2$, $W_1, W_2 \in \mathbb{W}^0(\Pi)$, then $D^+_{W_2}(\Pi)$ has finite index in $D^+_{W_1}(\Pi)$.

Corollary. The iso class of $E((X)) \otimes_{E[[X]]} D^+_W(\Pi)$ doesn't depend on W.

Pf (of lemma): Let $m \geq 0$ s.t. $g_i\sigma_0 \in B(\sigma_0, m)$ (i.e $d(gg_i\sigma_0, g\sigma_0) \leq m$).
$\left(W_2 \subset \sum_{i=1}^{N} g_i W_1\right.$ b/c both W_1, W_2 are fin. dim).

(cont)
And do an argument on the tree to see that

$$I_{B^c}(W_2) \subset I_{B^c}(W_1) + \overbrace{\sum_{d(g\sigma_0,\sigma_0)\leq m} gW_1}^{\text{finite sum}} \Rightarrow I_{B^c}(W_1) \text{ has finite index in } I_{B^c}(W_2). \quad \boxtimes$$

Rmk: Let $D_W^{\natural}(\Pi) = \left(I_B(W)\right)^{+}$. Then the map

$$D_W^+(\Pi) \hookrightarrow D_W^{\natural}(W)$$

$$\mu \longmapsto \mu|_{I_B(W)}$$

gives an iso $D_W^{\natural}(\Pi) \otimes \mathcal{E}((T)) \cong D_W^+(\Pi) \otimes \mathcal{E}((T))$.

So if we are only interested in the action of T, we may use this one instead.

Prop: $D(\Pi)$ is étale.

Pf Need to show that the map $D(\Pi)^p \longrightarrow D(\Pi)$ $\cancel{(1+T)}^i$

$$(\mu_0, \dots, \mu_{p-1}) \mapsto \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \varphi(\mu_i)$$

is surjective.

It is enough to prove that, if A is the map:

$$A: D_W^+(\Pi)^p \longrightarrow D_W^+(\Pi)$$

then im A has finite index in $D_W^+(\Pi)$.

And this is true because the space of functions that are zero on $W + \sum \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} W$ is included in $\mathrm{Im}(A)$.

Recall the Classical Langlands corresp.

Irr rep. of $GL_2(F)$	Rep of G_F
a) $\mathrm{Ind}_B^{GL_2(F)} \chi_1 \otimes \chi_2$, $\left\{ \begin{array}{l} \chi_1 \lvert\cdot\rvert_F \neq \chi_2 \cdot \lvert\cdot\rvert_F^{-1} \\ (\chi_1 \neq \chi_2) \end{array} \right.$ ← depends on normalization.	a) $\chi_1 \lvert\cdot\rvert_F \oplus \chi_2$ $N=0$ (2-dim)
b) $\mathbb{1}$	b) $\lvert\cdot\rvert_F \oplus 1$ $N=0$ (2-dim)
c) St	c) $\lvert\cdot\rvert_F \oplus 1$, $N \neq 0$ ← nilpotent operator
d) Supercuspidal ⟶	d) 2 dim $\underline{\underline{\text{irr}}}$ rep of G_F

Let's consider the (φ,Γ)-module associated to a principal series $\mathrm{Ind}_B^{GL_2}(\delta_1 \otimes \delta_2)$ ← think of it as smooth functions on $\mathbb{P}^1(\mathbb{Q}_p)$

Colmez introduces the space $B(\delta_1,\delta_2)$ = locally constant functions $\phi : \mathbb{Q}_p \to E$ such that the map

$$x \mapsto \left(\frac{\delta_1}{\varpi\delta_2}\right)(x)\,\phi\left(\frac{1}{x}\right)$$

can be extended into a loc constant function on $\mathbb{Q}_p$.

(here ϖ = cyclotomic char mod p). (sending $x \mapsto x\lvert x\rvert^{-1}$ mod p).

Actually, $B(\delta_1,\delta_2) = \mathcal{C}_c^\infty(\mathbb{Q}_p, E) \oplus E\cdot\phi_\infty$, where

$$\phi_\infty = \begin{cases} \cancel{\frac{\delta_1}{\varpi\delta_2}(x)}\ \frac{\delta_1}{\varpi\delta_2}(x) & x \notin \mathbb{Z}_p \\ 0 & x \in \mathbb{Z}_p \end{cases}$$

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$, $(g\cdot\phi)(x) = \frac{\varpi}{\delta_1}(ad-bc)\,\frac{\delta_1}{\varpi\delta_2}(cx+d)\,\phi\left(\frac{ax+b}{cx+d}\right)$.

Also, $B(\delta_1,\delta_2) \cong \mathrm{Ind}_B^G(\delta_2 \otimes \delta_1\varpi^{-1})$ via

$x \mapsto v\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \longleftarrow\!\shortmid\ v$

Therefore $\mathrm{Ind}_B^G(\delta_1 \otimes \delta_2) \cong B(\varpi\delta_2, \delta_1)$.

To get a (φ,Γ)-module, we need W fin. dim, K-stable and generating $B(\varpi\delta_2, \delta_1)$.

We take for W the span of the characteristic functions and ϕ_∞

$$W = \langle \phi_\infty, \mathbb{1}_{i+p\mathbb{Z}_p} : i=0..p-1 \rangle_E$$

We need to understand $D^+(W) = \{\mu \in B(\varpi\delta_2,\delta_1)^\vee \text{ s.t } \mu|_{I_{B^c}(W)} = 0\}$.

Or equivalently, ← injects with finite index.

$$D^+(W) = I_B(W)^\vee$$

Note that $I_B(W) = \underbrace{\langle P^+(1_{i+p\mathbb{Z}_p}) \rangle}_{\mathcal{C}^\infty(\mathbb{Z}_p)} \oplus E\,\phi_\infty$

Therefore $I_B(W)^\vee = \underbrace{\mathcal{C}^\infty(\mathbb{Z}_p,E)^\vee}_{\cong E[[T]]} \oplus E\,\overset{\text{Dirac measure}}{\mathrm{Dir}_\infty}$

$1+T$ acts on Dir_∞ by:

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}(\mathrm{Dir}_\infty(f)) = \mathrm{Dir}_\infty(f) \quad \text{b/c} \quad \begin{pmatrix}1&1\\0&1\end{pmatrix} \text{ fixes } \infty.$$

Therefore $1+T$ acts on Dir_∞ trivially. So T acts by 0.

This gives $I_B(W)^\vee$ the structure of an $E[[T]]$-module. ($\cong E[[T]]/(T)$)

So $I_B(W)^\vee \otimes_{E[[T]]} E((T)) \cong E((T))$! 1-dim'l!!

If we repeat this for the trivial rep, we get 0. For St, we get $E((T))$ again. (In particular, D^+ is not faithful)

For Supersingular, to $\pi(r,0,\chi)$ we associate $D(\pi(r,0,\chi))$, and Fontaine gives

$$\chi \otimes \mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}.$$

Other topics

Rogawski: Hecke modules

$\mathcal{H} = \mathcal{H}_{\mathbb{Z}[q]}(G, I)$, $G = GL_n(F)$, F a p-adic field.

$\uparrow$ a free $\mathbb{Z}[q]$-module with basis $T_w \overset{\text{def}}{=} 1_{IwI}$, $w \in W = W_0 X = \langle \omega \rangle \ltimes W_0$, $X \simeq \mathbb{Z}^n$.

subject to:

$\to$ $T_w T_{w'} = T_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$.

$\to$ $T_s^2 = (q-1)T_s + q$ for $s \in S = \{s_1, \dots, s_{n-1}\}$.

$\mathcal{H}_{\mathbb{C}} = \mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{C}$ -modules $\simeq$ Reps generated by their I-invariants.

If L is a Levi subgroup of G,

$\mathcal{H}_{\mathbb{Z}[q]}(L, I \cap L) \hookrightarrow \mathcal{H}_{\mathbb{Z}[q]}$

} Bushnell, Kutzko "Orange book".

Consider L^+ positive elts in L.

Example: if $L = T$ is the torus,

$$T^+ = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} : \text{val}(t_i) \text{ is decreasing} \right\}.$$

Prop: The $\mathbb{Z}[q]$-modules generated by $\{T_w, w \in W \cap L^+\}$ is stable under product, and it's called $\mathcal{H}(L^+)$

So we get an injection (as $\mathbb{Z}[q]$-algebras):

$$\theta_{L^+} : \mathcal{H}_{\mathbb{Z}[q]}(L^+, I \cap L) \hookrightarrow \mathcal{H}_{\mathbb{Z}[q]}$$

Moreover, θ_{L^+} extends ~~with~~ a unique way into (need to invert q!).

$$\theta_L : \mathcal{H}_{\mathbb{Z}[q]}(L, I\cap L) \otimes \mathbb{Z}[q^{\pm 1}] \longrightarrow \mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1}]$$

Note that if $a_L \in X \simeq \mathbb{Z}^n$ is strongly positive element (in $L\cap W$), then for each $w \in L\cap W$, $\exists k \in \mathbb{Z}$ s.t. $a_L^k w \in L^+$.

Then:

$$\theta_L(T^{\otimes}_{a_L^k w}) = \theta_L\left((T^{\otimes}_{a_L})^k T^{\otimes}_w\right) = \theta_L^+(T^{\otimes}_{a_L^k w}) = T_{a_L^k v} =$$

But from: $T_s^2 = (q-1)T_s + q$, we have T_s invertible $\Leftrightarrow$ q invertible.

Example: $L = T$ $\mathcal{H}(T, T\cap I) = \mathbb{Z}[q][X]$, so

$$\theta_T : \mathbb{Z}[q^{\pm 1}][X] \hookrightarrow \mathcal{H} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}[q^{\pm 1}]$$

$$\underset{yz^{-1}}{\overset{\shortparallel}{x}} \longmapsto T_y T_z^{-1} \qquad (y, z \in T^+)$$

Rmk: Usually the Bernstein presentation is normalized:

$$\tilde{\theta}_T = \delta^{1/2}(x)\,\theta(x), \quad \text{where} \quad \delta^{1/2}(x) = q^{-\frac{\ell(x)}{2}}$$

(and extend to $\mathbb{Z}[q^{\pm 1/2}]$

The Bernstein subalgebra is $A = \mathrm{Im}\,\tilde{\theta}_T$.

Thm (Bernstein): $\mathcal{H}_{\mathbb{Z}[q]} \otimes \mathbb{Z}[q^{\pm 1/2}]$ is free over A, with basis

$\{T_{w_0} : w_0 \in W_0\}$.

Moreover, the center of $\mathcal{H}_{\mathbb{Z}[q]} \otimes \mathbb{Z}[q^{\pm 1/2}]$ is A^{W_0}, and A is f.gen / A^{W_0}.

Let M be a simple $\mathcal{H}\otimes_{\mathbb{Z}[q]}\mathbb{C}$-module, with central action of the center of $\mathcal{H}\otimes\mathbb{C}$. So M is a finite-dimensional, therefore containing a character for A. So

$$\exists\, \chi : A \to \mathbb{C}, \text{ and } \quad \text{standard modules. (of dim} = n!)$$

$$\boxed{\mathcal{H}\otimes_A \chi} \twoheadrightarrow M$$

If $\lambda : T \to \mathbb{C}^\times$ is a character of the torus and we assume it's unramified.

Also $(\mathrm{Ind}_B^G \lambda)^I$ has dim $n!$, and they should be related...

- <u>Satake isomorphism</u>

$$\begin{tikzcd}
\mathcal{H}(G,K)_{\mathbb{Z}[q^{\pm 1/2}]} \arrow[r, "\overset{\text{Satake iso.}}{\sim}"] & \left(\mathbb{Z}[q^{\pm 1/2}][X]\right)^{W_0} \arrow[d, "\simeq"', "\tilde{\theta}_T"] \\
\mathcal{H}\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{\pm 1/2}] \arrow[u, "1_K*(-)"] & Z\left(\mathcal{H}\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{\pm 1/2}]\right) \arrow[l, hook']
\end{tikzcd}$$