NUMERICAL EXPERIMENTS WITH PLECTIC STARK–HEEGNER POINTS

LFANT SEMINAR

April 6, 2021

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Universitat Autònoma de Barcelona

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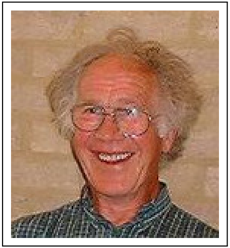
  - Functional equation relating $s \leftrightarrow 2 - s$.

The BSD conjecture and Heegner points



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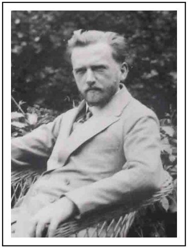
Sir P. Swinnerton-Dyer



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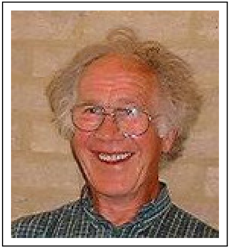

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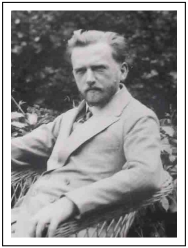

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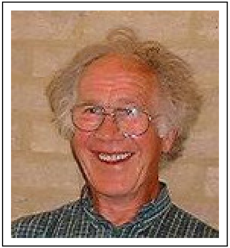

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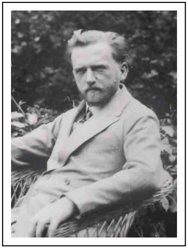

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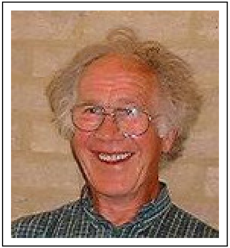

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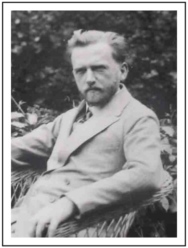

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- Simplest setting: $F = \mathbb{Q}$ (and $K/\mathbb{Q}$ imaginary quadratic), and $\ell \mid \mathfrak{N} \implies \ell$ split in K.

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Theorem (Gross–Zagier)

$$P_K = \mathrm{Tr}_{H_\tau/K}(P_\tau) \text{ nontorsion } \iff L'(E/K, 1) \neq 0.$$

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Lemma

$\iota_\mathfrak{p}$ induces bijections

$$\Gamma/\Gamma_0^B(\mathfrak{m}) \cong \mathcal{V}_0, \quad \Gamma/\Gamma_0^B(\mathfrak{p}\mathfrak{m}) \cong \mathcal{E}_0$$

$\mathcal{V}_0$ (resp. $\mathcal{E}_0$) are the even vertices (resp. edges) of the BT tree.

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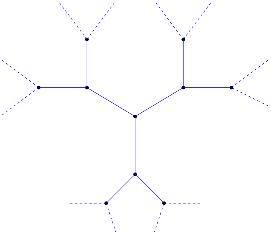

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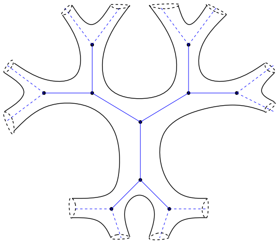

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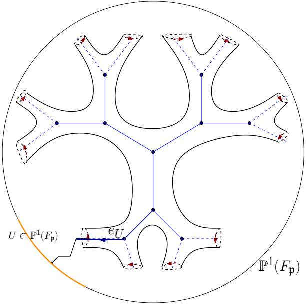

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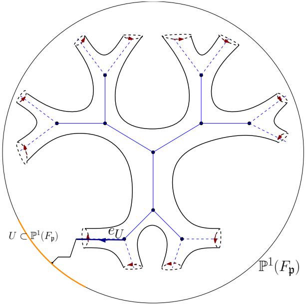
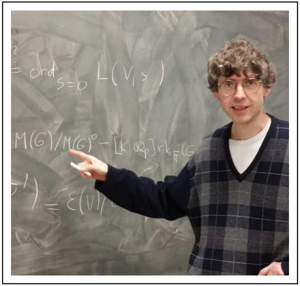


Plectic conjectures



Jan Nekovář

Tony Scholl

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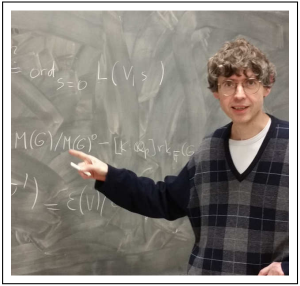

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> “ $L^{(r)}(E/K, 1)$ should be related to CM-points on a r-dimensional quaternionic Shimura variety. ”

Goal : Construct $Q \in \wedge^r(E(K))$ such that

$$Q \text{ non-torsion} \iff L^{(r)}(E/K, 1) \neq 0.$$

p-adic Plectic invariants

Michele Fornea

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$F_S = \prod_{\mathfrak{p}\in S} F_\mathfrak{p}$, $\mathbb{P}^1(F_S) = \prod_{\mathfrak{p}\in S} \mathbb{P}^1(F_\mathfrak{p})$, and $\mathcal{H}_S = \prod_{\mathfrak{p}\in S} \mathcal{H}_\mathfrak{p}$.

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  - $\mu\left(\mathbb{P}^1(F_\mathfrak{p}) \times U_{S^\mathfrak{p}}\right) = 0$, for all $\mathfrak{p} \in S$, all $U_{S^\mathfrak{p}} \subseteq \mathbb{P}^1(F_{S^\mathfrak{p}})$.

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p-adic Plectic invariants

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Plectic invariant attached to E, K and S

$$J := \langle \Phi_E, \Theta_\psi \rangle \in \bigotimes_{\mathfrak{p}\in S} K_\mathfrak{p}.$$

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- Therefore we can define Φ_E, unique up to sign.

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  Ideally, we'd like to define a class attached to $\bigotimes_{\mathfrak{p}\in S} \tau_\mathfrak{p}$.

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- Due to the fact that $a_{\mathfrak{p}_1}(E)a_{\mathfrak{p}_2}(E) = -1 \implies$ extra vanishing of L_p.

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  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 99.2-c1, $\beta = -8w + 17$, $-16w + 9$, $-20w + 29$, $-9w + 14$, $-12w + 29$, $-32w + 41$, $-12w - 7$, $-35w + 17$.

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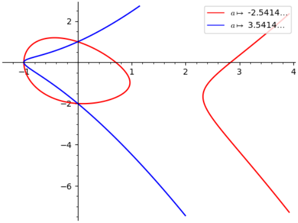
- In one of the examples, we obtain what seems to be zero. We expect that this is due to the low working precision...

A pretty example



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$E/F : y^2 + xy + y = x^3 + wx^2 + (w+1)\,x + 2$,

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[1] https://www.lmfdb.org/EllipticCurve/2.2.37.1/63.2/d/1

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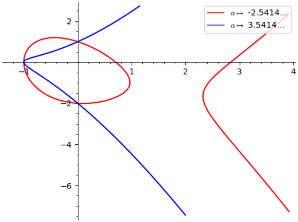

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- $E(K) \otimes \mathbb{Q} = \langle P, Q \rangle$, with
  $P = (3 - w, 4 - w)$ and $Q = (8 - \frac{25}{9}w, (\frac{-23}{27}w + \frac{17}{6})\sqrt{\beta} + \frac{25}{18}w - \frac{9}{2})$.

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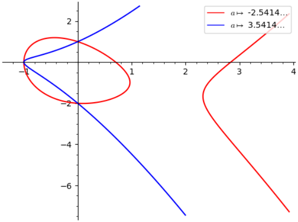

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[1] https://www.lmfdb.org/EllipticCurve/2.2.37.1/63.2/d/1

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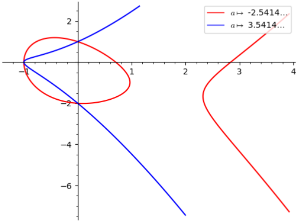

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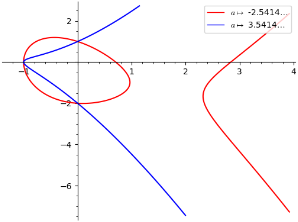

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- This matches our computation of $J = 2\cdot 3^2+3^6+O(3^7)$.

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- We don't know whether there are coset representatives that allow for that in our setting.

Lifting to $\mathrm{H}^1(\Gamma_S, \mathrm{HC}(\mathbb{Z}))$

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- Knowing the functions f_1 and f_2 to some fixed radius allows to find ϕ such that $\nu(\phi) = (f_1, f_2)$, by solving a linear system of equations.

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- To compute plectic Heegner points, need fundamental domains for Bruhat–Tits trees acted on by groups attached to totally definite quaternion algebras (work in progress).

Merci !

http://www.mat.uab.cat/~masdeu/