Lectures on Affine Algebraic Geometry

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Disclaimer

These notes were taken during a course given by Prof. Adrian Iovita during the 2007 Fall semester at Concordia University (Montréal). It is meant to be a first course in Algebraic Geometry from the scheme-theory point of view, introducing the commutative algebra when it is needed in the discussion.

The course is mainly self-contained, and assumes a basic knowledge of commutative algebra, although most of the results are proved in the course.

It is somewhat non-standard in the choice of topics, as it moves fast towards étale cohomology, which will be the subject of the second course. This means that a lot of fundamental theory is skipped, but also it allows one to see things more difficult to find in other introductory texts. If it followed an available text, that would be [2], but it really goes its own way.

I have tried to write almost all that was said in the lectures, maybe omitting some simple proofs that are best done as exercises. It is very advisable as well to do all the exercises that are suggested, in order to follow the exposition.

I would like to thank Dr. Shahab Shahabi for fixing so many typos.

Contents

Chapter 1

Motivation

Let $E/\mathbb{Q}$ be an elliptic curve given by $y^2 = x^3 + ax + b$, with $\Delta_E \neq 0$, embedded in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$, and suppose that $a, b \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.

Consider $\sigma \in G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$, and let $E^\sigma/\overline{\mathbb{Q}}$ be defined by $y^2 = x^3 + a^\sigma x + b^\sigma$, where $(\cdot)^\sigma = \sigma(\cdot)$. Then $\Delta_{E^\sigma} = (\Delta_E)^\sigma \neq 0$ because $\Delta_E \neq 0$, so $E^\sigma/\overline{\mathbb{Q}}$ is another elliptic curve.

Note that there is a map

$$\begin{aligned} E(\overline{\mathbb{Q}}) &\longrightarrow E^\sigma(\overline{\mathbb{Q}}) \\ (x, y) = P &\longmapsto P^\sigma = (x^\sigma, y^\sigma) \end{aligned}$$

Question 1. Does there exist a map (as varieties) $E^\sigma \to E$ such that on points it is the previously given one?

To see how this would go, take affine patches (that is, remove the point at ∞) and we want a polynomial $F(x, y) = \sum a_{ij} X^i Y^j$, such that

$$\begin{aligned} \tfrac{\overline{\mathbb{Q}}[X,Y]}{(Y^2 - X^3 - aX - b)} &\longrightarrow \tfrac{\overline{\mathbb{Q}}[X,Y]}{(Y^2 - X^3 - a^\sigma X - b^\sigma)} \\ \overline{F(X, Y)} &\longmapsto \overline{F^\sigma(X, Y)} \end{aligned}$$

(the automorphism σ acts on F by acting on its coefficients).

So we have a morphism:

$$\begin{array}{ccc} E^\sigma & \xrightarrow{\varphi} & E \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{\mathbb{Q}}) & \xrightarrow{\sigma} & \mathrm{Spec}(\overline{\mathbb{Q}}), \end{array}$$

which is *not* what we wanted, because it's not the identity on the base.

However, if we now consider the commutative diagram, coming from the embedding $\mathbb{Q} \subseteq \overline{\mathbb{Q}}$,

$$\begin{array}{ccc} \operatorname{Spec}(\overline{\mathbb{Q}}) & \xrightarrow{\sigma} & \operatorname{Spec}(\overline{\mathbb{Q}}) \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}(\mathbb{Q}), & \end{array}$$

we conclude that there's a morphism of *schemes* over $\mathbb{Q}$, $\varphi \colon E^\sigma \to E$. But note that E (and E^σ) are *not* varieties over $\mathbb{Q}$!

So even to study varieties, one needs to look at schemes.

1.1 An example

Consider the ring of integers, $\mathbb{Z}$, and define

$$\operatorname{Spec}(\mathbb{Z}) \stackrel{\text{def}}{=} \{2, 3, 5, 7, 11, 13, \ldots\} \cup \{0\}.$$

We want it to be a geometric object. So we first put a topology on it by defining a subset $M \subseteq \operatorname{Spec}(\mathbb{Z})$ to be closed iff:

- $M = \operatorname{Spec}(\mathbb{Z})$,
- $M = \varnothing$,
- $0 \notin M$ and M is finite.

Remark. Every nonempty open $U \subseteq \operatorname{Spec}(\mathbb{Z})$ contains 0. So the set $\{0\}$ is dense in $\operatorname{Spec}(\mathbb{Z})$.

Exercise 1. (a) Show that $\operatorname{Spec}(\mathbb{Z})$ is *compact*.

(b) Show that $\operatorname{Spec}(\mathbb{Z})$ is *not* Hausdorff.

Now we want a function theory, that is, we want to define the functions on any open subset.

Let $P \in \operatorname{Spec}(\mathbb{Z})$. Define the *residue field* of P:

$$\kappa(P) \stackrel{\text{def}}{=} \begin{cases} \mathbb{F}_p & \text{if } p \text{ is a (nonzero) prime,} \\ \mathbb{Q} & \text{if } p = 0. \end{cases}$$

An element $a \in \mathbb{Z}$ can be considered as a function:

$$\begin{aligned} a\colon \operatorname{Spec}(\mathbb{Z}) &\longrightarrow \textstyle\coprod_{P \in \operatorname{Spec}(\mathbb{Z})} \kappa(P) \\ P &\longmapsto \text{image of } a \text{ in } \kappa(P). \end{aligned}$$

So to $\mathbb{Z}$ we have associated a geometric object, the pair $\big(\operatorname{Spec}(\mathbb{Z}), \{\kappa(P)\}_{P \in \operatorname{Spec}(\mathbb{Z})}\big)$, much like to $k[X]$ there is associated $\mathbb{A}^1_k$ (for $k = \overline{k}$ an algebraically closed field).

Chapter 2

Review on Commutative Algebra

Let A be a ring. By this, unless otherwise stated, we will mean a commutative ring with identity, and with $1 \neq 0$.

Let $\mathfrak{a} \subseteq A$ be an ideal. We can consider the object $A/\mathfrak{a}$, which is defined to be:

$$(A, +)/(\mathfrak{a}, +)$$

as abelian group. Multiplication is defined as $\overline{x} \cdot \overline{y} \stackrel{\text{def}}{=} \overline{xy}$. One checks that this is well-defined.

Proposition 2.0.1. *We have the following properties:*

- *If $\mathfrak{a} \subseteq A$ is an ideal, then $\exists f \colon A \to A/\mathfrak{a}$ canonical and surjective, with $\ker(f) = \mathfrak{a}$.*

- *If $f \colon A \to B$ is a ring homomorphism, then $f(A) \subseteq B$ is a subring of B, and f induces an isomorphism $A/\ker(f) \simeq f(A)$.*

Let $f \colon A \to B$ be a ring homomorphism. Let $\mathcal{I}_A, \mathcal{I}_B$ denote the set of ideals in each ring. We have a map $f^* \colon \mathcal{I}_B \to \mathcal{I}_A$, defined by $f^*(\mathfrak{b}) \stackrel{\text{def}}{=} f^{-1}(\mathfrak{b}) \stackrel{\text{def}}{=} \{x \in A \mid f(x) \in \mathfrak{b}\} \subseteq A$.

We define now three important subsets of elements of a ring A.

Definition 2.0.2. 1. An element $x \in A$ is a *zero divisor* if $\exists y \in A, y \neq 0$ such that $xy = 0$. (note that 0_A is a zero divisor).

2. An element $x \in A$ is called *nilpotent* if $x^n = 0$ for some $n \geq 1$.

3. An element $x \in A$ is called a *unit* if there exists $y \in A$ such that $xy = 1$. The set of all units is written $A^\times$, and $(A^\times, \cdot)$ is a multiplicative group, called the *group of units*.

Remark. 1. If $x \in A$ is nilpotent, then x is a zero divisor.

2. If $x \in A$ is a unit, then x is **not** a zero divisor (and hence not a nilpotent, either).

Lemma 2.0.3. *Let* $\mathfrak{N}(A) \stackrel{def}{=} \{$*nilpotent elements*$\}$. *Then* $\mathfrak{N}(A)$ *is an ideal, called the* nilradical. *Also* $\mathfrak{N}\left(A/\mathfrak{N}(A)\right) = \{0\}$.

Proof. Everything follows easily (Hint: $(x+y)^{n+m} = \sum \binom{n+m}{i} x^i y^{n+m-i}$). □

2.1 Prime and Maximal Ideals

Let A be a ring, and let $\mathfrak{p} \subset A$ be a proper (i.e., $1 \notin \mathfrak{p}$) ideal of A.

Definition 2.1.1. 1. We say that $\mathfrak{p}$ is a *prime ideal* if whenever $xy \in \mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

2. We call $\mathfrak{p}$ a *maximal ideal* if $\forall \mathfrak{a} \subseteq A$ such that $\mathfrak{p} \subsetneq \mathfrak{a}$, then $\mathfrak{a} = A$.

Definition 2.1.2. 1. A ring A is an *integral domain* if 0 is the only zero divisor.

2. A ring A is a *field* if $A^\times = A \setminus \{0\}$, that is, every nonzero element is a unit.

We leave the proof of the following lemma as an easy exercise:

Lemma 2.1.3. *Let A be a ring, and $\mathfrak{p} \subsetneq A$. Then:*

1. $\mathfrak{p}$ is prime $\iff$ $A/\mathfrak{p}$ is an integral domain.

2. $\mathfrak{p}$ is maximal $\iff$ $A/\mathfrak{p}$ is a field. □

Write $X = \operatorname{Spec}(A)$ for the set of prime ideals of A. If $x \in X$, write $\mathfrak{p}_x$ for the prime ideal of A corresponding to x. So really $x = \mathfrak{p}_x$, but sometimes we want to emphasize the fact that it is a prime, and sometimes just an element in a set.

Lemma 2.1.4. *If* $\mathfrak{N}(A)$ *is the* nilradical *of* A, *then we have:*

$$\mathfrak{N}(A) = \bigcap_{x \in \operatorname{Spec}(A)} \mathfrak{p}_x.$$

Proof. We prove it by double inclusion, so let

$$I \stackrel{\text{def}}{=} \cap_{\mathfrak{p}\in\operatorname{Spec}(A)}\mathfrak{p}.$$

Then:

- $\subseteq$: Let $x \in \mathfrak{N}(\mathfrak{a})$, so $x^n = 0$ for some $n \geq 1$. If $\mathfrak{p} \in \operatorname{Spec}(A)$, then $0 = x^n \in \mathfrak{p} \implies x \in \mathfrak{p}$.

- $\supseteq$: Suppose that $I \nsubseteq \mathfrak{N}(A)$. Then $\exists f \in I$ such that $f^n \neq 0$ $\forall n \geq 1$. Let $S \stackrel{\text{def}}{=} \{1, f, f^2, \dots, f^n, \dots\} \not\ni 0$. Let $\mathcal{I} \stackrel{\text{def}}{=} \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \cap S = \varnothing\} \ni (0)$. One can apply then Zorn's lemma ($\mathcal{I}$ is an inductive set), to conclude that $\mathcal{I}$ has a maximal element, call it $\mathfrak{p}$. Note that $\mathfrak{p} \subsetneq A$, and that $\mathfrak{p}$ is a prime ideal (why?) not containing f. But this contradicts the fact that $f \in I$(!) ☐

Definition 2.1.5. The *Jacobson ideal* is defined to be:

$$\mathfrak{J}(A) \stackrel{\text{def}}{=} \bigcap_{\mathfrak{m}\subset A} \mathfrak{m},$$

where $\mathfrak{m}$ runs over the set of all maximal ideals of A.

Clearly, $\mathfrak{J}(A)$ is an ideal containing $\mathfrak{N}(A)$.

Lemma 2.1.6. *We have the following characterization of the elements in the Jacobson ideal:*

$$x \in \mathfrak{J}(A) \iff \forall a \in A\ ,\ 1 + ax \in A^\times.$$

Proof. Let $x \in \mathfrak{J}(A)$, and $a \in A$. Suppose that $1 + ax$ is not a unit. Then $(1 + ax)$ is a proper ideal in A, so that $\exists\mathfrak{m}$ a maximal ideal s.t. $(1 + xa) \subseteq \mathfrak{m}$. But $x, ax \in \mathfrak{m} \implies (1 + ax) - (ax) = 1 \in \mathfrak{m}$, which is a contradiction.

Conversely, suppose that $\forall a \in A$, $1 + ax$ is a unit, and that $x \notin \mathfrak{m}$ for some $\mathfrak{m}$ maximal. Then $1 = t - ax,$ for some $t \in \mathfrak{m} \implies 1 + ax = t \in \mathfrak{m} \implies 1 + ax$ is not a unit, which is a contradiction with our assumption. ☐

2.2 Nakayama's Lemma

In this section we state and prove an important result in commutative algebra, and discuss some of its applications.

Theorem 2.2.1 (Nakayama's Lemma). *Suppose that M is a finitely-generated A-module, and let $\mathfrak{a} \subseteq \mathfrak{J}(A)$ be an ideal of A contained in the Jacobson ideal. Under these conditions,* ***if*** *$M = \mathfrak{a}M$,* ***then*** *$M = 0$.*

Proof. Recall the Jacobson ideal, $\mathfrak{J}(A) = \bigcap_{\mathfrak{m}\in M(A)} \mathfrak{m}$, and that $x \in \mathfrak{J}(A) \iff 1 + ax \in A^\times$, $\forall a \in A$. Also, $\mathfrak{a}M$ is the submodule of M generated by $\{ax \mid a \in \mathfrak{a}, x \in M\}$.

Let $x_1, \ldots, x_n$ be generators for M. Let $1 \leq i \leq n$.

$$x_i \in M = \mathfrak{a}M \implies x_i = \sum_{j=1}^{n} \alpha_{ij} x_j \quad \alpha_{ij} \in \mathfrak{a}$$

so $(\alpha_{ii} - 1)x_i + \sum_{j\neq i} \alpha_{ij} x_j = 0$ for each $i = 1 \ldots n$. This can we written in a matrix:

$$\begin{pmatrix} \alpha_{11} - 1 & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - 1 & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

Call S the matrix, and S^* its adjoint matrix. So we have $SS^* = S^*S = \det(S)I_n$. Also, from the matrix identity we get that $\det(S)$ annihilates x_i for each i. As they generate M, this means that $\det(S)$ annihilates M.

But finally, note that $\det(S) = (-1)^n + u$, where $u \in \mathfrak{a}$. As $\mathfrak{a} \subseteq \mathfrak{J}(A)$, we must have $\det(S) \in A^\times$, so $M = 0$. □

Let $(A, \mathfrak{m})$ be a local ring. So $\mathfrak{m} = \mathfrak{J}(A)$. Let M be a finitely-generated A-module. Then note that $\mathfrak{m}M$ is an A-submodule of M.

Define $\overline{M} \overset{\text{def}}{=} M/\mathfrak{m}M$, which is an A-module. Moreover, it's k-vector space, with $k \overset{\text{def}}{=} A/\mathfrak{m}$, the residue field of A.

Theorem 2.2.2. *Let $(A, \mathfrak{m})$ be a local ring. Let M be a finitely-generated A-module, and let $x_1, x_2, \ldots, x_n \in M$ be such that $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ form a basis (or generate) of $\overline{M}$ over $k = A/\mathfrak{m}$. Then $x_1, x_2, \ldots, x_n$ generate M (as an A-module).*

Proof. Write $\kappa \overset{\text{def}}{=} A/\mathfrak{m}$, and $\overline{M} \overset{\text{def}}{=} M/\mathfrak{m}M$.

Let N be the A-submodule of M generated by $\{x_1, \ldots, x_n\} \subseteq M$. We want to show that $N = M$, so consider $M/N \overset{\text{def}}{=} \hat{M}$, and denote $x + N$ as $\hat{x}$.

Claim. $\mathfrak{m}\hat{M} = \hat{M}$

Proof. The fact that $\mathfrak{m}\hat{M} \subseteq \hat{M}$ is trivial.

Conversely, let $\hat{x} \in \hat{M}$ for some $x \in M$. Then $\overline{x} = \overline{a_1x_1} + \cdots + \overline{a_nx_n}$, with $a_i \in A$. Then $\overline{x} = \overline{a_1x_1 + \cdots + a_nx_n}$, which means that $x = a_1x_1 + \cdots + a_nx_n + t$ for some $t \in \mathfrak{m}M$. Hence $\hat{x} = \widehat{a_1x_1 + \cdots + a_nx_n} + \hat{t}$. But as N is generated by $\{x_1, \ldots, x_n\}$, the first term is 0. So $\hat{x} = \hat{t} \in \mathfrak{m}\hat{M}$, as wanted. $\square$

Apply now Nakayama's lemma to prove that $\hat{M} = 0$, so that $M = N$. $\square$

2.3 Tensor Products

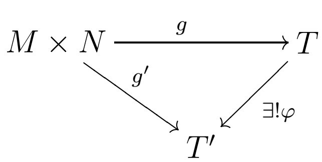
Let M,N,P be A-modules, and let $f\colon M \times N \to P$ be a map.

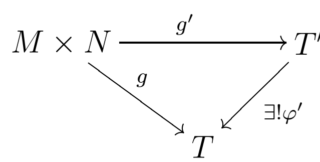
Definition 2.3.1. The map f is called *bilinear* if f is A-linear in each argument, that is, if for all $a, b \in A$, and $x, y \in M$ and $n \in N$, $f(ax + by, n) = af(x, n) + bf(y, n)$, and for all $x, y \in N$ and $m \in M$, $f(m, ax + by) = af(m, x) + bf(m, y)$.

Theorem 2.3.2. *Let M, N be A-modules. Then ther is a pair (T, g) consisting of an A-module T and an A-linear map $g\colon M \times N \to T$ such that, for every A-module P and any A-bilinear map $f\colon M \times N \to P$, there exists a unique A-module map $\varphi\colon T \to P$ such that $\varphi \circ g = f$.*

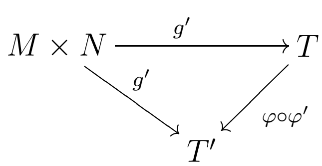
The pair (T, g) is unique up to unique isomorphism, and it is denoted $M \otimes_A N$, the tensor product of M and N over A. *The map g is written as $(m, n) \mapsto m \otimes n$.*

Proof. The uniqueness is guaranteed by universality, but we will illustrate how this proof goes anyway. So suppose that (T, g) and (T', g') both satisfy the conditions. Apply them to P being T and T', to get:





Note then that $\varphi \circ \varphi'\colon T' \to T'$ is such that, when composed after g', equals g', so it solves the diagram:



But $\mathrm{Id}_{T'}$ solves also the diagram, and hence by uniqueness, $\varphi \circ \varphi' = \mathrm{Id}_{T'}$. Similarly, we would conclude that composing in the other order gives Id_T, and hence we get a (unique) isomorphism $T \simeq T'$.

For the existence, we also construct T, as follows.

Let C be the free A-module with basis $M \times N$ (that is, $C = A^{M\times N}$). Let D be the submodule of C generated by all the elements of the form:

$$\begin{aligned}
(x_1 + x_2, y) - (x_1, y) - (x_2, y) &\quad \forall x_1, x_2 \in M, \forall y \in N, \\
(x, y_1 + y_2) - (x, y_1) - (x, y_2) &\quad \forall x \in M, \forall y_1, y_2 \in N, \\
(ax, y) - a(x, y) &\quad \forall a \in A, \forall x \in M, \forall y \in N, \\
(x, ay) - a(x, y) &\quad \forall a \in A, \forall x \in M, \forall y \in N.
\end{aligned}$$

We let then $T \stackrel{\text{def}}{=} C/D$, and let g be the obvious map sending $(m, n) \in M \times N$ to the class of $(m, n) \in C$ in C/D.

We leave it as an exercise that g is bilinear and that the pair (T, g) satisfies the required property. □

Example 2.3.3. Sometimes we can get extrange results. For instance, show that $\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/5 = 0$.

Example 2.3.4. If M is an A-module, then $M \otimes_A A \simeq M$.

Example 2.3.5. Let $\varphi \colon A \to B$ be a ring homomorphism, and consider $B \otimes_A A[x]$ (make B into an A-module through φ). Then $B \otimes_A A[x] \simeq B[x]$ as $A - modules$.

In general, if we have two ring homomorphisms $\varphi_i \colon A \to B_i$ $(i = 1, 2)$ making B_i into A-algebras, then $B_1 \otimes_A B_2$ has a ring structure, defined by $(b \otimes b')(b_1 \otimes b_1') = bb_1 \otimes b'b_1'$. Check it as an exercise.

The tensor product has functorial properties: let M, M_1, N, N_1 be A-modules, and $f \colon M \to M_1$, $g \colon N \to N_1$. We then get a map $f \otimes g \colon M \otimes_A N \to M_1 \otimes_A N_1$, defined by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

We leave the proof of the following theorem as an (easy) exercise.

Theorem 2.3.6. *Let M, N, and P be A-moduels. Then there exists a canonical A-linear isomorphism:*

$$\hom_A(M, \hom_A(N, P)) \simeq \hom_A(M \otimes_A N, P)$$

so that $\hom_A(N, -)$ *and* $(-) \otimes_A N$ *are* adjoint *functors.*

Proposition 2.3.7. *If the sequence of A-modules:*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \tag{2.1}$$

is exact, then for any A-module N the following is exact, too:

$$M' \otimes_A N \xrightarrow{f\otimes Id} M \otimes_A N \xrightarrow{g\otimes Id} M'' \otimes_A N \longrightarrow 0 \tag{2.2}$$

Proof. We show a slick proof, that doesn't deal with the individual elements of the modules.

Let P be any A-module. Consider the following sequence:

$$0 \longrightarrow \hom(M'' \otimes N, P) \xrightarrow{(g\otimes \mathrm{Id})^*} \hom(M \otimes N, P) \xrightarrow{(f\otimes \mathrm{Id})^*} \hom(M' \otimes N, P)$$

By adjunction, this is the same as the sequence:

$$0 \longrightarrow \hom(M'', \hom(N, P)) \xrightarrow{g^*} \hom(M, \hom(N, P)) \xrightarrow{f^*} \hom(M', \hom(N, P))$$

Claim (A). *If the sequence* (2.1) *is exact, then for all X an A-module, the sequence*

$$0 \longrightarrow \hom(M'', X) \longrightarrow \hom(M, X) \longrightarrow \hom(M', X)$$

is also exact.

Claim (B). *If for any A-module X the sequence*

$$0 \longrightarrow \hom(M'', X) \longrightarrow \hom(M, X) \longrightarrow \hom(M', X)$$

is exact, then the sequence (2.1) *is exact, too.*

Now, apply the "Claim A" to the module $X = \hom(N, P)$, for varying P. Then the sequence appearing in the proof is exact, and to remove the hom's, we apply "Claim B".

The proof of the two claims is left as an exercise. □

Example 2.3.8. Let M be an A-module, and $\mathfrak{a} \subseteq A$ an ideal. Then $A/\mathfrak{a}$ is an A-module. We will show that, for any A-module M, $M \otimes_A A/\mathfrak{a} \simeq M/(\mathfrak{a}M)$.

By the previous proposition, the following is exact:

$$M \otimes \mathfrak{a} \longrightarrow M \otimes A \longrightarrow M \otimes (A/\mathfrak{a}) \longrightarrow 0$$

and we have the other natural sequence, which is exact by construction:

$$0 \longrightarrow \mathfrak{a}M \longrightarrow M \longrightarrow M/\mathfrak{a}M \longrightarrow 0$$

We then use that $M \otimes A \simeq M$ and that $M \otimes \mathfrak{a} \twoheadrightarrow \mathfrak{a}M$ via the map $m \otimes a \mapsto am$. By diagram chasing, we conclude $M \otimes (A/\mathfrak{a}) \simeq M/\mathfrak{a}M$, as wanted.

Example 2.3.9. If $0 \to M' \to M$ is exact and N is an A-module, then the resulting sequence $0 \to M' \otimes N \to M \otimes N$ is not exact in general.

As an easy counterexample, consider $f\colon \mathbb{Z} \to \mathbb{Z}$ sending $x \mapsto 5x$, which is injective. Tensor it by $\mathbb{Z}/5\mathbb{Z}$, and get $f \otimes id\colon \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$, which maps $\overline{x} \mapsto \overline{5x} = \overline{0}$, clearly not injective.

2.4 Flatness

Definition 2.4.1. An A-module N is called *flat* (or A-flat) if, for any injective A-module map $f\colon M' \hookrightarrow M$, the resulting map $f \otimes \mathrm{Id}\colon M' \otimes N \to M \otimes N$ is still injective.

(equivalently, the tensor functor preserves exact sequences).

Example 2.4.2. The ring A itself, as an A-module, is flat. Hence any free A-module is flat, too.

Example 2.4.3. Let $A = \mathbb{Z}$, and $M = \mathbb{Q}$. Prove, as an exercise, that:

- $\mathbb{Q}$ is flat,
- $\mathbb{Q}$ is not a free $\mathbb{Z}$-module, and
- $\mathbb{Q}$ is not a finitely-generated $\mathbb{Z}$-module.

2.5 Localization

Let A be a ring, $S \subseteq A$ a set. We want to construct a new ring $S^{-1}A$, together with a map $f\colon A \to S^{-1}A$, such that if $s \in S$, then $f(s) \in (S^{-1}A)^{\times}$.

Definition 2.5.1. A set $S \subseteq A$ is called *multiplicatively closed* if:

- $1_A \in S$,
- $s, t \in S \implies st \in S$.

To construct the localization at S, consider $A \times S$, and define in it an equivalence relation, written $\sim$, as:

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ such that } u(at - bs) = 0.$$

Now, take the quotient $S^{-1}A \overset{\text{def}}{=} (A \times S)/\sim$, and write $\frac{a}{s} \overset{\text{def}}{=}$ class of (a, s).

Claim. *$S^{-1}A$ has a natural ring structure: if $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$, then:*

$$\frac{a}{s} + \frac{b}{t} \overset{def}{=} \frac{at+bs}{st}, \quad \frac{a}{s}\frac{b}{t} \overset{def}{=} \frac{ab}{st},$$

and the special elements $1_{S^{-1}A} \overset{def}{=} \frac{1_A}{1_A}$, and $0_{S^{-1}A} \overset{def}{=} \frac{0_A}{1_A}$.

We have the natural map $f\colon A \to S^{-1}A$, which sends $a \mapsto \frac{a}{1}$ ($= \frac{as}{s}$ for any $s \in S$).
Finally, note that $\forall s \in S, f(s) = \frac{s}{1}$ is a unit: $\frac{s}{1}\frac{1}{s} = \frac{s}{s} = 1_{S^{-1}A}$.

Remark. The map $f\colon A \to S^{-1}A$ is not always injective:

$$\ker f = \{a \in A \mid \frac{a}{1} = 0_{S^{-1}A}\} = \{a \in A \mid \exists u \in S \text{ with } ua = 0\}.$$

Also, note that $s, t \in S$ such that $st = 0$, then $0 \in S$, and then $S^{-1}A$ becomes the trivial ring. In fact, if $S \cap \mathfrak{N}(A) \neq \varnothing$, then $S^{-1}A$ is trivial. We will often avoid this situation.

Example 2.5.2.

- Let A be an integral domain. Take then $S = A \setminus \{0\}$. Then $S^{-1}A$ is a field. Moreover, $A \hookrightarrow S^{-1}A$ is injective. It is called the *fraction field of* A, and denoted either by $\mathrm{Frac}(A)$ or $\mathrm{Q}(A)$.
- Let A be any ring, and $S = \{\text{non-zero-divisors}\}$ (this is a generalization of the previous case). Then $A \hookrightarrow S^{-1}A$ is also injective, and it is called the *total ring of fractions of* A.
- Let A be any ring, $\mathfrak{p} \subsetneq A$ A prime ideal. Let then $S = A \setminus \mathfrak{p}$ (S is mult.closed $\iff$ $\mathfrak{p}$ is prime). We write then $A_\mathfrak{p} \overset{\text{def}}{=} S^{-1}A$, and we have:

  Lemma 2.5.3. *$A_\mathfrak{p}$ is a local ring, with maximal $\mathfrak{p}^e = f(\mathfrak{p}) = \mathfrak{p}A_\mathfrak{p}$.*

  Proof. Let $\mathfrak{m}_\mathfrak{p} \overset{\text{def}}{=} \{\frac{a}{s} \in A_\mathfrak{p} \mid a \in \mathfrak{p}\}$. Then $\mathfrak{m}_\mathfrak{p}$ is a maximal ideal in $A_\mathfrak{p}$ ($\mathfrak{m}_\mathfrak{p} = S^{-1}\mathfrak{p}$ as modules), and is the unique maximal, too: if $\frac{a}{s} \notin \mathfrak{m}_\mathfrak{p}$, then $a \notin \mathfrak{p}$. So $a \in S$, and hence $\frac{s}{a} \in A_\mathfrak{p}$, which means that $\frac{a}{s}$ is a unit in $A_\mathfrak{p}$. □

Example 2.5.4. We have two ways, given a ring A and a prime $\mathfrak{p} \subsetneq A$, to obtain a field:

1. Consider $A_\mathfrak{p} \supsetneq \mathfrak{m}_\mathfrak{p}$. Then $\kappa(\mathfrak{p}) \overset{\text{def}}{=} A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$.
2. Take the quotient $A/\mathfrak{p}$ which is an integral domain, and so $\kappa'(\mathfrak{p}) \overset{\text{def}}{=} \mathrm{Frac}(A/\mathfrak{p})$.

Prove, as an exercise, that $\kappa(\mathfrak{p}) \simeq \kappa'(\mathfrak{p})$.

As a consequence, if $\mathfrak{p} \subsetneq A$ is actually a maximal ideal, then $\kappa(\mathfrak{p}) \simeq A/\mathfrak{p}$.

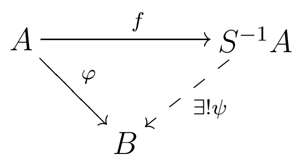
Example 2.5.5. Let $f \in A \setminus \mathfrak{N}(A)$. Let $S_f \overset{\text{def}}{=} \{1, f, f^2, \dots, f^n, \dots\}$. Then we write $A_f \overset{\text{def}}{=} S_f^{-1}A$, or sometimes $A[\frac{1}{f}]$.

We have:

$$\frac{a}{f^n} = \frac{b}{f^m} \iff \exists f^r \in S \text{ such that } f^{r+m}a = f^{r+n}b.$$

Localization can be described as well through a universal property:

Proposition 2.5.6. *Let A be a ring, $S \subseteq A$ a multiplicatively-closed set. Let B be another ring, and $\varphi\colon A \to B$ such that $\forall s \in S$, $\varphi(s) \in B^\times$. Then $\exists!$ ring homomorphism $\psi\colon S^{-1}A \to B$ such that the following diagram commutes:*



Proof. Define $\psi(\frac{a}{s}) \overset{\text{def}}{=} \varphi(a)\varphi(s)^{-1}$ and check that all works. □

2.6 Localization of A-modules

Let A be a ring, $S \subseteq A$ a multiplicatively closed set, and M and A-module. We construct another module, $S^{-1}M$, which is actually and $S^{-1}A$-module. The first way of defining it is straightforward: $S^{-1}M \overset{\text{def}}{=} M \otimes_A S^{-1}A$.

The following is an alternative (and equivalent) construction. Start with $M \times S$, and consider in it the equivalence relation given by $(m, s) \sim (n, t) \iff \exists u \in S$ such that $u(tm - sn) = 0 \in M$. We define then $S^{-1}M$ to be the set of equivalence classes, and write $\frac{m}{s}$ for the class of (m, s).

Again, we define its module structure mimicking the construction of the rationals: $\frac{m}{s} + \frac{n}{t} \overset{\text{def}}{=} \frac{tm+sn}{st}$, and $\frac{a}{s}\frac{m}{t} \overset{\text{def}}{=} \frac{am}{st}$. One checks that these operations are well-defined and make $S^{-1}M$ into an $S^{-1}A$-module.

Via the map $A \to S^{-1}A$, we can view $S^{-1}M$ also as an A-module. We have also an A-linear map $g\colon M \to S^{-1}M$, sending $m \mapsto \frac{m}{1}$.

Again, it satisfies a universal property: Let N be an $S^{-1}A$-module, together with an A-linear map $h\colon M \to N$. Then there exists a unique $S^{-1}A$-linear morphism $\varphi\colon S^{-1}M \to N$, making the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\;g\;} & S^{-1}M \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \exists!\varphi} & \\ & N & \end{array}$$

(just define $\varphi(\frac{m}{s}) \overset{\text{def}}{=} \frac{1}{s}h(m)$).

Moreover, if M and N are both A-modules with $f\colon M \to N$, and $S \subseteq A$ is a multiplicatively closed set; then there exists a unique $\overline{f}\colon S^{-1}M \to S^{-1}N$ map of $S^{-1}A$-modules such that the following diagram commutes:

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{\;\overline{f}\;} & S^{-1}M \\ \uparrow & & \uparrow \\ M & \xrightarrow{\;f\;} & N, \end{array}$$

and so we get the localization functor, from the category of A-modules to the category of $S^{-1}A$-modules.

Proposition 2.6.1. *Localization is exact (that is, it preserves exact sequences).*

Proof. Exercise. □

Proposition 2.6.2. *Let M be an A-module, $S \subseteq A$ a mult.set. Then there is a canonical $S^{-1}A$-module isomorphism: $\alpha\colon S^{-1}M \xrightarrow{\sim} S^{-1}A \otimes_A M$.*

Proof. It's easy. Just define $\alpha(\frac{m}{s}) \overset{\text{def}}{=} \frac{1}{s} \otimes m$, and $\alpha^{-1}(\frac{a}{s} \otimes m) \overset{\text{def}}{=} \frac{am}{s}$. □

Corollary 2.6.3. *The A-module $S^{-1}A$ is flat (we say that it's a flat A-algebra).*

Corollary 2.6.4. *Let $N \subseteq M$ be an A-submodule of M. Then $S^{-1}N \subseteq S^{-1}M$, and, as $S^{-1}A$-modules, we have:*

$$\frac{S^{-1}M}{S^{-1}N} \simeq S^{-1}(M/N)$$

We ommit the proofs of the following facts about localization, which can be checked as an exercise.

Proposition 2.6.5. *Let $N, P \subseteq M$ be submodules of an A-module M. Then:*

- $S^{-1}(N+P) = (S^{-1}N) + (S^{-1}P)$ *in* $S^{-1}M$.
- $S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P)$ *in* $S^{-1}M$.

Proposition 2.6.6. *If M, N are A-modules, there is a canonical isomorphism of $S^{-1}A$-modules:*

$$S^{-1}(M \otimes_A N) \simeq S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$

If A is a ring and $\mathfrak{p} \in \operatorname{Spec}(A) = X$, then the image of $f \in A$ in $A_\mathfrak{p}$ is the "germ" of f near $\mathfrak{p}$, as we will see in the following example.

Example 2.6.7. Let k be any algebraically closed field (e.g. $k = \mathbb{C}$). Let $A = k[X,Y]/(Y^2 - X^3 - 1) = k[x,y]$. Let $\mathfrak{p} = (x+1, y)$, which is a maximal ideal of A.

The ring $A_\mathfrak{p} = (k[X,Y]/(Y^2 - X^3 - 1))_{(x+1,y)}$ is local, with maximal $\mathfrak{m}_\mathfrak{p}$. The ideal $\mathfrak{m}_\mathfrak{p}$ is generated by the images of $x+1$ and y in the localization $A_\mathfrak{p}$.

Note now that $y^2 = x^3 + 1 = (x+1)(x^2 - x + 1)$, and the second factor is not in $\mathfrak{m}_\mathfrak{p}$, so it's a unit $u \in A_\mathfrak{p}$. Hence we can write $x + 1 = y^2 u^{-1}$, and thus $\mathfrak{m}_\mathfrak{p} = (x+1, y) = yA_\mathfrak{p}$ is actually principal in $A_\mathfrak{p}$.

Take now any function (that is, an element of A). Say, for instance, $f(x,y) = xy^2 + y^2x^3 + x^4 - 1$. Let $f_\mathfrak{p}$ be the image of f in $A_\mathfrak{p}$. Then:

$$\begin{aligned} f_\mathfrak{p} &= xy^2 + y^2x^3 + (x+1)(x-1)(x^2+1) = xy^2 + y^2x^3 + y^2u^{-1}(x-1)(x^2+1) = \\ &= y^2(x + x^3 + u^{-1}(x-1)(x^2+1)) = y^2 v \end{aligned}$$

where $v \in A_\mathfrak{p}^\times$. So we say that f vanishes to order two at $\mathfrak{p}$ (because $f_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}^2 \setminus \mathfrak{m}_\mathfrak{p}^3$).

2.7 Local Properties

Let A be a ring, and M an A-module.

For (P) a property of A (or of M), we say that (P) is a local property if: A has (P) if, and only if, $A_\mathfrak{p}$ has (P) for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

Proposition 2.7.1. *Let M be an A-module. Then the property of M being 0 is local. More precisely, TFAE:*

1. $M = 0$
2. $M_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec}(A)$.

3. *$M_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in M(A)$.*

Proof. The only nontrivial implication is (3) $\implies$ (1): suppose that $M \neq 0$. Let $x \in M$ be a nonzero element. Let $\mathfrak{a} \stackrel{\text{def}}{=} \{a \in A \mid ax = 0\} = \operatorname{Ann}(x)$. Then $\mathfrak{a} \neq A$ because $x \neq 0$. Then $\mathfrak{a} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. But $M_{\mathfrak{m}} = 0$, so $\frac{x}{1} = \frac{0}{1}$ in $M_{\mathfrak{m}}$, that is, there is some $s \notin \mathfrak{m}$ such that $sx = 0$, which is a contradiction. □

Let M, N be A-modules, $f\colon M \to N$ an A-linear map.

Proposition 2.7.2. *Injectivity and surjectivity are local properties. More precisely, TFAE:*

1. *f is injective (resp. surjective).*
2. *$f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (resp. surjective) for each $\mathfrak{p} \in \operatorname{Spec}(A)$.*
3. *$f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (resp. surjective) for each $\mathfrak{m} \in \operatorname{Spec}(A)$.*

Proof. Again, the only nontrivial implication is (3) $\implies$ (1): Let $P \stackrel{\text{def}}{=} \ker(f)$. We have an exact sequence:

$$0 \to P \to M \xrightarrow{f} N$$

Let $\mathfrak{m} \in M(A)$ be a maximal ideal of A. We get then an exact sequence:

$$0 \to P_{\mathfrak{m}} \to M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}}$$

If $f_{\mathfrak{m}}$ is injective, we get that $P_{\mathfrak{m}} = 0$. So $P_{\mathfrak{m}} = 0$ for each $\mathfrak{m} \in M(A) \implies P = 0$ (by the previous proposition).

For surjectivity, we would do it similarly, but with the cokernel instead of the kernel. □

Corollary 2.7.3. *Flatness is a local property. More precisely, TFAE:*

- *M is a flat A-module.*
- *$M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for each $\mathfrak{p} \in \operatorname{Spec}(A)$.*
- *$M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module for each $\mathfrak{m} \in M(A)$.*

Chapter 3

Spec and the Zariski Topology

Let A be a ring.

Definition 3.0.4. The *spectrum* of the ring is $X \stackrel{\text{def}}{=} \operatorname{Spec}(A) \stackrel{\text{def}}{=} \{\text{prime ideals of } A\}$

We will write $x \in \operatorname{Spec}(A)$ to think of them as points, or $\mathfrak{p}_x$ if we think of them as prime ideals in A. We could say $x = \mathfrak{p}_x$, though.

Let $\mathfrak{a} \subseteq A$ be an ideal. Define

$$V(\mathfrak{a}) \stackrel{\text{def}}{=} \{x \in X \mid \mathfrak{a} \subseteq \mathfrak{p}_x\}$$

Write $\mathcal{F} \stackrel{\text{def}}{=} \{V(\mathfrak{a}) \mid \mathfrak{a} \text{ an ideal of } A\} \subseteq \mathcal{P}(X)$.

Proposition 3.0.5. *With the previous definitions, the following holds:*

1. *$\varnothing$ and X belong to $\mathcal{F}$,*
2. *if $F_i \in \mathcal{F}$, for $i \in I$, then $\bigcup_{i \in I} F_i \in \mathcal{F}$,*
3. *if F_1 and F_2 belong to $\mathcal{F}$, then so does $F_1 \cap F_2$.*

Proof.

1. $\varnothing = V(A)$, and $X = V(\{0\})$.
2. Write $F_i = V(\mathfrak{a}_i)$, with $\mathfrak{a}_i \subseteq A$ ideals. Then $\bigcup_{i \in I} F_i = V(\sum_{i \in I} \mathfrak{a}_i)$.
3. $F_1 \cup F_2 = V(\mathfrak{a}_1 \mathfrak{a}_2)$, if $F_i = V(\mathfrak{a}_i)$. Check it as an exercise.

□

With the characterization given so far of the topology, we would say that:

$$U \subseteq \operatorname{Spec}(A) \text{ is open} \iff U = X - V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{a} \not\subseteq \mathfrak{p}\}$$

We want a better description for the opens. Concretely, we search for a basis for the topology. So let $f \in A$, and define $X_f \overset{\text{def}}{=} X \setminus V((f))$, which is an open.

$$X_f = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$$

Lemma 3.0.6. *The family $\{X_f \mid f \in A\}$ is a basis for the Zariski topology on X.*

Proof. Need to show that, for all $\mathfrak{p} \in X$, and for all U open such that $\mathfrak{p} \in U$, there exists an $f \in A$ such that $\mathfrak{p} \in X_f \subseteq U$.

Let $\mathfrak{p} \in U = X \setminus V(A)$. So $\mathfrak{a} \not\subseteq \mathfrak{p}$. Then exists $f \in \mathfrak{a}$ such that $f \notin \mathfrak{p}$. Hence $\mathfrak{p} \in X_f$.

Also, if $f \notin \mathfrak{q}$, then $\mathfrak{q} \not\supseteq \mathfrak{a}$, so $\mathfrak{q} \in U$, which proves that $X_f \subseteq U$. □

Suppose now that $f_1 \neq f_2$. If both f_1 and f_2 are in the nilradical of A, then $X_{f_1} = \varnothing = X_{f_2}$. So, in general, the X_f's are not all distinct.

Lemma 3.0.7. *Let $x \in X$. Then $\overline{\{x\}} = \{x\} \iff \mathfrak{p}_x$ is a maximal ideal.*

Proof. Easily, check that $\overline{\{x\}} = V(\mathfrak{p}_x)$. Then,$\overline{\{x\}} = \{x\}$ is equivalent to $V(\mathfrak{p}_x) = \{\mathfrak{p}_x\}$, which is the same to saying that $\mathfrak{p}_x$ is a maximal ideal. □

3.1 Functoriality of Spec

Let A, B be two rings, and $f\colon A \to B$ a ring homomorphism.

Lemma 3.1.1. *The map $f^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is* ***continuous****.*

Proof. Let $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$.

First, note that $f^*(\mathfrak{q}) \overset{\text{def}}{=} f^{-1}(\mathfrak{q})$ is a prime of A. To check continuity, let $a \in A$, and consider $(f^*)^{-1}(X_a)$. We check that this equals $Y_{f(a)}$. □

Example 3.1.2. Let k be a field. Then $\operatorname{Spec}(k) = \{(0)\}$.

Example 3.1.3. Consider $f\colon \mathbb{Z} \to \mathbb{F}_p$ be the reduction homomorphism (modulo p). Then f^* sends the only point in $\operatorname{Spec}(\mathbb{F}_p)$ to the ideal (p) of $\mathbb{Z}$.

Example 3.1.4. Consider the inclusion $f\colon \mathbb{Z} \subseteq \mathbb{Q}$. Then f^* sends the unique point of $\operatorname{Spec}(\mathbb{Q})$ to $(0) \in \operatorname{Spec}(\mathbb{Z})$.

Example 3.1.5. Note that $\mathrm{Spec}(\mathbb{F}_3) \simeq \mathrm{Spec}(\mathbb{F}_5)$ (as both are one-point topological spaces), but there is no ring homomorphism $f : \mathbb{F}_3 \to \mathbb{F}_5$.

Example 3.1.6. Let p be a prime, and $A = \mathbb{Z}/p^n\mathbb{Z}$. Consider the canonical map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$.

The ideals of A correspond to those of $\mathbb{Z}$ containing $p^n\mathbb{Z}$. Thus A has only one prime ideal, and so $\mathrm{Spec}(A)$ is a one-point topological space.

So even with A not being a field, we can get $\mathrm{Spec}(A)$ to have only one point. And again, $\mathrm{Spec}(A) \simeq \mathrm{Spec}(k)$, but A is not even a field.

Example 3.1.7. Let $A = \frac{\mathbb{Z}}{2^3 3^2 5\mathbb{Z}}$. Then $\mathrm{Spec}(A) = \{\overline{2A}, \overline{3A}, \overline{5A}\}$, with the discrete topology.

Example 3.1.8. Let $A = \mathbb{Z}_{(2)} = \{\frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, (a,b) = 1, \quad 2 \nmid b\} \subseteq \mathbb{Q}$.

Then $\mathrm{Spec}(A) = \{(0), (2)\}$.

Claim. *Every ideal of A is of the form $2^n A$.*

Note also that (2) is maximal (and thus closed in $\mathrm{Spec}(A)$). The ideal (0) is then open, and $\overline{\{(0)\}} = \mathrm{Spec}(A)$.

Chapter 4

Towards the Sheaf of Functions

Let A be a ring, and consider $X = \operatorname{Spec}(A)$, which is so far a topological space with the Zariski topology. Every element $a \in A$ can be thought of as a function on X, with values in $\coprod_{x \in X} \kappa(x)$ (where $\kappa(x) = \kappa(\mathfrak{p}_x) = A_{\mathfrak{p}_x}/\mathfrak{m}_{\mathfrak{p}_x}$), in the following way: if $x \in X$, then $a(x) \overset{\text{def}}{=}$ image of a in $\kappa(x)$.

If $U \subseteq X$ is an open, we want to define functions $U \to \coprod_{x \in U} \kappa(x)$.

As a first case, what if $U = X_f$, for some $f \in A$?

Example 4.0.9. Let A be a ring, and $S \subseteq A$ be a multiplicatively closed set. Let $\varphi\colon A \to S^{-1}A$ be the canonical morphism. It induces a continuous map as usual, that is: $\varphi^*\colon \operatorname{Spec}(S^{-1}A) = Y \to \operatorname{Spec}(A) = X$. We study this map more closely:

Lemma 4.0.10.

1. *φ^* is injective.*

2. *The image of φ^* is*

$$X_S \overset{def}{=} \bigcap_{f \in S} X_f = \{x \in X \mid \mathfrak{p}_x \cap S = \varnothing\}$$

3. *If we put on X_S the topology induced from X, then φ^* is a homeomorphism $Y \simeq X_S$.*

Proof. Let $Y = \operatorname{Spec}(S^{-1}A)$. It is easy to check that $\varphi^*(Y) \subseteq X_S$. We will construct an inverse map $\psi\colon X_S \to Y$.

Given $\mathfrak{p} \in X_S$, define $\psi(\mathfrak{p}) \overset{\text{def}}{=} S^{-1}\mathfrak{p} \subseteq S^{-1}A$.

One checks that $S^{-1}\mathfrak{p}$ is prime (it is a proper ideal because $\mathfrak{p} \cap S = \varnothing$) just from the definition.

Also, we need that $\psi \circ \varphi^* = \mathrm{Id}_Y$, $\varphi^* \circ \psi = \mathrm{Id}_{X_S}$.

Let $\mathfrak{q} \subseteq S^{-1}A$. $\psi(\varphi^*(\mathfrak{q})) = \psi(\mathfrak{p})$, where $\mathfrak{p} = \{a \in A \mid \frac{a}{1} \in \mathfrak{q}\}$. In general, it is true that $S^{-1}\mathfrak{p} \subseteq \mathfrak{q}$, because $\mathfrak{q}$ is an ideal ($\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \in \mathfrak{q}$). But let $\frac{a}{s} \in \mathfrak{q}$. And hence:

$$s\frac{a}{s} \in \mathfrak{q} \Rightarrow \frac{a}{1} \in \mathfrak{q} \Rightarrow a \in \mathfrak{p} \Rightarrow \frac{a}{s} \in S^{-1}\mathfrak{p}$$

The other composition is checked similarly.

Lastly, we want to show that ψ is continuous. So let $\frac{a}{s} \in S^{-1}A$. Easily, one checks that $\psi^{-1}(Y_{\frac{a}{s}}) = X_a \cap X_S$, which is an open in X_S, as wanted. □

Example 4.0.11. Let A be an integral domain, and $S = A \setminus 0$. Then $S^{-1}A = Q(A)$ is a field. In the corresponding specs, the image of its unique point is $X_S = \{x \in X \mid \mathfrak{p}_x \cap (A \setminus 0) = \varnothing\} = \{(0)\}$.

Example 4.0.12. Let A be a ring, and $\mathfrak{p} \subsetneq A$ a prime ideal. Let $S = A \setminus \mathfrak{p}$. Then $S^{-1}A = A_{\mathfrak{p}}$, and we get that $\mathrm{Spec}(A_{\mathfrak{p}}) \simeq X_S = \{\mathfrak{q} \in X \mid \mathfrak{q} \subseteq \mathfrak{p}\}$.

Example 4.0.13. Let A be a ring, and $f \in A$. Let $S = \{1, f, f^2, \dots\}$ (assume that f is not nilpotent, to avoid trivial cases).

In this case, $S^{-1}A = A_f$, and we get $\mathrm{Spec}(A_f) \simeq X_f \subseteq X$.

This last example is important: every element $\alpha \in A_f$ can be seen as a function on $\mathrm{Spec}(A_f) \simeq X_f$:

$$X_f \simeq \mathrm{Spec}(A_f) \to \coprod_{\mathfrak{q} \in \mathrm{Spec}(A_f)} \kappa(\mathfrak{q})$$

Also, $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$, where $\mathfrak{p} = \varphi^*(\mathfrak{q})$.

We want to define, for $U \subseteq X$ open, $\mathcal{O}_X(U) \overset{\text{def}}{=} \{$ "special" functions $U \to \coprod_{x \in U} \kappa(x)\}$.

So far, we have set:

- For $U = X$, $\mathcal{O}_X(X) = A$.
- For $U = X_f$, $\mathcal{O}_X(X_f) = A_f$.

We will develop first the general theory of sheaves in order to define $\mathcal{O}_X(U)$ for general U.

Chapter 5

Some Sheaf Theory

Let X be a topological space.

Definition 5.0.14. A *presheaf* (of abelian groups, although it can be of rings, modules, or actually of any abelian category) is a rule $\mathcal{F}$ which assigns:

- To every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$.
- To every pair of opens $V \subseteq U \subseteq X$, a group homomorphism $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$, called *restriction*, satisfying:
  - $\rho_{UU} = \mathrm{Id}_{\mathcal{F}(U)}$,
  - If $W \subseteq V \subseteq U \subseteq X$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

The following is the "standard" example to keep in mind.

Example 5.0.15. Let X be a topological space, and define $\mathcal{F}$ to be the presheaf of complex-valued continuous functions on X. That is, for a open $U \subseteq X$, one defines $\mathcal{F}(U) \overset{\text{def}}{=} \{f \colon U \to \mathbb{C} \mid f \text{ is continuous}\}$.

The maps ρ_{UV} are the restriction of a continuous map to a smaller open.

Because of this example, one sometimes writes $\rho_{UV}(f)$ as $f|_U$

The presheaf $\mathcal{F}$ defined in the previous example has an interesting property: given any open set U, and any open covering of it, then a function f on U is determined uniquely by its restrictions to the opens in the cover. Moreover, given the *local* description of a function, it is possible to glue it to a *global* function, provided that the local data is compatible.

This is called the *sheaf property*, and we will make it more precise in the following.

Definition 5.0.16. Let X be a topological space, $\mathcal{F}$ a presheaf on X. Then $\mathcal{F}$ is a *sheaf of abelian groups* if for every $U \subseteq X$ open, $\{U_i\}_{i\in I}$ an open covering of U and $\{f_i\}_{i\in I}$ with $f_i \in \mathcal{F}(U_i)$, such that for all pairs $i, j \in I$, $\rho_{U_i,U_i\cap U_j}(f_i) = \rho_{U_j,U_i\cap U_j}(f_j)$, then:

$$\exists! f \in \mathcal{F}(U) \text{ such that for all } i \in I, \rho_{U,U_i}(f) = f_i$$

Another way of describing sheaves is through its *stalks*, which we now define.

So suppose that $\mathcal{F}$ is a presheaf on X, and fix $x \in X$. Let $I \stackrel{\text{def}}{=} \{U \subseteq X \mid U \text{ is open and } x \in U\}$.

The set I, ordered by inclusion, is a directed set ($U \geq V \iff U \subseteq V$).

Consider then the directed family (of abelian groups):

$$\{\{\mathcal{F}(U) \mid U \in I\}, \rho_{UV}\}$$

This is a directed family of abelian groups, and one can take its direct limit.

Definition 5.0.17. The *stalk* of $\mathcal{F}$ at x, is:

$$\mathcal{F}_x \stackrel{\text{def}}{=} \varinjlim_{U\in I} (\mathcal{F}(U), \rho_{UV})$$

We make this more down-to-earth:

$$\mathcal{F}_x = \left(\coprod_{U\in I} \mathcal{F}(U) \right) \Big/ \sim$$

where if $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, then $f \sim g \iff \exists W \in I, W \subseteq U \cap V$ such that $\rho_{UW}(f) = \rho_{VW}(g)$.

For every $U \in I$, we have a map

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad f \mapsto [f]$$

which is a morphism of abelian groups.

Also, given $[f], [g] \in \mathcal{F}_x$, with $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, we can define $[f] + [g] \stackrel{\text{def}}{=} [f' + g']$, where we let $W \stackrel{\text{def}}{=} U \cap V$, so that $f' \stackrel{\text{def}}{=} f|W$ and $g' \stackrel{\text{def}}{=} g|W$. It is easy to show that this is well-defined and it makes $\mathcal{F}_x$ into an abelian group.

In the exercises we will show that, if $\mathcal{F}$ is a sheaf of abelian groups on a topological space X, then $\mathcal{F}$ can be recovered from all its stalks.

5.1 Morphisms of (pre)sheaves

Given $\mathcal{F}$, $\mathcal{G}$ sheaves of abelian groups on X.

Definition 5.1.1. A morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a natural transformation of the functor $\mathcal{F}$ to $\mathcal{G}$.

That is, φ is a collection $\{\varphi_U\}_{U \subseteq X}$, such that $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ and such that, if $V \subseteq U \subseteq X$, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V), \end{array}$$

(or, in the alternate notation, $\varphi_U(f)|_V = \varphi_V(f|_U)$).

If $\mathcal{F}, \mathcal{G}$ are sheaves, then φ is a *morphism of sheaves* if it is one of presheaves.

Suppose that $\mathcal{F}, \mathcal{G}$ are sheaves, and $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism. Note that φ induces a morphism of directed sets $\{\varphi_U\}$, and thus a morphism on the stalks,

$$\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$$

for each $x \in X$, which is defined as:

$$\varphi_x([f]) \stackrel{\text{def}}{=} [\varphi_U(f)] \quad \text{if } f \in \mathcal{F}(U)$$

Definition 5.1.2. A morphism φ is *injective* (resp. *surjective, bijective*) if φ_x is injective (resp. surjective, bijective) for each $x \in X$.

Suppose that $\mathcal{F}$ is a *presheaf* of abelian groups on X.

Proposition 5.1.3. *There exists a unique sheaf of abelian groups $\mathcal{F}^+$ on X, together with a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{F}^+$ with the property that the induced map on the stalks $\varphi_x\colon \mathcal{F}_x \xrightarrow{\sim} \mathcal{G}_x$ is an isomorphism for all $x \in X$ (note that φ is injective, so we can think of $\mathcal{F} \subseteq \mathcal{F}^+$).*

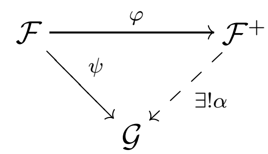
Proof. We define $\mathcal{F}^+$ explicitly as, given $U \subseteq X$ any open subset,

$$\mathcal{F}^+(U) \stackrel{\text{def}}{=} \Big\{ s\colon U \to \bigcup_{x \in U} \mathcal{F}_x \;\Big|\; \begin{array}{l} s(x) \in \mathcal{F}_x \forall x \in U, \text{ and} \\ \exists \text{ an open cover } \{U_i\}_{i \in I} \text{ of } U \text{ with} \\ \text{sections } t_i \in \mathcal{F}(U_i) \text{ such that } s|_{U_i} = t_i \end{array} \Big\}.$$

From this, it is an exercise to show that $\mathcal{F}^+$ is a sheaf satisfying the required property. □

The sheaf $\mathcal{F}^+$ whose existence is asserted in the previous proposition is often called the *sheafification* of $\mathcal{F}$.

Proposition 5.1.4 (Universal Property of the sheafification). *Let $\mathcal{F}$ be a presheaf, and $\mathcal{F}^+$ its sheafification. Then, for every sheaf $\mathcal{G}$ on X and morphism $\psi\colon \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\alpha : \mathcal{F}^+ \longrightarrow \mathcal{G}$ which makes the diagram*



commutative.

Remarks. (1) If $\mathcal{F}$ and $\mathcal{G}$ are bijective (i.e. bijective on the stalks) then they are isomorphic, and we will identify them.

(2) If $\varphi\colon \mathcal{F} \hookrightarrow \mathcal{G}$ is injective, then for all $U \subseteq X$, $\varphi(U)\colon \mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ is injective, and we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$.

Let $\varphi\colon \mathcal{F} \hookrightarrow \mathcal{G}$ be a subsheaf of $\mathcal{G}$. We define the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ as the **sheafification** of the presheaf that assigns:

$$U \to \mathcal{G}(U)/\mathcal{F}(U)$$

Note also that, if $x \in X$, then $(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x$.

5.2 Changing the Topological Space

Let X,Y be topological spaces, and let $f\colon X \to Y$ a continuous map.

Let $\mathcal{F}$ be a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Then one can push-forward the sheaf $\mathcal{F}$ (called then a direct image sheaf), or pull-back the sheaf $\mathcal{G}$ (called an inverse image). For now, we just see the direct image, as that is what we need for now.

Definition 5.2.1. The direct image sheaf of $\mathcal{F}$ by f is $f_*\mathcal{F}$, defined by:

$$(f_*\mathcal{F})(V) \stackrel{\text{def}}{=} \mathcal{F}\left(f^{-1}(V)\right) \qquad \text{if } V \subseteq Y \text{ is any open.}$$

5.3 $\boldsymbol{\mathcal{B}}$-sheaves

Let X be a topological space, and let $\boldsymbol{\mathcal{B}} = \{U_i\}$ be a *basis* for the opens in X.

Definition 5.3.1. A $\boldsymbol{\mathcal{B}}$-sheaf $\mathcal{F}$ on X is the giving of:

- $\forall U \in \boldsymbol{\mathcal{B}}$, $\mathcal{F}(U) \in$ **AbGrp**.
- $\forall V \subseteq U$ in $\boldsymbol{\mathcal{B}}$, $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ a group homomorphism, satisfying the identity and transitivity properties, as in the usual sheaves.
- The sheaf axiom for the opens in $\boldsymbol{\mathcal{B}}$:

  Let $U \in \boldsymbol{\mathcal{B}}$, $U = \bigcup_i U_i$, for $U_i \in \boldsymbol{\mathcal{B}}$, and let $t_i \in \mathcal{F}(U_i)$ such that for all $W \in \boldsymbol{\mathcal{B}}$ such that $W \subseteq U_i \cap U_j$, $t_i|_W = t_j|_W$.

  Then $\exists!\ t \in \mathcal{F}(U)$ such that $t|_{U_i} = t_i\ \forall i \in I$.

Proposition 5.3.2. *Let X be a topological space, $\boldsymbol{\mathcal{B}}$ a basis for the opens in X, and $\mathcal{F}$ a $\boldsymbol{\mathcal{B}}$-sheaf on X. Then $\mathcal{F}$ extends uniquely to a sheaf $\mathcal{F}$ on X.*

Moreover, if $\mathcal{F}, \mathcal{G}$ are $\boldsymbol{\mathcal{B}}$-sheaves and φ is a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of $\boldsymbol{\mathcal{B}}$-sheaves, then φ extends uniquely to the corresponding sheaves $\mathcal{F}$ and $\mathcal{G}$ on X.

Proof. It is left as an exercise. One defines a projective system for each open $U \subseteq X$, formed by those opens in $\boldsymbol{\mathcal{B}}$ contained in U, and then $\mathcal{F}(U)$ is defined as the projective limit of the corresponding abelian groups. □

Chapter 6

Affine Schemes

In this section we use the formality of $\boldsymbol{\mathcal{B}}$-sheaves to define the spectrum of a ring, the main object of study in this course. In this section, we fix a ring A, and $X \stackrel{\text{def}}{=} \operatorname{Spec}(A)$. Also, we let

$$\boldsymbol{\mathcal{B}} \stackrel{\text{def}}{=} \{X_f \mid f \in A\}.$$

We will define a $\boldsymbol{\mathcal{B}}$-sheaf of rings $\mathcal{O}_X$ that will extend to a sheaf of rings on X.

We start, for $f \in A$, to define $\mathcal{O}_X(X_f) \stackrel{\text{def}}{=} A_f$, that is, the localization of A at the set $S = \{1, f, f^2, \dots\}$ (recall that there is a natural homeomorphism $\operatorname{Spec}(A_f) \simeq X_f$).

Now, suppose that $X_f \subseteq X_g$. This is the same as to say that $X_f = X_f \cap X_g \iff X_f = X_{fg}$. So we replace X_f with X_{fg}. We then define:

$$\begin{aligned} \rho_{fg} \colon A_g = \mathcal{O}_X(X_g) &\longrightarrow \mathcal{O}_X(X_{fg}) = A_{fg} \\ \tfrac{a}{g^n} &\longmapsto \tfrac{af^n}{(fg)^n}, \end{aligned}$$

(or we can define it using the universal property of localization, which we leave as an exercise).

One can check that the restrictions behave as they should.

We proceed now to verify the sheaf property in detail. Let X_f be one basic open, and consider an arbitrary covering of it by basic opens. As X_f is quasi-compact, one can extract a finite sub-covering:

$$X_f = \bigcup_{i=1}^{m} X_{g_i}$$

So, $V(f) = \cap_{i=1}^m V(g_i) = V\left((g_1, \dots, g_n)\right)$.

This means that $\mathfrak{rad}(f) = \mathfrak{rad}\left((g_1 \ldots, g_n)\right)$, so that $f^s \in (g_1, \ldots, g_n)$. So $f^s = \sum_{i=1}^m c_i g_i$ for some $c_i \in A$.

More generally, as $X_{g_i^{n+1}} = X_{g_i}$, one can write:

$$f^s = \sum_{i=1}^m b_i g_i^{n+1}$$

Existence

So given a set of sections $t_i \in \mathcal{O}_X(X_{g_i})$, written as $t_i = \frac{a_i}{g_i}$ (note that, if the exponent in the denominator is greater than one, one can replace X_{g_i} with $X_{g_i^n}$ and reduce to this case).

The compatibility condition is

$$\frac{a_i g_j}{g_i g_j} = \frac{a_j g_i}{g_i g_j},$$

which is equivalent to the existence of $n \geq 1$ such that $(g_i g_j)^n (a_i g_j - a_j g_i) = 0$ (and n is chosen large enough so that it works for all pairs (i, j)). This, in turn, can be rewritten as $g_j^{n+1}(a_i g_i^n) = g_i^{n+1}(a_j g_j^n)$.

Define then $a \stackrel{\text{def}}{=} \sum_{i=1}^m a_i b_i g_i^n$. We have then:

$$a g_j^{n+1} = \sum_{i=1}^m b_i (a_i g_i^n g_j^{n+1}) = \sum_{i=1}^m b_i (a_j g_i^{n+1} g_j^n) = a_j g_j^n \sum b_i g_i^{n+1} = a_j g_j^n f^s$$

So $g_j^n(a g_j - f^s a_j) = 0$, so that in A_{g_j}, $t_j = \frac{a_j}{g_j} = \frac{a}{f^s}$, and then we can define $t \stackrel{\text{def}}{=} \frac{a}{f^s}$, which solves our existence problem.

Uniqueness

Suppose that $t \in \mathcal{O}_X(X_f) = A_f$, such that $t|_{X_{g_i}} = 0\, \forall i$. We want to prove that $t = 0$. Write $t = \frac{a}{f^n}$. So by hypothesis, one can find u large enough such that $g_i^u a = 0\, \forall i$.

We then have $f^s = \sum_{i=1}^m c_i g_i^u$, and so $a f^s = \sum_{i=1}^m c_i g_i^u a = 0 \implies a f^s = 0 \implies t = \frac{a f^s}{f^{s+u}} = 0$.

Definition 6.0.3. The pair $(X, \mathcal{O}_X)$ is called the *spectrum of A* (Spec(A)), and is called an *affine scheme*.

So far, given a ring A, we have defined a $\boldsymbol{\mathcal{B}}$-sheaf of rings $\mathcal{O}_X$ on $X = \text{Spec}(A)$. From now on, we will call the pair $(X, \mathcal{O}_X) = \text{Spec}(A)$. If one wants to distinguish the topological space from the pair, one uses $|X|$ for the space, and $\mathcal{O}_X$ for the sheaf.

Remark. We note that $\mathcal{O}_X(\varnothing) = \mathcal{O}_X(X_0) = A_0 = 0$, the zero ring. Also, $\mathrm{Spec}(0) = \varnothing$, so everything is still consistent.

Remark. Let $(X, \mathcal{O}_X) = \mathrm{Spec}(A)$, and let $x \in X$. Then:

$$\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U) = \varinjlim_{x \in U \in \boldsymbol{\mathcal{B}}} \mathcal{O}_X(U) = \varinjlim_{X_f \ni x} \mathcal{O}_X(X_f) =$$

$$= \varinjlim_{f \in A \setminus \mathfrak{p}_x} \mathcal{O}_X(X_f) = \varinjlim_{f \in A \setminus \mathfrak{p}_x} A_f \simeq A_{\mathfrak{p}_x}$$

The proof of the last equality is as follows: let $S \stackrel{\text{def}}{=} A \setminus \mathfrak{p}_x$. We are taking then

$$\varinjlim_{f \in S} A_f.$$

For $g \in S$, we want a map $A_g \to A_{\mathfrak{p}_x} = S^{-1}A$, and this we get from the universal property of localization. Also, we have a canonical map

$$A_g \to \varinjlim_{f \in S} A_f.$$

Given $g, h \in S$, the following diagram commutes:

$$\begin{array}{ccc} A_g & \xrightarrow{\frac{a}{g^n} \mapsto \frac{ah^n}{(gh)^n}} & A_{gh} \\ & \searrow \quad \swarrow & \\ & S^{-1}A. & \end{array}$$

So by the universal property of the direct limit,

$$\exists!\, F \colon \varinjlim_{f \in S} A_f \to S^{-1}A$$

such that $F([\frac{a}{g^n}]) = \frac{a}{g^n} \in S^{-1}A$.

Finally, we need to check that F is an isomorphism, and this is clearly checked (surjectivity is actually trivial).

6.1 Exercises

For all the problems in this homework, if X is a toplogical space and $\mathcal{F}$ is a sheaf of abelian groups on X, we set $\mathcal{F}(\phi) = 0$. Here ϕ is the void set seen as an open of X and 0 is the group with one element.

1. Let $X \stackrel{\text{def}}{=} \{0,1\}$ and define on X the topology on which the open sets are $\phi, \{0\}, \{1\}, \{0,1\}$.
   (a) Describe all sheaves of abelian groups on X.
   (b) Find a ring A such that X and $\mathrm{Spec}(A)$ are homeomorphic as topological spaces.
   (c) If $\mathcal{F}$ is a sheaf of abelian groups on X, describe its stalks.

2. Let $X \stackrel{\text{def}}{=} \{0,1\}$ and define on X the topology in which the open sets are $\phi, \{1\}, \{0,1\}$.
   (a) Describe all sheaves of abelian groups on X.
   (b) Find a ring A such that X and $\mathrm{Spec}(A)$ are homeomorphic as topological spaces.
   (c) If $\mathcal{F}$ is a sheaf of abelian groups on X, describe its stalks.

3. Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. Let

$$Y \stackrel{\text{def}}{=} \amalg_{x \in X} \mathcal{F}_x.$$

For all pairs (U, s), where $U \subset X$ is open and $s \in \mathcal{F}(U)$, let

$$s(U) \stackrel{\text{def}}{=} \{s_x | x \in U\} \subset Y.$$

Define a topology on Y in which a basis of open sets is given by the sets $s(U)$, for all pairs (U, s) as above. We also define $\pi : Y \to X$ by $\pi(s_x) = x$, for all $s_x \in \mathcal{F}_x \subset Y$.

Prove that for all open subset U of X, we have

$$\mathcal{F}(U) \simeq \{s : U \to \pi^{-1}(U) \text{ continuous } \mid \pi \circ s = \mathrm{Id}\}.$$

4. Let $X \stackrel{\text{def}}{=} \mathbb{C} - \{0\}$, where $\mathbb{C}$ is the field of complex numbers. Let $\mathcal{F}$ be the sheaf on X defined by: $\mathcal{F}(U) \stackrel{\text{def}}{=} \{f : U \to \mathbb{C} - \{0\} \text{ continuous }\}$, where $\mathcal{F}(U)$ is an abelian group with the multiplication of functions as operation. Let $\varphi : \mathcal{F} \to \mathcal{F}$ be the morphism defined by $\varphi_U : \mathcal{F}(U) \to \mathcal{F}(U)$ by $\varphi_U(f) = f^2$.
   (a) Show that $\mathcal{F}$ is a sheaf and φ is a morphism of sheaves of abelian groups.
   (b) Show that φ is surjective.

(c) Show that φ_U is not surjective for all U open of X.

5. Let $X = \mathbb{C} \cup \{\infty\}$ be the complex Riemann sphere. Define the sheaves $\mathcal{O}_X, \mathcal{F}_0, \mathcal{F}_\infty$ on X by: if $U \subset X$ is an open,

$$\mathcal{O}_X(U) \stackrel{\text{def}}{=} \{f : U \to \mathbb{C} \text{ analytic }\}$$

$$\mathcal{F}_0(U) \stackrel{\text{def}}{=} \begin{cases} \{f : U \to \mathbb{C} \text{ analytic }\} & \text{if } 0 \notin U \\ \{f : U \to \mathbb{C} \text{ analytic and such that } f(0) = 0\} & \text{if } 0 \in U \end{cases}$$

The sheaf $\mathcal{F}_\infty$ is defined similarly, replacing 0 by ∞.

We let $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}_0 \oplus \mathcal{F}_\infty$ and define $\varphi : \mathcal{F} \to \mathcal{O}_X$ by: if U is an open of X then $\varphi_U(f, g) \stackrel{\text{def}}{=} f + g$ for all $(f, g) \in \mathcal{F}_0(U) \oplus \mathcal{F}_\infty(U)$.

(a) Show that φ is surjective.

(b) Show that φ_U is not surjective for all U open of X.

6. Describe the affine schemes i.e. describe the topological spaces and the sheaves of rings on them (no need to draw pictures):

(a) $\mathrm{Spec}(\mathbb{C}[X]/(X^2))$

(b) $\mathrm{Spec}(\mathbb{C}[X]/(X^2 - X))$

(c) $\mathrm{Spec}(\mathbb{C}[X]/(X^3 - X^2))$

(d) $\mathrm{Spec}(\mathbb{C}[X]/(X^2 + 1))$

7. Let X be a topological space, $\mathcal{F}, \mathcal{G}$ sheaves of abelian groups on X and $\varphi \colon \mathcal{F} \to \mathcal{G}$ a morphism. Show that φ is injective if and only if φ_U is injective for all U open of X.

Chapter 7

Locally Ringed Spaces and Schemes

With the previous discussion, we have proved:

Lemma 7.0.1. *For* $(X, \mathcal{O}_X) = \mathrm{Spec}(A)$*, and for all* $x \in X$*,* $\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}_x}$*.*

In particular, the stalks $\mathcal{O}_{X,x}$ *are local rings.*

Definition 7.0.2. A pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf on X such that $\mathcal{O}_{X,x}$ is a local ring (for each $x \in X$) is called a *locally ringed space*.

Definition 7.0.3. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is "locally affine".

That is, for each $x \in X$, $\exists U \ni x$ such that $(U, \mathcal{O}_X|_U) \simeq \mathrm{Spec}(A)$, for some ring A.

The meaning of $\simeq$ in the previous definition will be made clear in the next section.

7.1 Morphisms of locally ringed spaces

Let $(X, \mathcal{O}_X))$, $(Y, \mathcal{O}_Y)$ be locally ringed spaces.

Definition 7.1.1. A *morphism* from $(X, \mathcal{O}_X))$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ such that:

- $f\colon X \to Y$ is a continuous map, and
- $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a *local* morphism of sheaves (on Y).

We should explain what *local* means in the previous definition. Fix $x \in X$, and let $y = f(x) \in Y$ be its image. Then $f^\sharp$ induces a map $f^\sharp_y \colon \mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$. Furthermore, note that the target of this map is the direct limit of the sections over the preimages of the opens in Y that contain y. Hence there is a natural map to the direct limit over *all* the opens in X containing x, that is, there is a canonical map $\psi \colon (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$.

With this settings, $f^\sharp$ is said to be *local* if $(\psi \circ f^\sharp_y)(\mathfrak{m}_{Y,y}) \subseteq \mathfrak{m}_{X,x}$.

Theorem 7.1.2. *Suppose that $(X, \mathcal{O}_X)$ is a scheme, and $(Y, \mathcal{O}_Y)$ is an* affine *scheme. Then there is a natural bijection:*

$$\{(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\} \overset{\textit{1:1}}{\leftrightarrow} \mathrm{Hom}_{\textit{rings}}(A, \mathcal{O}_X(X)).$$

Corollary 7.1.3. *Suppose that $(X, \mathcal{O}_X) = \mathrm{Spec}(B)$, and $(Y, \mathcal{O}_Y) = \mathrm{Spec}(A)$. Then there is a natural bijection:*

$$\{\textit{Morphisms of schemes } \mathrm{Spec}(B) \to \mathrm{Spec}(A)\} \overset{\textit{1:1}}{\leftrightarrow} \mathrm{Hom}_{\textit{rings}}(A, \mathcal{O}_X(X)).$$

This means that there is a contravariant equivalence of categories (aka anti-equivalence) between **Rings** and **AffSch**.

Proof (of the theorem):
Given a morphism of schemes $(f, f^\sharp)$, we map it to the morphism of rings given by $f^\sharp_Y \colon \mathcal{O}_Y(Y) = A \to (f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$.

Conversely, let $\psi \colon A \to \mathcal{O}_X(X)$ be a given ring homomorphism. We want a pair $(f, f^\sharp)$. Let $x \in X$. Consider the composition:

$$A \xrightarrow{\psi} \mathcal{O}_X(X) \xrightarrow{\rho_x} \mathcal{O}_{X,x}.$$

Then one can define $\mathfrak{p}_x \overset{\text{def}}{=} (\rho_x \circ \psi)^{-1}(\mathfrak{m}_{X,x})$. Then $\mathfrak{p}_x \in \mathrm{Spec}(A) = Y$, and one defines $f(x) \overset{\text{def}}{=} y$.

To define $f^\sharp$, note that as $Y = \mathrm{Spec}(A)$, one can define it locally, by saying what it is on $\boldsymbol{\mathcal{B}} = \{Y_g\}_{g \in A}$. So we define:

$$f^\sharp_{Y_g} \colon \mathcal{O}_Y(Y_g) = A_g \to (f_*\mathcal{O}_X)(Y_g) = \mathcal{O}_X\left(f^{-1}(Y_g)\right) = \mathcal{O}_X(X_{\psi(g)})$$

by sending $\frac{a}{g^n} \mapsto \psi(a)|_{X_{\psi(g)}} \left(\psi(g)|_{X_{\psi(g)}}\right)^{-n}$.

We leave as an exercise to check continuity of f, "locality" of $f^\sharp$, and that the two given maps are inverse to each other.

It is also a good exercise to redo the proof in the setting of the corollary. □

Chapter 8

Sheaves of Modules

We just introduce here the terminology, but we won't prove any interesting results for now.

Let A be a ring, M an A-module. Let then $(X, \mathcal{O}_X) = \operatorname{Spec}(A)$. From now on, we will abuse notation and call it just $X = \operatorname{Spec}(A)$.

We will define a sheaf of $\mathcal{O}_X$-modules on X, called $\mathcal{F}_M$. Again, we use the trick of the $\boldsymbol{\mathcal{B}}$-sheaves: let $\boldsymbol{\mathcal{B}} = \{X_f\}_{f \in A}$, and define $\mathcal{F}_M(X_f) \overset{\text{def}}{=} M_f \simeq M \otimes_A A_f$.

Then, if $f, g \in A$, $X_{fg} \subseteq X_g$, we have the restriction maps:

$$\rho_{fg} \colon \mathcal{F}_M(X_g) = M_g \to \mathcal{F}_M(X_{fg}) = M_{fg}, \qquad \frac{m}{g^n} \mapsto \frac{mf^n}{(fg)^n},$$

(equivalently, if we think of $M_g = M \otimes_A A_g$, the map is actually $\operatorname{Id} \otimes \rho_{fg}$).

Remark. M_f is an A_f-module, that is, an $\mathcal{O}_X(X_f)$-module.

With the previous definition, the sheaf axiom is clearly satisfied (from the fact that it holds for $\mathcal{O}_X$). So $\mathcal{F}_M$ extends to a sheaf on X, and it is a *sheaf of $\mathcal{O}_X$-modules*, that is:

- $\forall U \subseteq X$ open, $\mathcal{F}_M(U)$ is a $\mathcal{O}_X$-module.
- $\forall V \subseteq U \subseteq X$ opens, then ρ_{UV} is a $\mathcal{O}_X(U)$-linear map $\mathcal{F}_M(U) \to \mathcal{F}_M(V)$ (note that $\mathcal{F}_M(V)$ has a $\mathcal{O}_X(U)$-module structure via the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$).

Definition 8.0.4. A sheaf $\mathcal{F}_M$ coming from an A-module M is called a *quasi-coherent* sheaf of $\mathcal{O}_X$-modules.

If M is a finitely-generated A-module, then $\mathcal{F}_M$ is called *coherent*.

Let X be *any* scheme. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

Definition 8.0.5. The sheaf $\mathcal{F}$ is *quasi-coherent* if, for all $x \in X$, there is an open neighborhood $x \in U \subseteq X$ such that:

- $U = (U, \mathcal{O}_X|_U)$ is affine, say isomorphic to $\mathrm{Spec}(A)$.
- $\mathcal{F}|_U$ is isomorphic (as a sheaf on U) to $\mathcal{F}_M$, for some A-module M.

If M is a finitely-generated A-module for each U, then $\mathcal{F}$ is said to be *coherent*.

Chapter 9

Subschemes of Affine Schemes

We will study the two main types of subschemes that we will deal with, namely open and closed subschemes (of affine schemes). For us, then, when we say "subscheme" we mean one of those two types (and usually we will be specify which one we mean).

9.1 Closed Subschemes

Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $Y \subseteq X$ be a *closed subspace* (that is, topologically). So $Y = V(\mathfrak{a}) \subseteq \mathrm{Spec}(A)$.

Then $V(\mathfrak{a}) \simeq \mathrm{Spec}(A/\mathfrak{a})$. From the map $A \to A/\mathfrak{a}$ we get a morphism of schemes $\varphi\colon Y = \mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A) = X$ such it is the inclusion (set-theoretically, if one wants).

In this case, the map φ is called a *closed immersion*.

But note that $V(\mathfrak{a}) = V(\mathfrak{b}) \iff \mathfrak{rad}(\mathfrak{a}) = \mathfrak{rad}(\mathfrak{b})$, and so if $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{rad}(\mathfrak{a})$, then we get another closed immersion $Z = \mathrm{Spec}(A/\mathfrak{b}) \hookrightarrow \mathrm{Spec}(A)$. Topologically, $Z \simeq Y$. However, as schemes they are not necessarily isomorphic!

We can take the *reduced* subscheme $(Y, \mathcal{O}_{\mathrm{red}}) \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A/\,\mathfrak{rad}(\mathfrak{a}))$, and this gives a "canonical" choice for the scheme structure of a closed subset.

Remark. Every closed subscheme of an affine scheme is affine.

9.2 Open Subschemes

Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $U \subseteq X$ be an open subset. Then we get an scheme $(U, \mathcal{O}_X|_U)$ (see homework #5), and we have a natural morphism (of schemes!) $U \hookrightarrow X$, which is called an *open immersion*:

- $\varphi\colon U \hookrightarrow X$ is the inclusion of the open.
- $\varphi^\sharp\colon \mathcal{O}_X \to \varphi_*(\mathcal{O}_X|_U)$ is defined, on opens $V \subseteq X$ as follows:

$$\varphi^\sharp_V\colon \mathcal{O}_X(V) \to \varphi_*(\mathcal{O}_X)(V) = \mathcal{O}_X|_U(\varphi^{-1}(V)) = \mathcal{O}_X|_U(U \cap V) = \mathcal{O}_X(U \cap V)$$

is actually the restriction from V to $U \cap V$: $\varphi^\sharp = \rho_{V,U\cap V}$.

In this case, given an open $U \subseteq X$, there is a **unique** open immersion $U \hookrightarrow X$.

But given X an affine scheme, the new scheme $(U, \mathcal{O}_X|U)$ may not be affine! (see again HW #5).

Example 9.2.1. Let X be any scheme, and let k be a field. Recall that, given $x \in X$, we have defined the *residue field* of x, to be $\kappa(x) \overset{\text{def}}{=} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.

Lemma 9.2.2. *We have a 1:1 correspondence:*

$$\{\textit{Morphisms } (f, f^\sharp)\colon \operatorname{Spec}(k) \to (X, \mathcal{O}_X)\} \overset{\textit{1:1}}{\leftrightarrow} \{(x, \iota) \mid x \in X, \iota\colon \kappa(x) \hookrightarrow k\}.$$

Proof. Let $(f, f^\sharp)$ be a morphism of schemes $\operatorname{Spec}(k) \to (X, \mathcal{O}_X)$. The scheme $\operatorname{Spec}(k)$ has a single point, denoted $\star$. Its sheaf of functions is defined then by $\mathcal{O}(\{\star\}) = k$, and $\mathcal{O}(\varnothing) = 0$. Let then $x \overset{\text{def}}{=} f(\star) \in X$. Note that $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O} \implies f^\sharp_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_\star = k$, which is a local morphism of local rings, so it factors through $\mathfrak{m}_{X,x}$, giving the desired embedding $\iota\colon \kappa(x) \hookrightarrow k$.

Conversely, suppose given (x, ι). We want to get the pair $(f, f^\sharp)$. The map f is easy to define: $f(\star) \overset{\text{def}}{=} x$.

We now need a local morphism of sheaves $\mathcal{O}_X \to f_*\mathcal{O}$. Note that, given $U \subseteq X$, $(f_*\mathcal{O})(U) = \mathcal{O}(f^{-1}(U))$ is k if $x \in U$, and is 0 if $x \notin U$ (this is what is called a *skyscraper sheaf* on x). This implies the following on the stalks: $(f_*\mathcal{O})_a = k$ when $a = x$, and the stalk is 0 at any other point.

Given $U \subseteq X$, we define $f^\sharp_U\colon \mathcal{O}_X(U) \to f_*\mathcal{O}(U)$ as the zero map if $x \notin U$, and if $x \in U$, then define $f^\sharp_U \overset{\text{def}}{=} \iota \circ \pi \circ \rho_x$, where ι is the given map, π is the projection to the quotient, and ρ_x is the map to the stalk at x:

$$\begin{array}{ccc} \mathcal{O}_X(U) & \overset{f^\sharp_U}{\dashrightarrow} & k \\ \big\downarrow{\scriptstyle \rho_x} & & \big\uparrow{\scriptstyle \iota} \\ \mathcal{O}_{X,x} & \xrightarrow{\pi} & \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \end{array}$$

One can easily check that this gives a morphism of schemes, thus proving the lemma. ∎

This gives us some consequences: let $(X, \mathcal{O}_X)$ be a scheme.

- Given $x \in X$, one can think of the point as a closed *subset* of X, via $\operatorname{Spec}(\kappa(x))$.
- If k is a field, a morphism $\operatorname{Spec} k \to (X, \mathcal{O}_X)$ is called a k-valued point (so this involves not only giving $x \in X$, but also to give an embedding $\iota : \kappa(x) \hookrightarrow k$).
- Let $\xi = (0) \in \operatorname{Spec} \mathbb{Z}$. Fix an embedding $\mathbb{Q} = \mathcal{O}_{X,\xi} \hookrightarrow \overline{\mathbb{Q}}$. We get then a morphism $\psi\colon \operatorname{Spec}(\overline{\mathbb{Q}}) \to \operatorname{Spec}(\mathbb{Z})$. So such morphisms will correspond to embeddings $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$.

Chapter 10

Glueing Schemes

The following construction, whose proof is left as an easy but notation-intensive exercise, is very useful in constructing schemes, and it resembles other constructions. Although it is referred to in the literature as the *gluing lemma*, it is actually the amalgamated coproduct in the category of schemes.

Proposition 10.0.3 (The Gluing lemma). *Let $\{X_i\}_{i\in I}$ be an arbitrary collection of schemes, indexed by a set I. For each pair $(i,j) \in I \times I, i \neq j$, suppose given an open subscheme $X_{ij} \subseteq X_i$, (and so, to (j,i) there corresponds an open subscheme $X_{ji} \subseteq X_j$), together with isomorphisms $\varphi_{ij} \colon X_{ij} \xrightarrow{\simeq} X_{ji}$, such that:*

- *$\varphi_{ji} = \varphi_{ij}^{-1}$ for all $i \neq j$,*
- *For each triple $i, j, k \in I$, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ where this makes sense (that is, in $X_{ij} \cap X_{ik} \subseteq X_i$). This is called the* cocycle condition.

Then, there exists a scheme X, together with open immersions $X_i \subseteq X$ for each $i \in I$, such that $X = \amalg_{i\in I} X_i$, and such that $X_i \cap X_j \simeq X_{ij} \simeq X_{ji}$.

Example 10.0.4.
- If one sets $U_{ij} = \varnothing$ for all pairs $(i,j) \in I \times I$, this construction gives the *disjoint union* of schemes.
- The simplest case of the proposition is when $I = \{1, 2\}$. Then one takes two schemes X_1, X_2, chooses isomorphic open subschemes $U_1 \subseteq X_1$, and $U_2 \subseteq X_2$, and glues the original schemes X_1 and X_2 by identifying U_1 with U_2.

Chapter 11

Fibers of a Morphism of Schemes

If X, Y are *sets*, $F\colon X \to Y$ is a map, and $Z \subseteq Y$ is a subset, then $F^{-1}(Z) = \{x \in X \mid F(x) \in Z\}$. We want to mimic this for schemes.

So let X, Y be schemes, $F\colon X \to Y$ a morphism, and $Z \hookrightarrow Y$ a subscheme. We want to define $F^{-1}(Z) \hookrightarrow X$ as a subscheme, such that F induces a morphism $F^{-1}(Z) \to Z$ with some universal property.

We first deal with the easiest case:

11.1 Affine schemes, closed immersion

So let $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, and $Z \hookrightarrow Y$ closed (so $Z = \operatorname{Spec}(B/\mathfrak{b})$, and $Z \to Y$ is induced by the quotient map $B \twoheadrightarrow B/\mathfrak{b}$).

The morphism $X \to Y$ corresponds in turn to a ring homomorphism $f\colon B \to A$. Define then $f^{-1}(Z) \overset{\text{def}}{=} \operatorname{Spec}(A/\mathfrak{b}^e) = \operatorname{Spec}(A/f(\mathfrak{b})A) \hookrightarrow \operatorname{Spec}(A)$, by considering the two diagrams:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ F^{-1}(Z) & \overset{?}{\dashrightarrow} & Z \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} A & \longleftarrow & B \\ \downarrow & & \downarrow \\ A/\mathfrak{b}^e & \dashleftarrow & B/\mathfrak{b} \end{array}$$

Example 11.1.1. Let Y be an affine scheme, $y \in Y = \operatorname{Spec}(B)$ a closed point. So we have $\operatorname{Spec}(\kappa(y)) \hookrightarrow Y$, and $F^{-1}(y) = \operatorname{Spec}(A/\mathfrak{m}_y^e)$.

Example 11.1.2. Suppose that $k = \overline{k}$ is algebraically closed. Consider the schemes $X = \operatorname{Spec}(k[x, y]/(xy))$, and $Y = \operatorname{Spec}(k[t])$. Define a map $F\colon X \to Y$ as the corresponding to the ring homomorphism:

$$f\colon k[t] \to k[x,y]/(xy) \qquad t \mapsto x+y$$

This corresponds to a projection of the plane to a line that is not parallel to any of the two axes.

Let $a \in Y$ be a closed point, corresponding to $\mathfrak{m}_a \subseteq k[t]$, so $\mathfrak{m}_a = (t-a)$ for some $a \in k$ (high abuse of notation here!). So

$$a = \operatorname{Spec}\left(k[t]/\mathfrak{m}_a\right) = \operatorname{Spec}(k) \hookrightarrow Y.$$

We can then compute

$$F^{-1}(a) = \operatorname{Spec}\left(\frac{k[x,y]/(xy)}{(\overline{x}+\overline{y}-a)}\right) \hookrightarrow X.$$

Using isomorphism theorems, one can change the expression inside Spec() to see that it is isomorphic to $k[x]/(x(a-x))$.

If $a \neq 0$, then (x) and $(x-a)$ are coprime ideals, so by the *Chinese Remainder Thm*, $k[x]/(x(a-x)) \simeq k \times k$, and so $F^{-1}(a) \simeq \operatorname{Spec}(k\times k) = \operatorname{Spec}(k) \coprod \operatorname{Spec}(k) \hookrightarrow X$ (two distinct points).

If $a = 0$, then $F^{-1}(0) = \operatorname{Spec}(k[x]/(x^2)) \hookrightarrow X$, and this is thought of as a double point.

We want to study more general inverse images, so we need to introduce *fiber products*.

Again, we first do this for sets. Consider the following diagram:

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{\psi} & S \end{array}$$

Define in this case $X \times_S Y \stackrel{\text{def}}{=} \{(x,y) \in X \times Y \mid \varphi(x) = \psi(y)\}$.

Example 11.1.3.

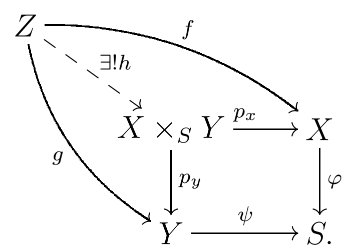
- If $S = \{x\}$, we get $X \times_S Y = X \times Y$, recovering the usual cartesian product.
- If φ, ψ are inclusions, then $X \times_S Y \simeq X \cap Y$.
- If $Y = \{y\}$ for some $y \in S$, then $X \times_S Y \simeq \varphi^{-1}(y)$. And, more generally, if $Y \subseteq S$, then $X \times_S Y \simeq \varphi^{-1}(Y)$.

11.2 Categorical Fiber Product

Given X, Y, S objects in some category, and morphisms $\varphi\colon X \to S$, and $\psi\colon Y \to S$, the *fiber product* of X and Y along S is defined, if it exists, to be an object $X \times_S Y$, together with two maps $p_x\colon X \times_S Y \to X$, $p_y\colon X \times_S Y \to Y$, such that the following commutes:

$$\begin{CD} X \times_S Y @>{p_x}>> X \\ @V{p_y}VV @VV{\varphi}V \\ Y @>{\psi}>> S, \end{CD}$$

and which is universal in the following sense: if Z is another object with morphisms $f\colon Z \to X$, $g\colon Z \to Y$ such that $\varphi \circ f = \psi \circ g$, then exists a unique $h\colon Z \to X \times_S Y$ making the whole diagram commutative:



This is what we do when working in the category of schemes (with morphisms of schemes). By general nonsense (aka category theory), if the fiber product exists, it is unique up to unique isomorphism.

11.3 The Affine Case

Let $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(R)$, and we have $\overline{\varphi}\colon R \to A$, $\overline{\psi}\colon R \to B$, corresponding to the morphisms of schemes. Then we can consider the tensor product $A \otimes_R B$:

$$\begin{CD} A \otimes_R B @<{i_A}<< A \\ @A{i_B}AA @AA{\overline{\varphi}}A \\ B @<<{\overline{\psi}}< R, \end{CD}$$

where $i_A(a) = a \otimes 1$, $i_B(b) = 1 \otimes b$. We can easily check that this commutes, and we have the following:

Lemma 11.3.1. *The fiber product of X and Y along S is $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$.*

Proof. Just use the universal product of the tensor product, which is just the same as the fiber product but with arrows reversed. □

11.4 For S affine, and X,Y any schemes

In this case, let $S = \operatorname{Spec}(R)$. We can write $X = \bigcup_i U_i$, with $U_i = \operatorname{Spec}(A_i)$ affine opens. For each of these i, and from the original setup, we get maps $\rho_i \circ \overline{\varphi} \colon R \to \mathcal{O}_X(U_i) = A_i$. So A_i acquires a natural structure of R-algebra.

Now, cover $Y = \bigcup_j V_j$, $V_j = \operatorname{Spec}(B_j)$ also open affines. In the same way, each of the B_j are R-algebras.

So, for each pair i, j, we get a fiber product $U_i \times_S V_j$. We now have to glue together each of those along their intersections (which should again be covered...), to get the global $X \times_S Y$.

For the most general case, that is, for X, Y, S general schemes, one can still construct the fiber product, so this exists always. For more details, check [5] or [2].

Example 11.4.1.

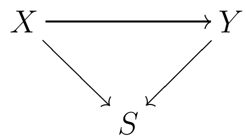
- Let X, Y, Z be schemes, $F \colon X \to Y$ a morphism, and $Z \hookrightarrow Y$ a subschemes. Then $F^{-1}(Z)$ is defined to be $X \times_Y Z$.
- If we have a k-valued point of Y, that is, $\xi = \operatorname{Spec}(k) \to Y$, then $F^{-1}(\xi) = X \times_Y \xi$ (although ξ need not be a subscheme of X!).
- Let Y be any scheme, and U_1, U_2 open (or closed) subschemes of Y. Then $U_1 \cap U_2 = U_1 \times_Y U_2$ (this is called the scheme-theoretic intersection).
- If $S = \operatorname{Spec}(R)$ is affine, then we sometimes write $X \times_R Y$ to mean $X \times_S Y$.

Example 11.4.2. If $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $F \colon Y \longrightarrow X$ a morphism and if we have a closed immersion $Z \hookrightarrow Y$, with $Z = \operatorname{Spec}(B/\mathfrak{b})$, then $F^{-1}(Z) = \operatorname{Spec}(A/\mathfrak{b}^e)$, as we have defined it previously. This should agree with our brand new definition, namely, with $\operatorname{Spec}(A \otimes_B B/\mathfrak{b})$, which is indeed true.

Note that, given a scheme X, there is always a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ (corresponding to the unique ring homomorphism $\mathbb{Z} \to \mathcal{O}_X(X)$). So we can always define an "unrestricted" fiber product, as $X \times Y \stackrel{\text{def}}{=} X \times_{\operatorname{Spec} \mathbb{Z}} Y$. This will not necessarily have nice properties, as the following example shows:

Example 11.4.3. Let $X = \operatorname{Spec}(\mathbb{Z}/5\mathbb{Z})$, and $Y = \operatorname{Spec}(\mathbb{Z}/3\mathbb{Z})$. Then $X \times Y = \operatorname{Spec}(\mathbb{Z}/5 \otimes_{\mathbb{Z}} \mathbb{Z}/3) = \varnothing$.

To avoid this situation, we will usually fix a base scheme S, and we'll work with the category of S-schemes (or schemes over S): the objects are diagrams $X \to S$, and a morphism $[X \to S] \to [Y \to S]$ is a commutative diagram:



The "unrestricted" product in this category is $X \times_S Y \to S$.

Example 11.4.4. If $S = \operatorname{Spec}(R)$, then $\{$affine S-schemes$\} \leftrightarrow \{R$-algebras$\}$.

Example 11.4.5. If $S = \operatorname{Spec}(k)$ (k a field), then $X \times_k Y = \varnothing \iff X = \varnothing$ or $Y = \varnothing$, so this looks much nicer!.

Chapter 12

Relation to Classical Algebraic Geometry

In this section we study the scheme theoretic version of affine space, and compare it with the analogous concept for classical algebraic geometry.

Let $k = \overline{k}$ be an algebraically-closed field.

Definition 12.0.6. The *n-dimensional affine scheme over* k is defined to be $\mathbb{A}_k^n \stackrel{\text{def}}{=} \operatorname{Spec}(k[x_1, \ldots, x_n])$.

12.1 Case $n = 1$

Let $\mathfrak{p} \in \mathbb{A}_k^1 = \operatorname{Spec}(k[x])$. As $\mathfrak{p} \subseteq k[x]$ is a prime ideal, either it is the zero ideal (non-closed, and its closure is the whole space), or it is of the form $\mathfrak{p} = (x - a)$ for some $a \in k$. In the latter case, $\mathfrak{p}$ is maximal, hence closed.

So $\mathbb{A}_k^1 = M(k[x]) \cup \{\xi\}$, where $M(k[x])$ is the maximal spectrum (the subspace consisting of the closed points), and ξ is the point corresponding to the zero ideal. Note that $M(k[x]) \simeq k$, and that $\mathbb{A}_{\text{var}}^1 = k$, by definition (in classical algebraic geometry, one defines the affine n-space as k^n).

Theorem 12.1.1 (Hilbert Nullstellensatz). *Let k be* any *field. If $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ is a maximal ideal, then $k \hookrightarrow k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite extension of k.*

Corollary 12.1.2. *For $k = \overline{k}$, the field k doesn't have nontrivial finite ($\implies$ algebraic) extensions, and hence $k[x_1, \ldots, x_n]/\mathfrak{m} \simeq k$.*

So we have an exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathfrak{m} \longrightarrow & k[x_1, \dots, x_n] & \longrightarrow & k \longrightarrow 0, \\ & x_i & \longmapsto & a_i. \end{array}$$

In this case, then, $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$.

12.2 Case $n = 2$

In this case, again we decompose $\mathbb{A}^2_k$ as the disjoint union $\mathrm{Spec}(k[x, y]) = M(k[x, y]) \cup \{\text{non-closed points}\}$.

For the closed points, note that if $\mathfrak{m} \subseteq k[x, y]$ is maximal, that happens if, and only if, $\mathfrak{m} = (x - a, y - b)$ for some $(a, b) \in k^2$ (the "usual" plane).

For the non-closed points: firstly, we have $\xi = (0)$, the *generic point* (its closure is the whole space). We want to classify the remaining non-closed points.

For this, let $f = f(x, y) \in k[x, y]$ be an irreducible polynomial. Then $\mathfrak{p} \overset{\text{def}}{=} (f)$ is a non-maximal prime, hence giving a non-closed point in $\mathbb{A}^2_k$. We have:

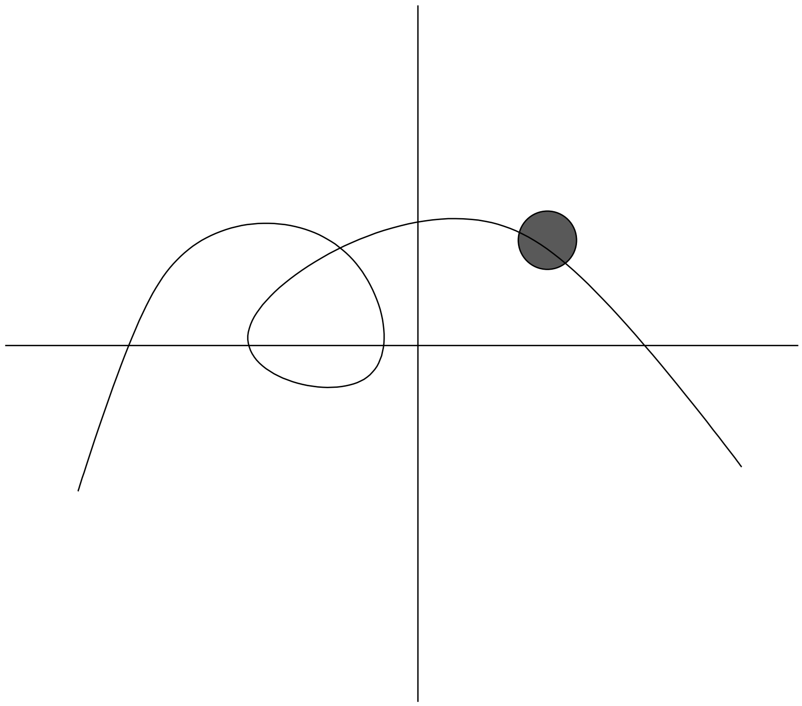
$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{p}\} \cup \{\mathfrak{m} \mid \mathfrak{m} \supseteq \mathfrak{p}\}.$$

Also, note that $(f) \subseteq (x - a, y - b) \iff f(a, b) = 0$, so that:

$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{p}\} \cup \{(a, b) \in k^2 \mid f(a, b) = 0\}.$$

So we get an irreducible curve in k^2 (given by the vanishing of $f(x, y)$), together with the prime $\mathfrak{p}$ itself, which should be thought of as the generic point of that curve.

One should check that these are all the primes in $k[x, y]$, and so we conclude that $\mathbb{A}^2_k = k^2 \cup \{\xi\} \cup \{\text{generic points of irreducible curves in } k^2\}$.



12.3 General case ($n \geq 3$)

We can now deduce the general case for affine n-space, that would be as follows:

$\mathbb{A}_k^n = k^n \cup \{\xi\} \cup \{\text{generic pts of all irred. subvars. } \Sigma \subseteq k^n \text{ of dimension } 0 < d < n\}$.

12.4 Irreducible subschemes of Affine n-space

Let now $A = k[x_1, \ldots, x_n]/I$, where I is an ideal, which we take reduced: $\mathfrak{rad}(I) = I$.

Let $X \stackrel{\text{def}}{=} \{x \in k^n \mid \forall f \in I, f(x) = 0\} \subseteq k^n$. Then X is an algebraic set, and the ring homomorphism $k[x_1, \ldots, x_n] \to A$ induces a morphism $\operatorname{Spec}(A) \stackrel{\text{closed immersion}}{\hookrightarrow} \mathbb{A}_k^n$.

In this case, $\operatorname{Spec}(A) = M(A) \cup \{\xi\} \cup \{\text{non-closed points}\}$, where $M(A) = X$, and the non-closed points, which correspond to primes containing I, are the generic points of irreducible subvarieties of X.

In fact, more is true: we have an equivalence of categories:

$$\left\{\begin{matrix}\text{(affine) schemes of} \\ \text{finite type} \\ \text{over } k = \overline{k}\end{matrix}\right\} \stackrel{\simeq}{\leftrightarrow} \left\{\begin{matrix}\text{(affine) algebraic} \\ \text{sets over } k\end{matrix}\right\},$$

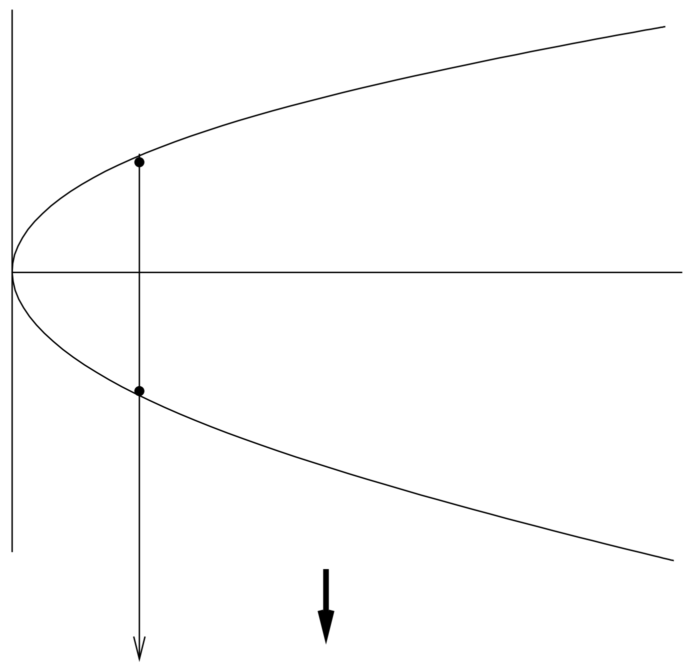
which assigns $\mathrm{Spec}(A) \mapsto M(A)$, and to a variety X the scheme $\mathrm{Spec}(A(X))$, where $A(X)$ is the ring of regular functions of X. Also, irreducible schemes corresponds to algebraic varieties (both corresponding to the case of I being prime).

Example 12.4.1. Let $k = \overline{k}$, and $\varphi\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ given by the corresponding hom. of rings $\overline{\varphi}$, $x \mapsto x^2$.

As varieties, we have $\varphi\colon k \to k$, such that

$$\begin{aligned}\varphi(a) = \varphi((x-a)) = \overline{\varphi}^{-1}((x-a)) = \{f(x) \mid \overline{\varphi}(f) \in (x-a)\} = \\ = \{f(x) \mid f(x^2) \in (x-a)\} = (x-a^2)\end{aligned}$$

Therefore, $\varphi(a) = a^2$. If $\mathrm{char}(k) \neq 2$, then $\varphi^{-1}(1) = \{+1, -1\}$, but $\varphi^{-1}(0) = \{0\}$, and we would like that the fibers to be somewhat consistent.



But scheme-theoretically,

$$\varphi^{-1}(0) = \varphi^{-1}((x)) = \mathrm{Spec}\,(k[x]/(\overline{\varphi}(x))) = \mathrm{Spec}\,\big(k[x]/(x^2)\big)$$

which represents a "double point".

12.5 Exercises

1. Let $(X, \mathcal{O}_X)$ be a scheme and $U \subset X$ an open subset. Prove that the pair $(U, \mathcal{O}_X|_U)$ is a scheme.

2. Let $(X, \mathcal{O}_X)$ be the affine scheme $\mathrm{Spec}(K[x, y])$, where K is a field. Let $U = X - \{\underline{m}_0\}$ where $\underline{m}_0$ is the ideal of $K[x, y]$ generated by x and y. Prove that the scheme $(U, \mathcal{O}_X|_U)$ is not an affine scheme.

3. Let $(X, \mathcal{O}_X)$ be a scheme and $f \in \mathcal{O}_X(X)$ a global section of its structure sheaf. Define $X_f \overset{\text{def}}{=} \{x \in X \quad | \quad f_x \text{ is not in } \mathfrak{m}_{X,x}\}$, where f_x denotes the stalk of f at x, i.e. the image of f in $\mathcal{O}_{X,x}$.

   (a) If $U \subset X$ is an open subset such that the open subscheme $(U, \mathcal{O}_X|_U)$ of $(X, \mathcal{O}_X)$ is affine, isomorphic to $\mathrm{Spec}(B)$, and $\overline{f} \in B = \mathcal{O}_X(U)$ is the restriction of f to U, show that $X_f \cap U = U_{\overline{f}} = \mathrm{Spec}(B_{\overline{f}})$. Deduce that X_f is open in X.

   (b) Assume that X is a quasi-compact topological space. Let $A = \mathcal{O}_X(X)$ and $a \in A$ an element such that $a|_{X_f} = 0$. Show that for some $n > 0$, $f^n a = 0$ in A.

   (c) Assume now that X has a finite cover by opens $(U_i)_{i=1,\dots,n}$, where each $U_i \cap U_j$ is quasi-compact and $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. Let $b \in \mathcal{O}_X(X_f)$. Show that there is an element $a \in A = \mathcal{O}_X(X)$ and $n > 0$ such that $f^n b = a|_{X_f}$.

   (d) With the hypothesis at iii) above conclude that $\mathcal{O}_X(X_f) \cong A_f$.

4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes and $F = (\varphi, \varphi^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism of schemes having the property that there is a covering $\{U_i\}_i$ of Y such that $(U_i, \mathcal{O}_Y|_{U_i})$ is an affine scheme for every i. Suppose that for every i, the morphism induced by $F\colon (\varphi^{-1}(U_i), \mathcal{O}_X|_{\varphi^{-1}(U_i)}) \to (U_i, \mathcal{O}_Y|_{U_i})$ is an isomorphism. Then F is an isomorphism (we recall that this means φ is a homeomerphism and $\varphi^\#$ is an isomorphism of sheaves.)

5. (a) Let $(X, \mathcal{O}_X)$ be a scheme. Prove that this scheme is affine if and only if there is a finite set of global sections $f_1, f_2, \dots, f_n \in A = \mathcal{O}_X(X)$ such that the open sets X_f (defined in problem 3)) are affine and the ideal of A generated by $f_1, f_2, \dots, f_n$ is A. Hint: use exercises 3) and 4) above.

   (b) Use 5) (a) to give a (another) proof of problem 2) above.)

6. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes and $F = (\varphi, \varphi^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism of schemes. We say that F is a finite morphism if there is a covering $\{U_i\}_i$ of Y by open subsets such that the schemes $(U_i, \mathcal{O}_Y|_{U_i}) = \operatorname{Spec}(B_i)$ are affine for all i, such that the schemes $(\varphi^{-1}(U_i), \mathcal{O}_X|_{\varphi^{-1}(U_i)})$ are affine isomorphic to $\operatorname{Spec}(A_i)$ and such that A_i is a finite B_i-algebra for every i, i.e. A_i is a finitely generated B_i-module.

   (a) Show that if the morphism of schemes F above is finite then for every open $U \subset Y$ such that $(U, \mathcal{O}_Y|_U)$ is affine isomorphic to $\operatorname{Spec}(B)$ the scheme $(V = \varphi^{-1}(U), \mathcal{O}_X|_V)$ is also affine, isomorphic to $\operatorname{Spec}(A)$ and B is a finite A-algebra.

   (b) Show that a finite morphism is closed, i.e. the image of every closed subset of X under φ is a closed subset of Y.

Chapter 13

Local Schemes

We begin with a definition:

Definition 13.0.1. A *local scheme* is Spec(A), where A is a local ring.

We study some examples.

Example 13.0.2.

- The easiest one is with $A = k[x]_{(x)}$. Let $X = \text{Spec}(A)$. Then $|X| = \{(x), (0)\}$. The point (x) is closed (and hence (0) is open). We have a canonical map $k[x] \to k[x]_{(x)}$, giving a morphism $X \to \mathbb{A}^1_k = \text{Spec}(k[x])$. Note also that $X = \mathcal{O}_{\mathbb{A}^1_k,0}$, the stalk at the origin.

  Remark. In general, we have: for any scheme Y, and $y \in Y$, a map $\text{Spec}(\mathcal{O}_{Y,y}) \to Y$. Just need to take an affine open neighborhood of y, and then compose the map $\text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_Y(U)) = U$ with the open immersion $U \hookrightarrow Y$.

- The second example is the obvious generalization to dimension 2: consider the ring $A = k[x,y]_{(x,y)}$, and let X be its spectrum, $X = \text{Spec}(A)$.

The ideal $(x, y) \subseteq k[x, y]$ corresponds to the (closed) point $(0, 0) \in k^2$. Again, we get a map $X \hookrightarrow \mathbb{A}^2_k$. We get:

$$\text{Spec}(A) = \{(0,0)\} \cup \{\text{non-closed points}\} \cup \{\xi\}$$

where the non-closed points are $\{\mathfrak{p} = (f) \mid f \in A \text{ irred}, f \in (x,y)\}$. That is, they are only the irreducible curves through the point $(0, 0)$.

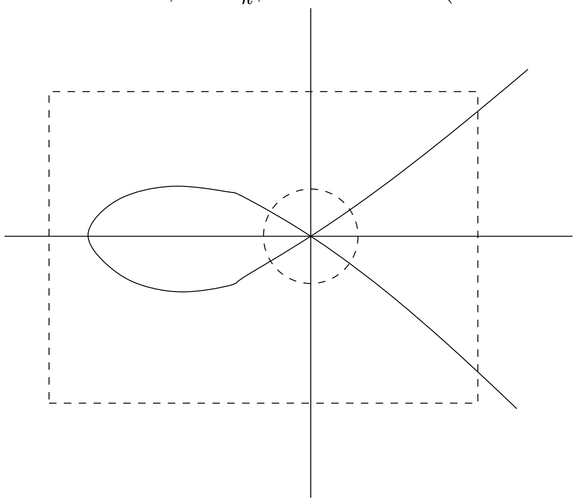
Sometimes, we will want smaller neighborhoods. Note the following chain of inclusions:

$$k[x,y] \hookrightarrow k[x,y]_{(x,y)} \hookrightarrow k[[x,y]]$$

(the last ring is the $\mathfrak{m} = (x, y)$-adic completion of $k[x, y]_{(x,y)}$).

To this there corresponds a (much smaller) neighborhood $Y \subseteq \mathbb{A}^2_k$:

For instance, in $\mathbb{A}^2_k$, consider the (non-closed) point $\alpha = \mathfrak{p} = (Y^2 - X^3 - X^2) \in \mathbb{A}^2_k$.



In X, the point α corresponds to $\mathfrak{p}' = (Y^2 - X^3 - X^2) \in X$, which is still irreducible.

Claim. *The curve $y^2 - y^3 - y^2$ is* ***not*** *irreducible in $k[[x, y]]$.*

Proof. Start by writing $y^2 - x^2(x + 1)$. The polynomial $x + 1$ has a square root (provided that $\mathrm{char}(k) \neq 2$):

$$(1 + x)^{1/2} = \sum_{k \geq 0} \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \frac{1/2(1/2 - 1)}{2}x^2 + \ldots \stackrel{\text{def}}{=} u(x)$$

So in $k[[x, y]]$, $y^2 - x^2(x + 1) = (y - xu(x))(y + xu(x))$, and hence our curve is not irreducible. ☐

Chapter 14

Reduced Schemes over Non-algebraically Closed Fields

Let k be a field, and let A be an algebra of finite type over k, which we assume reduced (that is $\mathfrak{N}(A) = 0$). We can always write $A \simeq k[x_1, \ldots, x_n]/I$, for some radical ideal I (i.e., $\mathfrak{rad}(I) = I$). Let $X = \mathrm{Spec}(A)$. We have a map $X \hookrightarrow \mathbb{A}^n_k$ (coming from the projection to the quotient on the polynomial rings).

Fix $\overline{k}$ an algebraic closure of k. The embedding $k \hookrightarrow \overline{k}$ gives a map $\varphi\colon \mathbb{A}^n_{\overline{k}} \to \mathbb{A}^n_k$.

Example 14.0.3. $k = \mathbb{R}$ and $\overline{k} = \mathbb{C}$, $[\mathbb{C} : \mathbb{R}] = 2$, and $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\bar{\cdot}, 1\}$, where $\bar{\cdot}$ represents complex conjugation.

For $n = 1$, $\mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[x]) \ni \mathfrak{m}$ a closed point, $\mathfrak{m} \subseteq \mathbb{R}[x]$ maximal. By Hilbert's Nullstellensatz, $\mathbb{R} \hookrightarrow \mathbb{R}[x]/\mathfrak{m}$, so

$$\mathbb{R}[x]/\mathfrak{m} \simeq \begin{cases} \mathbb{R}, \\ \mathbb{C}. \end{cases}$$

1. If $\mathbb{R}[x]/\mathfrak{m} \simeq \mathbb{R}$, then $\mathbb{R}[x] \twoheadrightarrow \mathbb{R}$ with kernel $\mathfrak{m}$ implies that $\mathfrak{m} = (x - a)$, where $a \in \mathbb{R}$ is the image of x under the mentioned surjection.

2. If $\mathbb{R}[x]/\mathfrak{m} \simeq \mathbb{C}$, then the surjection $\mathbb{R}[x] \twoheadrightarrow \mathbb{C}$ sends $x \mapsto \lambda \notin \mathbb{R}$. Let, in this case, $\alpha \overset{\text{def}}{=} \lambda + \overline{\lambda}$, and $\beta \overset{\text{def}}{=} \lambda\overline{\lambda}$. Then $\alpha, \beta \in \mathbb{R}$, and λ satisfies $\lambda^2 - \alpha\lambda + \beta = 0$ (and, moreover, the polynomial $x^2 - \alpha x + \beta$ is irreducible in $\mathbb{R}[x]$).

   This implies that $x^2 - \alpha x + \beta \in \ker(\psi) = \mathfrak{m} \implies \mathfrak{m} = (x^2 - \alpha x + \beta)$, because $(x^2 - \alpha x + \beta)$ is already maximal.

   We conclude that $\mathfrak{m}$ corresponds to a pair $\{\lambda, \overline{\lambda}\}$, with $\lambda \notin \mathbb{R}$ (it's an orbit of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on $\mathbb{C}$.

Putting the two cases together, we conclude that the closed points of $\mathbb{A}^1_{\mathbb{R}}$ corresponds to pairs $(\lambda, \overline{\lambda})$ with $\lambda \in \mathbb{C}$ (and now we don't impose the condition that $\lambda \notin \mathbb{R}$).

We next delve into the case $n = 2$. $\mathbb{A}^2_{\mathbb{R}} = \text{Spec}(\mathbb{R}[x, y])$. Let $\mathfrak{m}$ be a closed point. We have again the two cases, as before:

$$\mathbb{R}[x, y]/\mathfrak{m} \simeq \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$$

1. $\mathbb{R}[x, y]/\mathfrak{m} \simeq \mathbb{R} \implies \mathfrak{m} = (x - a, y - b)$, if $x \mapsto a$ and $y \mapsto b$. That is, $\mathfrak{m}$ corresponds to $(a, b) \in \mathbb{R}^2$.

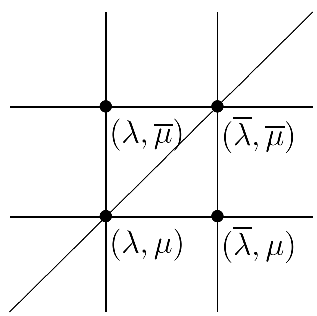
2. $\mathbb{R}[x, y]/\mathfrak{m} \simeq \mathbb{C}$, with $x \mapsto \lambda$,$y \mapsto \mu$.

   (a) $\lambda \in \mathbb{R}, \mu \notin \mathbb{R}$. In this case we conclude, if we define $\gamma = \mu + \overline{\mu}$ and $\delta \overset{\text{def}}{=} \mu\overline{\mu}$, that $\mathfrak{m} = (x - \lambda, y^2 - \gamma y + \delta)$.

   (b) $\lambda \notin \mathbb{R}, \mu \in \mathbb{R}$. This is the symmetrical case, and hence we get $\mathfrak{m} = (x^2 - \alpha x + \beta, y - \mu)$, with $\alpha = \lambda + \overline{\lambda}$, and $\beta = \lambda\overline{\lambda}$.

   (c) $\lambda \notin \mathbb{R}, \mu \notin \mathbb{R}$. In this case, let λ and μ satisfy the two quadratic equations $\lambda^2 - \alpha\lambda + \beta = 0$, and $\mu^2 - \gamma\mu + \delta = 0$. Then $I = (x^2 - \alpha x + \beta, y^2 - \gamma y + \delta) \subseteq \mathfrak{m}$. However, $\mathbb{R}[x, y]/I \simeq \mathbb{C} \times \mathbb{C}$, which is not a field (check it!).

   We need then to find more elements in $\mathfrak{m}$. Note that, over $\mathbb{C}$, $x^2 - \alpha x + \beta = (x - \lambda)(x - \overline{\lambda})$, and $y^2 - \gamma y + \delta = (y - \mu)(y - \overline{\mu})$.



   Notice the line through the points (λ, μ) and $(\overline{\lambda}, \overline{\mu})$. Finding its equation, we obtain the polynomial

$$h(x, y) = \Im(\mu)x - \Im(\lambda)\mu + \Im(\lambda\overline{\mu}) - bx - dy + s \in \mathbb{R}[x, y].$$

   Note that $h(x, y) \in \mathfrak{m}$. Also, $(h(x, y), x^2 - \alpha x + \beta) = (h(x, y), y^2 - \gamma y + \delta)$, and it is maximal.

In conclusion, the closed points in $\mathbb{A}^2_{\mathbb{R}}$ correspond to pairs $(\lambda, \mu), (\overline{\lambda}, \overline{\mu}) \in \mathbb{C}^2$; there is the generic point $\xi = (0)$, and for each $f(x, y) \in \mathbb{R}[x, y]$ irreducible, either we get a non-closed point (if f remains irreducible in $\mathbb{C}[x, y]$), or we obtain a pair $(\mathfrak{q}, \overline{\mathfrak{q}}) \in \mathbb{A}^2_{\mathbb{C}}$, in case that f splits as the product of two polynomials in $\mathbb{C}[x, y]$.

In general, if k is any field, and we fix $\overline{k}$ an algebraic closure, and denote $G = \mathrm{Gal}(\overline{k}/k)$, then the map:

$$\mathbb{A}^n_{\overline{k}} \to \mathbb{A}^n_k$$

gives a one-to-one correspondence of the points in $\mathbb{A}^n_k$ with the G-orbits of points in $\mathbb{A}^n_{\overline{k}}$. Under this correspondence, the closed points of $\mathbb{A}^n_k$ correspond to G-orbits of $\left(\overline{k}\right)^n$, which are always finite.

Chapter 15

Non-reduced Schemes

To simplify, in this section we take $k = \overline{k}$ to be an algebraically closed field.

Let A be a k-algebra of finite type, so $A \simeq k[x_1, \ldots, x_n]/I$, and $\mathfrak{N}(A) \neq 0$.

Let $X = \operatorname{Spec}(A)$, and we have then $X \hookrightarrow \mathbb{A}^n_k$. We want to study schemes supported at the origin of $\mathbb{A}^n_k$ (that is, which have only one closed point, corresponding to the maximal ideal $(x_1, \ldots, x_n) \subseteq k[x_1, \cdots, x_n]$).

Again, we start with the case of $n = 1$, $\mathbb{A}^1_k = \operatorname{Spec}(k[x])$. Then we must have a ring homomorphism

$$\begin{array}{ccc} k[x] & \xrightarrow{\varphi} & A \\ \uparrow & & \uparrow \\ (x) & \longrightarrow & \mathfrak{m} \end{array}$$

So as $\ker(\varphi) \subsetneq (x) \implies \ker(\varphi) = (x^m)$. For instance, suppose that $m = 2$. Then $A = k[x]/(x^2)$, $|X| = \{(x)\}$, and $(x) \mapsto 0 \in k$.

So let $f(x) \in k[x]$, $f = a_0 + a_1 x + \cdots + a_t x^t$. If we consider then $f|_X = f \pmod{x^2} = a_0 + a_1 x \pmod{x^2}$. Note that $a_0 = f(0)$, and $a_1 = f'(0)$. So X is an "infinitesimal neighborhood" of 0 in $\mathbb{A}^1_k$.

Next, let's consider the ring $k[\epsilon]/(\epsilon^2)$. It is called the *ring of dual numbers*. We want $X \simeq \operatorname{Spec}(k[\epsilon]/(\epsilon^2))$, with an embedding $X \hookrightarrow \mathbb{A}^2_k = \operatorname{Spec}(k[x,y])$. And still, we require that X has only one closed point. This means that we need a map:

$$\begin{array}{ccc} k[x,y] & \xrightarrow{\varphi} & k[\epsilon]/(\epsilon^2) \\ \uparrow & & \uparrow \\ (x,y) & \longrightarrow & (\epsilon) \end{array}$$

and we want $\mathfrak{m} = \ker(\varphi) \subsetneq (x, y)$. This will imply that $(x,y)^2 \subseteq \mathfrak{m}$. Note that $(x,y)^2 = (x^2, xy, y^2)$, and $k[x,y]/(x^2, xy, y^2)$ has dimension (as k-vector-space) 3, while our ring of dual numbers has dimension 2. This means that there are extra elements in $\mathfrak{m}$: $\mathfrak{m} = (x^2, xy, y^2, \alpha x + \beta y)$, for some $(\alpha, \beta) \in k^2$, and $(\alpha, \beta) \neq (0,0)$.

So we get a map:

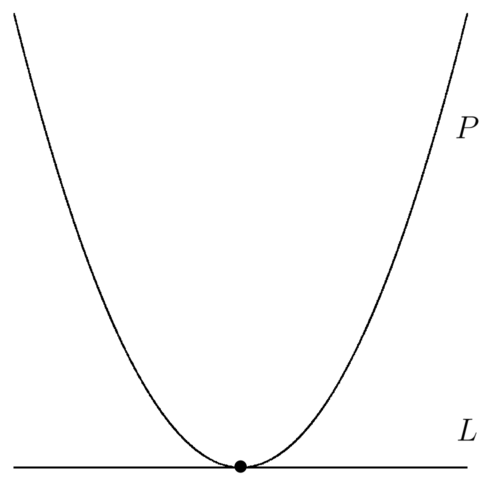
$$X_{(\alpha,\beta)} = \operatorname{Spec}\left(\frac{k[x,y]}{(x^2, xy, y^2, \alpha x + \beta y)}\right) \hookrightarrow \mathbb{A}^2_k$$

Let $f(x,y) \in k[x,y]$. Then $f|_{X_{(\alpha,\beta)}} = 0 \iff f \in (x^2, xy, y^2, \alpha x + \beta y)$. Write then $f(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + \ldots \pmod{\mathfrak{m}} \equiv a_0 + a_1 x + a_2 y \pmod{\alpha x + \beta y}$. Then $f(x,y) \equiv 0 \iff a_0 = 0$ and $a_1 x + a_2 y \in (\alpha x + \beta y)$.

Note that $a_0 = f(0,0)$, $a_1 = \frac{\partial f}{\partial x}(0,0)$, and $a_2 = \frac{\partial f}{\partial y}(0,0)$. So $f|_{X_{(a,b)}} = 0$ if, and only if, $f(0,0) = 0$, and the gradient of f at $(0,0)$ is parallel to the vector (α, β).

One might wonder when do these schemes appear. The following example explains it.

Example 15.0.4. Let $L = \operatorname{Spec}(k[x,y]/(y))$, and let $P = \operatorname{Spec}(k[x,y]/(x^2 - y))$, both seen as subschemes of $\mathbb{A}^2_k$.



We can then take $P \cap L \stackrel{\text{def}}{=} P \times_{\mathbb{A}^2_k} L$. It is again affine, and we need to compute the tensor product $\frac{k[x,y]}{(y)} \otimes_{k[x,y]} \frac{k[x,y]}{(x^2-y)}$.

Exercise 2. Let A be a ring, $\mathfrak{a}, \mathfrak{b} \subseteq A$ two ideals. Then:

$$A/\mathfrak{a} \otimes_A A/\mathfrak{b} \simeq \frac{A/\mathfrak{a}}{\mathfrak{b}(A/\mathfrak{a})} \simeq \frac{A/\mathfrak{a}}{(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}} \simeq A/(\mathfrak{a}+\mathfrak{b}).$$

So $P \cap L = \operatorname{Spec}\left(\frac{k[x,y]}{(y, x^2-y)}\right) = \operatorname{Spec}\left(\frac{k[x,y]}{(y, x^2)}\right) = \operatorname{Spec}\left(\frac{k[x,y]}{(y, x^2, xy, y^2)}\right) = X_{(0,1)}$.

The schemes $X_{(\alpha,\beta)}$ also appear as "limits" of schemes. This is what we will see in the next section.

Chapter 16

Families of Schemes

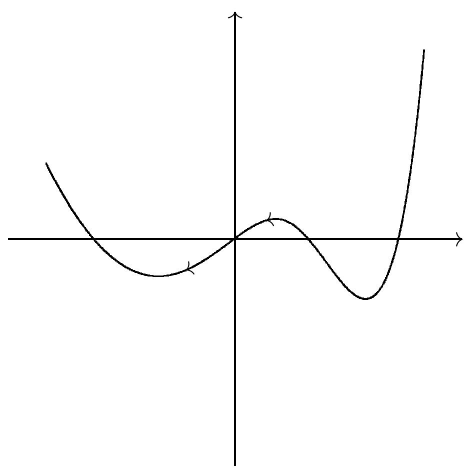
We start with an intuitive example, and we'll proceed with a more rigorous approach.

We start with the scheme $X = \{(0,0),(a,b)\} \hookrightarrow \mathbb{A}^2_k$, with $(a,b) \neq (0,0)$.

This comes from the closed subscheme $X = V((x,y)) \cup V((x-a,y-b))$, which can be computed to be $V((x(x-a), x(y-b), y(x-a), y(y-b))$. So we set

$$X = \operatorname{Spec}\left(\frac{k[x,y]}{(x^2-ax, xy-bx, xy-ay, y^2-by)}\right).$$

Now, we want to make (a,b) approach $(0,0)$ through some curve C, which will be given parametrically by $(a(t), b(t))$, such that when $t = 0$, $a(0) = b(0) = 0$.



This corresponds to a morphism $\overline{\varphi}\colon k[x,y] \to k[t]$, sending $x \mapsto a(t)$ and $y \mapsto b(t)$, which induces the corresponding morphism of schemes $\varphi\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$.

Consider now the "family" of schemes

$$X_t = \{(0,0),(a(t),b(t))\}, \quad t \in k \setminus \{0\}$$

(actually, we want that $(a(t),b(t)) \neq (0,0)$ for all the t considered).

We want to compute the "$\lim_{t\to 0} X_t$".

For $t \neq 0$, we have $I_t \subseteq k[x,y]$. Think of those ideals as k-subspaces of $k[x,y]$. Then $\dim_k\left(\frac{k[x,y]}{I_t}\right) = \dim_k(k^2) = 2$. So the I_t have codimension 2.

Again, write $I_t = (x^2 - a(t)x, xy - b(t)x, xy - a(t)y, y^2 - b(t)y)$.

So we should get $I_0 = (x^2, xy, y^2)$. But this is of codimension 3!. This means that there is some missing element.

Note that as both $xy - b(t)x$ and $xy - a(t)y$ are in I_t, their difference is there, too. Hence $a(t)y - b(t)x \in I_t$. Also, as $a(0) = b(0) = 0$, one can factor t out of both $a(t)$ and $b(t)$, to get:

$$a(t)y - b(t)x = t\left(-b_1x + a_1y + t(\ldots)\right).$$

As $t \neq 0$, it is a unit, and so $-b_1x + a_1y \in I_0$. Adding this new element to I_0 makes it of codimension 2, as we wanted.

We would set then $X_0 = \operatorname{Spec}(k[x,y]/I_0) = X_{(-b_1,a_1)}$.

We now proceed to a more rigorous aproach.

Definition 16.0.5. A *family of schemes* (indexed by Y) is a morphism of schemes $\varphi\colon X \to Y$.

So for each closed point $y \in Y$, there is $X_y \overset{\text{def}}{=} \varphi^{-1}(y) \hookrightarrow X$.

A family $\varphi\colon X \to Y$ is "good" if φ is *flat*, and in this case it is called a *flat family*.

Let's recall what this means: if for $x \in X$ we let $y = \varphi(x) \in Y$, we have $\varphi_y^\sharp\colon \mathcal{O}_{Y,y} \to (\varphi_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$. We require that $\mathcal{O}_{X,x}$ be then a *flat* $\mathcal{O}_{Y,y}$-algebra (note that this is a local property).

If $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine, then φ is flat if, and only if, A is a flat B-algebra (via $\overline{\varphi}$).

We must now interpret the schemes we saw, X_t, as a family (at least, for $t \in k \setminus \{0\}$).

Let then $\mathbb{G}_{m,k} \overset{\text{def}}{=} \mathbb{A}^1_k \setminus \{0\} = \operatorname{Spec}(k[t]_{(t)})$. We want a scheme X' together with a morphism of $X' \to \mathbb{G}_{m,k}$. So take

$$X' \overset{\text{def}}{=} \operatorname{Spec}\left(\frac{k[t]_{(t)}[x,y]}{I'}\right), \qquad I' = (x^2 - a(t)x, xy - b(t)x, xy - a(t)y, y^2 - b(t)y).$$

Note then that we get a map $\varphi\colon X' \to \mathbb{G}_{m,k}$ coming from the canonical ring homomorphism of the corresponding rings.

Claim. *If* $\alpha \in k \setminus \{0\}$, $\varphi^{-1}(\alpha) = X_\alpha$.

Proof. Exercise. □

We want to find $\psi\colon X \to \mathbb{A}^1_k$ a **flat family** making the following diagram commutative:

$$\begin{array}{ccc} X' & \dashrightarrow & X \\ \downarrow & & \downarrow \psi \\ \mathbb{A}^1_k \setminus \{0\} & \hookrightarrow & \mathbb{A}^1_k, \end{array}$$

and then we will set $X_0 \overset{\text{def}}{=} \psi^{-1}(0) \overset{\text{def}}{=} \lim_{\alpha \to 0} X_\alpha$.

We want $X = \operatorname{Spec}\left(\frac{k[t][x,y]}{I}\right)$, with an ideal I such that:

- $k[t][x,y]/I$ is a flat $k[t]$-algebra, and
- The extension of I, $I^e = Ik[t]_{(t)}[x,y]$ should equal I'.

So just define $I \overset{\text{def}}{=} I' \cap k[t][x,y]$ (that is, $I = (I')^c$). Then, by definition of I', we have a map

$$\frac{k[t][x,y]}{I} \hookrightarrow \frac{k[t]_{(t)}[x,y]}{I'}$$

and the latter ring is free as a $k[t]_{(t)}$-module (of rank 2), so it is torsion-free, and then it is flat as a $k[t]$-module.

We can finally compute $X_0 = \psi^{-1}(0)$, and this turns out to be the same as we had computed before, that is:

$$\operatorname{Spec}\left(\frac{k[x,y]}{(x^2, y^2, xy, -b_1x + a_1y)}\right).$$

Furthermore, we know that it will have the right dimension, by flatness.

16.1 Exercises

In all these problems K denotes an algebraically closed field of characteristic 0.

1. Let $\varphi : X = \operatorname{Spec}(K[x,y]/(xy)) \to Y = \operatorname{Spec}(K[t])$ be the morphism of schemes corresponding to the K-algebra homomorphism $\psi : K[t] \to K[x,y]/(xy)$ defined by $t \to x + y$. Calculate the inverse image under φ of the generic point ξ of Y.

2. Let

$$\varphi : X = \mathrm{Spec}\big(K[x, y, u, v]/((x, y) \cap (u, v))\big) \to Y = \mathrm{Spec}(K[s, t])$$

be the morphism of schemes corresponding to the K-algebra homomorphism $\psi : K[s, t] \to K[x, y, u, v]/((x, y) \cap (u, v))$ defined by $s \to x + u$ and $t \to y + v$. If $(a, b) \in K^2$ corresponds to the closed point $P_{(a,b)} \in Y$, calculate $\varphi^{-1}(P_{(a,b)})$.

3. If L, L' are fields such that $L \subset L'$ then we have a morphism of schemes $\mathbb{A}^n_{L'} \to \mathbb{A}^n_L$ defined by the inclusion of rings $L[x_1, ..., x_n] \to L'[x_1, ..., x_n]$ for $n \geq 1$. If $n = 2$, $L = \mathbb{Q}, L' = \overline{\mathbb{Q}}$ find the images of the following points in $\mathbb{A}^2_{\mathbb{Q}}$:

   (a) $(\sqrt{2}x - \sqrt{3}y - 1) \in \mathbb{A}^2_{\overline{\mathbb{Q}}}$,

   (b) $(x - \zeta, y - \zeta^{-1}) \in \mathbb{A}^2_{\overline{\mathbb{Q}}}$, where ζ is a pth root of unity.

4. Let us consider $X = \mathrm{Spec}(\mathbb{C}[x, y]/(y^3 - x^5 + x^3))$.

   (a) Calculate explicitly the affine scheme $X_0 \stackrel{\text{def}}{=} X \times_{\mathrm{Spec}(\mathbb{C}[x,y])} \mathrm{Spec}(\mathbb{C}[[x, y]])$ and deduce that "near the closed point $(0, 0) = (x, y) \in \mathbb{A}^2_{\mathbb{C}}$" X is the intersection of three curves passing through $(0, 0)$.

   (b) How does the scheme $Y \stackrel{\text{def}}{=} \mathrm{Spec}(\mathbb{R}[x, y]/(y^3 - x^5 + x^3))$ look like near the origin $(0, 0) = (x, y) \in \mathbb{A}^2_{\mathbb{R}}$?

5. Consider the sub-schemes $X_t \stackrel{\text{def}}{=} \{(0, 0), (t, 0), (0, t)\} \hookrightarrow \mathbb{A}^2_K$, where $t \in K - \{0\}$.

   (a) Show that the limit scheme as $t \to 0$ is $X_0 \stackrel{\text{def}}{=} \mathrm{Spec}(K[x, y]/(x^2, xy, y^2))$.

   (b) Show that X_0 is contained in (i.e. is a closed sub-scheme of) the union of any distinct two lines through $(0, 0) \in \mathbb{A}^2_K$.

   (c) Show that the restriction to X_0 of any function $f \in K[x, y]$ is determined by the values of f at $(0, 0)$ and its first derivatives in every direction; thus we think of X_0 as a first order infinitesimal neighborhood of of the point $(0, 0)$ in $\mathbb{A}^2_K$.

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