Diophantine Approximation.

Conjecture (Mordell): Let K be a # field, and C a curve over K of genus ≥ 2. Then $\# C(K) < \infty$ (proved).

Example: $\{ [x:y:z] \in \mathbb{P}^2 \mid x^p + y^p = z^p \}$, $p \geq 5$.

Goal of the course: prove this conjecture.

We won't follow Falting's (83) proof, as it's too difficult.
We will instead follow the proof due to Vojta / Faltings / Bombieri.

Step 1: Embed $C \hookrightarrow J$, its Jacobian variety, assuming $C(K) \neq \emptyset$.

Step 2: For any smooth projective variety X/K and any divisor class $[D]$ on X, there is a height function $h : X(\bar{K}) \longrightarrow \mathbb{R}$, that has many nice properties ("height machine").

In particular, if $X = J$ and $[D] = [\Theta]$, then we obtain the "Canonical Neron-Tate Height" $\hat{h}_{J,\Theta} : J(\bar{K}) \longrightarrow \mathbb{R}$ with the properties:

a) $\hat{h}$ extends to a positive definite quadratic form on $J(\bar{K}) \otimes_{\mathbb{Z}} \mathbb{R}$

b) $\{ x \in J(K) \mid \hat{h}(x) < C \}$ is finite $\forall C > 0$.

We will write $\|x\| = \sqrt{\hat{h}(x)}$, and $\langle x, y \rangle$ for the euclidean metric associated to $\hat{h}$.

Step 3: Prove that $J(K) / 2J(K)$ is finite

Together w/ step 2, get the Mordell-Weil theorem: $J(K)$ is a f.gen ab. group.

Step 4: Using techniques from Diophantine Approximation (on $C \times C$) we prove Vojta's inequality: $\exists K_1, K_2 > 0$ (K_1 depends on C, K_2 only on $g(C)$) s.t. if $x, y \in C(\bar{K})$ w/ $\|x\| > K_1$, $\|y\| > K_2 \cdot \|x\|$, then $\langle x, y \rangle \leq \frac{3}{4} \|x\| \|y\|$.

To prove Vojta's inequality is a hard step.

<u>Step 5</u>: Suppose that $\# C(K) = \infty$.

By Step 2 (b), for each $N \in \mathbb{N}$ we can find $x_1, \dots, x_N \in C(K)$ such that $\|x_1\| > \kappa_1$, $\|x_{i+1}\| > \kappa_2 \|x_i\|$.

By Vojta's inequality, in $J(K) \otimes \mathbb{R}$, the angle $\measuredangle(x_i, x_j) \geq \frac{\pi}{6}$ $\forall i \neq j$

For $N >> 0$, this is a contradiction.

Therefore $\# C(K) < \infty$.

<u>To Do</u>:

A) Abelian Varieties, Jacobians, Theta divisor.

B) Heights and height machine, heights on Abelian varieties. Mordell-Weil thm.

C) Diophantine approximation, Roth's theorem. Siegel's subspace thm.

D) Diophantine approximation on $C \times C$. Vojta's inequality.

Abelian Varieties

Let k be a field.

<u>Def</u>: A variety (scheme) G over k is called a <u>group variety</u> (group scheme) if

1) The functor $\underline{k\text{-alg}} \longrightarrow \underline{\text{Sets}}$, $R \longmapsto G(R)$ takes values in groups.

$\Leftrightarrow$

2) There are algebraic morphisms $G \times_k G \xrightarrow{\mu} G$, $G \xrightarrow{i} G$, $\mathrm{Spec}(k) \xrightarrow{e} G$ s.t. some diagrams commute

(e.g.
$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(\mu, id)} & G \times G \\ \downarrow {\scriptstyle (id, \mu)} & & \downarrow {\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array}$$
)

Def: A group variety A/k is called abelian if A is complete (⇒projective)
proper

Fact: The group operation on an abelian variety is commutative.

Example: If $k = \mathbb{C}$, then (as a complex manifold) $A = \mathbb{C}^g/\Lambda$ for $\Lambda \subseteq \mathbb{C}^g$ a complete lattice. (i.e $\Lambda \simeq \mathbb{Z}^{2g}$ and $\Lambda \otimes \mathbb{R} \simeq \mathbb{C}^g$).

Warning: if $g > 1$, then $\mathbb{C}^g/\Lambda$ is not necessarily an algebraic variety!

Example (Jacobian): Let C be a complex algebraic curve (i.e. a compact Riemann surface).

Then $$J_C := \frac{H^0(C, \Omega^1_{hol})^\vee}{H_1(C, \mathbb{Z})} \qquad H^0(C, \Omega^1_{hol})^\vee \simeq H^1(C, \mathcal{O}_{hol})$$ is an abelian variety, called the Jacobian of C.

Ref: Milne, "Abelian Varieties" in Arithmetic Geometry (Cornell & Silverman ed).
Mumford, "Abelian Varieties".

Over $\mathbb{C}$: it's a connected, compact, cpx Lie group A.

So:
- it's commutative
- V = tgt space of A at the identity. $\exp: V \to A$ is surjective with Kernel Λ, discrete.

So $A \simeq V/\Lambda$.

But, not all complex tori are abelian varieties.

Need an embedding into Projective space.

For this, one needs a positive definite Hermitian form on V, taking integral values on Λ. (Riemann form).

The conditions on the matrix for this form are called the Riemann relations.

Let the N-torsion be the kernel of multiplication by N.

For $\mathbb{C}^g/\Lambda$, $\Lambda \simeq \mathbb{Z}^{2g}$, so the N-torsion is $\frac{1}{N}\mathbb{Z}^{2g}/\mathbb{Z}^{2g} \simeq (\mathbb{Z}/N\mathbb{Z})^{2g}$.

Over $K \neq \mathbb{C}$, it's not a "torus" (check for a good definition of torus).

A/K is complete if $A \to \operatorname{Spec}(K)$ is proper.

It's a group variety: $\exists\, m: A \times A \to A$, $i: A \to A$, $e: \operatorname{Spec} K \to A$ s.t. ...

<u>Properties</u>

- It's commutative
- If $K = \bar{K}$ (or at least $K = K_s$ (sep. closure)), then N-torsion on A for char $K \nmid N$ is $\simeq (\mathbb{Z}/N\mathbb{Z})^{2g}$ where $\dim A = g$.

Let $\operatorname{Pic}(A) :=$ set of <u>algebraic</u> equivalence classes of invertible sheaves on A. ← generalization of lin. equiv.

$\operatorname{Pic}^0(A) :=$ sheaves in $\operatorname{Pic}(A)$ alg. equiv to the trivial bundle.

They both are functors on AbVar.

<u>Fact</u>: $\operatorname{Pic}^0(-)$ is representable: $\exists$ an abelian variety whose points over $\bar{K}$ are in bijection w/ $\operatorname{Pic}^0(A)$.

<u>Def</u>: The dual abelian variety of A is $A^\vee$ not defined, so $\operatorname{Pic}^0(A) =: A^\vee$.

We have a canonical invertible sheaf P on $A \times A^\vee$ s.t. if $a \in A^\vee$, then the pullback of P to $A \times \{a\} \simeq A$ is a representative of a on $A^\vee$. P is called the <u>Poincare sheaf</u>.

<u>Def</u>: An isogeny is a surjective morphism of abelian varieties w/ finite kernel.

<u>Example</u>: multiplication by N is an isogeny.

Def The <u>degree</u> of an isogeny is the order of its kernel.

Def: An isogeny $\lambda : A \to A^\vee$ is called a <u>polarization</u>.

If $\deg \lambda = 1$, λ is called a <u>principal polarization</u>.

(<u>Note</u>: ppal polarization is an isomorphism $A \xrightarrow{\sim} A^\vee$).

Polarization "rigidify" ab. vars (e.g. there are finitely many automorphisms fixing ~~the~~ given polarization).

<u>Theorem</u>: Any abelian variety is <u>isogenous</u> to a principally polarized ab. var.

$$\left(\text{i.e. } (A, \lambda) \longmapsto (A', \lambda') \text{ with } \deg \lambda' = 1\right).$$

Another def of a Polarization: λ is an embedding into projective space.

For jacobians, the polarization associated to the theta divisor is principal

So $J \cong J^\vee$

For a curve C, we have $C \hookrightarrow J(C)$ by $P \mapsto [P - D]$

where D is a fixed degree-1 divisor.

For an elliptic curve (E, ∞), the map $P \longmapsto [P - \infty]$ is an iso:

<u>Proof</u>

- <u>injectivity</u>: if $[P - \infty] \sim [Q - \infty]$ then $[P - Q] \sim \mathrm{div}(f)$, $f \in K(E)$.
  - in curves, linear eq. = alg. equiv

  But $f : E \to \mathbb{P}^1$ has degree 1, so $E \cong \mathbb{P}^1$, contradicting $\mathrm{genus}(E) = 1$.

- <u>Surjectivity</u>: Riemann-Roch says that, for any D with $\deg D = 0$, $\ell(D + \infty) = 1$.

  Let $f \in \mathcal{L}(D + \infty)$ generate $\mathcal{L}(D + \infty)$. Then $\mathrm{div}(f) \geq -D - \infty$

  $\deg(\mathrm{div}(f)) = 0$. So $\mathrm{div}(f) = -D - \infty + P$ for some P.

  Hence $D \sim [P - \infty]$ //

· Divisors

Let X be a scheme $/k$.

Def: A divisor (Weil divisor) on X is a formal sum $\sum a_i [Z_i]$, where $Z_i \subseteq X$ are irreducible subvarieties of pure codimension 1, $a_i \in \mathbb{Z}$.

The group of Weil divisors is $\mathrm{Div}(X)$.

Assume X is integral (reduced + irreducible) and smooth (at least regular in codim 1), and $k(X)$ the field of rat'l functions on X.

(i.e. $k(X) = \mathcal{O}_{X,\eta}$).

If $f \in k(X)$, we can define a divisor (f) as

$$(f) = \sum_{Z \in X^{(1)}} v_Z(f)[Z] \qquad \left(X^{(1)} = \text{set of irreducible varieties of pure codim } 1\right)$$

and where v_Z is the valuation corresponding to the DVR $\mathcal{O}_{X,\eta_Z}$ (as X is reg. in codim 1).

Ex: X/k a smooth curve, then a divisor is a formal sum of closed points.

e.g. if $X = \mathbb{P}^1$, and x is a coordinate function, $(x) = (0) - (\infty)$.

Def: We say that a Weil divisor of the form (f) is rational, or "linearly equiv." to 0.

Def: $Cl(X) := \mathrm{Div}(X) / \text{rational divisors}$.

Example: $Cl(\mathbb{P}^1) = \mathbb{Z}$ (if $k = \bar{k}$, we have an easy proof by degree)

Def: A Cartier divisor on X is a global section of the quotient sheaf $\mathcal{K}^\times / \mathcal{O}^\times$ (if X is integral, $\mathcal{K}^\times = k(X)^\times$).

(equivalently, a Cartier divisor is given by a cover $X = \bigcup U_i$, for each U_i a rat'l function $f_i \in \mathcal{K}^\times(U_i)$ s.t. $\frac{f_i}{f_j} \in \mathcal{O}^\times(U_i \cap U_j)$ invertible morphisms.

If X is regular in codimension 1, then we get a homomorphism:

$$\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$$
$$((U_i)_i, f_i) \longmapsto \sum_{Z \in X^{(1)}} v_Z(f_i)\,[Z]$$

(well-defined!)

↑ this is well-defined!

Def $\mathrm{CaCl}(X) := \dfrac{\mathrm{CaDiv}(X)}{\mathrm{im}\ \mathcal{K}^\times(X)}$.

We get a diagram:

$$\begin{tikzcd}
\mathcal{K}^\times(X) \arrow[r] \arrow[d] & \mathrm{CaDiv}(X) \arrow[r] \arrow[d] & \mathrm{CaCl}(X) \arrow[d] \\
(\text{rational divisors}) \arrow[r] & \mathrm{Div}(X) \arrow[r] & \mathrm{Cl}(X)
\end{tikzcd}$$

Thm: if X is "nice" (locally factorial...) (e.g. X smooth) then
$$\mathrm{CaCl}(X) \simeq \mathrm{Cl}(X).$$

Line Bundles

Let X be any scheme.

Def A line bundle $\mathcal{L}$ on X is a locally-free coherent sheaf of rank 1 on X.

Line bundles can be described by "transition functions", so they are connected with Cartier divisors:

Fact (1) If $\mathcal{L}, \mathcal{L}'$ are line bundles, so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$.

(2) If $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \xrightarrow{\simeq} \mathcal{O}_X$

Def: $\mathrm{Pic}(X) :=$ group of isomorphism classes of line bundles on X.

Example: if $X = \mathrm{Spec}\, R$, R a Dedekind domain, then
$$\{\text{line bundles on } X\}/\simeq \;\longleftrightarrow\; \{\text{fractional ideals of } R\}/\{\text{principal fractional ideals}\}.$$

(in the example, we've seen that $\mathrm{Pic}(X) \cong \mathcal{Cl}(R)$).

Thm: $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$

Pf Describe line bundles using transition function. //

Corollary: $\mathrm{Pic}(X) \cong \mathrm{Ca}\,\mathcal{Cl}(X)$.

Pf Exact sequence of sheaves:

$$0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times / \mathcal{O}_X^\times \to 0.$$

Taking cohomology, get a long exact seq.

$$0 \to H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to \underbrace{H^0(X, \mathcal{K}_X^\times)}_{\text{rational divisors}} \to \underbrace{H^0(X, \mathcal{O}_X^\times)}_{\mathrm{CaDiv}(X)} \to$$

$$\to \underbrace{H^1(X, \mathcal{O}_X^\times)}_{\cong \mathrm{Pic}(X)} \to \underbrace{H^1(X, \mathcal{K}_X^\times)}_{0} \quad \text{because } \mathcal{K}_X^\times \text{ is flasque.}$$ //

Curves: A curve X/k will mean a projective, smooth, connected scheme of dimension 1 over k.

Def: Let $D = \sum_{x \in X^{(1)}} n_x [x]$

Then the degree of D is $\sum n_x \cdot [k(x):k] \in \mathbb{Z}$

(note that $[k(x):k]$ is finite by Hilbert's Nullstellensatz).

We get a gp. hom. $\deg : \mathrm{Div}(X) \to \mathbb{Z}$, and $\deg((f)) = 0$. for $f \in k(X)^\times$.

Therefore, get a homomorphism $\mathcal{Cl}(X) \to \mathbb{Z}$.

The kernel of it is $\mathcal{Cl}^0(X)$.

Using $\mathrm{Pic}(X) \cong \mathcal{Cl}(X)$, get a group $\mathrm{Pic}^0(X)$.

Asside: $X/\mathbb{C}$ a compact Riemann surface. We have the exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0 \qquad \text{(of sheaves on } X^{an}\text{)}$$

sheaf of holomorphic functions

which induces:

$$H^0(X,\mathcal{O}_X^*) \to H^1(X^{an},\mathbb{Z}) \hookrightarrow H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X^{an},\mathbb{Z})$$

because (on H^0) exp is surjective

$H^1(X,\mathcal{O}_X^*) \cong \mathrm{Pic}(X)$ (GAGA); $\delta = c_1$ first chern class; $H^2(X^{an},\mathbb{Z}) \cong \mathbb{Z}$; deg.

So $\mathrm{Pic}^0(X) \cong \dfrac{H^1(X^{an},\mathcal{O}_X^{hol})}{H^1(X^{an},\mathbb{Z})}$.

Let $D \in \mathrm{Div}(X)$. We can define a line bundle $\mathcal{L}(D)$ as:

$$\mathcal{L}(D)(U) := \{ f \in k(U) = k(X) : (f|_U) \geq -D \}$$ ← this is a line bundle.

Fact: if $D \sim D'$ (i.e. $D - D' = (g)$ for some g), then $\mathcal{L}(D) \cong \mathcal{L}(D')$.

Def: Let $L(D) := H^0(X, \mathcal{L}(D))$.

Def: The projective space $\mathbb{P}(L(D))$ is called the <u>complete linear system</u> of D.

$$\mathbb{P}(L(D)) \xleftrightarrow{1:1} \{ D' \sim D : D' \geq 0 \}.$$

(let $f \in L(D)$. Then $D' := D + (f)$ is effective, $D' \sim D$).

<u>Jacobians</u>

Let C/k be a curve. Then J (Jacobian) is a variety s.t.

$J(F)$ $(k \subseteq F)$ is isomorphic to $\mathrm{Pic}^0(C \times_k F) = Cl^0(C \times_k F)$

Assume there exists $x \in C(k)$, and let $n \geq 1$. Then we get the map

$$\mathrm{Sym}^n C \to J \quad \text{by} \quad x_1 + \cdots + x_n \longmapsto (x_1) + \cdots + (x_n) - n(x)$$

Why is this map algebraic? Because it's a transformation of functors.
We go now more formal.

Jacobian of a Curve

Def Let C/k be a (smooth, proj.) curve. The Jacobian variety J of C is a variety representing the following functor:

$$T \longmapsto P_C^0(T) = \left\{ [\mathcal{L}] \in \mathrm{Pic}(C\times T) \text{ s.t. } \deg \mathcal{L}_t = 0 \ \forall \text{ fibers} \right\} \Big/ \left\{ \mathcal{L} : \mathcal{L} \cong q^*\overline{\mathcal{L}} \text{ for some } \overline{\mathcal{L}} \in \mathrm{Pic}(T) \right\}$$

$\Leftrightarrow \mathcal{L}_t \in \mathrm{Pic}^0(C\times\{t\})$

line bundles that come from T $= pr_2^*(\mathrm{Pic}(T))$

In particular, if $T = \mathrm{Spec}\, F$, then $P_C^0(F) = \mathrm{Pic}^0(C \times_k F)$.

Construction

Assume $C(k) \neq \emptyset$ and that J exists. Choose $x_0 \in C(k)$.
Then we can evaluate the functor at J itself.

So $P_C^0(J) = \mathrm{Mor}(J, J) \ni id_J$.

Hence $id_J \leftrightarrow [M] \in \mathrm{Pic}(C\times J)$ s.t. (a) $M|_{\{x_0\}\times J}$ is trivial
(b) $[M|_{C\times\{0\}}] = 0 \in P_C^0(C)$
(i.e. $M|_{C\times\{0\}}$ is trivial).

Thm: Suppose J exists. Let T be a pointed k-scheme w/ base point $t_0 \in T(k)$. There is a 1-1 correspondence

$$P_C^0(T) \overset{1:1}{\longleftrightarrow} \left\{ [\mathcal{L}] \in \mathrm{Pic}(C\times T) \text{ s.t. } \begin{array}{l} \mathcal{L}|_{\{x_0\}\times T} \\ \mathcal{L}|_{C\times\{t_0\}} \end{array} \text{ are trivial} \right\}.$$

by sending $[(id_C \times f)^* M]$ to $f: T \to J$.

Construction.

Step 1: $\mathrm{Sym}^r C$ is a smooth variety (take r copies of C, and mod out by the action of the group Sym_r on the components).

Then $T \longmapsto \mathrm{Div}^r_C(T) \sim$ relative effective Cartier divisor on $C \times T/T$.

Def: A relative effective Cartier divisor on $C \times T$ of constant degree r over T is a Cartier divisor $D = (U_i, f_i)$ on $C \times T$ s.t.:

a) $f_i \in \mathcal{O}(U_i)$ $\forall i$ (i.e. D is effective).

b) $|D| = \bigcup V(f_i)$ $\left(V(f_i) = \mathrm{Spec}\left(\mathcal{O}_{U_i}/(f_i)\right)\right)$ is flat over T.

c) $\forall t \in T$, $D_t := D \times_T \{t\}$ on $C \times \{t\}$ has degree r.

Write $\mathrm{Div}^r_C(T) = \{D \in \mathrm{CaDiv}(C \times T) : D$ is relat. effective of degree $r\}$.

Fact: Div^r_C is a contravariant functor on k-schemes.

Now, let D_{can} be (the canonical) the relative effective Cartier divisor on $\mathrm{Sym}^r C \times C/\mathrm{Sym}^r C$, obtained as follows: $D_{can} = \left(\sum s_i(C^r)\right)/\Sigma_r$

where $s_i : C^r \to C \times C^r$
$(P_1, \dots, P_r) \mapsto (P_i, P_1, \dots, P_r)$

Note that the fiber $D_{can, P_1 + \dots + P_r} = (P_1) + \dots + (P_r)$.

Theorem: Let T be a scheme. Then there is a (natural) 1-1 correspondence between morphisms $\varphi : T \to \mathrm{Sym}^r C$ and ~~relative effective div~~ $\mathrm{Div}^r_C(T)$, given by $\varphi \longmapsto (1 \times \varphi)^*(D_{can})$

(note $1 \times \varphi : C \times T \to C \times \mathrm{Sym}^r C$).

Pf (of Thm)

This obviously defines a transformation $\pi : \mathrm{Sym}^r C \to \mathrm{Div}^r_C$.

and π is injective (exercise). We need to prove surjectivity.

Assume D is split, i.e. $D = \sum n_i s_i(T)$, for some $s_i : T \to C \times T$

sections to pr_2.

In this case, define $\tilde{\varphi} : T \to C^r$ by $\tilde{\varphi} = (\overbrace{\bar{s}_1, \bar{s}_1, \dots}^{n_1}, \overbrace{\bar{s}_2, \bar{s}_2, \dots}^{n_2}, \dots \overbrace{\bar{s}_r, \bar{s}_r \dots}^{n_r})$

($\sum n_i = r$, and let $\bar{s}_i := pr_1 \circ s_i$ ($\bar{s}_i : T \to C$).

Then let $\varphi := (T \xrightarrow{\tilde{\varphi}} C^r \to \mathrm{Sym}^r C)$

Exercise: $(1 \times \varphi)^*(D_{can}) = D$.

flat surjective morphism

Fact: There is a flat cover $T' \xrightarrow{q} T$ s.t. $(1 \times q)^*(D)$ is split.

So if we believe this, let $\varphi' : T' \to \mathrm{Sym}^r C$ correspond to $(1 \times q)^*(D)$, constructed as above.

$$
\begin{tikzcd}
T' \times_T T' \arrow[r, "pr_2"] \arrow[d, "pr_1"] & T' \arrow[d] \\
T' \arrow[r] & \mathrm{Sym}^r C
\end{tikzcd}
$$

why?

by the univ. property of the fiber product

$$(1 \times (\varphi' \circ pr_1))^*(D_{can}) = (1 \times (q \circ pr_1))^*(D)$$

$$\|$$

$$(1 \times (\varphi' \circ pr_2))^*(D_{can}) = (1 \times (q \circ pr_2))^*(D)$$

But as π is injective, this gives $\varphi' \circ pr_1 = \varphi' \circ pr_2$. ~~by flat descent~~.

By flat descent, $\exists\ \varphi : T \to \mathrm{Sym}^r C$ s.t. $(1 \times \varphi)^*(D_{can}) = D$.

look it up!

Let $P_C^r(T) = \{ \mathcal{L} \in \mathrm{Pic}(C \times T) : \deg \mathcal{L}_t = r \} / \sim$

(~~[illegible]~~ so that $\mathcal{L} \sim \mathcal{L}' \iff \exists \mathcal{M}$ st $\mathcal{L} \otimes pr_2^* \mathcal{M} \simeq \mathcal{L}'$ $\quad (pr_2 : C \times T \to T)$.

We can identify P_C^0 with P_C^r via:

the chosen point $\in C$.

$$P_C^0 \ni \mathcal{L} \longmapsto \mathcal{L} \otimes pr_1^* \mathcal{L}(r \cdot x_0).$$

Fact. There is a natural transformation $f : \mathrm{Div}_C^r \to P_C^r$

$$D \mapsto \mathcal{L}(D)$$

Note: $f^{-1}(\mathcal{L}) = \mathbb{P}(H^0(C, \mathcal{L}))$ ← complete linear system of $\mathcal{L}$

Lemma: Assume f has a section $g : P_C^r \to \mathrm{Div}_C^r$. Then P_C^r ~~is~~ representable.

Pf Let $\varphi = g \circ f$, $\varphi : \mathrm{Div}_C^r \to \mathrm{Div}_C^r \leftrightarrow \varphi : \mathrm{Sym}^r C \to \mathrm{Sym}^r C$ (Yoneda)

Set J' to be the pull-back:

$$\begin{tikzcd} \mathrm{Sym}^r C \arrow[d, "(1,\varphi)"] & J' \arrow[l] \arrow[d] \\ \mathrm{Sym}^r C \times \mathrm{Sym}^r C & \mathrm{Sym}^r C \arrow[l, "\Delta"'] \end{tikzcd}$$

Exercise: $J' \cong P_C^r$.

Let now $r > 2g$, and let γ be an $(r-g)$-tuple in $C(k)$.

Set $D_\gamma := \sum_{p \in \gamma} p$, $\mathcal{L}_\gamma := \mathcal{L}(D_\gamma)$.

Theorem: a) $\mathrm{Div}_C^\gamma(T) := \{ D \in \mathrm{Div}_C^r(T) : \ell(D_t - D_\gamma) = 1 \quad \forall t \}$.

~~b)~~ is represented by an open nonempty subvariety $C^\gamma \subseteq \mathrm{Sym}^r C$.

Moreover, if $k = k^{sep}$, then $\mathrm{Sym}^r C = \bigcup_\gamma C^\gamma$.

b) Let $P^\gamma(T) = \{ \mathcal{L} \in P_C^r(T) : \ell(\mathcal{L}_t \otimes \mathcal{L}_\gamma^{-1}) = 1 \}$.

Then $P^\gamma \subseteq P_C^r$ is a subfunctor, and $f : C^\gamma \to P^\gamma$ has a section.

Pf of Thm: Use semicontinuity. (see it in Cornell-Silverman)

Now we need to show that the constructed scheme is complete.

Choose $P \in C(L)$.

Observe: $\Delta - \{P\}\times C - C\times\{P\} \in \mathrm{Div}(C\times C)$ defines a divisorial correspondence from (C,P) to itself.

This corresponds to $f^P: C \to J$, which on points is $f^P(Q) = \mathcal{L}(Q-P)$.

For any $r \geq 1$, set $f^r : C^r \to J$, $f^r = \underbrace{f+f+\dots+f}_{r \text{ times}}$

So on points, $f^r(Q_1,\dots,Q_r) = \mathcal{L}(\sum Q_i - rP)$.

As this map is symmetric, it factors through

$f^{(r)}: \mathrm{Sym}^{(r)} C \to J$.

Write $W^{(r)} := f^{(r)}(\mathrm{Sym}^{(r)} C) = f^{r}(C^r)$.

Thm:

(a) If $r \leq g$, then $f^{(r)}: \mathrm{Sym}^r C \to W^r$ is birational.

(b) Suppose $D \in \mathrm{Sym}^r C(k)$; let $F := \mathrm{Sym}^r C \times_J \{f^{(r)}(D)\}$

Then $0 \to T_D(F) \to T_D(\mathrm{Sym}^r C) \xrightarrow{df^{(r)}_D} T_{f^{(r)}(D)} J$ is exact

(on vectorspaces).

Pf (of (a)):

It's not hard to see (using semicontinuity) that there's a nonempty open $U \subseteq \mathrm{Sym}^r C$ such that $|D| =$ point if $D \in U(k^{sep})$ (if $r \leq g$).

So $f^{(r)}$, on U, is purely inseparable. This proves (a) if char $k=0$.

If char $k = p$, then need to prove (b).

//

Corollary: $f^{(g)}: \mathrm{Sym}^g C \to J$ is birational and surjective.

Pf want to see $W^g = J$.

As $\dim W^g = g$ (birational to $\mathrm{Sym}^g C$), it's enough to show that $\dim J = g$.

Thm: $T_0 J \cong H^1(C, \mathcal{O}_C)$ ← has dimension g. $\Rightarrow$ ✓. (as $\dim J = \dim T_0 J$)

Pf For any algebraic group G,

$$T_e G = \mathrm{Ker}\left(G(k[\varepsilon]) \to G(k)\right) \quad \text{where } k[\varepsilon] = k[t]/(t^2)$$

So we need to compute $\mathrm{Ker}\left(P_C^0(k[\varepsilon]) \to P_C^0(k)\right)$

Now $\mathbb{Z} \times P_C^0(k[\varepsilon]) \simeq H^1(C_{k[\varepsilon]}, \mathcal{O}^\times_{C_{k[\varepsilon]}})$ $\left(C_{k[\varepsilon]} = \text{pullback of } C \text{ to } \mathrm{Spec}\, k[\varepsilon]\right)$

Note that $\mathcal{O}^\times_{C_{k[\varepsilon]}} = \mathcal{O}_C^\times \oplus \varepsilon \mathcal{O}_C$ (indeed, $R[\varepsilon]^\times = R^\times \oplus \varepsilon R$).

and therefore we are computing $\mathrm{Ker}\left(H^1(C, \mathcal{O}_C^\times \oplus \varepsilon\mathcal{O}_C) \to H^1(C, \mathcal{O}_C^\times)\right)$

$\cong H^1(C, \varepsilon \mathcal{O}_C)$ //

We want to fix an embedding of J into projective space (we know that there's one, we need just to fix it!)

Let Θ be the divisor $[W^{g-1}]$

Theorem: Θ is an ample, irreducible divisor on J.

Pf Irreducible is ok. We need to see that it's ample.

↓

Step 1: For an abelian variety A, let $\mathrm{Pic}^\circ(A) = \{\mathcal{L} : t_a^*\mathcal{L} \cong \mathcal{L}, a \in A(\bar{k})\}/\cong$

where $t_a : A_{\bar{k}} \to A_{\bar{k}}$
$b \mapsto a+b$

There is an abelian variety $A^\vee$ representing $\mathrm{Pic}^\circ A$ (the dual abelian variety).

Suppose $\mathcal{L} \in \mathrm{Pic}\, A$. Then $\mathcal{L}$ defines a homomorphism

$$\varphi_{\mathcal{L}} : A \longrightarrow A^\vee, \quad a \longmapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

(homomorphism by the thm of the square) of abelian varieties.

Moreover, $\mathcal{L}$ is ample $\Leftrightarrow$ $\ker \varphi_{\mathcal{L}} = \{a \in A(\bar{k}) : t_a^*\mathcal{L} \cong \mathcal{L}\}$ is finite.
($\varphi_{\mathcal{L}}$ is an isogeny).

Note: if $\mathcal{L}$ is ample, $\varphi_{\mathcal{L}}$ is called a polarization.
(there are always polarizations, because A is projective!)
if $\varphi_{\mathcal{L}}$ is an isomorphism, then it is called a principal polarization.

Step 2: Want to show that $\varphi_{\mathcal{L}(\Theta)}$ is an isomorphism.

On $A \times A^\vee$, there's a universal line bundle $\mathcal{P}$ (Poincaré bundle)
(i.e. it corresponds to $\mathrm{id}_{A^\vee}$ under the representing property of $A^\vee$).

It's easy to see that $(f^P \times 1)^*\mathcal{P}$ (on $C \times J^\vee$)
is a divisorial correspondence (C, P) to $(J^\vee, 0)$, which
in turn corresponds to a morphism $f^\vee : (J^\vee, 0) \to (J, 0)$.

Step 3: One proves that $-f^\vee$ is inverse to $\varphi_{\mathcal{L}(\Theta)}$ (long).

Hence Θ is ample.

Heights: Let K be a number field, $([K:\mathbb{Q}] < \infty)$.

we want a function $h : \mathbb{P}^n(K) \to \mathbb{R}_{\geq 0}$

such that $\#\{\underline{x} : h(\underline{x}) \leq C\} < \infty \quad \forall C$ constant.

Then, if we have a projective variety, by choosing an embedding $X \subseteq \mathbb{P}^n$, we will get a height on $X(K)$.

$h(\underline{x})$ depends on embedding $\longleftrightarrow$ very ample divisor on X

In fact, it will only depend additively on to the rat. equiv class of X up to $O(1)$.

If $X = A$ an abelian variety, choose a very ample divisor D s.t

h_D is a quadratic form on $A(K)$, (up to $O(1)$)

This can be used to prove:

- $A(K)$ is finitely-generated.
- $A(K) \otimes \mathbb{R}$ into euclidean vectorspace.

Then can study the lattice $A(K)/\text{torsion} \subseteq A(K) \otimes \mathbb{R}$

• Absolute values on fields

Let K be a field.

An absolute value is $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that:

i) $|x| = 0 \Leftrightarrow x = 0$

ii) $|x+y| \leq |x| + |y|$

iii) $|xy| = |x||y|$

If (ii) can be replaced by

ii)' $|x+y| \leq \max\{|x|, |y|\}$

then it is called non-Archimedean (otherwise it is called archimedean).

Recall: If K is a #field with ring of integers $\mathcal{O}_K$, then we have the following absolute values:

- For each maximal prime $\mathfrak{p} \subseteq \mathcal{O}_K$, have $|x|_{\mathfrak{p}} = N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$

  (where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$, and $(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}$

  Note: H-S calls $|\cdot|_{\mathfrak{p}} = \|\cdot\|_{\mathfrak{p}}$.

- One archimedean for each conjugate pair $\overset{(-)}{\sigma}: K \hookrightarrow \mathbb{C}$

  $|x|_{\sigma} := \|\sigma(x)\|$.

  Write $M_K = \underbrace{M_K^{\infty}}_{\text{archimedean}} \sqcup \underbrace{M_K^{0}}_{\text{finite (non-archimedean)}}$.

Proposition: Let $x \in K^*$, then:

$$\prod_{v \in M_K} |x|_v = 1$$

Pf $|N_{K/\mathbb{Q}}(x)|_v = \prod_{w|v} |x|_w$ for $v \in M_{\mathbb{Q}}$. Hence we can assume $K = \mathbb{Q}$, and then this is easy, using unique factorization. //

- **Heights on Projective Space.**

Def: Let K be a #field, $\underline{x} = [x_0 : \dots : x_n] \in \mathbb{P}^n(K)$.

Define
- $H_K(\underline{x}) := \prod_{v \in M_K} \max\{|x_0|_v, |x_1|_v, \dots, |x_n|_v\}$
- $h_K(\underline{x}) := \log H_K(\underline{x})$.

Lemma: H_K, h_K are well-defined

Pf Use product formula. //

Lemma:

1) $H_K(\underline{x}) \geq 1$

2) Let $[K':K] = n < \infty$. Then $H_{K'}(x) = H_K(\underline{x})^n$.

Pf Exercise //

Def: Let $\underline{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Define then $H(\underline{x}) := H_K(x)^{\frac{1}{[K:\mathbb{Q}]}}$ (if $\underline{x} \in \mathbb{P}^n(K)$). This is the absolute multiplicative height.
The absolute logarithmic height is $\frac{1}{[K:\mathbb{Q}]} h_K(x)$.

Corollary of lemma: H, h are well-defined, independently of choice of coordinates and K.

Exercise: Find all points $\underline{x} \in \mathbb{P}^n(\mathbb{Q})$ such that $H(\underline{x}) = 1$.

Proposition: Let $\sigma \in G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\underline{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$.
Then $H(x) = H(\sigma(x))$.

Pf Choose a Galois field $K/\mathbb{Q}$ s.t. $\underline{x} \in \mathbb{P}^n(K)$. Then σ permutes the absolute values of K, and $|\sigma(x_i)|_{\sigma v} = |x_i|_v$ if $\underline{x} = (x_0, \dots, x_n)$. //

Theorem: Let $B, D \geq 0$. Then the set
$$\{\underline{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : H(\underline{x}) \leq B \text{ and } [\mathbb{Q}(\underline{x}):\mathbb{Q}] \leq D\}$$ is finite.

Pf Choose coordinates $[x_0 : \dots : x_n]$ s.t. $x_0 = 1$, say. Then:

(a) $H([1:x_i]) \leq H(\underline{x}) \quad \forall i$

(b) $[\mathbb{Q}(x_i):\mathbb{Q}] \leq [\mathbb{Q}(\underline{x}):\mathbb{Q}]$

$\Rightarrow$ may assume $n = 1$.

So we will show that: $\#\{x \in \overline{\mathbb{Q}} : H([1:x]) \leq B,\ [\mathbb{Q}(x):\mathbb{Q}] = d\} < \infty$

Let $x_1, \dots, x_d$ be the conjugates of $x = x_1$, and:

$$\prod_{i=1}^{d} (T - x_i) = \sum_{r=0}^{d} (-1)^r S_r(x_1, \dots, x_d) T^{d-r}$$

Let $v \in M_K$, where $K = \mathbb{Q}(x)$.

We have:

$$|S_r(x_1, \dots, x_d)|_v = \Big| \sum_{1 \le i_1 < \dots < i_r \le d} x_{i_1} \cdots x_{i_r} \Big|_v \le C(v,r,d) \cdot \max_{1 \le \dots \le d} |x_{i_1} \cdots x_{i_r}|_v \le$$

$$\le C(v,r,d) \max_{1 \le i \le d} |x_i|_v^r \quad \text{with } C(v,r,d) \le \begin{cases} 1 & \text{if } v \in M_K^0 \\ C(d) & \text{if } v \in M_K^\infty \end{cases}$$

Hence: $\bar{x} = (x_1, \dots, x_d) \leftarrow$ the conjugates of x!

$$\max\{ |S_0(\bar{x})|_v \dots |S_d(\bar{x})|_v \} \le C(v,d) \prod_{i=1}^{d} \max\{ |x_i|_v, 1\}^d$$

$$\Rightarrow H([S_0 : \dots : S_d]) \le \underbrace{\prod_v C(v,d)}_{\approx C} \prod_{i=1}^{d} H([x_i : 1])^d \le C \cdot H([x:1])^{d^2}$$

So we are reduced to the case $K = \mathbb{Q}$ which is an easy exercise.

Corollary: If K is a #field, and $P = [x_0 : \dots : x_n] \in \mathbb{P}^n(K)$.

Fix $x_i \neq 0$. Then $H(P) = 1 \iff \frac{x_j}{x_i} = \begin{cases} 0 \\ \text{root of unity} \end{cases} \forall j$.

Pf $\Leftarrow$) is obvious

$\Rightarrow$) Suppose $H(P) = 1$. For $r \in \mathbb{N}$, set $P^r := [x_0^r : \dots : x_n^r]$.

Then $H(P^r) = H(P)^r = 1$.

As there are only finitely many points over K of ht 1 (by Thm), then

$P^r = P^s$ for some $r \neq s$

We want, given X projective variety $/\bar{\mathbb{Q}}$, and D a divisor on X, define a height $h_D : X(\bar{\mathbb{Q}}) \to \mathbb{R}$ such that:

- $h_D \approx h_E$ if $D \underset{\text{rat}}{\sim} E$
  i.e. $h_D = h_E + O(1)$ independent of $P \in X(\bar{\mathbb{Q}})$. (so $|h_D(P) - h_E(P)| \leq C$).
- $h_{D+E} \approx h_D + h_E$
- $h_H \approx h$ on $\mathbb{P}^n$ with H = hyperplane.

+ more properties!

<u>Theorem 1</u>: Let $S_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ be the Segre embedding taking $([x_0 : \dots : x_n], [y_0 : \dots : y_m]) \longmapsto ([x_i y_j]_{i,j})$.

Then (a) $S_{n,m}^* \mathcal{O}_{\mathbb{P}^N}(1) \cong p^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes q^* \mathcal{O}_{\mathbb{P}^m}(1)$ $\quad \begin{cases} p : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n \\ q : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \end{cases}$

(b) $h(S_{n,m}(x,y)) = h(x) + h(y)$

(c) If $\Phi_d : \mathbb{P}^n \to \mathbb{P}^N$ is the d-uple embedding, then $h(\Phi_d(x)) = d\,h(x)$

Pf/ Exercise, or look up in book //

<u>Theorem 2</u>: Let $\Phi : \mathbb{P}^n \to \mathbb{P}^m$ be a rational map, given by homogeneous polynomials $(f_0, \dots, f_m)$ of degree d.
Let $Z := V(f_0, \dots, f_m)$ (so Φ is defined on $\mathbb{P}^n \setminus Z$).

<u>Then</u>: (a) $\forall P \in (\mathbb{P}^n \setminus Z)(\bar{\mathbb{Q}})$, $h(\Phi(P)) \leq d\,h(P) + O(1)$ indep. of P.

(b) If $X \subseteq \mathbb{P}^n$ is closed and $X \cap Z = \phi$, then $\forall P \in X(\bar{\mathbb{Q}})$,
$h(\Phi(P)) = d\,h(P) + O(1)$ (so bounded above & below!).

Pf (a) Write $f_i(\underline{X}) = \sum_{|\underline{e}|=d} a_{i,\underline{e}} \underline{X}^{\underline{e}}$ where $\underline{e} = (e_0, \dots, e_n) \in \mathbb{N}^{n+1}$, $|\underline{e}| = \sum e_i$

Note that this sum has $\binom{n+d}{n}$ summands. Fix $K/\mathbb{Q}$, $v \in M_K$

<u>Notation</u>: • $|P|_v := \max\{|x_i|_v \mid i=0..n\}$, for $P = [x_0 : \dots : x_n] \in \mathbb{P}^n(K)$

(<u>Rk</u>: depends on the choice of coordinates).

• $|f|_v := \max\{|a_{\underline{e}}|_v \mid \underline{e} \text{ multiindices}\}$ for $f(X) = \sum_{\underline{e}} a_{\underline{e}} X^{\underline{e}}$.

• $e_v(r) := \begin{cases} r & \text{if } r \in M_K^\infty \\ 1 & \text{if } r \in M_K^0 \end{cases}$ (<u>note</u> $\prod_v e_v(r) < \infty$).

Consider $P \in \mathbb{P}^n(K)$, $P = [x_0 : \dots : x_n]$ such that Φ is defined $/K$.

We have $|f_i(P)|_v = \left|\sum_{\underline{e}} a_{i,\underline{e}} \underline{x}^{\underline{e}}\right|_v \leq e_v\left(\binom{n+d}{n}\right) \cdot |f_i|_v |P|_v^d$

whence $|\Phi(P)|_v \leq C \left(\max_i |f_i|_v\right) \cdot |P|_v^d$

Multiplying over all $v \in M_K$, we get:

$$H_K(\Phi(P)) \leq C^{\# M_K^\infty} H_K(\Phi) \cdot H_K(P)^d \quad \text{where } H_K(\Phi) = H_K\left([a_{i,\underline{e}}]_{i,\underline{e}}\right).$$

Take log and divide by $[K:\mathbb{Q}]$ to get (a).

(b) We need an inequality in the opposite direction.

Choose homogeneous equations defining X, $X = V(p_1, \dots, p_s)$

By assumption, $V(f_0, \dots, f_m, p_1, \dots, p_s) = X \cap Z = \emptyset$.

So by Hilbert's Nullstellensatz, $\sqrt{(f_0, \dots, f_m, p_1, \dots, p_s)} = (X_0, \dots, X_n)$

That is, there $\exists\, t \geq d$, and $\exists$ homogeneous polynomials g_{ij}'s, q_{ij}'s s.t.

$$g_{0j} f_0 + \dots + g_{mj} f_m + q_{1j} p_1 + \dots + q_{sj} p_s = X_j^t \quad \text{for } 0 \leq j \leq n.$$

We may assume that everything is defined over K (enlarge K, if necessary).

(cont pf)

Now let $P = [x_0 : \dots : x_n] \in X(K)$, (so that $P_i(\underline{x}) = 0$, $1 \le i \le s$).

Hence $g_{0j}(\underline{x}) f_0(\underline{x}) + \dots + g_{mj}(\underline{x}) f_m(\underline{x}) = x_j^t \qquad 0 \le j \le n$

Then $|P|_v^t = \max_j |g_{0j}(\underline{x}) f_0(\underline{x}) + \dots + g_{mj}(\underline{x}) f_m(\underline{x})|_v \le$

$\le e_v(m+1)\left(\max |g_{ij}(\underline{x})|_v\right)\left(\max |f_i(\underline{x})|_v\right) \le C \cdot |\underline{g}|_v \cdot |P|_v^{t-d} |\Phi(P)|_v$ (1 if v is finite.)

Taking $\prod_v$, we get, after taking logs:

$$dh(P) \le h(\Phi(P)) + O(1).$$

§3. Heights on Projective Varieties.

Def: Let $\Phi : X \to \mathbb{P}^n$ be a morphism, where $X/\bar{\mathbb{Q}}$ is a projective variety.
Then $h_\Phi(P) := h(\Phi(P))$, for $P \in X(\bar{\mathbb{Q}})$ (height on X).

Thm 3: X proj. smooth var., $\Phi : X \to \mathbb{P}^n$, $\Psi : X \to \mathbb{P}^n$ s.t. $\Phi^* \mathcal{O}_{\mathbb{P}^m}(1) \cong \Psi^* \mathcal{O}_{\mathbb{P}^m}(1)$
Then $h_\Phi = h_\Psi + O(1)$ (assume φ, ψ nonconstant).

Pf Choose $D \geq 0$ s.t. $\mathcal{L}(D) \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \psi^* \mathcal{O}_{\mathbb{P}^m}(1)$.
Then we can write $\Phi = (\text{linear map}) \circ f_{|D|}$, $\Psi = (\text{linear map}) \circ f_{|D|}$
and apply Thm 2 to get the result.

Thm 4 (Height Machine): To each smooth, projective X/K, and $D \in \mathrm{Div}(X)$, we can assign a function $h_{X,D} : X(\bar{K}) \to \mathbb{R}$ s.t.

a) If $X = \mathbb{P}^n$ and $D = H$ (hyperplane) then $h_{id,H} = h + O(1)$,

b) If $D \underset{rat}{\cong} E$, then $h_{X,D} = h_{X,E} + O(1)$

c) If $D, E \in \mathrm{Div}(X)$, then $h_{X,D+E} = h_{X,D} + h_{X,E} + O(1)$

d) If $f : X \to Y$, $D \in \mathrm{Div}(X)$, $E \in f^*([D])$, then $h_{X,E}(P) = h_{Y,D}(f(P)) + O(1)$

Moreover, $h_{X,D}$ is uniquely determined by (a)-(d), up to $O(1)$.

<u>Proof:</u>
Note that any divisor is the difference of two ample divisor, so we get uniqueness of $h_{X,D}$ given (a) – (d).

<u>Construction:</u> Let D be a divisor whose linear system is base point free, and let $\Phi_D : X \to \mathbb{P}^n$ be ~~the~~ any associated morphism (ie $\Phi_D^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}(D)$).

Set now $h_{X,D} := h_{\Phi_D}$, so $h_{X,D}(P) = h(\Phi_D(P))$.

By <u>thm 3</u>, the choice of Φ_D does not matter, up to a bounded function.

Part (a) follows, by choosing $\Phi_H = id$.

Moreover, (b) follows from Thm 3 if $|D| = |E|$ is basepoint free.

If D, E are base point free, part (c) follows from Thm 1, and (d) follows as well.

For general D, we write $D = D_1 - D_2$, D_i ample (so in particular, D_i are base point free), and we set

$$h_{X,D} := h_{X,D_1} - h_{X,D_2}.$$

If $D = E_1 - E_2$ is another such decomposition, then $D_1 + E_2 = D_2 + E_1$, and using (a), (c) for base-point free divisors we get

$$h_{X,D_1 - D_2} = h_{X,E_1 - E_2} + O(1).$$

Similarly, can check (b) – (d) for general divisors.

<u>Theorem 5</u>: The $h_{X,D}$ satisfy:

a) If $D \geq 0$ (i.e. is effective) and $\mathcal{B}$ is the base locus of $|D|$, then

$$h_{X,D} \geq O(1) \text{ on } (X \setminus \mathcal{B})(\bar{K}) .$$

$$\left(\mathcal{B} := \bigcap_{E \in |D|} \operatorname{Supp} E\right)$$

b) If D is ample, and $D \underset{alg}{\sim} 0$, then $\displaystyle\lim_{\substack{P \in X(\bar{K}) \\ h_{X,D}(P) \to \infty}} \frac{h_{X,E}(P)}{h_{X,D}(P)} = 0$

c) For any K' w/ $[K':K] < \infty$, D ample,

$$\#\{P \in X(K') \mid h_{X,D}(P) \leq C\} < \infty .$$

Pf

(a) write $D = D_1 - D_2$, w/ D_1, D_2 base-point free.

Choose a basis $\{f_0, \dots, f_n\}$ of $H^0(X, \mathcal{L}(D_2)) \subseteq K(X)$

Now $D_1 - D_2$ is effective, so $\{f_0, \dots, f_n\} \subseteq H^0(X, \mathcal{L}(D_1)) \Rightarrow$

$\Rightarrow$ can complete to get a basis $\{f_0, \dots, f_n, f_{n+1}, \dots, f_m\}$ of $H^0(X, \mathcal{L}(D_1))$.

We obtain morphisms

$$X \xrightarrow{\Phi_{D_1}} \mathbb{P}(H^0(X, \mathcal{L}(D_1))) \cong \mathbb{P}^m$$

$$x \longmapsto [f_0(x) : \dots : f_m(x)]$$

these isomorphisms change at each $x \in X$!!

and

$$X \xrightarrow{\Phi_{D_2}} \mathbb{P}(H^0(X, \mathcal{L}(D_2))) \cong \mathbb{P}^n \text{ using only } f_0, \dots, f_n.$$

If $P \notin \operatorname{Supp}(D_1)$, we conclude $h_{X,D}(P) = h_{X,D_1}(P) - h_{X,D_2}(P) + O(1) =$

because $h_{X,D}$ may be originally defined w/ some other decomposition

$= h(\Phi_{D_1}(P)) - h(\Phi_{D_2}(P)) + O(1) = h([f_0(P) : \dots : f_m(P)]) - h([f_0(P) : \dots : f_n(P)]) + O(1)$

$\geq O(1)$

Choose very ample divisors $H_1, \dots, H_s$ on X s.t.:

i) $D + H_i$ is base point free $\forall i$

ii) $\bigcap_{i=1}^{s} H_s = \emptyset$

$\leftarrow$ easy to do (see book, or think (works for $\geq 7 \dim X$))

Doing the previous calculation w/ $D_1 = D + H_i$, $D_2 = H_i$ we see that

$h_{X,D}(P) \geq O(1)$ $\forall P \notin \mathrm{Supp}(D)$.

Write $B = \bigcap_{i=1}^{r} D_r$, w/ $D_i \sim D$. Use the same argument $\Rightarrow h_{X,D}(P) \geq O(1)$

for $P \notin B$ //

(b) Since D is ample and $E \underset{alg}{\sim} 0$, $\exists m \in \mathbb{N}$ s.t. $\forall n \in \mathbb{Z}$, $mD + nE$ is base-point free (prove it as exercise).

Therefore, for any $n \in \mathbb{Z}$ $\exists$ $c > 0$ s.t. $m h_{X,D} - n h_{X,E} \geq -c$

c may depend on n !

So, for $n \geq 1$, we get:

$$\frac{m}{n} + \frac{c}{n h_{X,D}(P)} \geq \frac{h_{X,E}(P)}{h_{X,D}(P)} \geq -\frac{m}{n} - \frac{c}{n h_{X,D}(P)} \qquad \forall P \in X(\bar{k}).$$

Let $h_{X,D}(P) \to \infty$, giving:

$$\frac{m}{n} \geq \frac{h_{X,E}(P)}{h_{X,D}(P)} \geq -\frac{m}{n} \quad \Rightarrow \quad \left| \frac{h_{X,E}(P)}{h_{X,D}(P)} \right| \leq \frac{m}{n} \quad \forall n \in \mathbb{Z} \Rightarrow \checkmark //$$

- <u>Heights on Abelian Varieties.</u>

<u>Thm</u>: Let A/k be an abelian variety, $D \in \mathrm{Div}(A)$. Then:

a) Suppose $m \in \mathbb{Z}$. Then, $\forall P \in A(\bar{k})$,

$$h_{A,D}([m]P) = \frac{m^2+m}{2} h_{A,D}(P) + \frac{m^2-m}{2} h_{A,D}(-P) + O(1)$$

b) If $D \sim [-1]^* D$ (i.e. D is symmetric), then

$$h_{A,D}([m]P) = m^2 h_{A,D}(P) + O(1)$$

$$h_{A,D}(P+Q) + h_{A,D}(P-Q) = 2h_{A,D}(P) + 2h_{A,D}(Q) + O(1)$$

c) If $-D \sim [-1]^* D$ (i.e. D is antisymmetric), then:

$h_{A,D}([m]P) = m\, h_{A,D}(P) + O(1)$

$h_{A,D}(P+Q) = h_{A,D}(P) + h_{A,D}(Q) + O(1)$.

$\leftarrow$ the divisors in $\mathrm{Pic}^0(A)$.

Pf (Thm):

(a) The theorem of the cube implies that $[m]^* D \sim \frac{m^2+m}{2} D + \frac{m^2-m}{2}[-1]^* D$

Now use the properties of the height machine, noting:

$$h_{A,[m]^*D}(P) = h_{A,D}([m]P) + O(1).$$

(b) The first statement is clear from (a).

For the second part (paralelogram law), we need to prove: (and use the height machine)

$$s^* D + d^* D \sim 2\, pr_1^* D + 2\, pr_2^* D \qquad (\text{on } A\times A)$$

where $s, d, pr_1, pr_2 : A\times A \to A$ s.t. $s(P,Q) = P+Q$, $pr_1(P,Q) = P$

$d(P,Q) = P-Q$, $pr_2(P,Q) = Q$

So we need to see that

$$\Gamma_D := s^* D + d^* D - pr_1^* D - pr_2^* D \sim 0, \text{ i.e. } \mathcal{L}(\Gamma_D) \cong \mathcal{O}_{A\times A}.$$

Thm (Seesaw principle): Let X be a complete variety, and T an integral scheme $/K$. Let $\mathcal{L}$ be a line bundle on $X\times T$.

Write $\mathcal{L}_t$ = pullback of $\mathcal{L}$ to $X \times_T K(t)$, $t\in T$.

Spse $\mathcal{L}_t \cong \mathcal{O}_{X\times_T K(t)}$ $\forall t\in T$, and that $\exists x\in X(K)$ s.t. $\mathcal{L}_x \cong \mathcal{O}_T$

Then $\mathcal{L}$ is trivial.

Remark: By semicontinuity, it suffices to check it for t in some dense subset.

In our application, $X = T = A$, $t = a \in A(\bar{K})$, and $x = 0 \in A$.

Let $j : A \to A\times A$, $a \mapsto (0, a)$. Need to check that $j^*(\mathcal{L}(\Gamma_D)) \cong \mathcal{O}_A$

we should pullback the line bundle but abuse notation

$$j^*(s^* D + d^* D - 2pr_1^* D - 2 pr_2^* D) = (s\circ j)^* D + (d\circ j)^* D - 2(pr_1\circ j)^* D - 2(pr_2\circ j)^* D \sim$$

$$\sim D + [-1]^* D - 2\cancel{[0]} - 2D \sim [-1]^* D - D \sim 0 \quad ✓.$$

Let $i_a : A \to A\times A$, $b \mapsto (b, a)$. Need to check $i_a^* \mathcal{L}(\Gamma_D) \cong \mathcal{O}_{A_{K(a)}}$

↓