# An invitation to Katz's modular forms

## Fall 2008 seminar notes

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**Abstract**

These are notes that the authors wrote during the Fall semester of 2008, to complement several lectures given as part of the graduate student seminar at McGill. The authors wish to thank Bruno Joyal for pointing out some mistakes in a previous version of the notes.

# Contents

# 1 Hypercohomology

## 1.1 First approach: double complexes

We begin by recalling the definition of a (first quadrant cohomological) **double complex**. Let $\mathcal{A}$ be an abelian category with enough injectives. A double complex is a collection of objects $E^{pq}$ in $\mathcal{A}$ indexed by $\mathbb{N} \times \mathbb{N}$, along with horizontal and vertical differentials $d_h$ and $d_v$. More precisely, these are collections of morphisms in $\mathcal{A}$:

$$d_h^{pq} \colon E^{pq} \to E^{p+1,q},$$

and

$$d_v^{pq} \colon E^{pq} \to E^{p,q+1}.$$

We will typically omit the superscripts. The differentials are required to satisfy:

$$d_h \circ d_h = d_v \circ d_v = d_h \circ d_v + d_v \circ d_h = 0.$$

All of this information can be arranged in a diagram:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
E^{02} & \longrightarrow & E^{12} & \longrightarrow & E^{22} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
E^{01} & \longrightarrow & E^{11} & \longrightarrow & E^{21} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
E^{00} & \longrightarrow & E^{10} & \longrightarrow & E^{20} & \longrightarrow & \cdots
\end{array}$$

The rows and columns form complexes, and each square is *anti*-commutative. It becomes commutative if one replaces $d_v$ by $(-1)^p d_v$. We will often write $E^{\bullet\bullet}$ to denote a double complex, $E^{p\bullet}$ for the complex formed by the $p$-th column of $E^{\bullet\bullet}$, and $E^{\bullet q}$ for the complex formed by the $p$-th row of $E^{\bullet\bullet}$.

A morphism of double complexes $f \colon E^{\bullet\bullet} \to F^{\bullet\bullet}$ is defined naturally to be a collection of morphisms in $\mathcal{A}$

$$f^{pq} \colon E^{pq} \to F^{pq}$$

such that the obvious anti-commutation relations hold.

Given a double complex $E^{\bullet\bullet}$, we define the associated **total complex** by summing along diagonals:

$$\mathrm{Tot}(E)^n = \bigoplus_{p+q=n} E^{pq}$$

Define a differential $D$ on $\mathrm{Tot}(E)^\bullet$ by putting $D = d_h + d_v$. One easily checks that this does in fact make $\mathrm{Tot}(E)^\bullet$ into a complex:

$$D^2 = d_h \circ d_h + (d_h \circ d_v + d_v \circ d_h) + d_v \circ d_v = 0.$$

Let $C(\mathcal{A})$ denote the category of (cohomological) complexes of objects in $A$ indexed by $\mathbb{N}$. Given an object $A \in \mathcal{A}$, associating the complex:

$$A \to 0 \to 0 \to \cdots$$

yields an embedding $\mathcal{A} \hookrightarrow C(\mathcal{A})$. Let $F\colon \mathcal{A} \to \mathbf{Ab}$ be a left exact additive functor, and let $A$ be an object in $\mathcal{A}$. Since $\mathcal{A}$ has enough injectives, there exists an injective resolution:

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots .$$

The $n$-th right derived functor of $F$, denoted $R^nF$, associates to $A$ the $n$-th cohomology group of the complex:

$$F(I^0) \to F(I^1) \to F(I^2) \to \cdots$$

Hypercohomology extends this construction to $C(\mathcal{A})$. Towards this end we introduce the notion of resolution in $C(\mathcal{A})$.

Let $A^\bullet$ be a complex in $C(\mathcal{A})$. A **Cartan-Eilenberg (injective) resolution** of $A^\bullet$ is a double complex $I^{\bullet\bullet}$ of injective objects, along with a chain map $\varepsilon\colon A^\bullet \to I^{\bullet 0}$, satisfying the following two axioms:

1. If $A^p = 0$, then the corresponding column $I^{p\bullet}$ is the zero complex.

2. Let $Z^p(I, d_h)$ denote the subcomplex of $I^{p\bullet}$ consisting of the kernel of the horizontal differential $d_h$. Then $\varepsilon$ induces an augmentation $Z^p(A) \to Z^p(I, d_h)$; this is assumed to be an injective resolution. Similarly for the subcomplex $B^p(I, d_h)$ of coboundaries, and for the complex formed by the horizontal cohomology $H^p(I, d_h)$.

**Remark.** In condition two above, it suffices to assume that two of the three assumptions hold. The third is then a consequence of the other two.

A key result is that, since $\mathcal{A}$ has enough injectives, every complex in $C(\mathcal{A})$ has a Cartan-Eilenberg resolution. To prove this, we recall the Horseshoe lemma of homological algebra:

**Lemma 1.1.** *Suppose that*

$$0 \to A' \to A \to A'' \to 0$$

*is an exact sequence in $\mathcal{A}$. Let $A' \to (I')^\bullet$ and $A'' \to (I'')^\bullet$ be injective resolutions. Then $I^\bullet = (I')^\bullet \oplus (I'')^\bullet$ yields an injective resolution $A \to I^\bullet$ of $A$ and natural chain maps such that:*

$$0 \to (I')^\bullet \to I^\bullet \to (I'')^\bullet \to 0$$

*is exact.*

*Proof.* We first show how to define the augmentation $A \to I^0$. We begin with the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (I')^0 & \longrightarrow & I^0 & \longrightarrow & (I'')^0 & \longrightarrow & 0 \\
 & & {\scriptstyle f}\uparrow & & & & {\scriptstyle g}\uparrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Composition thus yields a natural map $A \to A'' \to (I'')^0$; injectivity of $(I')^0$ yields an extension of $f$ to a map $A \to (I')^0$. The direct sum of these two maps gives $h\colon A \to I^0$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (I')^0 & \longrightarrow & I^0 & \longrightarrow & (I'')^0 & \longrightarrow & 0 \\
 & & {\scriptstyle f}\uparrow & & {\scriptstyle h}\uparrow & & {\scriptstyle g}\uparrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Since $f$ and $g$ are injective, the snake lemma (or the 5-lemma) implies that $h$ is also injective. One continues this process inductively. □

**Lemma 1.2.** *Every object $A^\bullet \in C(\mathcal{A})$ admits a Cartan-Eilenberg resolution.*

*Proof.* The proof is taken from [Wei94] (Lemma 5.7.2). For each $p$, let $B^p(A)$, $Z^p(A)$ and $H^p(A)$ denote the $p$-th coboundaries, cocycles and cohomology of $A^\bullet$. There is thus an exact sequence for each $p$:

$$0 \to B^p(A) \to Z^p(A) \to H^p(A) \to 0.$$

Choose injective resolutions $I_B^{p\bullet}$ and $I_H^{p\bullet}$ of $B^p(A)$ and $H^p(A)$, respectively. By the previous lemma, there exists an injective resolution $I_Z^{p\bullet}$ of $Z^p(A)$ such that:

$$0 \to I_B^{p\bullet} \to I_Z^{p\bullet} \to I_H^{p\bullet} \to 0,$$

is an exact sequence of complexes that sits under the previous exact sequence. If $A^p = 0$ then we take the trivial resolutions. In this case, also $B^{p+1}(A) = 0$ and we can take $I_B^{p+1,\bullet}$ to be trivial.

Now consider the exact sequence:

$$0 \to Z^p(A) \to A^p \to B^{p+1}(A) \to 0$$

for each $p$. Apply the previous lemma a second time to obtain a resolution $I_A^{p\bullet}$ of $A^p$ for each $p$ such that:

$$0 \to I_Z^{p\bullet} \to I_A^{p\bullet} \to I_B^{p+1,\bullet} \to 0,$$

is an exact sequence of complexes that sits over the previous exact sequence. Note that if $A^p = 0$, then the last remarks above show that $I_A^{p\bullet}$ is the zero complex.

Define a double complex $I^{\bullet\bullet}$ by putting $I^{p\bullet} = I_A^{p\bullet}$, but with $d_v$ defined as the vertical differential of $I_A^{p\bullet}$, rescaled by a factor of $(-1)^p$ (so that we have an anti-commutative relation rather than a commutative one). Note that $d_v^2 = 0$ since $I_A^{p\bullet}$ is a complex. The horizontal differential $d_h$ is defined as the composition:

$$I_A^{p\bullet} \xrightarrow{d_h} I_B^{p+1,\bullet} \hookrightarrow I_Z^{p+1,\bullet} \hookrightarrow I_A^{p+1,\bullet}.$$

From the exactness of the two resolutions sequences above one deduces that $d_h^2 = d_h \circ d_v + d_v \circ d_h = 0$. We leave the verification of the second axiom of a Cartan-Eilenberg resolution to the reader. □

Let $I^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of a complex $A^\bullet \in C(\mathcal{A})$. The augmentation morphisms $A^p \to I^{0p}$ form a chain map $A^\bullet \to I^{0\bullet}$ which in turn defines a natural chain map:

$$\varepsilon\colon A^\bullet \to I^{0\bullet} \hookrightarrow \mathrm{Tot}(I)^\bullet.$$

One can show that this is a quasi-isomorphism, meaning that it induces an isomorphism of cohomology:

$$H(\varepsilon)\colon H^\bullet(A) \simeq H^\bullet(\mathrm{Tot}(I)).$$

This observation lies at the heart of a second definition of hypercohomology.

Any given morphism of chain complexes $f\colon A^\bullet \to B^\bullet$ extends to a map of double complexes between the chosen Cartan-Eilenberg resolutions. Moreover, one can define the notion of chain homotopy between two maps of double complexes in such a way that homotopic maps of complexes extend to homotopic maps of the chosen Cartan-Eilenberg resolutions:

**Definition 1.3.** Let $f, g\colon E^{\bullet\bullet} \to F^{\bullet\bullet}$ denote two maps between double complexes. A **chain homotopy** between $f$ and $g$ consists of maps $c_h^{pq}\colon E^{pq} \to F^{p+1,q}$ and $c_v^{pq}\colon E^{pq} \to E^{p,q+1}$ such that

$$g - f = (d_h c_h + c_h d_h) + (d_v c_v + c_v d_v).$$

A map $f\colon E^{\bullet\bullet} \to F^{\bullet\bullet}$ is said to be a **chain homotopy equivalence** if there exists a map $g\colon F^{\bullet\bullet} \to E^{\bullet\bullet}$ such that $fg$ and $gf$ are both chain homotopic to the respective identity maps.

As in homology, one can show that chain homotopic maps of double complexes $f, g\colon E^{\bullet\bullet} \to F^{\bullet\bullet}$ induce the same map in cohomology:

$$H(f) = H(g)\colon H^\bullet(\mathrm{Tot}(E)) \to H^\bullet(\mathrm{Tot}(F)).$$

It turns out that any two Cartan-Eilenberg resolutions of a complex are chain homotopy equivalent (the identity map of the complex lifts to a chain homotopy equivalence between the resolutions). These observations imply that the following is well-defined:

**Definition 1.4.** Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F\colon \mathcal{A} \to \mathbf{Ab}$ be a left exact additive functor. Given a complex $A^\bullet \in C(\mathcal{A})$, let $I^{\bullet\bullet}$ be a Cartan-Eilenberg resolution and put:

$$(\mathbb{R}^n F)(A^\bullet) = H^n(\mathrm{Tot}(F(I))).$$

Then $\mathbb{R}^n F\colon C(\mathcal{A}) \to \mathbf{Ab}$ is a functor, called the $n$-th **right hyperderived functor** of $F$.

Recall the embedding $\mathcal{A} \hookrightarrow C(\mathcal{A})$. A Cartan-Eilenberg resolution of $A \in \mathcal{A}$ is just an injective resolution, and one easily sees that the definition above yields the usual derived functor cohomology of $F$. So the notion of a hyperderived functor really is a generalisation of the notion of derived functor. One can show that $\mathbb{R}^n F$ is the $n$-th derived functor of the left exact functor $H^0 F = H^0 \circ F$, where $F$ is extended naturally to all of $C(\mathcal{A})$. Thus, the long exact sequence of cohomology yields a long exact sequence for hypercohomology:

**Lemma 1.5.** *Let $\mathcal{A}$ and $F$ be as above. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence in $C(\mathcal{A})$. Then there is a long exact sequence of hypercohomology:*

$$0 \to \mathbb{R}^0F(A) \to \mathbb{R}^0F(B) \to \mathbb{R}^0F(C) \xrightarrow{\delta} \mathbb{R}^1F(A) \to \mathbb{R}^1F(B) \to \mathbb{R}^1F(C) \xrightarrow{\delta} \cdots .$$

## 1.2 Another approach

Let $A \in \mathcal{A}$. The data of an injective resolution $I^\bullet$ of $A$ is equivalent to the existence of a quasi-isomorphism of chains:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

Given a chain $A^\bullet \in \mathcal{A}$, one can show that there always exists a quasi-isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

with the $I^n$'s injective. This can be proved directly, but we instead appeal to the existence of a Cartan-Eilenberg resolution $J^{\bullet\bullet}$ for $A^\bullet$. Then by our work in the previous section, one can take:

$$I^n = \mathrm{Tot}(J)^n.$$

Given such a quasi-isomorphism, one simply puts $\mathbb{R}^n F(A) = H^n(F(I)^\bullet)$. Taking the particular quasi-isomorphism induced by a Cartan-Eilenberg resolution shows that this definition agrees with that from the previous section. This definition is closer to the derived category approach to hypercohomology.

# 2 Algebraic de Rham cohomology

## 2.1 Definition

We begin by recalling the definition of the de Rham complex of a scheme. For simplicity we assume that $X/R$ is a smooth separated scheme of finite type over a noetherian ring $R$. In our applications, $R$ will typically be a field, or a ring of Witt vectors.

Recall the definition of the sheaf of Kahler differentials $\Omega^1_{X/R}$ of $X$. For $i \geq 2$ put $\Omega^i_{X/R} = \Lambda^i \Omega^1_{X/R}$. The differential $d\colon \mathcal{O}_X \to \Omega^1_{X/R}$ induces maps:

$$d\colon \Omega^i_{X/R} \to \Omega^{i+1}_{X/R}$$

for each $i$. For instance, let $U \subset X$ be affine, say $U = \operatorname{Spec}(A)$. Elements of $\Omega^i_{X/R}(U)$ are of the form:

$$\sum_j f_j da_{1j} \wedge da_{2j} \wedge \cdots \wedge da_{ij}$$

where $f_k, a_{mn} \in A$ for all $k, m, n$. The differential $d$ is defined locally by putting:

$$d\left(\sum_j f_j da_{1j} \wedge \cdots \wedge da_{ij}\right) = \sum_j df_j \wedge da_{1j} \wedge \cdots \wedge da_{ij}.$$

The same proof as in the case of the usual de Rham complex shows that these local maps can be glued to give a well-defined map $d$ of $R$-modules, which satisfies the Leibniz rule. That $d^2 = 0$ follows since it holds locally. This defines the **de Rham complex** $\Omega^\bullet_{X/R}$ of $X/R$:

$$\Omega^\bullet_{X/R}\colon 0 \to \mathcal{O}_X \to \Omega^1_{X/R} \to \Omega^2_{X/R} \to \Omega^3_{X/R} \to \cdots$$

Let $\Gamma = \Gamma(X, -)$ be the global section functor for sheaves on $X$. The **de Rham cohomology groups** of $X/R$ are simply the hypercohomology of $\Gamma$ evaluated on the de Rham complex:

$$H^i_{dR}(X/R) = (\mathbb{R}^i\Gamma)(\Omega^\bullet_{X/R}).$$

Since the global sections of the sheaves in the de Rham complex are all $R$-modules, it follows that the de Rham cohomology groups are $R$-modules as well.

**Remark** (Relative de Rham cohomology)**.** More generally, let $f\colon X \to Y$ be a morphism of schemes. We can then compute the right hyperderived functors of $f_*$ on the relative de Rham complex:

$$\mathcal{H}^i_{\mathrm{dR}}(X/Y) \stackrel{\mathrm{def}}{=} \mathbb{R}^i f_*(\Omega^\bullet_{X/Y})$$

The sheaf that we obtain is called the *relative de Rham cohomology.* If $U \subseteq Y$ is an affine open, say $U = \operatorname{Spec} R$, then it is easy to see that:

$$\mathcal{H}^i_{\mathrm{dR}}(X/Y)(U) = H^i_{\mathrm{dR}}(X/R)$$

so that this construction really generalises the previously given one.

## 2.2 The Čech resolution

The derived functor approach to cohomology is often poorly suited for computations. Thus, for practical applications, other cohomology theories are frequently used. In this section we will define the Čech resolution of a complex of sheaves. It is a fact that, under our hypotheses, the Čech complex can be used in place of a Cartan-Eilenberg resolution to compute the hypercohomology groups.

Assume that $X$ is a separated noetherian scheme. Take an open affine covering $\{U_i\}_{i \in I}$ of $X$ and impose a well-ordering on $I$. For indices $i_1, \ldots, i_n \in I$ put:

$$U_{i_1,\ldots,i_n} = U_{i_1} \cap \cdots \cap U_{i_n}.$$

Given a complex of sheaves:

$$\mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \cdots$$

let:

$$E^{p,q} = \prod_{i_0 < \cdots < i_q} \mathcal{F}^p(U_{i_0,\ldots,i_q}) = \prod_{i_0 < \cdots < i_q} \Gamma\left(U_{i_0,\ldots,i_q}, \mathcal{F}^p\right)$$

The differentials of the sheaf complex induce obvious maps:

$$d\colon E^{p,q} \to E^{p+1,q}.$$

These satisfy $d^2 = 0$ since $\mathcal{F}^\bullet$ is a complex. The classical Čech construction yields vertical differentials:

$$\delta\colon E^{p,q} \to E^{p,q+1}$$

where for $s \in \prod_{i_0 < \cdots < i_q} \mathcal{F}^p(U_{i_0,\ldots,i_q})$, the $i_0 < \cdots < i_{q+1}$ component of $\delta(s)$ is defined via the formula:

$$\delta(s)_{i_0 < \cdots < i_{q+1}} = \sum_{k=0}^{q+1} (-1)^{k+p} s_{i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_{q+1}}.$$

Then $\delta^2 = 0$, and one can easily check that $d \circ \delta + \delta \circ d = 0$. Thus $E^{\bullet\bullet}$ forms a double complex with the $d$'s as horizontal differentials and the $\delta$'s as vertical differentials. Analogously to classical sheaf cohomology, one has the following important comparison theorem:

**Theorem 2.1.** *Let $X$ be a noetherian separated scheme, and let $\mathcal{F}^\bullet$ be a complex of coherent sheaves. Then the sheaf hypercohomology of $\mathcal{F}^\bullet$ is isomorphic to the cohomology of the total complex of $E^{\bullet\bullet}$:*

$$(\mathbb{R}^i\Gamma)(\mathcal{F}^\bullet) \simeq H^i(\mathrm{Tot}(E)^\bullet).$$

*In particular, if $X/R$ is as above, the de Rham cohomology can be computed via the Čech resolution:*

$$H^i_{dR}(X/R) \simeq H^i(\mathrm{Tot}(E)^\bullet).$$

*where here $E^{\bullet\bullet}$ is obtained from the de Rham complex.*

For later computations, we describe the first two de Rham cohomology groups explicitely using the Čech resolution. Our initial work with hypercohomology shows that $H^0_{dR}(X/R)$ is just the kernel of the map between sheaf cohomology groups:

$$H^0_{dR}(X/R) = \ker(d\colon \mathcal{O}_X(X) \to \Omega^1_X(X)).$$

To see this via the Čech complex, we must consider the diagram:

$$\begin{array}{ccc} E^{01} & & \\ \uparrow{\scriptstyle\delta} & & \\ E^{00} & \xrightarrow{d} & E^{10} \end{array}$$

since $H^0_{dR}(X/R) = \ker(d+\delta) = \ker(d) \cap \ker(\delta)$. An element of $E^{00} = \prod_{i\in I} \mathcal{O}_X(U_i)$, where $\{U_i\}_{i\in I}$ is an affine covering of $X$, is in the kernel of $\delta$ if and only if it comes from a global section of $\mathcal{O}_X$. This follows from the sheaf axiom for $\mathcal{O}_X$. Under the identification $\ker\delta = \mathcal{O}_X(X)$, restricting $d$ to $\ker\delta$ gives the map $d\colon \mathcal{O}_X(X) \to \Omega^1_X(X)$. We thus see again that $H^0_{dR}(X/R) = \ker(d\colon \mathcal{O}_X(X) \to \Omega^1_X(X))$.

A similar analysis allows one to express $H^1_{dR}(X/R)$ as $Z^1/B^1$, where $Z^1$ and $B^1$ are the 1-**hypercocycles** and 1-**hypercoboundaries**, respectively:

$$\begin{aligned} Z^1 &= \Bigg\{ (\omega_i, f_{ij}) \in \Big(\prod_i \Omega^1_X(U_i)\Big) \times \Big(\prod_{i<j} \mathcal{O}_X(U_{ij})\Big) \mid d\omega_i = 0,\ \omega_i|_{U_{ij}} - \omega_j|_{U_{ij}} = df_{ij}, \\ &\qquad \text{and the following cocycle condition holds: } f_{ij}|_{U_{ijk}} - f_{ik}|_{U_{ijk}} + f_{jk}|_{U_{ijk}} = 0 \Bigg\}, \\ B^1 &= \Bigg\{ \big(dx_i, x_i|_{U_{ij}} - x_j|_{U_{ij}}\big) \in \Big(\prod_i \Omega^1_X(U_i)\Big) \times \Big(\prod_{i<j} \mathcal{O}_X(U_{ij})\Big) \mid (x_i) \in \prod_i \mathcal{O}_X(U_i) \Bigg\}. \end{aligned}$$

## 2.3 Hodge filtration for a curve

Let now $C/R$ be a curve over a noetherian ring $R$; this means that $C$ is smooth, connected, integral, proper and of relative dimension 1 over $R$. For curves it is always possible to find a covering by two open affines, say $\{U, V\}$. The Čech complex associated to such a covering is considerably simplified, as $\Omega^2_{C/R} = 0$ and there is only one intersection $U \cap V$ to consider. Thus $E^{pq} = 0$ whenever $p > 1$ or $q > 1$. We immediately deduce that $H^i_{dR}(C/R) = 0$ whenever $i > 2$. Note that even when $C$ is of relative dimension 1 over $R$, it is not true that $H^2_{dR}(C/R) = 0$. This phenomenon should not come as a surprise, however, if one is familiar with the analytic de Rham cohomology of a Riemann surface.

The hypercocycles and hypercoboundaries are much simplified in the case of a curve:

$$\begin{aligned} Z^1 &= \{(\omega_U, \omega_V, f) \in \Omega^1_{C/R}(U) \times \Omega^1_{C/R}(V) \times \mathcal{O}_C(U\cap V) \mid \omega_U - \omega_V = df \text{ on } U \cap V\}, \\ B^1 &= \{(dx_U, dx_V, x_U|_{U\cap V} - x_V|_{U\cap V}) \in \Omega^1_{C/R}(U) \times \Omega^1_{C/R}(V) \times \mathcal{O}_C(U\cap V) \mid x_U \in \mathcal{O}_C(U),\ x_V \in \mathcal{O}_C(V \end{aligned}$$

There is a natural map:

$$H^0(C, \Omega^1_{C/R}) \to H^1_{dR}(C/R)$$

which takes a global section $\omega \in \Omega^1_{C/R}(C)$ to the (class of) the triple $(\omega|_U, \omega|_V, 0)$ in $H^1_{dR}(C/R)$. If this is a 1-hypercoboundary, say $(\omega|_U, \omega|_V, 0) = (dx_U, dx_V, x_U - x_V)$, then $x_U$ and $x_V$ agree on $U \cap V$ and so can be glued to give a global section $x \in \mathcal{O}_C(C)$. However, since $C/R$ is connected, $\mathcal{O}_C(C) \simeq R$. We see that $\omega = dx = 0$, which shows that the map above is injective.

The cokernel of this map can be naturally identified with the sheaf cohomology group $H^1(C, \mathcal{O}_C)$. In order to do this, we first give an explicit description of $H^1(C, \mathcal{O}_C)$ using Čech cohomology relative to the cover $\{U, V\}$ of $C$. In this case the Čech complex takes the form:

$$0 \to \mathcal{O}_C(U) \times \mathcal{O}_C(V) \to \mathcal{O}_C(U \cap V) \to 0,$$

where the nontrivial map takes $(f, g) \mapsto f|_{U\cap V} - g|_{U\cap V}$. Thus:

$$H^1(C, \mathcal{O}_C) \simeq \frac{\mathcal{O}_C(U \cap V)}{\{f|_{U\cap V} - g|_{U\cap V} \mid f \in \mathcal{O}_C(U),\ g \in \mathcal{O}_C(V)\}}.$$

There is hence an obvious map $Z^1 \to H^1(C, \mathcal{O}_C)$ taking $(\omega_U, \omega_V, f)$ to the class of $f \in \mathcal{O}_C(U \cap V)$. The explicit descriptions of $H^1(C, \mathcal{O}_C)$ and $B^1$ show that this map vanishes on $B^1$ and induces a map:

$$H^1_{dR}(C/R) \to H^1(C, \mathcal{O}_C).$$

In order to show that this map is surjective, we will in fact show that the map:

$$Z^1 \to \mathcal{O}_C(U \cap V)$$

is surjective. We must show that if $f$ is a regular function on $U \cap V$, then $df = \omega_U - \omega_V$ where $\omega_U$ is regular on $U$ and $\omega_V$ is regular on $V$. In order to simplify the combinatorics of this problem, we will suppose that $U = C - \{P\}$ and $V = C - \{Q\}$. Removing a point from a curve yields an affine scheme, and so this assumption does not actually limit us. We will apply the Riemann-Roch theorem to solve this problem.

Recall that a (Weil) divisor on a curve is a finite formal sum of closed points of the curve. If $D$ is a divisor, then $\deg(D)$ is the sum of the coefficients of $D$. Functions and differentials define divisors via their orders at each point. Thus, if $f$ is an element of the function field $K(C)$ of a curve $C$, then we write:

$$\mathrm{Div}(f) = \sum_{P \in C} \mathrm{ord}_P(f) P.$$

Such a divisor is called **principal**. Similarly for differentials. Since curves are one dimensional and, for us, smooth, the sheaf of regular differentials $\Omega^1_{C/R}$ is a line bundle. This means that it is a locally free $\mathcal{O}_C$-module of rank 1. There exists a global regular differential $\omega$ on $C$. All other regular differentials are of the form $\omega' = f\omega$ for some $f$ in the function field of $C$ (note that not all $f$'s will yield a regular differential!). Thus $\mathrm{Div}(\omega')$ differs from $\mathrm{Div}(\omega)$ by a principal divisor (namely $\mathrm{Div}(f)$).

Given a divisor $D$ on $C$, we let:

$$\mathcal{L}(D) = \{f \in K(C) \mid \mathrm{Div}(f) + D \geq 0\} \cup \{0\}.$$

The inequality means that each coefficient of the divisor $\mathrm{Div}(f) + D$ is non-negative. Multiplication by $g \in K(C)$ induces an isomorphism:

$$\mathcal{L}(D) \simeq \mathcal{L}\left(D + \mathrm{Div}(g)\right).$$

Thus if we write $K = \mathrm{Div}(\omega)$, then $\mathcal{L}(K)$ is independent of the choice of regular differential $\omega$.

Scaling a function by an element of $R$ does not change the order of the function; there is hence an action of $R$ on $\mathcal{L}(D)$. In fact, $\mathcal{L}(D)$ is an $R$-module. It is a fact (?) that $\mathcal{L}(D)$ is a finitely generated projective module of finite rank. We will be working over nice rings (fields or PIDs), and so $\mathcal{L}(D)$ is a free $R$-module of finite rank in this case. Let $l(D)$ denote the $R$-rank of $\mathcal{L}(D)$. The Riemann-Roch theorem implies that:

$$l(D) - l(K - D) = \deg(D) + 1 - g,$$

where $g$ is the genus of $C$.

We return to our problem; so $U = C - \{P\}$, $V = C - \{Q\}$ and $f \in K(C)$ is regular on $U \cap V$. We must find differentials $\omega_U$ and $\omega_V$ regular on $U$ and $V$, resp., such that $df = \omega_U - \omega_V$ on $U \cap V$. If $C$ is of genus zero, then it has functions with a single pole of order 1 at any specified point, and regular otherwise (apply Riemann-Roch). One can hence take $\omega_U = d(f - g)$ and $\omega_V = d(g)$ for some appropriately chosen function $g$. We may thus suppose that $g \geq 1$.

Consider the divisor $D = K + nP$, where $K$ is the canonical divisor of $C$ associated to some regular differential $\omega$. The Riemann-Roch theorem gives:

$$l(K + nP) - l(-nP) = \deg(D) + 1 - g.$$

Now, if $\deg(D) = 0$ then $l(D) = 0$. Also, the Riemann-Roch theorem shows that $\deg(K) = 2g - 2$. The above becomes:

$$l(K + nP) = 2g - 2 + n + 1 - g = n + g - 1.$$

Thus, since $g \geq 1$, $l(K + nP)$ is positive. As $n$ increases, $l(K + nP)$ increases by 1 at each step. Taking $n$ to be the order of the pole of $df$ at $Q$, it follows that there exists a function $g \in K(C)$ such that $df + g\omega$ is regular on $U$ and $g\omega$ is regular on $V$. We may hence take $\omega_U = df + g\omega$ and $\omega_V = g\omega$ and:

$$(\omega_U, \omega_V, f) \mapsto f.$$

This shows that the natural map $H^1_{dR}(C/R) \to H^1(C, \mathcal{O}_C)$ is surjective.

The resulting sequence:

$$0 \to H^0(C, \Omega^1_{C/R}) \to H^1_{dR}(C/R) \to H^1(C, \mathcal{O}_C) \to 0$$

is exact. It is called the **Hodge filtration** for $C/R$. We have shown that the first map is injective, and that the second is surjective. From the explicit descriptions given for these groups, one immediately sees that the composition of the maps is zero. To conclude the proof of exactness, let us take an element in the kernel of the second map. It can be represented by a 1-hypercocycle of the form $(\omega_U, \omega_V, f_U - f_V)$, where $f_U$ is regular on $U$ and $f_V$ is regular on $V$. This hypercocycle differs from $(\omega_U - d(f_U), \omega_V - d(f_V), 0)$ by a 1-hypercoboundary. Now simply note that on $U \cap V$:

$$\omega_U - d(f_U) - (\omega_V - d(f_V)) = \omega_U - \omega_V - d(f_U - f_V) = 0.$$

By the sheaf axiom, there is thus a global regular differential $\omega$ that restricts to $\omega_U + d(f_U)$ on $U$ and $\omega_V + d(f_V)$ on $V$. This $\omega$ maps to the class of $(\omega_U, \omega_V, f_U - f_V)$ in $H^1_{dR}(C/R)$, which concludes the proof of the exactness of the Hodge filtration.

By Serre duality, the outer terms of the Hodge filtration are dual. The $R$-rank of $H^0(C, \Omega^1_{C/R})$ is $g$, the genus of $C/R$. Hence $H^1_{dR}(C/R)$ is a free $R$-module of rank $2g$.

# 3 The Poincaré Pairing

Let $C/k$ be a smooth proper curve. We assume, for simplicity, that $k$ is algebraically closed (otherwise, just minor modifications need to be done). We will define a perfect, alternating pairing:

$$\langle \cdot, \cdot \rangle_{\text{Poinc}} \colon H^1_{\text{dR}}(C/k) \times H^1_{\text{dR}}(C/k) \to k$$

Let $K = k(C)$ be the function field of $C$, and let $\Omega_K \stackrel{\text{def}}{=} \Omega^1_{K/k}$ be the $K$-vectorspace of *global meromorphic* differentials on $C$. Consider also the sheaves $\mathcal{O}_C$ and $\Omega^1_{C/k}$.

For each closed point $P \in C(k)$, we have a discrete valuation:

$$\text{ord}_P \colon K^\times \to \mathbb{Z}$$

Let $K_P$ be the completion of $K$ at $\text{ord}_P$, and $\mathcal{O}_P$ be the completion of $\mathcal{O}_{C,P}$ (the stalk at $P$, which is seen as a subring of $K$). Define also $\Omega_{K_P} \stackrel{\text{def}}{=} \Omega^1_{K_P/k}$, and $\Omega_P$ the completion of the stalk $\Omega_{C,P}$.

**Example.** *If $t$ is a uniformizer at $P$ (that is, $\text{ord}_P(t) = 1$), then:*

$$K \simeq k(t) = \left\{ \frac{p(t)}{q(t)} \mid p(t), q(t) \in k[t],\ q(t) \neq 0 \right\} \qquad \Omega_K \simeq k(t) \cdot dt$$

$$\mathcal{O}_{C,P} \simeq k[t]_{(t)} = \left\{ \frac{p(t)}{q(t)} \in k(t) \mid q(0) \neq 0 \right\} \subseteq K$$

$$\mathcal{O}_P \simeq k[\![t]\!] \qquad K_P \simeq k((t))$$

$$\Omega_P \simeq \mathcal{O}_P \cdot dt \qquad \Omega_{K_P} \simeq K_P \cdot dt$$

We have also, for each $P \in C(k)$, the *residue map*:

$$\operatorname{res}_P \colon \Omega_K \to k$$

defined as follows: if $\omega \in \Omega_K$, let $\omega_P$ be its image in $\Omega_{K_P}(= k((t)) \cdot dt)$. Then $\operatorname{res}_P(\omega) \stackrel{\text{def}}{=} \operatorname{res}_P(\omega_P) \stackrel{\text{def}}{=} a_{-1}$ (if $\omega = (\frac{a_{-n}}{t^n} + \cdots + \frac{a_{-1}}{t} + a_0 + a_1 + \cdots)dt$). One needs to check that this is actually well defined (independent of the choice of uniformizer $t$), and we ommit the tedious proof, specially when in positive characteristic (see [Har77], III.7, pg. 247ff for details, and [Ser59] for a proof).

Denote by $\Omega_K^{\text{II}}$ the $k$-vectorspace of meromorphic differential forms of the second kind:

$$\Omega_K^{\text{II}} \stackrel{\text{def}}{=} \{\omega \in \Omega_K \mid \operatorname{res}_P \omega = 0 \,\forall P \in C(k)\}$$

**Remark.** If $\omega \in \Omega_K^{\text{II}}$, then we can locally integrate it: for each $P \in C(k)$, there exists some $\gamma_P \in K_P(= k((t))\,)$ such that $d\gamma_P = \omega_P$.

In order to define the Poincaré pairing on $H^1_{\text{dR}}(C/k)$, we need to represent its elements as differentials of the second kind:

**Proposition 3.1.** *The deRham cohomology of $C$ can be computed as:*

$$H^1_{dR}(C/k) \simeq \frac{\Omega_K^{II}}{dK}$$

*Proof.* We work, as usual, with an open affine covering given by the two open sets

$$U \stackrel{\text{def}}{=} C \setminus \{P\}$$
$$V \stackrel{\text{def}}{=} C \setminus \{Q\}$$

First, we define a map $\varphi$ from the de Rham cohomology $H^1_{\text{dR}}(C/k)$ to $\Omega_K^{\text{II}}$. To a 1-hypercocycle $(\omega_U, \omega_V, f_{UV})$, the map $\phi$ associates $\omega_U$, thought now as a *meromorphic* differential, which is actually regular $U$. It is of the second kind, as around the point $P$ it can be written as $\omega_V + df_{UV}$ ($\omega_V$ is regular at $P$, and an exact differential has no residue). Note that if instead we had chosen the second component $\omega_V$, its image in $\frac{\Omega_K^{\text{II}}}{dK}$ wouldn't change, as they differ by an element in $dK$.

For the well-definedness of the map, just note that hypercoboundaries are mapped to exact forms (that is, in $dK$).

Let $(\omega_U, \omega_V, f_{UV})$ be a hypercocycle representing a class in $H^1_{\text{dR}}(C/k)$, and suppose that $\omega_U \in dK$. That is $\omega_U = dg$, for some $g \in K$, and note that then $g$ is regular on $U$. We also have the equality:

$$\omega_V = \omega_U - df_{UV} = d(g - f_{UV}) \quad \text{on } U \cap V$$

As $U \cap V$ is Zariski-dense in $V$, this equality actually holds on $V$, and hence the hypercocycle we started is actually a hypercoboundary. This implies that $\varphi$ is injective.

As for surjectivity, let $\omega \in \Omega^{\text{II}}_K$ be given. We want to find a triple $(\omega_U, \omega_V, f_{UV})$ such that $\omega_U$ differs from $\omega$ by an exact differential. We will of course use Riemann-Roch to do that. First, we want to modify $\omega$ by an exact differential so that the new form has only poles at $P$ and $Q$. Suppose that $\omega$ has a pole of order $n+1 \geq 2$ at some point $R$. Then there exists a function $f \in K(C)$ such that:

$$f \in \mathcal{L}\left(nR + mP\right) \setminus \mathcal{L}\left((n-1)R + mP\right)$$

for some $m$ large enough (by Riemann-Roch). So by adding an appropriate multiple of $df$ to $\omega$, we can reduce the order of the pole at $R$. Iterating this procedure a finite number of times, we can assume that $\omega$ is regular on $U \cap V$. A similar procedure allows us to find $g$ and $h$, both of them with poles only on $P$ and $Q$, such that $\omega + dg$ is regular on $U$, and $\omega + dh$ is regular on $V$. We thus obtain the hypercocycle:

$$(\omega + dg, \omega + dh, g - h)$$

mapping to the given $\omega$. □

**Remark.** If $C = E$ is an elliptic curve, we can consider the sheaf $\Omega^1(2\infty) \stackrel{\text{def}}{=} \Omega^1 \otimes \mathcal{L}(2\infty)$. This is the sheaf of forms which are regular outside $\infty$, and have a pole of order at most 2 at $\infty$. By the residue theorem, such a form is of the second kind, and so we get a canonical inclusion:

$$\Omega^1(2\infty) \hookrightarrow \Omega^{\text{II}}_K$$

One can easily check that the composition of the previous inclusion with the projection to the quotient induces an isomorphism:

$$\Omega^1(2\infty) \simeq \Omega^{\text{II}}_K / dK$$

and so for elliptic curves, we have an even more concrete description of the first de Rham cohomology.

Representing the elements of the deRham cohomology through differentials of the second kind, as in the previous proposition, we define $\langle \cdot, \cdot \rangle_{\text{Poinc}}$ as follows: given $[\alpha], [\beta] \in H^1_{\text{dR}}(C/k)$, with $\alpha, \beta \in \Omega^{\text{II}}_K$, for each $P \in C(k)$, let $\alpha_P = d\gamma_P$ for some $\gamma_P \in K_P$. Then:

$$\langle [\alpha], [\beta] \rangle_{\text{Poinc}} \stackrel{\text{def}}{=} \sum_{P \in C(k)} \text{res}_P(\gamma_P \beta_P) \quad \in k$$

**Proposition 3.2.** *The pairing $\langle \cdot, \cdot \rangle_{Poinc}$ is well-defined, alternating, and non-degenerate.*

*Proof.* We prove that it is alternating. Well-definedness is easy, and we will ommit the proof of non-degeneracy.

Let, for each $P \in C(k)$, write $\beta_P = d\eta_P$, for some $\eta_P \in K_P$. Then, by definition,

$$\langle [\beta], [\alpha] \rangle_{\text{Poinc}} + \langle [\alpha], [\beta] \rangle_{\text{Poinc}} = \sum_{P \in C(k)} \text{res}_P(\gamma_P \beta_P) + \text{res}_P(\eta_P \alpha_P) = \text{res}_P(\gamma_P \beta_P + \eta_P \alpha_P)$$

But note that, if $U$ is a neighborhood of $P$ to which both $\gamma_P$ and $\eta_P$ can be extended, then:

$$(d(\gamma\eta))_P = \alpha_P\eta_P + \gamma_P\beta_P$$

and so the residue at $P$ will be 0. Hence the sum of all the residues will be 0, too, as we wanted to show. □

**Example.** *Let $E/k$ be an elliptic curve over a field $k$,* $\operatorname{char} k = 0$*. Consider the standard equation for $E$:*

$$E: \quad y^2 = 4x^3 - g_2x - g_3$$

*where $x, y$ are chosen in such a way that, if $T$ is a parameter at $\infty$, then:*

$$x_\infty = \frac{1}{T^2} + a_0 + a_1T + \cdots$$
$$y_\infty = \frac{2}{T^3} + b_0 + b_1T + \cdots$$

*(note that the 2 in the expression for $y_\infty$ is there so that we get the 4 in the equation for $E$).*

*Then $\omega = dx/y, \eta = xdx/y$, and we will compute the Poincaré pairing. As it is alternating, $\langle\omega,\omega\rangle_{Poinc} = \langle\eta,\eta\rangle_{Poinc} = 0$, and so it is enough to calculate $\langle\eta,\omega\rangle_{Poinc}$:*

$$\langle\eta,\omega\rangle_{Poinc} = \sum_{P\in E(k)} \operatorname{res}_P(\gamma_P\omega_P) = \operatorname{res}_\infty(\gamma_\infty\omega_\infty)$$

*We compute the expansions of $\omega$ and $\eta$ at $\infty$:*

$$\omega_\infty = \frac{\frac{-2}{T^3} + a_1 + \cdots}{\frac{2}{T^3} + b_0 + b_1T + \cdots} = -1 + c_0 + c_1T + \cdots$$
$$\eta_\infty = (\frac{1}{T^2} + a_0 + a_1T + \cdots)(-1 + c_0 + c_1T + \cdots) = \frac{-1}{T_2} + d_0 + d_1T + \cdots$$
$$\gamma_\infty = \frac{1}{T} + d_0T + \frac{d_1}{2}T^2 + \cdots$$
$$\omega_\infty \cdot \gamma_\infty = \frac{-1}{T} + c_1 + \cdots$$

*So that* $\operatorname{res}_\infty(\gamma_\infty\omega_\infty) = -1$*, and hence we have found that $\langle\eta,\omega\rangle_{Poinc} = -1$ (which in turn implies that $\langle\omega,\eta\rangle_{Poinc} = 1$).*

# 4 The Gauss-Manin Connection

We let $k$ be a field, and $S/k$ a smooth curve of finite type. Let $f\colon X \to S$ a smooth, proper morphism. The goal is to define a connection on the sheaf $\mathcal{F} = \mathcal{H}^i_{\mathrm{dR}}(X/S)(\overset{\text{def}}{=} \mathbb{R}^if_*(\Omega^\bullet_{X/S}))$, following quite closely [KO68].

If all the involved schemes were affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$, then we would get one of the fundamental exact sequences of $B$-modules (see [Har77], chapter II):

$$\Omega^1_{R/k} \otimes_R B \longrightarrow \Omega^1_{B/k} \longrightarrow \Omega^1_{B/R} \longrightarrow 0$$

In the global situation, we get as well the fundamental exact sequence, where we recall that $f^*\Omega^1_{S/k} \stackrel{\mathrm{def}}{=} f^{-1}\Omega^1_{S/k} \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X$:

$$f^*\Omega^1_{S/k} \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0$$

Thanks to $X \to S$ being smooth, the first map is actually injective (so we get an extra 0 on the left). On the affine level, this can be seen because in that case $\Omega^1_{B/R}$ is $B$-flat, so $\mathrm{Tor}_1(\Omega^1_{B/R}, B) = 0$. The terms in this exact sequence are locally-free, and so we get a canonical filtration of the complex $\Omega^\bullet_{X/k}$ :

$$\Omega^\bullet_{X/k} = F^0(\Omega^\bullet_{X/k}) \supseteq F^1(\Omega^\bullet_{X/k}) \supseteq F^2(\Omega^\bullet_{X/k}) \supseteq \cdots$$

with

$$F^i = F^i(\Omega^\bullet_{X/k}) = \mathrm{img}[\Omega^{\bullet-i}_{X/k} \otimes_{\mathcal{O}_X} f^*\Omega^i_{S/k} \to \Omega^\bullet_{X/k}]$$

and such that:

$$\mathrm{gr}^i = \mathrm{gr}^i(\Omega^\bullet_{X/k}) \stackrel{\mathrm{def}}{=} F^i/F^{i+1} = f^*\Omega^i_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\bullet-i}_{X/S}$$

(see [Har77], Exercise $II.5.16d$). We only need the first two terms in this filtration. There is then an obvious exact sequence of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{gr}^1 & \longrightarrow & F^0/F^2 & \longrightarrow & \mathrm{gr}^0 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & f^*\Omega^1_{S/k} \otimes_{\mathcal{O}_X} \Omega^{(\bullet-1)}_{X/S} & \longrightarrow & F^0/F^2 & \longrightarrow & \Omega^\bullet_{X/S} & \longrightarrow & 0 \end{array}$$

which yields a long exact sequence in hypercohomology (that is, taking the hyperderived functors of $f_*$). In particular.

It is enough to define the Gauss-Manin connection on arbitrarily small open affine subsets of $S$, as the deRham sheaf $\mathcal{H}^i_{\mathrm{dR}}(X/S)$ is the sheaf associated to the presheaf:

$$U \mapsto \mathbb{H}^i(U, \Omega^\bullet_{X/S}|_U) \stackrel{\mathrm{def}}{=} \left(\mathbb{R}^i f_*(\Omega^\bullet_{X/S})\right)(U)$$

So from now on we will assume that $S$ is affine, and write $\mathbb{H}^i(\Omega^\bullet_{X/S})$ instead of $\mathbb{H}^i(S, \Omega^\bullet_{X/S})$. The long exact sequence in hypercohomology yields boundary maps:

$$\mathcal{H}^i_{\mathrm{dR}}(X/S) = \mathbb{H}^i(\Omega^\bullet_{X/S}) \xrightarrow{\delta} \mathbb{H}^{i+1}\left(f^{-1}\Omega^1_{S/k} \otimes_{f^{-1}\mathcal{O}_S} \Omega^{\bullet-1}_{X/S}\right)$$

Rs $f^{-1}\Omega^1_{S/k}$ is locally free and the differential of the complex is $f^{-1}\mathcal{O}_S$-linear, the term on the right is isomorphic to:

$$\Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathbb{H}^{i+1}(\Omega^{\bullet-1}_{X/S}) = \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathbb{H}^i(\Omega^\bullet_{X/S}) = \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^i_{\mathrm{dR}}(X/S)$$

and so the connecting homomorphism can be seen as a morphism:

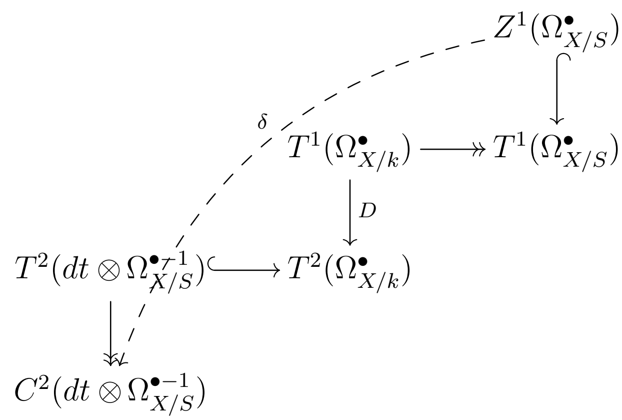
$$\nabla_i \colon \mathcal{H}^i_{\mathrm{dR}}(X/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^i_{\mathrm{dR}}(X/S)$$

**Definition 4.1.** The **Gauss-Manin connection** is $\nabla_i$.

**Example.** *We compute $\nabla_1 \colon \mathcal{H}^1_{dR}(X/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^1_{dR}(X/S)$. Let's assume that $S$ is affine, $S = \operatorname{Spec} R$. After localizing, we can even assume that $\Omega^1_{S/k} \simeq R \cdot dt$, for some $t \in R$.*

*Consider an affine cover of $X$, $X = \cup U_i$, with $U_i = \operatorname{Spec} B_i$.*

*Let $x \in \mathcal{H}^1_{dR}(X/S)$, represented by $((\omega_i)_{i \in I}, (f_{ij})_{i<j}) \in T^1\left(\Omega^\bullet_{X/S}\right)$, with $\omega_i \in \Omega^1_{B_i/R}$ and $f_{ij} \in B_{ij}$ satisfying the conditions for being a 1-hypercocycle (as we have worked out before). To compute $\nabla_1(x)$, we just need to follow the definition of the connecting homomorphism, induced by the morphism on the terms in the total complex (we denote by $Z$ the kernel of the differential on the corresponding complex):*




*That is, choose lifts $\overline{\omega_i}$ of $\omega_i$, apply $D$ to the element $((\overline{\omega_i})_i, (f_{ij})_{i<j})$, and we will be able to write:*

$$D\left((\overline{\omega}_i), (f_{ij})\right) = dt \otimes ((\eta_i), (g_{ij})_{i<j})$$

*and then*

$$\nabla_1(x) = dt \otimes [\ ((\eta_i), (g_{ij})_{i<j})\ ] \in \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^i_{dR}(X/S)$$

# 5 An Example: the Gauss-Manin Connection on the Legendre family

Let $k$ be a field. Let $S = \operatorname{Spec}(A)$, where $A$ is the ring $A \overset{\text{def}}{=} k[t][\frac{1}{t(t-1)}]$. Note that $S$ is a smooth curve over $k$, obtained from $\mathbb{P}^1$ by removing the points $\{0, 1, \infty\}$. Consider the family of elliptic curves $X$ over $S$ defined by (the projectivization of) the equation:

$$y^2 = P(x,t) = x(x-1)(x-t)$$

The ring $A$ has been defined so that $X$ is smooth over $S$. So $X$ is a smooth projective curve over $S$, which is known as the **Legendre family**. If we see $X$ as a scheme over $\operatorname{Spec} k$, then it is a smooth surface, what is called an elliptic surface. Another way of thinking of $X$ is as a *family* of elliptic curves: for each point $t_0$ of $S$ (which corresponds to an element $t_0 \in \overline{k}$, $t_0 \neq 0, 1$), we obtain $X_{t_0}$, which is an elliptic curve over $\overline{k}$, with equation $y^2 = P(x, t_0)$.

Fix the open cover $\mathcal{U} = \{U_0, U_1\}$, where:

$$U_0 \stackrel{\text{def}}{=} X \setminus \{(0,0)_{t_0} \mid t_0 \in S\}$$
$$U_1 \stackrel{\text{def}}{=} X \setminus \{\infty_{t_0} \mid t_0 \in S\}$$

We choose a basis $\{\omega, \eta\}$ for the relative deRham cohomology $\mathcal{H}^1_{\text{dR}}(X/S)$, represented by the Čhech hypercocycles $\underline{\omega}$ and $\underline{\eta}$. These are defined as:

$$\underline{\omega} = ((\omega_0, \omega_1), f_\omega) = \left(\left(\frac{dx}{y}, \frac{dx}{y}\right), 0\right) \tag{1}$$
$$\underline{\eta} = ((\eta_0, \eta_1), f_\eta) = \left(\left(\frac{tdx}{xy}, \frac{xdx}{y}\right), \frac{2y}{x}\right) \tag{2}$$

**Remark.** Note that

$$d\left(\frac{2y}{x}\right) = \left(\frac{2dy}{x} - \frac{2ydx}{x^2}\right) dx = \left(\frac{3x^2 - 2(t+1)x + t}{xy} - \frac{2ydx}{x^2}\right) dx = \frac{x^3 - tx}{x^2 y} dx = \eta_1 - \eta_0$$

so that $\underline{\eta}$ is effectively a 1-hypercocycle. Also note that $\eta_1$ has the same form as we used in a previous section. Note however that the equation that we are using now is not the one that we used to compute the Poincaré pairing, so that these forms are not exactly the same (they differ from them by a constant factor).

We will compute the matrix of the Gauss-Manin connection $\nabla$, in terms of the basis $\{\omega, \eta\}$ defined above.

## 5.1 Image of $\omega$

Recall that $\omega$ is represented by the 1-hypercocycle:

$$\underline{\omega} = ((\omega_0, \omega_1), f_\omega) = \left(\left(\frac{dx}{y}, \frac{dx}{y}\right), 0\right) \in E^{0,1} \oplus E^{1,0} = T^1\left(\mathcal{U}, \Omega^\bullet_{X/S}\right) = T^1_{X/S}$$

According to the definition of the connection, we lift this to an absolute hypercocycle (that is, in $T^1\left(\mathcal{U}, \Omega^\bullet_{X/k}\right)$). This operation doesn't change anything visually, so we don't write again the same expression.

Next, we take the $D$ operator, which acts as:

$$E^{0,1} \xrightarrow{d} E^{0,2}, \quad E^{0,1} \xrightarrow{-\check{d}} E^{1,1}, \quad E^{1,0} \xrightarrow{d} E^{1,1}, \quad E^{1,0} \xrightarrow{\check{d}} E^{2,0} = 0$$

Note that $E^{2,0} = 0$, because we are only considering a cover by two opens subsets, so there are no triple intersections. Also, our element on $E^{1,0}$ is 0, so the two right-most maps don't need to be computed. Also, as the two components on $E^{0,1}$ are equal, the map $-\check{d}$ has image 0 in our case.

Because of the previous argument, we just need to compute the left-most map. For that, we compute the total differential of the two one-forms. As they have the same expression, we just consider the 0-th component:

$$d(\omega_0) = d\left(\frac{1}{y}dx\right) = \frac{-1}{y^2}dy \wedge dx$$

This expression can be simplified by noting that the equation of the *surface* $X/k$ gives:

$$2ydy = P_x dx + P_t dt \implies 2ydy \wedge dx = P_t dt \wedge dx$$

(we write $P_x$ and $P_t$ for the corresponding partial derivatives of $P(x,t) = x(x-1)(x-t)$).

So using the previous equality we obtain:

$$d(\omega_0) = \frac{-P_t}{2y^3}dt \wedge dx$$

This is then the 2-hypercocycle that we get after applying $D$:

$$\left(\left(\frac{-P_t}{2y^3}dt \wedge dx, \frac{-P_t}{2y^3}dt \wedge dx\right), 0\right)$$

Notice that this effectively is a 1-hypercocycle wedged by $dt$, as we knew a priori. Hence we can already write the image of the Gauss-Manin connection:

$$\underline{\nabla(\omega)} = dt \otimes \left(\left(\frac{-P_t}{2y^3}dt \wedge dx, \frac{-P_t}{2y^3}dt \wedge dx\right), 0\right)$$

We need to express this hypercocycle in terms of the basis elements. It is not clear a priori how to change the representative we have got by a 1-coboundary so to get a visible combination, but the procedure has been worked out by Tedlaya and can be made into an algorith. We will manage to do it in this case. For this, let $A = A(x,t)$ and $B = B(x,t)$ be polynomials such that:

$$-P_t = AP + BP_x$$

(this is possible because $P$ doesn't have repeated roots, as we know). Then note that:

$$\frac{-P_t}{2y^3} = \frac{AP + BP_x}{2y^3} = \frac{A}{2y} + \frac{BP_x}{2y^3}$$

Next, note that:

$$d\left(\frac{B}{y}\right) = \frac{B_x}{y}dx + \frac{-B}{y^2}dy = \frac{B_x}{y}dx + \frac{-BP_x}{2y^3}dx \implies \frac{BP_x}{2y^3}dx \equiv \frac{B_x}{y}dx$$

and so:

$$\frac{-P_t}{2y^3} \equiv \frac{A + 2B_x}{2y}dx$$

Making the polynomials explicit, we see that $A$ and $B$ are:

$$A(x,t) = \frac{1}{t(t-1)}(-3x + 2 - t) \qquad\qquad B(x,t) = \frac{1}{t(t-1)}x(x-1)$$

so we get:

$$\left(\underline{\nabla(\omega)}\right)_0 = \frac{-P_t}{2y^3}dx \equiv \frac{1}{t(t-1)}\frac{x-t}{2y}dx = \frac{1}{2t(t-1)}\left(-t\omega_0 + \eta_0\right)$$

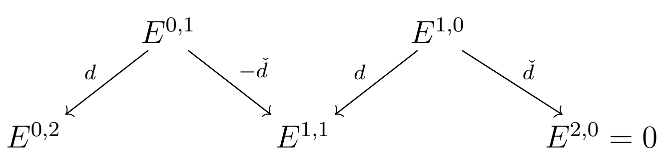
## 5.2 Image of $\eta$

Recall that $\eta$ is represented by the 1-hypercocycle:

$$\underline{\eta} = \left(\left(\frac{tdx}{xy}, \frac{xdx}{y}\right), \frac{2y}{x}\right) \in E^{0,1} \oplus E^{1,0} = T^1\left(\mathcal{U}, \Omega^\bullet_{X/S}\right) = T^1_{X/S}$$

According to the definition of the connection, we lift this to an absolute hypercocycle (that is, in $T^1(\mathcal{U}, \Omega^\bullet_{X/k})$). This operation doesn't change anything visually, so we don't write again the same expression.

Next, we take the $D$ operator, which acts as:




Again, $E^{2,0} = 0$, because we are only considering a cover by two opens subsets, so there are no triple intersections.

We start with the left-most map. For that, we need to compute the total differential of the two one-forms:

$$d(\eta_0) = d\left(\frac{t}{xy}dx\right) = \frac{1}{xy}dt \wedge dx + \frac{-t}{xy^2}dy \wedge dx$$

$$d(\eta_1) = d\left(x\frac{dx}{y}\right) = \frac{-x}{y^2}dy \wedge dx$$

We simplify this expressions as before, using the same equation:

$$2ydy \wedge dx = P_t dt \wedge dx$$

So using the previous equality we obtain:

$$d(\eta_0) = \frac{2y^2 - tP_t}{2xy^3} dt \wedge dx$$
$$d(\eta_1) = \frac{-xP_t}{2y^3} dt \wedge dx$$

These expressions will later be simplified, but for now let's keep up with the computation of $D$. It remains to do the middle maps. Note that:

$$df_\eta = \frac{2}{x} dy + \frac{2y}{x^2} dx$$

which simplifies to:

$$df_\eta = \frac{xP_x - 2y^2}{x^2 y} dx + \frac{P_t}{xy} dt$$

so that, if we compute:

$$\eta_0 - \eta_1 + df_\eta = \frac{P_t}{xy} dt$$

So finally we have obtained the 2-hypercocycle represented by:

$$\left( \left( \frac{2y^2 - tP_t}{2xy^3} dt \wedge dx, \frac{-xP_t}{2y^3} dt \wedge dx \right), \frac{P_t}{xy} dt \right)$$

Notice that this effectively is 1-hypercocycle wedged by $dt$, as we knew a priori. This is the image of the Gauss-Manin connection:

$$\underline{\nabla(\eta)} = dt \otimes \left( \left( \frac{2y^2 - tP_t}{2xy^3} dx, \frac{-xP_t}{2y^3} dx \right), \frac{P_t}{xy} \right)$$

We look now at the 0-th component.

$$\left( \underline{\nabla(\eta)} \right)_0 = \frac{2y^2 - tP_t}{2xy^3} dx = \frac{1}{xy} dx + \frac{-tP_t}{2xy^3} dx$$

For this, we repeat the same argument, but now we find $A, B$ such that:

$$-tP_t = AP + BP_x$$

and explicitly,

$$A(x,t) = \frac{1}{t-1}(-3x + 2 - t) \qquad B(x,t) = \frac{1}{t-1} x(x-1)$$

The same argument as before yields:

$$\frac{-tP_t}{2xy^3}dx = \frac{AP + BP_x}{2xy^3}dx = \frac{A}{2xy}dx + \frac{BP_x}{2xy^3}dx$$

and again, note that:

$$d\left(\frac{B}{xy}\right) = \frac{xB_x - B}{x^2y}dx + \frac{-BP_x}{2xy^3}dx \implies \frac{BP_x}{2xy^3}dx \equiv \frac{xB_x - B}{x^2y}dx$$

so we have:

$$\frac{-tP_t}{2xy^3}dx \equiv \frac{A + 2B_x}{2xy}dx - \frac{B}{x^2y}dx = \frac{1}{t-1}\frac{-x+2-t}{2xy}dx$$

Putting everything together, we get:

$$\left(\underline{\nabla(\eta)}\right)_0 = \left(\frac{1}{xy} + \frac{-tP_t}{2xy^3}\right)dx \equiv \left(\frac{1}{xy} + \frac{1}{t-1}\frac{-x+2-t}{2xy}\right)dx = \frac{1}{2(t-1)}\left(-\omega_0 + \eta_0\right)$$

.

Let $A = A(x,t)$ and $B = B(x,t)$ be polynomials such that:

$$-xP_t = AP + BP_x$$

(this is possible because $P$ doesn't have repeated roots, as we know). Then note that:

$$\frac{-xP_t}{2y^3} = \frac{AP + BP_x}{2y^3} = \frac{A}{2y} + \frac{BP_x}{2y^3}$$

Next, note that:

$$d\left(\frac{B}{y}\right) = \frac{B_x}{y}dx + \frac{-B}{y^2}dy = \frac{B_x}{y}dx + \frac{-BP_x}{2y^3}dx \implies \frac{BP_x}{2y^3}dx \equiv \frac{B_x}{y}dx$$

and so:

$$\frac{-xP_t}{2y^3} \equiv \frac{A + 2B_x}{2y}dx$$

Making this explicit, we see that $A$ and $B$ have the form:

$$A(x,t) = \frac{1}{t-1}(-3x+1) \qquad\qquad B(x,t) = \frac{1}{t-1}x(x-1)$$

so we get:

$$\left(\underline{\nabla(\eta)}\right)_1 = \frac{-xP_t}{2y^3}dx \equiv \frac{1}{t-1}\frac{x-1}{2y}dx = \frac{1}{2(t-1)}\left(-\omega_1 + \eta_1\right)$$

which agrees with the computation of the 0-th component.

## 5.3 Conclusion

After all the previous calculations, we conclude that the matrix of $\nabla$ with respect to the basis given by $\{\omega, \eta\}$ is:

$$\frac{1}{2t(t-1)}\begin{pmatrix} -t & -t \\ 1 & t \end{pmatrix}$$

# 6 Reinterpreting Modular Forms

## 6.1 Classical Modular Forms

Let $\Gamma = \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ be the classical modular group, and let $\mathcal{H}$ be the upper half-plane. The group $\Gamma$ acts on $\mathcal{H}$ on the left by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau \stackrel{\text{def}}{=} \frac{a\tau + b}{c\tau + d}$$

One can quotient out $\mathcal{H}$ by $\Gamma$, and then compactify (by adding one point) both $\mathcal{H}$ and its quotient by $\Gamma$. We call $Y(1) \stackrel{\text{def}}{=} \Gamma(1)\backslash\mathcal{H}$, and $X(1) \stackrel{\text{def}}{=} \overline{Y(1)} = \Gamma(1)\backslash\overline{\mathcal{H}}$. Note then that both $Y(1)$ and $X(1)$ are Riemann surfaces (and thus smooth algebraic curves), and that $X(1)$ is also compact (so a projective curve), while $Y(1)$ is an affine curve.

Given $f\colon \mathcal{H} \to \mathbb{C}$ a modular form of *even* weight $k$, define:

$$\omega_f \stackrel{\text{def}}{=} f(\tau)d\tau^{\otimes\frac{k}{2}} \in \left(\Omega^1_{\mathcal{H}/\mathbb{C}}\right)^{\otimes\frac{k}{2}}(\mathcal{H})$$

If $\gamma \in \Gamma$, then we can compute:

$$\gamma^*\omega_f = f(\gamma\tau)d(\gamma\tau)^{\otimes\frac{k}{2}} = (c\tau + d)^k f(\tau)\left(\frac{d}{d\tau}\frac{a\tau + b}{c\tau + d}\right)^{\frac{k}{2}}(d\tau)^{\otimes\frac{k}{2}} = \omega_f$$

and hence $\omega_f$ is $\Gamma$-invariant, so it can be seen as a differential on $Y(1)$:

$$\omega_f \in \left(\Omega^1_{Y(1)/\mathbb{C}}\right)^{\otimes\frac{k}{2}}(Y(1))$$

This module is the algebra of Kähler differentials on $Y(1)$ (recall that it's an affine curve!).

This is a nice geometric interpretation of modular forms of even weight. Unfortunately, one cannot make sense of modular forms of odd weight in such a simple-minded way. Thankfully, there is a nice geometric way to interpret modular forms of all weights. Moreover, this description makes it possible to define modular forms over rings other than $\mathbb{C}$.

We will reinterpret the classical case to motivate the upcoming definition of a geometric modular form.

### 6.1.1 Functions on lattices

Let $\mathcal{R}$ be the set of lattices in $\mathbb{C}$ (recall that a subgroup $L \subseteq \mathbb{C}$ is a lattice if it is a free $\mathbb{Z}$-module of rank 2, such that $L \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}$).

If $L \in \mathcal{R}$ then $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, with $\omega_1, \omega_2$ two $\mathbb{R}$-linearly independent complex numbers. Then $\mathbb{C}/L$ is a compact torus, and it becomes an elliptic curve over $\mathbb{C}$ by taking $\mathcal{O} \stackrel{\text{def}}{=} \overline{0} \in \mathbb{C}/L$ as the identity element. The group law on $\mathbb{C}/L$ is just the natural quotient group structure inherited from $\mathbb{C}$.

Note that $\mathbb{C}^\times$ acts on $\mathcal{R}$ by scaling (homothety).

Consider the map $\beta \colon \mathcal{H} \to \mathcal{R}$ sending:

$$\tau \mapsto L_\tau,$$

where $L_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$.

Next let $\mathcal{E}$ denote the set of isomorphism classes of elliptic curves over $\mathbb{C}$. There are maps $u \colon \mathcal{R} \to \mathcal{E}$ and $\beta \colon \mathcal{H} \to \mathcal{E}$, which send $L \mapsto (\mathbb{C}/L, \mathcal{O})$ and $\tau \mapsto E_\tau = (\mathbb{C}/L_\tau, \mathcal{O})$.

**Proposition 6.1.** *The map $u \colon \mathcal{R} \to \mathcal{E}$ factors through $\mathbb{C}^\times$ and induces an isomorphism $\mathcal{R}/\mathbb{C}^\times \simeq \mathcal{E}$.*

*Proof.* Surjectivity follows from the fact that for each elliptic curve $E/\mathbb{C}$, there is an isomorphism $E(\mathbb{C}) \simeq \mathbb{C}/L$ for some lattice $L$. This lattice can be computed by fixing an invariant differential $\omega$ on $E$, and then:

$$L = \left\{ \int_\gamma \omega \mid \gamma \in H_1(E, \mathbb{Z}) \right\} \subseteq \mathbb{C}$$

For injectivity, suppose that $L_1, L_2 \in \mathcal{R}$ are two lattices such that

$$\mathbb{C}/L_1 \stackrel{\varphi}{\simeq} \mathbb{C}/L_2$$

As $\mathbb{C}$ is the universal covering space for $\mathbb{C}/L_i$, the isomorphism $\varphi$ can be lifted to a holomorphic map $\phi \colon \mathbb{C} \to \mathbb{C}$, such that $\phi(0) = 0$. This lift satisfies:

$$\phi(z + l_1) - \phi(z) \in L_2,$$

for each $z \in \mathbb{C}$ and $l_1 \in L_1$. Since $L_2$ is discrete, this implies:

$$\phi(z + l_1) - \phi(z) = c \text{ (a constant)}.$$

Taking a derivative shows $\phi'$ is invariant under $L_1$ and that it is holomorphic; the maximum modulus principle thus implies then that $\phi' = b \in \mathbb{C}$ is a constant. Hence $\phi(z) = bz + c$ for some $b, c \in \mathbb{C}$. But we know that $\phi(0) = 0$, so we must have $c = 0$. Hence $\phi(z) = bz$, and $L_2 = bL_1$, as we wanted to show. □

The map $\beta$ induces an isomorphism $\mathcal{R}/\mathbb{C}^\times \simeq \Gamma\backslash\mathcal{H}$, since two lattices $L_\tau$ and $L_{\tau'}$ are homothetic if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) = \Gamma$. Giving the appropriate complex-analytic structure on $\mathcal{R}/\mathbb{C}^\times$ makes $\beta$ an analytic isomorphism.

Consider now a function $F\colon \mathcal{R} \to \mathbb{C}$ such that

$$F(\lambda L) = \lambda^{-k} F(L)$$

for $\lambda \in \mathbb{C}^\times$. Given such a "homogeneous" function, define $f\colon \mathcal{H} \to \mathbb{C}$ by

$$f(\tau) \overset{\text{def}}{=} F(L_\tau) = F(\tau\mathbb{Z} \oplus \mathbb{Z})$$

Note that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have:

$$\begin{aligned} f(\gamma\tau) = F(L_{\gamma\tau}) = F\left(\mathbb{Z}\left(\frac{a\tau+b}{c\tau+d}\right) \oplus \mathbb{Z}\right) &= \\ = F\left(\frac{1}{c\tau+d}(\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d))\right) &= F((c\tau+d)^{-1}L_\tau) = \\ = (c\tau+d)^k F(L_\tau) = (c\tau+d)^k f(\tau) & \end{aligned}$$

so that if, in addition, $f$ satisfies holomorphicity on $\mathcal{H}$ and at $\infty$, then $f$ is a weight-$k$ modular form for $\Gamma$.

### 6.1.2 Functions on Elliptic Curves

The set of lattices $\mathcal{R}$ of the previous subsection is a fibration over $\mathcal{R}/\mathbb{C}^\times$. We would like to describe a fibration $\mathcal{E}'$ over $\mathcal{E}$, and a map $\mathcal{R} \to \mathcal{E}'$ such that:

$$\begin{array}{ccc} \mathcal{R} & \dashrightarrow & \mathcal{E}' \\ \downarrow & & \downarrow \\ \mathcal{R}/\mathbb{C}^\times & \xrightarrow{\simeq} & \mathcal{E} \end{array}$$

commutes. Then we can reinterpret modular forms as functions on $\mathcal{E}'$, analogously to viewing them as functions on $\mathcal{R}$.

Let $\mathcal{E}'$ be the set of isomorphism classes of pairs $(E, \omega)$, where $E/\mathbb{C}$ is an elliptic curve, and $\omega$ is a basis for $H^0(E, \Omega^1_{E/\mathbb{C}})$ (note that this has dimension 1, so one can take any globally holomorphic differential for $\omega$ ). The isomorphism condition for such pairs is:

$$(E, \omega) \simeq (E', \omega') \iff \exists \text{ an isomorphism } \varphi\colon E \to E' \text{ such that } \varphi^*\omega' = \omega$$

There is an obvious map $\mathcal{E}' \to \mathcal{E}$ which forgets $\omega$, and another map $\mathcal{R} \to \mathcal{E}'$ which takes a lattice $L$ to the pair $(\mathbb{C}/L, dz)$ (where $z$ is the natural coordinate function induced from $\mathbb{C}$). Note that if $\lambda \in \mathbb{C}^\times$, then $\lambda L$ is mapped to:

$$[(\mathbb{C}/(\lambda L), dz)] = [(\mathbb{C}/L, \lambda dz)]$$

We thus can let $\mathbb{C}^\times$ act on $\mathcal{E}'$ compatibly by setting

$$\lambda[(E,\omega)] \stackrel{\text{def}}{=} [(E,\lambda\omega)].$$

Again, for $k$ an integer, consider a function $G\colon \mathcal{E}' \to \mathbb{C}$ such that $G(E,\lambda\omega) = \lambda^{-k}G(E,\omega)$ for all $\lambda \in \mathbb{C}^\times$. A similar computation as before yields that the function $g\colon \mathcal{H} \to \mathbb{C}$ defined by

$$g(\tau) \stackrel{\text{def}}{=} G([(\mathbb{C}/L_\tau, dz)])$$

satisfies a modular transformation, and so it is a weight-$k$ modular form (assuming it satisfies the necessary holomorphicity conditions). This shows that we can think of modular forms as functions of pairs $(E,\omega)$ of elliptic curves and holomorphic differentials.

In the next subsection, we introduce the tool that will allow us impose holomorphicity conditions at $\infty$.

## 6.2 The Tate Curve

Let $\tau \in \mathcal{H}$, and consider its associated elliptic curve $E_\tau \stackrel{\text{def}}{=} \mathbb{C}/(\tau\mathbb{Z} \oplus \mathbb{Z})$. We have an (analytic) isomorphism:

$$\begin{array}{ccc} E_\tau & \xrightarrow{\simeq} & \mathbb{C}^\times/q_\tau^{\mathbb{Z}} \stackrel{\text{def}}{=} \mathrm{Tate}_{\mathbb{C}}(q) \\ z & \longmapsto & e^{2\pi i z} \end{array}$$

where $q_\tau \stackrel{\text{def}}{=} e^{2\pi i\tau}$, and $q_\tau^{\mathbb{Z}}$ is defined to be the multiplicative subgroup of $\mathbb{C}^\times$ generated by $q_\tau$.

We want to derive equations for $\mathrm{Tate}_{\mathbb{C}}(q)$. For this, let

$$L \stackrel{\text{def}}{=} L_\tau \stackrel{\text{def}}{=} 2\pi i(\tau\mathbb{Z} \oplus \mathbb{Z})$$

and let $X$ and $Y$ be:

$$X \stackrel{\text{def}}{=} \wp(2\pi iz, L), \quad Y \stackrel{\text{def}}{=} \wp'(2\pi iz, L)$$

We get the equations:

$$Y^2 = 4X^3 - g_2(L)X - g_3(L) = 4X^3 - \frac{E_4(q)}{12}X - \frac{E_6(q)}{216}$$

where:

$$E_4(q) = 12(2\pi i)^4 g_4(\tau) = 12g_2(L) = 1 + 240\sum \sigma_3(n)q^n$$
$$E_6(q) = 216(2\pi i)^6 g_6(\tau) = 216g_3(L) = 1 - 504\sum \sigma_5(n)q^n$$

(note that the $q$-expansions are in $\mathbb{Z}[\![q]\!]$, and hence the equation for $\mathrm{Tate}_{\mathbb{C}}(q)$ is defined over $\mathbb{Z}[1/6]((q))$. We want to remove also these denominators. So replace $X = x + 12$, and $Y = x + 2y$, and we obtain:

$$y^2 + xy = x^3 + B(q)x + C(q)$$

where:

$$B(q) = -5\frac{E_4(q)-1}{240} = -5\sum \sigma_3(n)q^n$$
$$C(q) = \frac{1}{12}\left(-5\frac{E_4(q)-1}{240} - 7\frac{E_6(q)-1}{504}\right) = -\sum \frac{5\sigma_3(n)+7\sigma_5(n)}{12}q^n$$

It's an elementary number theoretic calculation to show that $C(q)$ has coefficients in $\mathbb{Z}$ as well.

So we arrive at the definition of the Tate curve:

**Definition 6.2.** The **Tate curve** is the elliptic curve over $\mathbb{Z}((q))$ given by the equation $y^2 + xy = x^3 + B(q)x + C(q)$, together with the canonical differential $\omega_{\mathrm{can}} \stackrel{\mathrm{def}}{=} \frac{dx}{x+2y}$.

**Definition 6.3.** Given any ring $R$, the **Tate curve over** $R$ is defined as

$$\mathrm{Tate}_R(q) \stackrel{\mathrm{def}}{=} \mathrm{Tate}(q) \times_{\mathrm{Spec}\,\mathbb{Z}} R,$$

which is an elliptic curve over $R \otimes_{\mathbb{Z}} \mathbb{Z}((q))$.

**Remark.** Note that there is a canonical map $R \otimes_{\mathbb{Z}} \mathbb{Z}((q)) \to R((q))$ but it need not be surjective. For concreteness take $R = \mathbb{Q}$. Then $\sum_{n\geq 0} 2^{-n}q^n$ is not in the image of this map.

The Tate curve will be used to impose holomorphicity conditions on the modular forms, once we reinterpret them in a more geometric way.

## 6.3 Geometric Modular Forms

We give now the definition of modular forms that can be found in [Kat73]. Given a ring $R$ and an elliptic curve $E$ defined over $\mathrm{Spec}(R)$, consider the sheaf on $E$ of regular differentials $\Omega^1_{E/R}$, and let $\underline{\omega}_{E/R} \stackrel{\mathrm{def}}{=} p_*(\Omega^1_{E/R})$. This invertible sheaf is called **Katz canonical sheaf**.

**Definition 6.4** (due to N.Katz)**.** Fix an integer $k$, and let $R_0$ be a (commutative, unital) ring. A **(geometric) modular form** of weight $k$ and level 1, defined over $R_0$, is a rule $f$ which assigns to every $R_0$-algebra $R$ and elliptic curve $E/R$ defined over $\mathrm{Spec}(R)$, a section

$$f(E/R) \in \underline{\omega}_{E/R}^{\otimes k},$$

in such a way that:

1. $f(E/R)$ depends only on the isomorphism class of $E/R$.
2. $f$ commutes with arbitrary base change. That is, if $\varphi\colon R \to R'$ is an $R_0$-algebra homomorphism, and $E/R$ is a pair over $R$, then we can consider the curve $E'/R' \stackrel{\mathrm{def}}{=} E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \stackrel{p}{\to} E$. We require then that:

$$f(E'/R') = \varphi\left(f(E/R)\right)$$

This definition is equivalent to:

**Definition 6.5.** Fix an integer $k$, and let $R_0$ be a (commutative, unital) ring. A **(geometric) modular form** of weight $k$ and level 1, defined over $R_0$, is a rule $g$ which assigns to every pair $(E/R, \omega)$ a value

$$g(E/R, \omega) \in R,$$

where $R$ is an $R_0$-algebra, $E/R$ is an elliptic curve over $R$, and $\omega$ is a basis for $H^0(E, \Omega^1_{E/R})$, in such a way that:

1. $g(E/R, \omega)$ depends only on the isomorphism class of $(E/R, \omega)$.
2. $g(E/R, \lambda\omega) = \lambda^{-k} f(E/R, \omega)$ for all $\lambda \in R^\times$.
3. $g$ commutes with arbitrary base change. That is, if $\varphi\colon R \to R'$ is an $R_0$-algebra homomorphism, and $(E/R, \omega)$ is a pair over $R$, then we can consider the pair $E'/R' \stackrel{\text{def}}{=} E \times_{\text{Spec}(R)} \text{Spec}(R') \stackrel{p}{\to} E$ and $\omega' \stackrel{\text{def}}{=} p^*\omega$. We require then that:
$$g(E'/R', \omega') = \varphi\left(g(E/R, \omega)\right)$$

To convert from one to the other note that given $g$, one can define $f$ as in the first definition:

$$f(E/R) \stackrel{\text{def}}{=} g(E/R, \omega)\omega^k,$$

where $\omega \in \underline{\omega}_{E/R}$ is any basis. The weight-$k$ modularity of $g$ ensures that this definition does not depend on the choice of $\omega$.

Conversely, given a rule $f$ as in the first definition, one can define $g$:

$$g(E/R, \omega) \stackrel{\text{def}}{=} f(E/R)/\omega^k,$$

where the notation means the unique element $x$ of $R^\times$ such that $f(E/R) = x\omega^k$.

**Remark.** The last condition ensures meromorphicity at $\infty$, at least.

**Definition 6.6.** The **$q$-expansion of** $f$ at $\infty$ is:

$$f\left(\text{Tate}_{R_0}(q), \omega_{\text{can}}\right) \in \mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$$

**Definition 6.7.** We say that $f$ is a **(geometric) holomorphic modular form** if its $q$-expansion at $\infty$ actually belongs to $\mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} R_0$. We say that $f$ is a **(geometric) cusp form** if its $q$-expansion at $\infty$ belongs to $q\mathbb{Z}[\![q]\!] \otimes_{\mathbb{Z}} R_0$.

## 6.4 Comparing Katz's and Classical Modular Forms

Consider now the set $\mathcal{H} \times \mathbb{C}$, with its canonical projection $p_1$ onto $\mathcal{H}$. Then $\Gamma$ and $\mathbb{Z}^2$ both act on $\mathcal{H} \times \mathbb{C}$ via the rules:

$$(\tau, v) \cdot (\alpha, \beta) \stackrel{\text{def}}{=} (\tau, v + \alpha\tau + \beta)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, v) \stackrel{\text{def}}{=} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau, (c\tau + d)^{-1} v \right)$$

With these actions, the projection $p_1$ is $\Gamma$-equivariant (and also $\mathbb{Z}^2$-equivariant, with the trivial action on $\mathcal{H}$). Consider the quotient $\mathbb{E} \stackrel{\text{def}}{=} (\mathcal{H} \times \mathbb{C})/\mathbb{Z}^2$, together with the induced map $p \colon \mathbb{E} \to \mathcal{H}$. For any $\tau \in \mathcal{H}$, the fiber $p^{-1}(\tau)$ above it is isomorphic to $E_\tau$.

By regarding $\mathbb{E}$ as a fibration over $\mathcal{H}$, we are thinking of $\mathbb{E}$ complex analytically. One should replace $\mathcal{H}$ by $Y(1) = \text{Spec}(R)$, where $R = \mathbb{C}[j]$ is the $j$-line, in order to work algebraically. In this light we have:

$$\mathbb{E} = \underline{\text{Proj}} \left( R[X, Y, Z]/(Y^2 Z = 4X^3 - g_2(\tau) X Z^2 - g_3(\tau) Z^3) \right)$$

where $g_2(\tau), g_3(\tau) \in R$ are the classical Eisenstein series. Note that the maximal ideals of $R$ correspond bijectively to the points of $\mathcal{H}$. If $\tau \in \mathcal{H}$ then the corresponding maximal ideal is:

$$\mathfrak{m}_\tau = \{ f \in R \mid f(\tau) = 0 \}.$$

Thus $\mathbb{E}$ is an elliptic curve over $R$.

We will now show that any geometric modular form $F$ over $R_0 = \mathbb{C}$ gives a classical modular form $f$, by the rule

$$f(\tau) \stackrel{\text{def}}{=} F(E_\tau, dz), \quad \tau \in \mathcal{H}.$$

For each $\tau \in \mathcal{H}$ there is a natural $\mathbb{C}$-algebra map:

$$\text{ev}_\tau \colon R \to \mathbb{C},$$

given by evaluating functions at $\tau$. Geometrically we have the following picture:

$$\begin{array}{ccc} E_\tau & \longrightarrow & \mathbb{E} \\ \downarrow & & \downarrow \\ \text{Spec}\,\mathbb{C} & \longrightarrow & \text{Spec}\,R \end{array}$$

This diagram is cartesian, realising $E_\tau$ as the fiber of the map $\mathbb{E} \to \text{Spec}(R)$ over the maximal ideal $\mathfrak{m}_\tau$ of $R$. Thus, property (3) of a geometric modular form gives:

$$\text{ev}_\tau(F(\mathbb{E}, dz)) = F(E_\tau, dz) = f(\tau).$$

But $F(\mathbb{E}, dz) \in R$, and so this equality says precisely that $f = F(\mathbb{E}, dz)$ is holomorphic on $\mathcal{H}$.

Next we show that $f$ is weight $k$-invariant. For $\tau \in \mathcal{H}$ and:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

note that:

$$(E_{\gamma\tau}, dz) \simeq (E_\tau, (c\tau + d)^{-1} dz).$$

Properties (1) and (2) for a geometric modular form thus give:

$$f(\gamma\tau) = F(E_{\gamma\tau}, dz) = F(E_\tau, (c\tau+d)^{-1}dz) = (c\tau+d)^k F(E_\tau, dz) = (c\tau+d)^k f(\tau),$$

which is the weight $k$-invariance property for $f$.

The final thing to check is that $f$ as defined is holomorphic at infinity. But the Tate curve and the holomorphicity of a geometric modular form at infinity are cooked up precisely so that this is true. Indeed, consider the map $\varphi$ which takes a 1-periodic function $f$ to its associated Fourier series:

$$\begin{aligned} \varphi \colon R &\to \mathbb{C}((q)) \\ f &\mapsto \sum_{n \geq n_0} a_n q^n, \qquad q = e^{2\pi i \tau} \end{aligned}$$

This gives a diagram

$$\begin{array}{ccc} \mathrm{Tate}(q) & \longrightarrow & \mathbb{E} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C}((q)) & \longrightarrow & \operatorname{Spec} R \end{array}$$

and so, by the compatibility under base change, we have

$$\varphi\left(F(\mathbb{E}, dz)\right) = F(\mathrm{Tate}(q), dz) \in \mathbb{C}[[q]],$$

since $F$ is holomorphic at $\infty$ as a geometric modular form.

Thus, $f$ as defined is a holomorphic modular form in the classical sense. In fact, if one begins with a classical modular form, one can also construct a geometric holomorphic modular form. In this way one sees that geometric modular forms over $\mathbb{C}$ are the same as classical modular forms. We will henceforth drop the appelations "geometric" and often even "holomorphic" without confusion.

## 6.5 The Final Interpretation

In this section we will sacrifice some rigour in favour of ease of exposition. Basically, we will assume that there exists a univeral elliptic curve $\mathcal{E}$ over $Y(1)$, parametrising isomorphism classes of elliptic curves over $\mathbb{C}$ (as a fine moduli scheme). This is true for higher levels, but not for levels 1 or 2.

Recall that $Y(1)$, the $j$-line, is an affine scheme over $\mathbb{Z}$. Consider the Katz canonical sheaf $\underline{\omega}$ on $Y(1)$ obtained by viewing $\mathcal{E}/Y(1)$ as an elliptic curve. In this case, a modular form of weight $k$ over a $\mathbb{Z}$-algebra $R_0$ is just a global section of $\underline{\omega}^{\otimes k}$ on $Y(1)$. Also, if one considers the compactification $X(1)$ of $Y(1)$, and extend $\underline{\omega}^{\otimes k}$ to $X(1)$, a holomorphic modular form of weight $k$ over $R_0$ is a global section of $\underline{\omega}^{\otimes k}$ on $X(1)$.

**Remark.** The authors experienced some confusion distinguishing when to work with symmetric powers and when to work with tensor powers. We have tried to keep our notation consistent by using either $\mathrm{Symm}^k(\cdot)$ or $(\cdot)^{\otimes k}$. Note however that, for one dimensional locally-free sheaves or modules, the two concepts agree (they are both still 1-dimensional). In the following section we will need to consistently work with $\mathrm{Symm}^k(\cdot)$ because we will be dealing with the two-dimensional module $H^1_{\mathrm{dR}}(\mathcal{E}/Y(1))$.

# 7 The Kodaira-Spencer Map

Given an elliptic curve $E/R$ with structure morphism $p$ (where $R$ is a ring which is one-dimensional and smooth over a base ring $R_0$), we have defined so far:

- The Hodge filtration: $0 \longrightarrow H^0(E, \Omega^1_{E/R}) \longrightarrow H^1_{\mathrm{dR}}(E/R) \xrightarrow{\beta} H^1(E, \mathcal{O}_E) \longrightarrow 0$.
- The Poincaré pairing $\langle \cdot, \cdot \rangle_{\mathrm{Poinc}} \colon H^1_{\mathrm{dR}}(E/R) \times H^1_{\mathrm{dR}}(E/R) \to R$.
- The Gauss-Manin connection $\nabla \colon H^1_{\mathrm{dR}}(E/R) \to \Omega^1_{R/R_0} \otimes_R H^1_{\mathrm{dR}}(E/R)$.

Consider the sheaf of regular differentials $\Omega^1_{E/R}$, and let

$$\underline{\omega} = \underline{\omega}_{E/R} \overset{\mathrm{def}}{=} \pi_* \Omega^1_{E/R}$$

be the Katz canonical sheaf.

**Definition 7.1.** The **Kodaira-Spencer map** is the sheaf homomorphism:

$$\begin{aligned} \varphi_{\mathrm{KS}} \colon \qquad \underline{\omega}^{\otimes 2} &\longrightarrow \Omega^1_{R/R_0} \\ \omega_1 \otimes \omega_2 &\longmapsto \langle \omega_1, \nabla \omega_2 \rangle_{\mathrm{Poinc}} \end{aligned}$$

where $\omega_1, \omega_2$ are differentials on some open subset $U \subseteq \mathrm{Spec}(R)$, and we pair $\omega_1$ with the deRham part of $\nabla\omega_2$, getting an element in $\Omega^1_{R/R_0}(U)$).

We say that $E/R$ is *almost modular* if $\varphi_{\mathrm{KS}}$ is an isomorphism.[1]

[1]This notation is due to N.Katz (see [Kat73, A1.3.17]). The reason for it is that in this case the classifying map from $R$ to the modular stack is étale. Therefore $\mathrm{Spec}\, R$ locally looks like a modular curve, and $E/R$ like the universal elliptic curve over it.

**Example.** *Consider the Tate curve* $(\mathrm{Tate}(q), \omega_{can})$ *over* $\mathbb{C}((q))$ *(that is, analytically). We compute the image of the Kodaira-Spencer map on* $\omega_{can}$*. Recall that* $\omega_{can}$ *is* $dt/t$*, and that* $\eta_{can}$ *is the differential of the second kind which is dual to* $\omega_{can}$ *with respect to the Poincaré pairing. Set* $\theta \overset{def}{=} q\frac{d}{dq} \in \Omega^1_{\mathbb{C}((q))/\mathbb{C}}$*. Then we can compute the Gauss-Manin connection on* $H^1_{dR}(\mathrm{Tate}(q), \mathbb{C}((q)))$*, which has matrix:*

$$\nabla(\theta)\begin{pmatrix}\omega_{can}\\ \eta_{can}\end{pmatrix} = \begin{pmatrix}\frac{-P}{12} & 1\\ \frac{P^2-12\theta P}{144} & \frac{P}{12}\end{pmatrix}\begin{pmatrix}\omega_{can}\\ \eta_{can}\end{pmatrix}$$

*where* $P$ *is defined as:*

$$P(q) \overset{def}{=} 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n, \quad \sigma_1(n) \overset{def}{=} \sum_{d|n} d.$$

*This shows that*

$$\varphi_{KS}(\omega_{can}\otimes\omega_{can})(\theta) = \langle\omega_{can}, \nabla(\theta)(\omega_{can})\rangle_{Poinc} = 1$$

*because* $\langle\omega_{can},\omega_{can}\rangle_{Poinc} = 0$ *and* $\langle\omega_{can},\eta_{can}\rangle_{Poinc} = 1$*.*
*From this we obtain*

$$\left(\varphi_{KS}(\omega_{can}\otimes\omega_{can})\right)(\theta) = 1$$

*and hence:*

$$\varphi_{KS}(\omega_{can}\otimes\omega_{can}) = \frac{dq}{q}.$$

# 8 Derivations of Modular Forms

We would like to use all this machinery to define differential operators on spaces of modular forms. For this, assume that $\mathcal{E}/Y(1)$ is almost modular (this is in fact true, although we ommit any proof). We have identified the modular forms of weight $k$ over $R_0$ with

$$H^0\left(Y(1), \underline{\omega}_{\mathcal{E}/Y(1)}\right)^{\otimes k} = H^0\left(\mathcal{E}, (\Omega^1_{\mathcal{E}/Y(1)})^{\otimes k}\right)$$

so let $R = \Gamma(Y(1), \mathcal{O}_{Y(1)})$ be the $j$-line. Also, notice that, as $\Omega^1_{\mathcal{E}/R}$ is locally-free of rank 1, the symmetric power commutes with taking global sections, so that we can identify modular forms of weight $k$ over $R_0$ with

$$H^0\left(\mathcal{E}, \Omega^1_{\mathcal{E}/R}\right)^{\otimes k} = \mathrm{Symm}^k H^0\left(\mathcal{E}, \Omega^1_{\mathcal{E}/R}\right) \overset{\iota}{\hookrightarrow} \mathrm{Symm}^k H^1_{\mathrm{dR}}(\mathcal{E}/R).$$

The $k$-th tensor power of the connection $\nabla$ is a connection

$$\nabla\colon \mathrm{Symm}^k(H^1_{\mathrm{dR}}(\mathcal{E}/R)) \to \Omega^1_{R/R_0}\otimes \mathrm{Symm}^k(H^1_{\mathrm{dR}}(\mathcal{E}/R)),$$

and, since $R$ is affine, we can think of $\nabla$ as taking values actually in

$$R\otimes \mathrm{Symm}^k(H^1_{\mathrm{dR}}(\mathcal{E}/R))$$

Suppose now we had a retraction

$$r\colon H^1_{\mathrm{dR}}(\mathcal{E}/R) \to H^0\left(\mathcal{E}, \Omega^1_{\mathcal{E}/R}\right) = H^0\left(Y(1), \underline{\omega}\right)$$

to the first map in the Hodge filtration. This induces a map

$$r\colon \operatorname{Symm}^k(H^1_{\mathrm{dR}}(\mathcal{E}/R)) \to \operatorname{Symm}^k H^0\left(Y(1), \underline{\omega}\right).$$

The composition

$$\partial \stackrel{\text{def}}{=} (\varphi_{\mathrm{KS}}^{-1} \otimes r) \circ \nabla \circ \iota$$

(where $\varphi_{\mathrm{KS}}^{-1}$ is thought as a map on global sections) gives a map

$$\partial\colon H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R})^{\otimes k} \to H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R})^{\otimes (k+2)}$$

Since the flanking maps in the definition of $\partial$ respect the Hodge filtration, the Leibniz rule satisfied by $\nabla$ makes $\partial$ a weight-2 derivation on

$$M \stackrel{\text{def}}{=} \bigoplus_{k\in\mathbb{Z}} H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R})^{\otimes k}.$$

**Remark.** We will require that the retraction $r$ preserves any extra structure that the objects in the Hodge filtration may have. In this way, the derivation $\partial$ will respect this structure as well.

# 9 The Unit Root Splitting and Serre's Operator

## 9.1 The Canonical Subgroup

Let's assume (for simplicity) that $R_0$ is a complete DVR, with residue characteristic $p > 3$, and generic characteristic 0. Set $\operatorname{ord}(p) \stackrel{\text{def}}{=} 1$. Recall that an elliptic $E/R$, where $R$ is a $R_0$-algebra, is said to be **ordinary** if its special fiber has invertible Hasse invariant.

**Theorem 9.1** (Lubin, ordinary case)**.** *There is one and only one way to attach to every* ordinary *elliptic curve $E/R$ (where $R$ is a $p$-adically complete $R_0$-algebra) a finite flat rank-$p$ subgroup scheme $H \subseteq E$ (the **canonical subgroup** of $E/R$) such that:*

1. *$H$ depends only on the isomorphism class of $E/R$;*
2. *The formation of $H$ commutes with arbitrary base-change $R \to R'$;*
3. *For $R_N \stackrel{def}{=} R_0/p^N R_0$, $R \stackrel{def}{=} R_N((q))$, and $E/R$ the (base change of the) Tate curve $Tate_{R_N}(q)$ over $R_N((q))$, then $H$ is the subgroup $\mu_p$ of $Tate_{R_N}(q)$.*

*Moreover, the elliptic curve $E' \stackrel{def}{=} E/H$ is also ordinary.*

*Proof.* Consider the Frobenius morphism

$$F\colon \tilde{E} \to \tilde{E}^{(p)},$$

where $\tilde{E}^{(p)}$ is obtained from $\tilde{E}$ by applying the Frobenius endomorphism $\sigma_p$ to the equation defining $\tilde{E}$.

Let $\tilde{H}$ be the kernel of $F$. Since $E$ is ordinary at $p$, the subgroup $\tilde{H}$ is a finite cyclic group of order $p$. The dual isogeny to $F$, called the Verschiebung $V$, is separable (see [Sil86]) and, since $R$ is $p$-adically complete, we can uniquely lift $V$ to an isogeny $V^{\uparrow}\colon E^{(p)} \to E$. Here, $E^{(p)}$ is a lift to $R$ of $\tilde{E}^{(p)}$. Finally, we let $H$ be the kernel of the dual isogeny to $V^{\uparrow}$. This subgroup $H$ is the *canonical subgroup*, and it lifts $\tilde{H}$ to $R$. More details may be found in [Col05].

We need to check the case of the Tate curve. So let $R_N$ and $R$ be as in (3), and let $E$ be the Tate curve over $R_N((q))$. Reducing modulo $p$, the Frobenius map has kernel $\mu_p$, which is the reduction of $\mu_p$ as a subgroup scheme of $\mathrm{Tate}_{R_N}(q)$. Hence, by uniqueness, $H = \mu_p$ in this case. □

## 9.2 The action of Frobenius

Let $R$ be a $p$-adically complete ring, and let $E/R$ be an *ordinary* elliptic curve. Let $H \subseteq E$ be its canonical subgroup, and $E' \stackrel{\mathrm{def}}{=} E/H$, with $\pi\colon E \to E'$ the projection. By functoriality, we get an $R$-morphism:

$$\pi^*\colon H^1_{\mathrm{dR}}(E'/R) \to H^1_{\mathrm{dR}}(E/R)$$

.

Let now choose a power $q$ of $p$ such that $q \equiv 1 \pmod{n}$, and choose for $R$ the ring of $p$-adic modular *functions* of level $n$, defined over $W(\mathbb{F}_q)$ (the Witt vectors over $\mathbb{F}_q$), which we write:

$$R \stackrel{\mathrm{def}}{=} M\left(W(\mathbb{F}_q), 1, n, 0\right)$$

(the 1 in the parameter list refers to the growth 1).

Let $E/R$ be the *universal* elliptic curve with level-$n$ structure, such that it has invertible Hasse invariant modulo $p$. By universality, there is a unique homomorphism $\varphi\colon R \to R$ such that

$$E' = E^{(\varphi)}$$

and the morphism $\varphi$ is precisely the Frobenius morphism on $M(W(\mathbb{F}_q), 1, n, 0)$ (a lifting to char 0 called the "Deligne-Tate map"). As we mentioned above, this induces a homomorphism

$$\pi^*\colon H^1_{\mathrm{dR}}(E'/R) = H^1_{\mathrm{dR}}(E^{(\varphi)}/R) = \left(H^1_{\mathrm{dR}}(E/R)\right)^{(\varphi)} \to H^1_{\mathrm{dR}}(E/R)$$

By composition, we get a $\varphi$-linear endomorphism on $H^1_{\mathrm{dR}}(E/R)$, $F(\varphi) \stackrel{\mathrm{def}}{=} \pi^* \circ \varphi^{-1}$. As $\pi^*$ is induced by an $R$-morphism, the action $F(\varphi)$ respects the Hodge filtration:

$$0 \to H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R}) \to H^1_{\mathrm{dR}}(\mathcal{E}/R) \to H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) \to 0$$

and so it induces actions on $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R})$ and $H^1(\mathcal{E}, \mathcal{O}_\mathcal{E})$. We want to compute these actions, in terms of $\varphi$:

**Lemma 9.2.** *On $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R})$, $F(\varphi) = p\varphi$ and, on $H^1(\mathcal{E}, \mathcal{O}_\mathcal{E})$, $F(\varphi) = \varphi$.*

*Proof.* □

If we take the basis $\omega, \eta$ adapted to the Hodge filtration, then the matrix of $F(\varphi)$ is:

$$\begin{pmatrix} p/\lambda & 0 \\ C & \lambda \end{pmatrix}$$

with $\lambda \in R$ invertible, and $C \in R$.

We want to construct a splitting of the Hodge filtration. As the map $H^1_{\mathrm{dR}}(E/R) \to H^1(\mathcal{E}, \mathcal{O}_\mathcal{E})$ is of the form $a\omega + b\eta \mapsto b\eta$, this amounts to finding $f \in R$ such that $\eta \mapsto f\omega + \eta$. To make it canonical in some way, we impose that it is $F(\varphi)$-linear. This is the same as imposing that:

$$F(\varphi)(f\omega + \eta) \in R \cdot (f\omega + \eta)$$

Using the fact that $R$ is closed with respect to the $p$-adic topology, we can find such an $f$. It is given as:

$$f \stackrel{\mathrm{def}}{=} \frac{c}{\lambda} + \sum_{n \geq 1} p^n \frac{\varphi^{\frac{n(n-1)}{2}}(1/\lambda) \cdot \varphi^n(c)}{\varphi^{\frac{n(n+1)}{2}}(\lambda)}$$

Now, recall that the giving of a splitting is equivalent to the giving of a retraction $r \colon H^1_{\mathrm{dR}}(\mathcal{E}/R) \to H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/R})$. This retraction will yield, as we have seen, a weight-2 derivation on the algebra of modular forms.

## 9.3 Serre's $\theta$ operator

Let now $f(q)$ be the $q$-expansion of a modular form of weight $k$, and let $f \cdot \omega^{\otimes k}$ be the corresponding global section in $\underline{\omega}^k$. Recall the differential operator

$$\theta = q\frac{d}{dq}.$$

We compute $\partial f$. For that, we think of $f \cdot \omega^{\otimes k}$ as an element of $\mathrm{Symm}^k H^1_{\mathrm{dR}}(\mathcal{E}/R)$, and apply $\nabla$ to it:

$$\nabla(f \cdot \omega^{\otimes k}) = df \cdot \omega^{\otimes k} + kf\omega^{\otimes(k-1)}\nabla(\omega) = df \cdot \omega^{\otimes k} + kf\omega^{\otimes(k-1)}\left(-\frac{P}{12} + \eta\right).$$

Next, note that

$$\left.\begin{array}{c} df = \frac{df}{dq}dq = \theta(f)\frac{dq}{q} \\ \varphi_{\mathrm{KS}}(\omega^{\otimes 2}) = \frac{dq}{q} \end{array}\right\} \implies \varphi_{\mathrm{KS}}^{-1}(df) = \theta(f) \cdot \omega^{\otimes 2}$$

So if we apply $\varphi_{\mathrm{KS}}^{-1}$ to the previous expression and recall that

$$\nabla(\omega) = \frac{dq}{q} \otimes \left( -\frac{P}{12}\omega + \eta \right),$$

we get

$$\left( (\varphi_{\mathrm{KS}}^{-1} \otimes r) \circ \nabla \circ \iota \right) (f) = \theta(f) - kf\frac{P}{12},$$

so that if $f$ is modular of weight $k$, then

$$\partial(f) \stackrel{\text{def}}{=} \theta(f) - kf\frac{P}{12}$$

is modular of weight $k+2$.

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