Introduction.
The Beurling transform.

The Beurling transform of a function \( f \in L^p(\mathbb{C}) \) is:

\[
\mathcal{B}f(z) = c_0 \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} \, dm(w).
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It is essential to quasiconformal mappings because

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B(\bar{\partial}f) = \partial f \quad \forall f \in W^{1,p}.
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$$B(\bar{\partial} f) = \bar{\partial} f \quad \forall f \in W^{1,p}.$$

Recall that $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$. Also $B : W^{n,p}(\mathbb{C}) \to W^{n,p}(\mathbb{C})$ is bounded for $1 < p < \infty$ and $n > 0$. 
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In general, if $x \not\in \text{supp}(f) \subset \mathbb{R}^d$ then a convolution CZO of order $n$ is

$$Tf(x) = \int K(x-y)f(y)$$

with

$$|\nabla^j K(x)| \leq \frac{1}{|x|^{d+j}} \quad \text{for } j \leq n.$$
The problem we face.

If \( T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \),
The problem we face.

If $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $T : L^p(\Omega) \to L^p(\Omega)$. 
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For \( \Omega \) a rectangle, \( B \chi_\Omega \) is in every \( L^p(\Omega) \) but not in \( W^{1,p}(\Omega) \) for \( p \geq 2 \).
The problem we face.

If $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $T : L^p(\Omega) \to L^p(\Omega)$. But for $g \in W^{1,p}(\Omega)$ maybe not $\nabla T(g) \in L^p(\Omega)$. When is $T : W^{n,p}(\Omega) \to W^{n,p}(\Omega)$ bounded?
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If $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $T : L^p(\Omega) \to L^p(\Omega)$. But for $g \in W^{1,p}(\Omega)$ maybe not $\nabla T(g) \in L^p(\Omega)$. When is $T : W^{n,p}(\Omega) \to W^{n,p}(\Omega)$ bounded? We seek for answers in terms of test functions and in terms of the geometry of the boundary.
Test function conditions.
Theorem (Cruz, Mateu, Orobitg, 2013)

Given a $C^{1+\epsilon}$ domain $\Omega \subset \mathbb{R}^d$, $T$ even and $p > d$. If $T(\chi_{\Omega}) \in W^{1,p}(\Omega)$, then $T$ is bounded in $W^{1,p}(\Omega)$. 

If $n = 1$, the converse is true.
Results.

Theorem (Cruz, Mateu, Orobitg, 2013)

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Theorem (P., Tolsa, 2014)

*Given a Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $p > d$. If $T(P) \in W^{n,p}(\Omega)$ for polynomials $P \in \mathcal{P}^{n-1}(\Omega)$, then $T$ is bounded in $W^{n,p}(\Omega)$."

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For any $1 < p \leq d$, if $|\nabla^n T(P)(x)|^p dx$ is a $p$-Carleson measure in $\Omega$ for every $P \in \mathcal{P}^{n-1}(\Omega)$, then $T$ is bounded in $W^{n,p}(\Omega)$. 
Introduction.

Test function conditions.

A geometric condition.

Results.

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Theorem (P., Tolsa, 2014)

For any $1 < p \leq d$, if $|\nabla^n T(P)(x)|^p \, dx$ is a $p$-Carleson measure in $\Omega$ for every $P \in \mathcal{P}^{n-1}(\Omega)$, then $T$ is bounded in $W^{n,p}(\Omega)$. If $n = 1$, the converse is true.
Consider a Lipschitz domain $\Omega$. We perform a Whitney covering $W$ such that $\text{dist}(Q, \partial \Omega) \approx \ell(Q)$. \{5\} $Q \in W$ has finite superposition. We can choose a central cube. We can think on Carleson boxes (or shadows). We can think on Harnack chains.
Consider a Lipschitz domain $\Omega$. 

The Whitney covering.

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The key point: approximating by polynomials.

A new approach for the case $n = 1$:

**Key Lemma**

The following are equivalent:

1. $\|\nabla T f\|_{L^p(\Omega)}^p \leq C \|f\|^p_{W^{1,p}(\Omega)}$.
2. $\sum_{Q \in \mathcal{W}} |f_{3Q}|^p \|\nabla T(\chi_Q)\|_{L^p(Q)}^p \leq C \|f\|^p_{W^{1,p}(\Omega)}$.

Ingredients: bounds for the kernel, Poincaré inequality and Hölder.
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- $\| \nabla T f \|_{L^p(\Omega)}^p \leq C \| f \|_{W^{1,p}(\Omega)}^p$.
- $\sum_{Q \in \mathcal{W}} |f_Q|^p \| \nabla T(\chi_{\Omega}) \|_{L^p(Q)}^p \leq C \| f \|_{W^{1,p}(\Omega)}^p$.

Enough to prove

$$\sum_{Q} \| \nabla T(f - f_Q \chi_{\Omega}) \|_{L^p(Q)}^p \lesssim \| \nabla f \|_{L^p(\Omega)}^p.$$
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Break the local part and non-local part.
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Break the local part and non-local part. 
Local part is a good function, in $W^{1,p}(\mathbb{R}^d)$. 

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Local part is a good function, in $W^{1,p}(\mathbb{R}^d)$.
For the non-local part, we use a Harnack chain of cubes.
Ingredients: bounds for the kernel, Poincaré inequality and Hölder.
Proof of the $T(P)$ theorem ($p > d$).

We want to see that $T(\chi_\Omega) \in W^{1,p}(\Omega)$ implies $T$ bounded in $W^{1,p}(\Omega)$. 

\[ \sum_{Q \in W} |f_3^Q| p \|\nabla T(\chi_\Omega)\| L^p(Q) \leq \|f\| L^\infty \|\nabla T(\chi_\Omega)\| L^p(\Omega) \leq C \|f\| L^\infty. \] 

Since $p > d$, by the Sobolev Embedding Theorem $\|f\| L^\infty \leq C \|f\| W^{1,p}(\Omega)$. 

\[ \text{Introduction.} \quad \text{Test function conditions.} \quad \text{A geometric condition.} \]
Proof of the T(P) theorem \((p > d)\).

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\sum_{Q \in \mathcal{W}} |f_Q|^p \| \nabla T \chi_{\Omega} \|_{L^p(Q)}^p
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Proof of the $T(P)$ theorem ($p > d$).

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The Carleson measures.

According to carleson measures for Besov space of analytic functions $B_p(\rho)$,

**Definition**

We say that $\mu$ is $p$-Carleson for $\Omega \subset \mathbb{R}^d$ iff for every Whitney cube $P$,

$$\sum_{Q \subset \text{Sh}(P)} \mu(\text{Sh}(Q))^{p'} \ell(Q)^{\frac{p-d}{p-1}} \leq C \mu(\text{Sh}(P)).$$
Proof of Carleson $\Rightarrow$ boundedness ($p \leq d$).

Assume that $n = 1$ and

$$\mu(x) = |\nabla T \chi_\Omega(x)|^p dx$$

is $p$-Carleson for $\Omega$. We want

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$$\sum_{Q \in \mathcal{W}} |f_{3Q}|^p \mu(Q) \leq \sum_{Q \in \mathcal{W}} \left( \sum_{P: Q \subset \text{Sh}(P)} |f_{3P} - f_{3\mathcal{N}(P)}| \right)^p \mu(Q)$$
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But, by Poincaré inequalities

$$\sum_{Q \in \mathcal{W}} |f_{3Q}|^p \mu(Q) \leq \sum_{Q \in \mathcal{W}} \left( \sum_{P: Q \subset \text{Sh}(P)} |f_{3P} - f_{3N(P)}| \right)^p \mu(Q)$$

$$\leq \sum_{Q \in \mathcal{W}} \left( \sum_{P: Q \subset \text{Sh}(P)} \|\nabla f\|_{L^p(5P)} \ell(P)^{1 - \frac{d}{p}} \right)^p \mu(Q)$$
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But, by Poincaré inequalities and some $p$-Carleson measure properties,

$$\sum_{Q \in \mathcal{W}} |f_{3Q}|^p \mu(Q) \leq \sum_{Q \in \mathcal{W}} \left( \sum_{P: Q \subset \text{Sh}(P)} |f_{3P} - f_{3N(P)}| \right)^p \mu(Q)$$

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$$\leq C \sum_{Q \in \mathcal{W}} \|\nabla f\|^p_{L^p(5Q)} \leq C \|f\|^p_{W^{1,p}(\Omega)}.$$
The converse is true for $n = 1$: a duality argument.

Hypothesis: $T$ bounded in $W^{1,p}(\Omega)$. Then the averaging function

$$Af(x) := \sum_{Q \in \mathcal{W}} \chi_Q(x) f_Q,$$
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$$Af(x) := \sum_{Q \in \mathcal{W}} \chi_Q(x) f_Q,$$

by the Key Lemma, is also bounded $A : W^{1,p}(\Omega) \to L^p(\mu)$ for

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\]

by the Key Lemma, is also bounded \( A : W^{1,p}(\Omega) \rightarrow L^p(\mu) \) for

\[
\mu(x) = |\nabla T \chi_{\Omega}(x)|^p \, dx.
\]

By duality, \( A^* : L^p(\mu) \rightarrow (W^{1,p}(\Omega))^* \) is also bounded.
The converse is true for $n = 1$: a duality argument.

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By duality, $A^* : L^p(\mu) \to (W^{1,p}(\Omega))^*$ is also bounded.

$(p = 2)$ For $g = \chi_{Sh(P)}$,

$$\sum_{Q \subset Sh(P)} \mu(Sh(Q))^2 \lesssim \cdots \lesssim \|A^* g\|_{W^{1,2}(\Omega)}^2 \lesssim \|g\|_{L^2(\mu)}^2 = \mu(Sh(P)).$$
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$W^{1,2}(\Omega)$ is Hilbert, there is $A^*(g) \in W^{1,2}(\Omega)$. 
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$W^{1,2}(\Omega)$ is Hilbert, there is $A^*(g) \in W^{1,2}(\Omega)$. $A^*(g)$ solves a Neumann problem $\Delta h = \tilde{g}$. 
A geometric condition.
Results.

Theorem (P., 2013)

For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to the boundary of $\Omega$ is in the Besov space $B_{p,p}^{n-1/p}(\partial \Omega)$ then $B(\chi_\Omega) \in W^{n,p}(\Omega)$, with

$$\|\nabla^n B(\chi_\Omega)\|_{L^p(\Omega)} \lesssim \|N\|_{B_{p,p}^{n-1/p}(\partial \Omega)}^p + C_{\text{length}}(\partial \Omega).$$
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Theorem (P., 2013)

For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to the boundary of $\Omega$ is in the Besov space $B^{n-\frac{1}{p}}_{p,p}(\partial \Omega)$ then $\mathcal{B}(\chi_\Omega) \in W^{n,p}(\Omega)$, with

$$\|\nabla^n \mathcal{B}(\chi_\Omega)\|_{L^p(\Omega)}^p \lesssim \|N\|_{B^{n-1/p}_{p,p}(\partial \Omega)}^p + C \text{length}(\partial \Omega).$$

V. Cruz and X. Tolsa proved the case $n = 1$.

Tolsa proved a converse for $n = 1$ and $\Omega$ smooth enough.
Ingredients for the proof.

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For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to $\partial \Omega$ is in the Besov space $B_{p,p}^{n-\frac{1}{p}}(\partial \Omega)$ then $\mathcal{B}(\chi_\Omega) \in W^{n,p}(\Omega)$, with

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Ingredients:

- Generalized Peter Jones’ betas (using polynomials instead of lines).
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- Generalized Peter Jones’ betas (using polynomials instead of lines).
- Equivalence between Besov $B^s_{p,p}$ norm and a sum of betas (Dorronsoro).
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For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to $\partial \Omega$ is in the Besov space $B_{p,p}^{n-1/p}(\partial \Omega)$ then $\mathcal{B}(\chi_\Omega) \in W^{n,p}(\Omega)$, with

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- Generalized Peter Jones’ betas (using polynomials instead of lines).
- Equivalence between Besov $B_{p,p}^s$ norm and a sum of betas (Dorronsoro).
- Beurling of characteristic functions of circles, half-planes, polynomials, ...
Conclusions.

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- Next steps:
  - Proving analogous results for any $s \in \mathbb{R}_+$.
  - Looking for a more general set of operators where the Besov condition on the boundary implies Sobolev boundedness.
  - Sharpness of all those results.
The end.

Moltes gràcies!!
Děkuji!!
Defining some generalized betas of David-Semmes.

A measure of the flatness of a set $\Gamma$: 

$$\beta_\Gamma(Q) = \inf_{V} w(V) \ell(Q)$$

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**Definition (P. Jones)**

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Ending
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$$\text{Definition } \beta_\infty(I, A) = \inf_{P \in \mathcal{P}^1} \| A - P \|_\infty(I)$$

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\beta_p(I, A) = \inf_{P \in \mathcal{P}_1} \frac{1}{\ell(I)^{1/p}} \left\| \frac{A - P}{\ell(I)} \right\|_p
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$$\beta_n(I, A) = \inf_{P \in \mathcal{P}^n} \frac{1}{\ell(I)} \left\| \frac{A - P}{\ell(I)} \right\|_1$$

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...
Geometric condition in terms of betas: The Besov space.

**Definition**

For $0 < s < \infty$, $1 \leq p < \infty$, $f \in B_{p,p}^s(\mathbb{R})$ if

$$
\|f\|_{B_{p,p}^s} = \|f\|_{L^p} + \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\Delta_h^{[s]+1} f(x)}{h^s} \right|^p \frac{dm(h)}{|h|} dm(x) \right)^{1/p} < \infty.
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**Theorem (Dorronsoro)**

Let $f : \mathbb{R} \to \mathbb{R}$ be a function in the Besov space $B_{p,p}^s$. Then, for any $n \geq [s]$,

$$
\|f\|_{B_{p,p}^s}^p \approx \|f\|_{L^p}^p + \sum_{I \in \mathcal{D}} \left( \frac{\beta_n(I)}{\ell(I)^{s-1}} \right)^p \ell(I).
$$
Main idea: projecting cubes to the boundary.

\[ \int_{Q_k \cap \Omega} |\partial^\alpha B_{\chi(z)}(z)|^p \, dm(z) \]
Main idea: projecting cubes to the boundary.
Main idea: projecting cubes to the boundary.

\[
\int_{Q \cap \Omega} |\partial^n B \chi_{\Omega}(z)|^p dm(z) \leq \sum_{Q \in W} \int_Q |\partial^n B \chi_{\Omega}(z)|^p dm(z) \leq \sum_{Q \in W} m(Q) \|\partial^n B \chi_{\Omega}\|_{L^\infty(Q)}^p
\]
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\[ \chi_\Omega = \chi_{\Omega_0} + (\chi_\Omega - \chi_{\Omega_0}) \]

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\[ |\partial B(\chi_{\Omega} - \chi_{\Omega_0})(z)| \leq \int_{\Omega \Delta \Omega_Q} \frac{dm(w)}{|z - w|^3} \]

\[ \int_{Q \cap \Omega} |\partial B \chi_{\Omega}(z)|^p dm(z) \]

\[ \leq \sum_{Q \in W} \int_{Q} |\partial B \chi_{\Omega}(z)|^p dm(z) \]

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