Joint approximation in BMO

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Abstract

A characterization of Mergelyan sets for the space of harmonic functions whose boundary values are in BMO (VMO) is obtained. The main step of the proof is the use of certain VMO functions which are related to the sharpness of the John-Nirenberg inequality.

This note is a continuation of the work [N-O]. Our main purpose is to describe Mergelyan sets for BMO (VMO), as asked in [S], using notation, techniques and arguments from [N-O].

5 Mergelyan sets

A relatively compact subset $F \subset \mathbb{R}^{d+1}_+$ is called a Mergelyan set for BMO (VMO) if for any $f \in$ BMO (VMO) and uniformly continuous on F there exists a sequence of continuous functions $\{p_n\}$ tending to f in the weak-$*$ topology (norm topology) and $p_n \longrightarrow f$ uniformly on F.

Theorem 6. *Let F be a relatively compact set in the upper half space $\mathbb{R}^{d+1}_+$. Then, the following conditions are equivalent:*

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(a) *F is a Mergelyan set for* $\mathrm{BMO}(\mathbb{R}^{d+1}_+)$ *equipped with the weak-$*$ topology.*

(b) *F is a Mergelyan set for* $\mathrm{VMO}(\mathbb{R}^{d+1}_+)$ *equipped with the norm topology.*

(c) *Almost every point of $\overline{F} \cap \mathbb{R}^d$ is the non-tangential limit of points of F, that is, $|\overline{F} \cap \mathbb{R}^d \backslash F_{nt}| = 0$.*

Proof of Theorem 6. We first show that condition (c) is necessary. We will proceed as in the first proof of Theorem 3 of [N-O]. Assume $|\overline{F} \cap \mathbb{R}^d \backslash F_{nt}| > 0$, that is, $|\overline{F} \cap \mathbb{R}^d \backslash F_\eta| > 0$ for some $\eta > 0$. Let x be a density point of $\overline{F} \cap \mathbb{R}^d \backslash F_\eta$. Observe that $x \in \overline{F}_\eta$ and apply Theorem 5 of [N-O] with the set $A = F_\eta$. So, one obtains a non-negative function $g \in \mathrm{VMO}(\mathbb{R}^d)$, $g(y) = 0$ for all $y \in F_\eta$, and

$$\lim_{\delta \to 0} g_{Q(x,\delta)} = \infty. \tag{5.1}$$

Then, we claim that the harmonic extension of g in $\mathbb{R}^{d+1}_+$ is uniformly continuous on F. That is, given $\varepsilon > 0$ there is some $\delta > 0$ such that if $z, w \in F$ and $|z - w| < \delta$ then $|g(z) - g(w)| < \varepsilon$. First of all, we will see that $|g(z)| < \varepsilon/2$ if $z = (x, z_{d+1}) \in F$ and $z_{d+1} \leq \delta_1$ using $g \in \mathrm{VMO}$ and $g(y) = 0$ for all $y \in F_\eta$. Let R be the cube in $\mathbb{R}^d$ of center x and side length z_{d+1}, observe

$$\begin{aligned}
|g(z)| = |g(z) - g_R| &= \left| \int (g(y) - g_R) P(x - y, z_{d+1})\, dy \right| \\
&\leq C \sum_{k \geq 1} 2^{-k} |g - g_R|_{2^k R} \leq C \sum_{k \geq 1} k 2^{-k} M(g, 2^k z_{d+1}) \\
&\leq C M(g, 2^{k_0} z_{d+1}) \sum_{k=1}^{k_0} k 2^{-k} + C \|g\|_* \sum_{k > k_0} k 2^{-k} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\end{aligned}$$

if k_0 is large enough and $2^{k_0} z_{d+1}$ is sufficiently small. Consequently, if $z, w \in F$ and $z_{d+1}, w_{d+1} \leq \delta_1$ then $|g(z) - g(w)| < \varepsilon$. On the other hand, g is uniformly continuous on compact sets of $\mathbb{R}^{d+1}_+$. Thus, there exists $\delta_2 > 0$ such that if $z, w \in F$, $z_{d+1}, w_{d+1} \geq \delta_1/2$ and $|z - w| < \delta_2$ then $|g(z) - g(w)| < \varepsilon$. Finally, take $\delta = \min(\delta_2, \delta_1/2)$ and this proves the claim.

Now, if F is a Mergelyan set for BMO (or VMO) there is a sequence of continuous functions $\{P_n\}$ tending to g in the weak-$*$ topology and $P_n(z) \to g(z)$ uniformly on F.

Thus, for some absolute constant C and for all n, $|P_n(z)| \leq C$, for any $z \in F$, $\|P_n\|_* \leq C$ and, by Lemma 2.1 of [N-O], we have

$$P_n(z) \to g(z) \quad \text{for any} \quad z \in \mathbb{R}^{d+1}_+. \tag{5.2}$$

By continuity, $|P_n(y)| \leq C$ at every point $y \in \overline{F} \cap \mathbb{R}^d$. Next, using that x is a density point of $\overline{F} \cap \mathbb{R}^d$ and $\sup_n \|P_n\|_* \leq C$ we get

$$|(P_n)_{Q(x,\delta)}| \leq 2C$$

if δ is small. Then, from the estimate $|P_n(z) - (P_n)_{T(z)}| \leq C\|P_n\|_*$ we deduce that $|P_n(z)| \leq 4C$ for all n where $z = (x,t)$ and $0 < t < \delta$. This contradicts (5.2) because from (5.1) the values $g(z)$ are unbounded when $z = (x,t)$, and t tends to 0.

Conversely, assume $|\overline{F} \cap \mathbb{R}^d \backslash F_{nt}| = 0$ and let us show that F is a Mergelyan set for BMO (VMO). So, given $f \in$ BMO ($f \in$ VMO), $\|f\|_* = 1$ and f uniformly continuous on F, one has to find continuous functions P_k tending to f in the weak-$*$ (norm) topology and uniformly on F.

Observe that, by Lemma 2.4 and Theorem 2.5 of [N-O], one can assume that f is bounded. Since f is uniformly continuous on F we can extend f on $\overline{F}$, call it $\tilde{f}$. We assume, without loss of generality, that

$$\tilde{f} \equiv 0 \qquad \text{on } \overline{F} \cap \mathbb{R}^d. \tag{5.3}$$

We now claim that it is sufficient to prove that given $\varepsilon > 0$ there are continuous functions Φ_k tending to f in the weak-$*$ (norm) topology and also pointwise on F, $\|\Phi_k\|_\infty \leq C\|f\|_\infty$, where C is a universal constant, and satisfying

$$|\Phi_k(x)| < \varepsilon, \quad \text{for} \quad x \in \overline{F} \cap \mathbb{R}^d. \tag{5.4}$$

To establish this claim, observe that Lemma 2.1 of [N-O] gives that Φ_k tend to f uniformly on compact sets of $\mathbb{R}^{d+1}_+$. Hence, there exists $\eta_k > 0$, $\eta_k \to 0$ as $k \to \infty$, such that

$$|\Phi_k(z) - f(z)| < \varepsilon \quad \text{for any} \quad z \in F \quad \text{such that} \quad z_{d+1} \geq \eta_k.$$

On the other hand, since Φ_k are continuous, f is uniform continuous on F, vanishing on $\overline{F} \cap \mathbb{R}^d$, and $\overline{F} \cap \mathbb{R}^d$ is a compact set, condition (5.4) gives that there is $\delta_k > 0$ such that

$$|\Phi_k(z) - f(z)| < \varepsilon \quad \text{for any} \quad z \in F \quad \text{with} \quad z_{d+1} \leq \delta_k.$$

Consequently, $|\Phi_k(z) - f(z)| < \varepsilon$ for the points $z \in F$ satisfying either $z_{d+1} \le \delta_k$ or $z_{d+1} \ge \eta_k$. Considering a subsequence of Φ_k one may assume that $\eta_k > \delta_k > \eta_{k+1}$ for any $k = 1, 2, \dots$. Now, one can take

$$P_N = \frac{1}{N} \sum_{k=N+1}^{2N} \Phi_k .$$

It is clear that P_N tend to f in the weak-$*$ (norm) topology. Also, if $z \in F$ there is at most one k such that $\delta_k < z_{d+1} < \eta_k$ or $\eta_{k+1} < z_{d+1} < \delta_k$. Hence for points $z \in F$, one has

$$|P_N(z) - f(z)| \le \frac{N-1}{N}\varepsilon + \frac{\|\Phi_k\|_\infty + \|f\|_\infty}{N} < \varepsilon + \frac{2C\|f\|_\infty}{N}$$

and this would finish the proof. Therefore, one only has to find the functions Φ_k mentioned in the previous claim.

Since $|\overline{F} \cap \mathbb{R}^d \backslash F_{nt}| = 0$ and condition (5.3), Fatou's Theorem gives that $f(x) = 0$ at almost every point $x \in \overline{F} \cap \mathbb{R}^d$. Given $k = 1, 2, \dots$ let $D(k)$ denote the collection of dyadic cubes in $\mathbb{R}^d$ of length side 2^{-k}. Proposition 2.3 of [N-O] asserts that the functions

$$\varphi_k = \sum_{Q \in D(k)} a_Q \Psi_Q$$

tend to f in the weak-$*$ topology and if $f \in \mathrm{VMO}$ tend to f in norm. Now, we require the continuity of the functions Ψ_Q and therefore the functions φ_k will be continuous. However, the natural choice $\Phi_k = \varphi_k$ does not work because φ_k may not satisfy $|\varphi_k| < \varepsilon$ on $\overline{F} \cap \mathbb{R}^d$. The same trouble would appear if we took $\Phi_k(x) = f(x, k^{-1})$, the restriction of f at level k^{-1}.

Denote by $\mathcal{B} = \mathcal{B}(k)$ the subcollection of those cubes Q in $D(k)$ satisfying that $\overline{F} \cap \frac{5}{4}Q \neq \varnothing$ and $\mathcal{A} = \mathcal{A}(k, \varepsilon)$ those cubes in $\mathcal{B}$ such that $|a_Q| \ge \varepsilon$. If $x \in \overline{F} \backslash \bigcup_{\mathcal{A}} \frac{5}{4}Q$ then $|\varphi_k(x)| \le \sum_{Q} |a_Q| \Psi_Q(x) < \varepsilon$. Thus, we should modify φ_k on the points $x \in \overline{F} \cap \left(\bigcup_{\mathcal{A}} \frac{5}{4}Q \right)$. We next claim that

$$\sum_{Q \in \mathcal{A}} |Q| \to 0 \quad \text{as} \quad k \to \infty. \tag{5.5}$$

Note first that $\sum_{Q \in \mathcal{B}} \Psi_Q \to \chi_{\overline{F}}$ in $L^1(\mathbb{R}^d)$ as $k \to \infty$, because $\overline{F}$ is a compact set, and so $\sum_{Q \in \mathcal{B}} a_Q \Psi_Q$ tend to $f \chi_{\overline{F}} \equiv 0$ in $L^1(\mathbb{R}^d)$. Fix a cube $Q \in \mathcal{A}$. For all $x \in \frac{3}{4}Q$, one has

$$\left| \sum_{Q \in \mathcal{B}} a_Q \Psi_Q(x) \right| = |a_Q| \ge \varepsilon.$$

Consequently,

$$\left|\bigcup_{\mathcal{A}} \frac{3}{4}Q\right| \le \left|\left\{x \in \mathbb{R}^d : \left|\sum_{\mathcal{B}} a_Q \Psi_Q(x) - f(x)\chi_{\overline{F}}(x)\right| \ge \varepsilon\right\}\right| \underset{k\to\infty}{\longrightarrow} 0$$

and then we get (5.5).

Since $f(x) = 0$ at almost every $x \in \overline{F} \cap \mathbb{R}^d$ and a_Q is close to $f_{\frac{5}{4}Q}$ (that is, $|a_Q - f_{5/4Q}| \le C\|f\|_*$) from the John-Nirenberg Theorem we point out that there exist constants C_1, $C_2 > 0$ such that for any $Q \in \mathcal{B}$ one has

$$\left|\frac{5}{4}Q \cap \overline{F}\right| \le C_1 \exp(-C_2|a_Q|) \left|\frac{5}{4}Q\right|. \tag{5.6}$$

Moreover, if $f \in$ VMO one may take $C_2 = C_2(k) \to \infty$ as $k \to \infty$.

Fix a cube $Q \in \mathcal{A}$. Now, we apply the proof of the Main Lemma (finite case, because $\frac{5}{4}Q \cap \overline{F}$ is compact) and we obtain a non-negative continuous function $g = g_Q$ satisfying

$$\begin{aligned} &g \equiv 0 && \text{on} \quad (2Q)^c \\ &g \equiv |a_Q| && \text{on} \quad \frac{5}{4}Q \cap \overline{F}, \\ &⨍_Q g \le C, \\ &\|g\|_\infty \le C\|f\|_\infty, \\ &\|g\|_* \le m(C_2) \end{aligned}$$

where C is a constant independent of k and $m(C_2) \to 0$ as $C_2 \to \infty$. In particular, if $f \in$ VMO, $m(C_2) \to 0$ as $k \to \infty$. Denote by $\mathcal{A}^+$ those cubes in $\mathcal{A}$ such that $a_Q > 0$ and $\mathcal{A}^-$ those cubes in $\mathcal{A}$ such that $a_Q < 0$. Define

$$g_k = \sum_{Q \in \mathcal{A}^+} g_Q \qquad \text{and} \qquad h_k = -\sum_{Q \in \mathcal{A}^-} g_Q.$$

Again, $\|g_k\|_* \le C \max_{Q \in \mathcal{A}^+} \|g_Q\|_* \le Cm(C_2)$ and $\|h_k\|_* \le C \max_{Q \in \mathcal{A}^-} \|g_Q\|_* \le Cm(C_2)$. From (5.5) and the estimates $\|g_k\|_\infty \le C\|f\|_\infty$, $\|h_k\|_\infty \le C\|f\|_\infty$ we have $g_k \to 0$ in L^1 and $h_k \to 0$ in L^1. Thus, $g_k \to 0$ and $h_k \to 0$ in the weak-$*$ topology (or in norm if $f \in$ VMO). Then, by the Lemma 5.1 $(\varphi_k - g_k)^+ = \max(\varphi_k - g_k, 0)$ tends to f^+ in the weak-$*$ topology (or in norm if $f \in$ VMO) and $(\varphi_k - h_k)^- = \max(-\varphi_k + h_k, 0)$ tends to f^- in the weak-$*$ topology (or in norm if $f \in$ VMO). Finally, take

$$\Phi_k = (\varphi_k - g_k)^+ - (\varphi_k - h_k)^-$$

and the proof is completed. Note that if $x \in \overline{F} \cap \mathbb{R}^d$ then $(\varphi_k - g_k)^+(x) < \varepsilon$ and $(\varphi_k - h_k)^-(x) < \varepsilon$ $\square$

Lemma 5.1. *(a) Let $\{f_j\}$ be a sequence of functions in* $\mathrm{BMO}(\mathbb{R}^d)$ *and* $f \in \mathrm{BMO}(\mathbb{R}^d)$*. Assume that $f_j \to f$ in the weak-$*$ topology and $f_j \to f$ in L^1. Then $|f_j| \to |f|$ in the weak-$*$ topology.*

(b) Let $\{f_j\}$ be a sequence of functions in $\mathrm{VMO}(\mathbb{R}^d)$*. Assume that $f_j \to f$ in norm in* BMO *and $f_j \to f$ in L^1. Then $|f_j| \to |f|$ in norm in* BMO.

Remark: Since $f = f^+ - f^-$ and $|f| = f^+ + f^-$, one also gets $(f_j)^+ \to f^+$.

Proof. (a) There exists a subsequence $\{|f_k|\}$ tending to some $g \in \mathrm{BMO}$ in the weak-$*$ topology, because $\| \, |f_j| \, \|_* \leq C\|f_j\|_* < \infty$. Moreover, $|f_k| \to |f|$ in L^1. Clearly, $|f| = g$.

(b) Given $\varepsilon > 0$ we will show that $\| \, |f_j| - |f| \, \|_* \leq \varepsilon$ if $j \geq j_0$. Notice that $\{f_j\}$ and f are uniformly in VMO, therefore if $|Q| < \delta$ we have

$$\frac{1}{|Q|}\int_Q \big| \, |f_j| - |f| - (|(f_j)_Q| - |f_Q|) \, \big| \leq \frac{1}{|Q|}\int_Q |f_j - (f_j)_Q| \; + \frac{1}{|Q|}\int_Q |f - f_Q| < \varepsilon$$

When $|Q| \geq \delta$ we use the L^1-convergence to get

$$\frac{1}{|Q|}\int_Q \big| \, |f_j| - |f| \, \big| \leq \frac{\|f_j - f\|_1}{\delta} < \varepsilon$$

if j is bigger than some j_0. $\square$

Counterexample: One could guess that some hypothesis in the above Lemma are superfluous, that is, that f_j tending f in BMO always would imply $|f_j| \to |f|$ in BMO. But the following examples (provided to us by John Garnett) show that this is wrong.

On $\mathbb{R}$ take $f(x) = 3\sin(2\pi x)$. Given $j \in \mathbb{N}$ there is $g_j \in BMO$ with $\|g_j\|_* < 1/j$ and I_j^1, I_j^2 two disjoint intervals of unit length (whose spacing depends on j) such that $g_j = -1$ on I_j^1 and $g_j = 1$ on I_j^2. Thus $f + g_j \longrightarrow f$ in norm in BMO.

However, on the interval I_j^2 $|f + g_j| - |f| = \max[\min(1, 2f + 1), -1]$ (something similar happens for I_j^1 also), so that $\| |f + g_j| - |f| \|_*$ is big.

A modification of this example will give functions $f, \{g_j\} \in L^1 \cap \mathrm{BMO}$ such that $f + g_j \to f$ in norm in BMO and $f + g_j \to f$ in L^1, but $|f + g_j|$ does not converge to

$|f|$ in BMO. For each $j \in \mathbb{N}$ consider the intervals $I_j = [a_j, b_j]$ where $a_j = j - je^{-j^2}$ and $b_j = j + je^{-j^2}$. Define

$$g_j(x) = \min\left(1, \frac{1}{j^2}\log^+ |\frac{j}{j-x}|\right)$$

and $f(x) = 0$ if $x \notin \cup I_j$ and $f(x) = 3\sin(2\pi\frac{x-a_j}{b_j-a_j})$ if $x \in I_j$. Clearly, $\|g_j\|_1 \leq 2/j$ and $\|g_j\|_* \leq C/j^2$ and we obtain the example.

We finish this section remarking that any compact set of positive Lebesgue measure supports a VMO function.

Lemma 5.2. *For any compact set K of $\mathbb{R}^d$ of positive Lebesgue measure, $|K| > 0$, there is a function $f \in$ VMO satisfying*

(a) $0 \leq f \leq 1$.

(b) f *is supported in* K.

(c) $\displaystyle\int_K f \geq 2^{-d-2}|K|$.

Proof. Given $\varepsilon > 0$ (a small number to fix later), using that almost every point in K is a density point, we consider a finite family of cubes $\{Q_j\}_{j=1,\dots,N}$ pairwise disjoint (N depends on K and ε) such that

(a) $|Q_j \setminus K| < \varepsilon |Q_j|$.

(b) $\displaystyle\sum_{j=1}^N |Q_j \cap K| \geq 2^{-1}|K|$.

Now, to each Q_j we apply the Main Lemma of [N-O] getting a function $h_j \in$ VMO with the properties: $0 \leq h_j \leq 1$, $h_j \equiv \log(1/\varepsilon)$ on $Q_j \setminus K$, h_j is supported into $\frac{3}{2}Q_j$ and $|Q_j|^{-1}\int_{Q_j} h_j \leq C_1 = C_1(d)$. Define

$$f_j = 1 - \frac{h_j}{\log 1/\varepsilon},$$

so we have $0 \leq f_j \leq 1$, $f_j(x) = 0$ if $x \in Q_j \setminus K$ and

$$\begin{aligned}\int_{\frac{1}{2}Q_j} f_j = \int_{\frac{1}{2}Q_j} 1 - \frac{h_j}{\log 1/\varepsilon} &= |1/2Q_j| - (\log 1/\varepsilon)^{-1}\int_{\frac{1}{2}Q_j} h_j \\ &\geq |Q_j|\,(2^{-d} - (\log 1/\varepsilon)^{-1}C_1) \\ &= |Q_j|\,2^{-d-1},\end{aligned}$$

if one takes $\log 1/\varepsilon = C_1 2^{d+1}$. Let a_j be a continuous function supported in Q_j, $0 \le a_j \le 1$ and $a_j \ge 1/2$ on $\frac{1}{2}Q_j$. Clearly, $a_j f_j$ belongs to VMO because both are bounded and belong to VMO. Finally, the function

$$f = \sum_{j=1}^{N} a_j f_j$$

satisfies the claim:

$0 \le f \le 1$,

it is supported in K,

it is a finite sum of VMO functions and

$$\int_K f = \sum_{j=1}^{N} \int_{Q_j} a_j f_j \ge \frac{1}{2} \sum_{j=1}^{N} \int_{\frac{1}{2}Q_j} f_j$$
$$\ge 2^{-d-1} \sum_{j=1}^{N} |Q_j| \ge 2^{-d-2}|K|.$$

□

From this Lemma, J.J. Donaire [D] observed that compact sets in $\mathbb{C}$ of positive area are nonremovable for analytic functions in $\lambda_*(\mathbb{C})$, the little Zygmund space.

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