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The space of subgroups of an abelian group

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Abstract

We carry out the Cantor–Bendixson analysis of the space of all subgroups of any countable abelian group and we deduce a complete classification of such spaces up to homeomorphism.

Introduction

Let G denote a discrete group. The set $\mathcal{N}(G)$ of all normal subgroups of G has a natural topology called the Chabauty topology. It is a setting of interesting interplay between topological phenomena and algebraic properties of the group G. Introduced by Chabauty in [**2**], it reappeared in the work of Gromov [**7**] and Grigorchuk [**6**], where it proved to be a useful tool to understand asymptotic properties of discrete groups; for instance, see [**4**] for further applications. More precisely, consider the set of subsets of G, viewed as the product 2^G, endowed with the product (Tychonov) topology. The subset $\mathcal{N}(G)$ is easily seen to be closed. By construction, this is a compact, totally disconnected Hausdorff topological space, that is, a Boolean space. If G is countable, then it is metrizable. For this topology, a net (N_i) of normal subgroups converges to N if and only if for every $g \in N$ or $g \notin N$, eventually $g \in N_i$ or $g \notin N_i$ respectively.

In general, very little is known on the global structure of $\mathcal{N}(G)$ for a given group G; however, see [**3**, **5**], which especially deal with the case when G is free of finite rank. In this paper we treat the case where the group G is abelian (not necessarily finitely generated) and we preferably write A in place of G. We obviously have $\mathcal{N}(A) = \mathcal{S}(A)$, the space of subgroups of A. This is not unrelated to finitely generated groups: indeed, if A is any countable abelian group, then A embeds into the centre of a finitely generated group G, giving an obvious embedding of $\mathcal{S}(A)$ into $\mathcal{N}(G)$. The classification of the spaces $\mathcal{S}(A)$ for an abelian group A turns out to be much more tractable than both of its natural and difficult generalizations, namely the classification of either general Boolean spaces or general abelian groups; see, for instance, [**11**, **12**] concerning these problems. Indeed, we are able to provide a complete description of the spaces $\mathcal{S}(A)$ in terms of natural, and computable, invariants of the countable abelian group A.

A topological space X is called *perfect* if it has no isolated point and, at the other extreme, *scattered* if any non-empty subset has an isolated point. As a union of perfect subsets is perfect, every topological space has a unique largest perfect subset, called its *condensation part* and denoted by $\mathrm{Cond}(X)$, which is empty if and only if X is scattered. The subset $X - \mathrm{Cond}(X)$ is the largest scattered open subset, and is called the *scattered part* of X.

If A is an abelian group, its torsion elements form a subgroup denoted by T_A. Recall that an element of A is called divisible if it belongs to nA for all non-zero integers n. The set of divisible elements in A form a subgroup denoted by $\mathrm{Div}(A)$ and it is easy to check that it always has a direct complement in A. Given a prime p, we define $C_{p^\infty} = \mathbf{Z}[1/p]/\mathbf{Z}$; this is called a quasi-cyclic group. An abelian group is called *Artinian* if every non-increasing sequence of

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subgroups stabilizes; every such group is isomorphic to a direct sum $A = \bigoplus_{i=1}^{h} C_{p_i^\infty} \oplus F$ for some finite subgroup F; and the finite index subgroup $\bigoplus_{i=1}^{h} C_{p_i^\infty}$ coincides with $\mathrm{Div}(A)$. An abelian group A is called *minimax* if it has a finitely generated subgroup Z with A/Z Artinian. Such a subgroup Z is called a *lattice* in A.

PROPOSITION A (Corollaries 2.1.2 and 2.1.3). *Let A be an abelian group. Then $\mathcal{S}(A)$ is non-perfect (that is, contains an isolated point) if and only if A is minimax. In particular, we have the following.*

(1) *If A is countable and not minimax, then $\mathcal{S}(A)$ is a Cantor space.*
(2) *If A is uncountable, then $\mathcal{S}(A)$ is perfect.*

When the abelian group A is uncountable, we do not have any classification result for the perfect space $\mathcal{S}(A)$, except the following, which shows that the cardinality of A can be read out of the topology of $\mathcal{S}(A)$.

PROPOSITION B (see Subsection 2.2). *Suppose that the abelian group A has uncountable cardinal α. Then $\mathcal{S}(A)$ contains a subset homeomorphic to 2^α. Accordingly, α is the least cardinality of a basis for the topology of $\mathcal{S}(A)$. In particular, if A is uncountable, then $\mathcal{S}(A)$ is not metrizable.*

Our main result is the determination of the homeomorphism type of $\mathcal{S}(A)$ when A is a countable abelian group. Proposition A settles the case of non-minimax ones.

Recall that the *rank* of an abelian group A, denoted by $r(A)$, is the largest cardinal of a $\mathbf{Z}$-free family in A; this is also the $\mathbf{Q}$-dimension of the vector space $A \otimes_{\mathbf{Z}} \mathbf{Q}$. If A is minimax, then $r(A) < \infty$. We also have to introduce the notion of critical prime of a minimax abelian group, which plays a crucial role here. If A is a minimax abelian group and p a prime, then we define $\ell_p(A)$ as the largest integer k so that A maps onto $C_{p^\infty}^k$. Given a lattice Z in A, this is also the greatest integer k so that $C_{p^\infty}^k$ embeds into A/Z. The sum

$$h(A) = r(A) + \sum_{p \text{ prime}} \ell_p(A)$$

is finite and called the *height* of A. When A is not minimax, we set $h(A) = \infty$. A prime p is called *critical* for A if $\ell_p(A) \geqslant 2$. The set of critical and non-critical primes for A is denoted by $\mathrm{cr}(A)$ and $\mathrm{ncr}(A)$, respectively. The minimax abelian group A is called *critical* if $\mathrm{cr}(A) \neq \emptyset$. Finally, if A is a minimax abelian group, then $\mathrm{Div}(A)$ is contained in T_A as a subgroup of finite index; the number of subgroups of the finite group $T_A/\mathrm{Div}(A)$ is denoted by $n(A)$.

Let $[n]$ denote the set $\{0, \ldots, n-1\}$ with n elements and let $\omega = \bigcup_n [n]$. Let D denote the topological space $\omega \cup \{\infty\}$ consisting of the discrete sequence of points $(n)_{n \geqslant 0}$ converging to the limit ∞, which is homeomorphic to the subset $\{1/n | \, n \geqslant 1\} \cup \{0\}$ of $\mathbf{R}$. For any integer m, the space D^m is scattered, since D is scattered and the class of scattered spaces is closed under finite cartesian products.

THEOREM C (Theorem 4.5.1). *Let A be a non-critical minimax abelian group, and write $h = h(A)$ and $n = n(A)$. Then the space $\mathcal{S}(A)$ is countable and homeomorphic to $D^h \times [n]$.*

All the cases occur (for $h \geqslant 0$ and $n \geqslant 1$); for instance, the abelian group $A = \mathbf{Z}^h \oplus \mathbf{Z}/2^{n-1}\mathbf{Z}$ has $h(A) = h$ and $n(A) = n$, and as a finitely generated group, it is minimax and non-critical.

Let us now deal with critical minimax abelian groups. Again, we have to introduce some more definitions. Let A be a minimax abelian group and let V be a set of primes. First define

$$\ell_V(A) = \sum_{p \in V} \ell_p(A).$$

Let $\mathbf{Z}_V \subset \mathbf{Q}$ denote the ring of rationals whose denominator is divisible by no $p \in V$. Let $U_V(A) \leqslant A$ be the intersection of kernels of group homomorphisms $A \to \mathbf{Z}_V$. Set $a_V(A) = r(U_V(A))$ and $\gamma_{V^c}(A) = r(A/U_V(A))$. Note that $A/U_V(A)$ embeds into $\mathbf{Z}_V^{\gamma_{V^c}(A)}$.

On the other hand, let W be a 'dusty Cantor space'; namely a compact metrizable space consisting of the union of a Cantor space with an open dense countable discrete set. It is a consequence of Pierce's theorem (Theorem 1.3.3) that W is thus uniquely defined up to homeomorphism. For instance, W may be chosen as the union of the triadic Cantor set (which is the image of $\{0,2\}^{\mathbf{N}}$ in $\mathbf{R}$ by the injective continuous map $u \mapsto \sum_{n \geqslant 0} u(n)3^{-n-1}$) and the set of centres of all intervals in the complement, namely the set of reals of the form $\sum_{n \geqslant 0} v(n)3^{-n-1}$, where $v(n)$ is a sequence such that, for some $n_0 \geqslant 0$, we have $v(n) \in \{0,2\}$ for all $n < n_0$ and $v(n) = 1$ for all $n \geqslant n_0$.

Theorem D (Theorem 4.5.1). *Let A be a critical minimax abelian group. Then $\mathcal{S}(A)$ is uncountable and homeomorphic to $D^\sigma \times W$, where $\sigma = \sigma(A)$ is defined as follows:*

$$\begin{aligned}\sigma(A) &= \gamma_{\mathrm{ncr}(A)}(A) + \ell_{\mathrm{ncr}(A)}(A) \\ &= h(A) - (a_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{cr}(A)}(A)).\end{aligned}$$

Again, all the cases occur: the minimax group $(C_{2^\infty})^2 \times \mathbf{Z}^\sigma$ is critical and has $\sigma(A) = \sigma$. Even better, for a minimax group A with given $h(A)$, if $\mathcal{S}(A)$ is uncountable, then $0 \leqslant \sigma(A) \leqslant h(A) - 2$, and all these cases occur; taking $A = (C_{2^\infty})^{h-\sigma} \times \mathbf{Z}^\sigma$, we have $h(A) = h$ and $\sigma(A) = \sigma$ provided that $0 \leqslant \sigma \leqslant h - 2$.

From the conjunction of Theorems C and D, we get the following corollaries.

Corollary E (Boyer [1]). *Let A be an abelian group. Then $\mathcal{S}(A)$ is countable if and only if A is a non-critical minimax group.*

Note that the harder implication, namely the forward one, follows from Theorem C.

Corollary F (see Lemma 1.3.6). *Let A and B be countable abelian groups. The spaces $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are homeomorphic if and only if one of the following holds:*

(i) $h(A) = h(B) = \infty$;

(ii) *both A and B are minimax and non-critical, $h(A) = h(B)$ and $n(A) = n(B)$;*

(iii) *both A and B are minimax and critical, and $\sigma(A) = \sigma(B)$.*

Next, we answer the question as to where a given subgroup S of A lies in $\mathcal{S}(A)$.

Let X be a topological space. Let $X^{(1)}$ be its derived subspace, which is the set of its accumulation points, that is, non-isolated points. Define by induction $X = X^{(0)}$ and $X^{(n+1)} = (X^{(n)})^{(1)}$. If $x \in X^{(n)} - X^{(n+1)}$, we say that the Cantor–Bendixson rank† of x in X is n and we write it $\mathrm{CB}_X(x) = n$. Note that $\mathrm{Cond}(X) \subset X^{(n)}$ for all n. If X is a metrizable Boolean

†We are avoiding the use of ordinals in this introduction. See Subsection 1.1 for the extension of the Cantor–Bendixson as an ordinal-valued function defined on all the scattered part of X.

space, it can be checked that the Cantor–Bendixson rank of x in X is the integer n if and only if there exists a homeomorphism of a neighbourhood of x (which can be chosen clopen) to D^n mapping x to $(\infty, \dots, \infty)$. If there is some n such that $\mathrm{CB}_X(x) \leqslant n$ whenever $\mathrm{CB}_X(x)$ is defined[‡], we also define, for any $x \in X$, its *extended Cantor–Bendixson rank* as

$$\overline{\mathrm{CB}}_X(x) = \inf \sup\{\mathrm{CB}_X(y) | \, y \in V, \ \ \mathrm{CB}_X(y) < \infty\},$$

where the infimum ranges over all neighbourhoods of x. This extends the function CB_X (assuming $\sup \emptyset = -\infty$).

For a minimax abelian group A and a prime p, define

$$\kappa_p(A) = \ell_p(A/T_A) \quad \text{and} \quad \tau_p(A) = \ell_p(T_A);$$

and, for a set of primes V, we have

$$\kappa_V(A) = \sum_p \kappa_p(A) \quad \text{and} \quad \tau_V(A) = \sum_p \tau_p(A).$$

If $S \leqslant A$, define

$$\begin{aligned} d_A(S) &= \gamma_{\mathrm{ncr}(A)}(A/S) + \kappa_{\mathrm{ncr}(A)}(A/S) + \ell_{\mathrm{ncr}(A)}(S) \\ &= \sigma(A) - (\tau_{\mathrm{ncr}}(A/S) + \gamma_{\mathrm{ncr}(A)}(A) - \gamma_{\mathrm{ncr}(A)}(A/S)). \end{aligned}$$

THEOREM G (Theorem 4.5.1). *Let A be a minimax abelian group and let $S \leqslant A$. We have the following:*

(i) *S is in the scattered part of $\mathcal{S}(A)$ if and only if*

$$\kappa_{\mathrm{cr}(A)}(A/S) = \ell_{\mathrm{cr}(A)}(S) = 0;$$

(ii) *the extended Cantor–Bendixson rank of S in $\mathcal{S}(A)$ is $d_A(S)$.*

The outline of the paper is as follows. In Section 1, we establish some topological preliminaries, notably a characterization of the extended Cantor–Bendixson function by semi-continuity and hereditary properties (see Subsection 1.2) and a topological characterization of the spaces involved in Theorems C and D (see Subsection 1.3). The short Section 2 is devoted to general abelian groups, and we prove Propositions A and B there. Sections 3 and 4 are devoted to the study of $\mathcal{S}(A)$ when A is a minimax group. Section 3 contains enough to prove Theorem C, while both sections are necessary to obtain Theorems D and G.

Sections 3 and 4 contain a number of preliminary results, pertaining to the topology of commensurability classes in $\mathcal{S}(A)$ for an abelian minimax group A, which can be of independent interest.

1. *Topological preliminaries*

1.1. *Cantor–Bendixson analysis*

Let $X = X^{(0)}$ be a topological space. If $X^{(1)}$ is its derived subspace, by transfinite induction one can define $X^{(\alpha)}$ as the derived subspace of $\bigcap_{\beta<\alpha} X^{(\beta)}$. This is a decreasing family of closed subsets of X, and the first α is such that $X^{(\alpha)}$ is perfect, is called the Cantor–Bendixson rank of X and denoted by $\mathrm{CB}(X)$. The advantage of this ordinal-valued definition is that

[‡]This is the case if $X = \mathcal{S}(A)$ with A an arbitrary abelian group, as a consequence of Proposition A and Theorems C and D.

$\mathrm{Cond}(X) = \bigcap_\alpha X^{(\alpha)}$ (if we restrict to integers, this is only an inclusion $\subset$ in general). If $x \notin \mathrm{Cond}(X)$, then its Cantor–Bendixson rank is defined as

$$\mathrm{CB}_X(x) = \sup\{\alpha |\, x \in X^{(\alpha)}\}.$$

This function is extended to all of X by

$$\overline{\mathrm{CB}}_X(x) = \inf \sup\{\mathrm{CB}_X(y) |\, y \in V - \mathrm{Cond}(X)\},$$

where the infimum ranges over all neighbourhoods of x, assuming that $\sup \emptyset = -\infty$.

1.2. *Semi-continuity, heredity*

Definition 1.2.1. Denote by Ord the collection of ordinals. Let X be a topological space. A map $f : X \to \mathrm{Ord}$ is called *upper semi-continuous* at $x_0 \in X$ if f has a local maximum at x_0. It is *strictly* upper semi-continuous at x_0 if it is a strict local maximum, that is, if $f(x) < f(x_0)$ when x is close enough to x_0.

Definition 1.2.2. Let X be a topological space.

(i) If $Y \subset X$ is a dense subset, a map $X \to \mathrm{Ord}$ is Y-hereditary if for every $x \in X$ and for every neighbourhood V of x in X, we have

$$f(x) \leqslant \sup_{y \in V \cap Y} f(y).$$

(ii) A map $X \to \mathrm{Ord}$ is strictly hereditary if for every $x \in X$ and for every neighbourhood V of x in X, we have

$$f(x) \leqslant \sup_{x' \in V \setminus \{x\}} (f(x') + 1),$$

where we set $\sup \emptyset = 0$.

Lemma 1.2.3. *Let X be a topological space, with two subsets $I, C \subset X$ and a map $f : \overline{I} \longrightarrow \mathrm{Ord}$. Assume that the following conditions are satisfied:*

(i) *C has no isolated point;*
(ii) *$X = I \cup C$;*
(iii) *f is upper semi-continuous and I-hereditary on $\overline{I}$;*
(iv) *f is strictly upper semi-continuous and strictly hereditary on I.*

Then $I = \mathcal{I}(X)$, $C = \mathrm{Cond}(X)$ and f coincides with $\overline{\mathrm{CB}}_X$ on $\overline{I}$.

Proof. First, using (iv), by induction on α, if $x \in I$ and $f(x) = \alpha$, we get $x \in \mathcal{I}(X)$ and the Cantor–Bendixson rank of x in X is α. Hence $I \subset \mathcal{I}(X)$. Moreover, by (i) we have $C \subset \mathrm{Cond}(X)$. Therefore, using (ii), we get $I = \mathcal{I}(X)$ and $C = \mathrm{Cond}(X)$. Finally, using (iii), we get that f coincides with the extended Cantor–Bendixson rank on $C \cap \overline{I}$. □

1.3. *Characterization of some topological spaces*

It is very useful to have a characterization of some topological spaces. For instance, we already used in the introduction the classical fact (see [**8**, Theorem 7.4]) that if a non-empty topological space is metrizable, compact, perfect and totally disconnected, then it is a Cantor space. (By definition, a Cantor space is a space homeomorphic to the triadic Cantor set.)

The second case concerns scattered spaces. It is known [**10**] that a non-empty Hausdorff compact scattered topological space is characterized, up to homeomorphism, by its Cantor–Bendixson rank (an arbitrary ordinal, countable in the metrizable case), and the number of points of maximal Cantor–Bendixson rank (an arbitrary positive integer). For our purposes it is enough to retain the following.

PROPOSITION 1.3.1. *Let X be a Hausdorff compact scattered topological space of finite Cantor–Bendixson rank $m+1 \geqslant 1$, with $n \geqslant 1$ points of maximal Cantor–Bendixson rank (equal to m). Then X is homeomorphic to $D^m \times [n]$.*

This applies, for instance, to the ordinal $\omega^m \cdot n + 1$.

If X is a topological space, let us define

$$C_\alpha(X) = \{x \in \overline{\mathcal{I}(X)} \cap \mathrm{Cond}(X) \mid \overline{\mathrm{CB}}_X(x) \geqslant \alpha\}.$$

PROPOSITION 1.3.2. *Let X be a metrizable Boolean space and let $\sigma < \infty$. Assume that $\mathrm{CB}(X) < \infty$ and we have the following:*
(i) *$\mathrm{CB}(X) = \sigma + 1$ and $C_i(X)$ is a Cantor space for all $i \leqslant \sigma$;*
(ii) *$C_{i+1}(X)$ has empty interior in $C_i(X)$ for all i.*
Then X is homeomorphic to $D^\sigma \times W$.

We make use of the following result of Pierce [**11**, Theorem 1.1]. Here, we assume that a point not in the closure of the scattered part has extended Cantor–Bendixson rank $-\infty$.

THEOREM 1.3.3 (Pierce). *Let X and Y be metrizable Boolean spaces. Let ϕ be a homeomorphism $\mathrm{Cond}(X) \to \mathrm{Cond}(Y)$. Suppose that $\mathcal{I}(X)$ is homeomorphic to $\mathcal{I}(Y)$ and that ϕ preserves the extended Cantor–Bendixson rank. Then ϕ extends to a homeomorphism $X \to Y$.*

We also need the following lemma. Let us view D^{m-1} as the subspace $D^{m-1} \times \{\infty\}$ of D^m (D^0 being a singleton).

LEMMA 1.3.4. *Let $K = \{0,1\}^{\mathbf{N}}$ be the Cantor discontinuum and let $0 \leqslant \sigma < \omega$. Let $K = K_\sigma \supset K_{\sigma-1} \supset \cdots \supset K_0$ be subsets of K all homeomorphic to K and such that K_i has empty interior in K_{i+1} for each $0 \leqslant i \leqslant \sigma - 1$. Then there is a homeomorphism $K \to K \times D^\sigma$ mapping K_i to $K \times D^i$ for each $0 \leqslant i \leqslant \sigma$.*

The proof is an induction based on the following theorem.

THEOREM 1.3.5 [**9**, Theorem 2]. *Let K_i $(i = 1, 2)$ be Cantor spaces and let $C_i \subset K_i$ be closed subsets with empty interior in K_i. Assume that there is a homeomorphism $h : C_1 \to C_2$. Then there is a homeomorphism $\widetilde{h} : K_1 \to K_2$ extending h.*

Proof of Lemma 1.3.4. The proof is an induction on σ.

Step 1. The result is obvious if $\sigma = 0$ and follows from Theorem 1.3.5 if $\sigma = 1$.

Step 2. Assume now that $\sigma > 1$. Apply step 1 to get a homeomorphism $\phi : K \to K \times D$ with $\phi(K_{\sigma-1}) = K \times \{\infty\}$. By induction, there exists a homeomorphism $\psi : K \to K \times D^{\sigma-1}$ mapping $\phi(K_i)$ to $K \times D^i$ for all $0 \leqslant i \leqslant \sigma - 1$. Then, for $i \leqslant \sigma - 1$, the homeomorphism $(\psi \times \mathrm{Id}) \circ \phi$ of K to $K \times D^{\sigma-1} \times D = K \times D^\sigma$ maps K_i to $K \times D^i \times \{\infty\} = K \times D^i$. □

Proof of Proposition 1.3.2. The condition is obviously necessary.

Conversely, set $K_i = C_{\sigma-i}$ and use Lemma 1.3.4 to get a homeomorphism $\text{Cond}(X) \to \text{Cond}(Y)$ preserving the extended Cantor–Bendixson rank. The hypothesis on scattered parts and Pierce's theorem then allow one to get the desired homeomorphism. □

Lemma 1.3.6. *The spaces $D^m \times [n]$ $(m \geqslant 0,\ n \geqslant 1)$, $D^m \times W$ $(m \geqslant 0)$ and K (a Cantor space) are pairwise non-homeomorphic.*

Proof. The only perfect space here is K. The other uncountable ones are $D^m \times W$, which has Cantor–Bendixson rank $m+1$. The countable space $D^m \times [n]$ has Cantor–Bendixson rank $m+1$ and exactly n points of maximal Cantor–Bendixson rank (m). □

2. *Generalities*

2.1. *Isolated points*

Proposition 2.1.1. *Let A be an abelian group and let $S \in \mathcal{S}(A)$. Then S is isolated in $\mathcal{S}(A)$ if and only if S is finitely generated and A/S is Artinian.*

This follows from [**5**, Lemma 1.3, Proposition 2.1 and Lemma 4.1].

Corollary 2.1.2. *Let A be an abelian group. Then $\mathcal{S}(A)$ has isolated points (that is, non-perfect) if and only if A is minimax. In this case, isolated points in $\mathcal{S}(A)$ form exactly one commensurability class, namely the lattices in A.*

Corollary 2.1.3. *If a countable abelian group A is not a minimax group, then $\mathcal{S}(A)$ is a Cantor space.*

These two corollaries settle Proposition A.

2.2. *Uncountable groups*

Proof of Proposition B. If A/T_A has cardinality α, then A contains a copy of $\mathbf{Z}^{(\alpha)}$. Otherwise, T_A has cardinality α, and denoting by A_p the p-torsion in A, the direct sum $\bigoplus_p A_p$ can be written as a direct sum of α cyclic subgroups of prime order. Hence in both cases, A contains a subgroup isomorphic to a direct sum of non-trivial (cyclic) subgroups $\bigoplus_{i\in\alpha} S_i$. The mapping $2^A \to \mathcal{S}(A)$, $J \mapsto \bigoplus_{j\in J} S_j$ is the desired embedding. □

Lemma 2.2.1. *Let α be an infinite cardinal. The least cardinal for a basis of open sets in 2^α is α.*

Proof. The natural basis of the topological space 2^α has cardinality α. Conversely, if a basis has cardinality β, then it provides a basis of the dense subset $2^{(\alpha)}$. Thus $2^{(\alpha)}$ contains a dense subset $\mathcal{D}$ of cardinality not greater than β. The union of (finite) supports of all $f \in \mathcal{D}$ must be all of α, and so $\beta \geqslant \alpha$. □

PROPOSITION 2.2.2. *Let α be an infinite cardinal and let A be an abelian group of cardinal α. The least cardinal for a basis of open sets in $\mathcal{S}(A)$ is α.*

Proof. As we have topological embeddings $2^\alpha \subset \mathcal{S}(A) \subset 2^\alpha$ (the right-hand one being the inclusion $\mathcal{S}(A) \subset 2^A$, and the left-hand one following from Proposition B), this follows from Lemma 2.2.1. □

3. *The weight function on $\mathcal{S}(A)$*

In this section, all minimax groups are assumed abelian.

3.1. *Critical primes, idle subgroups and parallelism*

DEFINITION 3.1.1. Let A be a minimax group. Two subgroups S and S' are said to be *parallel* if they have a common lattice, and $\ell_p(S) = \ell_p(S')$ for every prime p.

Clearly, this is an equivalence relation. Commensurable implies parallel; the obstruction to the converse comes from what we call critical primes.

DEFINITION 3.1.2 (Strong Criticality). Let A be a minimax group and let $S \leqslant A$.
(i) The subgroup S is p-critical if p is a critical prime (that is, $\ell_p(A) \geqslant 2$) and $\ell_p(S) > 0$.
(ii) The subgroup S is strongly p-critical if $\ell_p(S) > 0$ and $\tau_p(A/S) > 0$.

DEFINITION 3.1.3. Let A be a minimax group. A subgroup S of A is idle if any subgroup parallel to S is commensurable to S.

LEMMA 3.1.4. *Let A be a minimax group and let $S \leqslant A$. The following are equivalent:*
(i) *S is idle;*
(ii) *S is not strongly p-critical for any critical prime p.*

Proof. Suppose that S is not strongly critical for any critical prime p. Let S' be a subgroup of A parallel to S. Replacing S and S' by S/Z and S'/Z, respectively, where Z is a common lattice, we can assume that S and S' are (Artinian) torsion subgroups of A.

Let us show that $\mathrm{Div}(S) = \mathrm{Div}(S')$, which clearly proves that S' is commensurable to S. Let p be a prime such that $\tau_p(A) > 0$.

If p is non-critical (that is, $\tau_p(A) = 1$), then either $\tau_p(S) = \tau_p(S') = 1$ and hence $\mathrm{Div}(S) \cap \mathrm{Div}(S')$ contains the divisible part $\mathrm{Div}(A_p)$ of the p-component A_p of A or $\tau_p(S) = \tau_p(S') = 0$ and neither $\mathrm{Div}(S)$ nor $\mathrm{Div}(S')$ contains this part.

If p is critical, then there are two cases (recall that S is not strongly p-critical for any critical prime p):

Case 1: $\tau_p(S) = 0$. Then $\tau_p(S') = 0$ and hence both $\mathrm{Div}(S)$ and $\mathrm{Div}(S')$ intersect trivially $\mathrm{Div}(A_p)$.

Case 2: $\tau_p(S) = \tau_p(A)$. Then $\tau_p(S') = \tau_p(A)$ and hence $\mathrm{Div}(S) \cap \mathrm{Div}(S') \supset \mathrm{Div}(A_p)$. All in all, this shows that $\mathrm{Div}(S) = \mathrm{Div}(S')$.

Conversely, suppose that S is strongly p-critical for some critical prime p. Let S' be the kernel of a homomorphism from S onto C_{p^∞}. Since S/S' is torsion, the natural map $A/S' \longrightarrow A/S$

maps $T_{A/S'}$ onto $T_{A/S}$. As $\ell_p(S/S') = 1$, we have $\tau_p(A/S') = \tau_p(A/S) + 1$. As S is strongly p-critical, we have $\tau_p(A/S') \geqslant 2$. Hence A/S' contains a subgroup L/S' isomorphic to C_{p^∞}, which is not equal (and therefore not commensurable) to S/S'. Thus L is parallel but not commensurable to S. □

3.2. *The weight function and semi-continuity*

LEMMA 3.2.1. *Let A be a minimax group and let p be a prime. Then, for every $S \leqslant A$, we have*

$$\tau_p(S) + \tau_p(A/S) \geqslant \tau_p(A),$$

with equality if S is torsion.

Proof. It is clearly an equality when S is torsion. Apply this to the exact sequence

$$0 \to T_A/(T_A \cap S) \to A/S \to A/(S + T_A) \to 0$$

to get (note that $T_A \cap S = T_S$)

$$\tau_p(A/S) = \tau_p(A/(S + T_A)) + \tau_p(T_A/T_S).$$

Again, using additivity in the torsion case, we have

$$\tau_p(T_A/T_S) = \tau_p(A) - \tau_p(S).$$

Thus

$$\tau_p(A/S) + \tau_p(S) = \tau_p(A) + \tau_p(A/(S + T_A)).$$

□

The following lemma is straightforward.

LEMMA 3.2.2. *Let A be an abelian group. The map $S \mapsto r(S)$ is lower semi-continuous on $\mathcal{S}(A)$. In particular, if $r(A) < \infty$, then the map $S \mapsto r(A/S)$ is upper semi-continuous on $\mathcal{S}(A)$.*

LEMMA 3.2.3. *Let A be an abelian group with finitely many elements of order p (for example, A is minimax). Then the map $S \mapsto \tau_p(S)$ is upper semi-continuous on $\mathcal{S}(A)$.*

Proof. Note that $\tau_p(S)$ makes sense, since the p-component of T_A, that is, the set of elements whose order is a power of p, is Artinian. Let $(T_A)_p$ be the p-component of the torsion in A, that is, the set of elements of p-prime order. Consider $S \leqslant A$ and let us show that τ_p is upper semi-continuous at S. There exists a finite subgroup M of $(T_S)_p$ such that $(T_S)_p/M$ is divisible. Hence, taking the quotient by M, we can assume that $(T_S)_p = S \cap (T_A)_p$ is divisible. Let F be the set of elements of order p in $A \setminus S$. Then S contains exactly $p^{\tau_p(S)} - 1$ elements of order p. Therefore, for any $S' \leqslant A$ with $S' \cap F = \emptyset$, we have $\tau_p(S') \leqslant \tau_p(S)$. □

LEMMA 3.2.4. *Let A be an abelian group with finitely many elements of order p, and with $r(A) < \infty$ (for example, A is minimax). Then the map $S \mapsto \tau_p(A/S)$ is lower semi-continuous on $\mathcal{S}(A)$.*

Proof. Note that the assumption on A is inherited by its quotients (we need $r(A) < \infty$ here), and so the map considered here makes sense. Let us check that this map is lower semi-continuous at S_0. We can suppose that S_0 is torsion. Indeed, let Z be a lattice in S_0. Then $\mathcal{S}(A/Z)$ can be viewed as an open subset in $\mathcal{S}(A)$, with S_0 corresponding to S_0/Z. Hence, assume that S_0 is torsion. We can write $\tau_p(A/S) = f(S) + g(S)$, with $f(S) = \tau_p(A/S) + \tau_p(S)$ and $g(S) = -\tau_p(S)$. By Lemma 3.2.3, we see that g is lower semi-continuous. By Lemma 3.2.1, since S_0 is torsion, f takes its minimal value $\tau(A)$ at S_0 and so is lower semi-continuous at S_0. □

DEFINITION 3.2.5. Let A be a minimax group. The weight of a subgroup S is

$$w_A(S) = r(A/S) + \ell(S) + \kappa(A/S).$$

The following lemma is straightforward.

LEMMA 3.2.6. *Let A be a minimax group. The map w_A is constant on each commensurability class in $\mathcal{S}(A)$.*

LEMMA 3.2.7. *Let A be a minimax group. The map w_A is upper semi-continuous on $\mathcal{S}(A)$.*

Proof. Observe that

$$w_A(S) = r(A/S) + \ell(A) - \tau(A/S),$$

and so w_A is upper semi-continuous as consequences of Lemmas 3.2.2 and 3.2.4. □

LEMMA 3.2.8. *Let A be a minimax group and let S be a torsion subgroup. If $S' \leqslant A$ is close enough to S and $\mathrm{Div}(S') \leqslant S$, then either $S' = S$ or $w_A(S') < w_A(S)$.*

Proof. First we can mod out by a finite subgroup of S and thus assume that S is divisible. We can write $w_A(S') = \ell(A) + r(A/S') - \sum_p \tau_p(A/S')$, which is a sum of upper semi-continuous functions. Hence if S' is close to S with $w_A(S') = w_A(S)$, we have $r(A/S') = r(A/S)$ by Lemma 3.2.2 (hence S' is torsion too) and $\tau_p(A/S') = \tau_p(A/S)$ for all p by Lemma 3.2.4, that is, $\ell_p(S') = \ell_p(S)$ for all p. As by assumption $\mathrm{Div}(S') \leqslant S$, it follows that $\mathrm{Div}(S') = S$. Thus S has a direct complement M in S'. Let F be the set of elements of prime order in $A \setminus S$. If S' is close enough to S, we have $S' \cap F = \emptyset$, and hence M cannot contain any element of prime order; thus $M = \{0\}$, and $S' = S$. □

PROPOSITION 3.2.9. *Let A be a minimax group, and let $S \in \mathcal{S}(A)$. If S' is close enough to S, then one of the following conditions holds:*
(1) *$w_A(S') < w_A(S)$;*
(2) *$w_A(S') = w_A(S)$ and S' is parallel and non-commensurable to S;*
(3) *$S' = S$.*

Proof. Let $S \leqslant A$ and let Z be a finitely generated subgroup of S. If S' is close enough to S, then $Z \leqslant S'$, and as $w_{A/Z}(S'/Z) = w_A(S')$ whenever S' contains Z, we can suppose that S is divisible. If S' is close to S and $w_A(S') = w_A(S)$, then $r(A/S') = r(A/S)$, so that S' is torsion, and $\tau_p(A/S') = \tau_p(A/S)$ for all p; hence $\ell_p(S) = \ell_p(S')$ for all p, that is, S and S' are parallel (argue as in the beginning of the proof of Lemma 3.2.8). Moreover, if S' is commensurable to S, then $\mathrm{Div}(S') = \mathrm{Div}(S)$. By Lemma 3.2.8, we get that if S' is close enough to S, then $S' = S$. □

As w_A is constant on commensurability classes, we get the following corollary.

Corollary 3.2.10. *Let A be a minimax group. Every commensurability class in $\mathcal{S}(A)$ is discrete.*

Corollary 3.2.11. *Let A be a minimax group and let $S \in \mathcal{S}(A)$. Then the function w_A is strictly upper semi-continuous at S if and only if S is idle.*

Indeed, the case (2) can occur only if S is non-idle.

If A has no critical primes, then every subgroup is idle and we thus get the following corollary.

Corollary 3.2.12. *If the minimax group A has no critical primes, then the map w_A is strictly upper semi-continuous on $\mathcal{S}(A)$.*

3.3. *Commensurable convergence*

Definition 3.3.1. Let H and S be two subgroups of A. We say that there is commensurable convergence from H to S if S belongs to the topological closure of the commensurability class of H.

Lemma 3.3.2. *Let A be a minimax group and let $S \leqslant T_A$ be a torsion subgroup. Let H be another subgroup of A. Then the following are equivalent:*

(i) *there is commensurable convergence from H to S;*

(ii) $\mathrm{Div}(H)$ *is contained in S.*

Proof. Suppose (ii). As a divisible subgroup, $\mathrm{Div}(H)$ has a direct complement L in H, and L contains a torsion-free subgroup of finite index L'. Let (F_n) be a non-decreasing sequence of subgroups of S containing $\mathrm{Div}(H)$ as a subgroup of finite index, with union all of S. Let (V_n) be a sequence of subgroups of finite index of L', with trivial intersection. Set

$$H_n = F_n \oplus V_n \subset S \oplus L'.$$

Then H_n has finite index in $F_n \oplus L$, which contains $\mathrm{Div}(H) \oplus L = H$ with finite index. Thus H_n is commensurable to H; clearly (H_n) tends to S.

Conversely, suppose that $T_H \cap S$ has infinite index in T_H. Then H contains a quasi-cyclic subgroup $P \simeq C_{p^\infty}$ such that $P \cap S$ is finite. Then every subgroup of A commensurable to H contains P. Therefore this remains true for every group in the closure of the commensurability class of H, which therefore cannot contain S. □

As a consequence, we get the following proposition.

Proposition 3.3.3. *Let A be a minimax group and let S be a subgroup of A. Let H be another subgroup of A. Then the following are equivalent:*

(i) *there is commensurable convergence from H to S;*

(ii) *H contains some lattice Z of S and, in A/Z, we find that $T_{H/Z}$ is virtually contained in S/Z.*

Proof. The implication (ii)⇒(i) is a direct corollary of Lemma 3.3.2. Suppose (i). Let Z_0 be a lattice of S and let (H_n) be a sequence of subgroups of A, commensurable to H and converging to S. Then $H_n \cap Z_0 \to Z_0$. As Z_0 is finitely generated, eventually H_n contains Z_0. Thus H contains a finite index subgroup Z of Z_0. Hence working in A/Z and applying Lemma 3.3.2, we get (ii). □

As an application, we have the two extreme cases.

PROPOSITION 3.3.4. *Let A be a minimax group. Then we have the following conditions.*

(i) *If Z is a lattice of A, then there is commensurable convergence from Z to any subgroup of A.*

(ii) *If $S = T_A$, then S belongs to the closure of any commensurability class in $\mathcal{S}(A)$.*

Hence, the lattices of a minimax group form a dense commensurability class. As it consists of isolated points, it is the unique dense commensurability class. At the opposite, the torsion subgroup is in the unique closed (and finite) commensurability class of $\mathcal{S}(A)$.

Another consequence of Proposition 3.3.3 is the following, which quite surprisingly is not an obvious consequence of the definition.

COROLLARY 3.3.5. *If there is commensurable convergence $H \to L$ and $L \to S$, then there is commensurable convergence $H \to S$.*

3.4. *Hereditary properties*

PROPOSITION 3.4.1. *Let A be a minimax group. Set*

$$\mathcal{I} = \{S \in \mathcal{S}(A) |\, \kappa_{\mathrm{cr}(A)}(A/S) = \ell_{\mathrm{cr}(A)}(S) = 0\}.$$

Then the map w_A is strictly hereditary on $\mathcal{I}$.

The proposition readily follows from the following lemma.

LEMMA 3.4.2. *Let A be a minimax group and let $S \in \mathcal{S}(A)$. Suppose that*

$$\kappa_{\mathrm{cr}(A)}(A/S) = 0 \quad \text{and} \quad r(A/S) + \ell_{\mathrm{ncr}(A)}(S) \geqslant 1$$

(if $S \in \mathcal{I}$, these two assumptions just mean that $w_A(S) \geqslant 1$). Then there exists $H \in \mathcal{S}(A)$ with commensurable convergence $H \to S$, with $w_A(H) = w_A(S) - 1$ and $\ell_{\mathrm{cr}(A)}(H) = \ell_{\mathrm{cr}(A)}(S)$.

Proof. We can assume that S is torsion. Indeed, let Z be a lattice of S. We can check that A/Z and S/Z satisfy the same hypotheses as A and S, and that $w_{A/Z}(H/Z) = w_A(H)$ for any $H \leqslant A$ containing Z.

(i) If $r(A) \geqslant 1$, then there exists a torsion-free subgroup $Q \leqslant A$ of rank one such that $\ell_{\mathrm{cr}(A)}(Q) = 0$ and $\tau(A/(S \oplus Q)) = \tau(A/S)$. Indeed, take a direct complement L of $\mathrm{Div}(A)$ in A and a torsion-free finite index subgroup L' of L, a cyclic subgroup $C \leqslant L'$, and the inverse image

$Q \leqslant L'$ of the torsion of L'/C. As $0 = \kappa_{\mathrm{cr}(A)}(A/S) = \ell_{\mathrm{cr}(A)}(L) = 0$, we have $\ell_{\mathrm{cr}(A)}(Q) = 0$. Now $\tau(A/(S \oplus Q)) = \tau(T_{A/S} \oplus L/Q) = \tau(A/S)$.

Setting $H = S \oplus Q$, we have $\ell_{\mathrm{cr}(A)}(H) = \ell_{\mathrm{cr}(A)}(S)$ and

$$\begin{aligned} w_A(H) &= r(A/H) + \ell(A) - \tau(A/H) \\ &= r(A/S) - 1 + \ell(A) - \tau(A/S) \\ &= w_A(S) - 1. \end{aligned}$$

Now S is the limit of the subgroups

$$H_k = S \oplus k!Q \leqslant H \quad (k \in \mathbf{N}),$$

which have finite index in H. Indeed, observe that we have the following:

* nQ is close to $\{0\}$ if n has some large prime divisor p such that no element of Q is divisible by every power of p;
* nQ has finite index in Q for any $n \geqslant 1$, because Q/nQ is Artinian with finite exponent.

(ii) If $r(A) = 0$, then A is Artinian, and by assumption this forces $\ell_{\mathrm{ncr}(A)}(S) \geqslant 1$. Hence S has a direct summand P isomorphic to C_{p^∞} for some non-critical prime p. Let H be a direct complement of P in S. We have $w_A(H) = w_A(S) - 1$ and $\ell_{\mathrm{cr}(A)}(H) = \ell_{\mathrm{cr}(A)}(S)$.

Denote by H_k the direct sum of H with the subgroup of P of order p^k. Then H_k is commensurable to H and H_k tends to S as k goes to infinity.

□

3.5. *Conclusion in the scattered case*

THEOREM 3.5.1. *Let A be a minimax group with no critical primes. Then $\mathcal{S}(A)$ is scattered, and the Cantor–Bendixson rank of $S \in \mathcal{S}(A)$ is given by $w_A(S)$. The Cantor–Bendixson rank of $\mathcal{S}(A)$ is $h(A) + 1$, and $\mathcal{S}(A)$ is homeomorphic to $D^{h(A)} \times [n(A)]$, where $n(A)$ is the number of subgroups commensurable to T_A.*

Proof. Suppose that A has no critical primes. Then the subset $\mathcal{I}$ of Proposition 3.4.1 coincides with all of $\mathcal{S}(A)$, so that w_A is strictly hereditary on $\mathcal{S}(A)$. Moreover, w_A is strictly upper semi-continuous on $\mathcal{S}(A)$ by Corollary 3.2.12. Hence we can apply Lemma 1.2.3 (with $C = \emptyset$) to obtain that $\mathcal{S}(A)$ is scattered and that the Cantor–Bendixson rank of an element $S \in \mathcal{S}(A)$ is $w_A(S)$.

Writing $w_A(S) = h(A) - (r(S) + \tau(A/S))$, we see that the maximal value of w_A is given by $h(A)$, and is attained exactly for subgroups commensurable to T_A. □

4. *The levelled weight and condensation on $\mathcal{S}(A)$*

Again, in this section, all Artinian and minimax groups are assumed to be abelian.

4.1. *The invariant γ*

Let V be a set of primes and let $\mathbf{Z}_V$ be the ring of rationals whose denominator has no divisor in V.

Let A be an abelian group. Then $\mathrm{Hom}(A, \mathbf{Z}_V)$ is a torsion-free $\mathbf{Z}_V$-module. Its rank, that is, the dimension of the $\mathbf{Q}$-vector space $\mathrm{Hom}(A, \mathbf{Z}_V) \otimes_{\mathbf{Z}_V} \mathbf{Q}$, is denoted by $\gamma_V(A)$. Note that $\gamma_\emptyset(A) = r(A)$.

Lemma 4.1.1. *If A has finite rank $r(A)$, then* $\mathrm{Hom}(A, \mathbf{Z}_V)$ *is a finitely generated $\mathbf{Z}_V$-module, free of rank $\gamma_V(A)$, and $\gamma_V(A) \leqslant r(A)$.*

Proof. If $n = r(A)$ and $(e_1, \dots, e_n)$ is a maximal $\mathbf{Z}$-free family in A, then the mapping $f \mapsto (f(e_1), \dots, f(e_n))$ embeds $\mathrm{Hom}(A, \mathbf{Z}_V)$ as a submodule of the free module of rank n. In particular, as $\mathbf{Z}_V$ is principal, $\mathrm{Hom}(A, \mathbf{Z}_V)$ is a free $\mathbf{Z}_V$-module of rank $\gamma_V(A)$ and $\gamma_V(A) \leqslant r(A)$. □

Define $a_V(A) = r(A) - \gamma_V(A)$. In addition, define $U_V(A)$ as the intersection of all kernels of homomorphisms $A \to \mathbf{Z}_V$; this is a characteristic subgroup of A.

Lemma 4.1.2. *Let A be an abelian group of finite rank and let $S \leqslant A$. Then we have:*

$$\gamma_V(A) \geqslant \gamma_V(A/S) \geqslant \gamma_V(A) - \gamma_V(S),$$
$$a_V(A) - r(S) \leqslant a_V(A/S) \leqslant a_V(A) - a_V(S).$$

In particular, we have $\gamma_V(A/S) = \gamma_V(A)$ and $a_V(A/S) = a_V(A)$ if S is torsion.

Proof. From the exact sequence $0 \to S \to A \to A/S \to 0$ we get the exact sequence of $\mathbf{Z}_V$-modules

$$0 \to \mathrm{Hom}(A/S, \mathbf{Z}_V) \to \mathrm{Hom}(A, \mathbf{Z}_V) \to \mathrm{Hom}(S, \mathbf{Z}_V),$$

so that $\mathrm{Hom}(A, \mathbf{Z}_V)$ lies in an extension of $\mathrm{Hom}(A/S, \mathbf{Z}_V)$ by some submodule of $\mathrm{Hom}(S, \mathbf{Z}_V)$. This gives us the first inequality, and the second one is equivalent to it. □

Lemma 4.1.3. *Let A be an abelian group of finite rank. We have $a_V(A) = r(U_V(A))$ and $\gamma_V(A) = r(A/U_V(A))$.*

Proof. The two statements are obviously equivalent; let us prove that $\gamma_V(A) = r(A/U_V(A))$.

Set $B = A/U_V(A)$. If $i : \mathbf{Z}^{r(B)} \to B$ is an embedding as a lattice, then, as $\mathrm{Hom}(B/\mathrm{Im}(i), \mathbf{Z}_V) = \{0\}$, it induces an embedding of $\mathrm{Hom}(B, \mathbf{Z}_V)$ into $\mathrm{Hom}(\mathbf{Z}^{r(B)}, \mathbf{Z}_V) = \mathbf{Z}_V^{r(B)}$, which is a $\mathbf{Z}_V$-module homomorphism. So $\gamma_V(A) \leqslant r(A/U_V(A))$.

Conversely, if $(f_1, \dots, f_m)$ is a maximal $\mathbf{Z}$-free family in $\mathrm{Hom}(A, \mathbf{Z}_V)$, with $m = \gamma_V(A)$, then $A/\bigcap \mathrm{Ker}(f_i)$ embeds into $\mathbf{Z}_V^m$; but $\bigcap \mathrm{Ker}(f_i)$ is reduced to $U_V(A)$ and so $r(A/U_V(A)) \leqslant \gamma_V(A)$. □

Corollary 4.1.4. *Let A be a minimax group. We have $a_V(A) = 0$ if and only if $\kappa_p(A) = 0$ for every $p \in V$.*

Proof. If $a_V(A) = 0$, then $U_V(A)$ is torsion and hence coincides with T_A. Therefore A/T_A embeds into $\mathbf{Z}_V^{r(A)}$. As a result $\kappa_p(A) = \ell_p(A/T_A) = 0$ for every $p \in V$. Conversely, suppose that $\kappa_p(A) = 0$ for every $p \in V$. We have an embedding of A/T_A into $\mathbf{Q}^{r(A)}$. Using Lemma 4.1.5 below, the assumption implies that the image is contained in a multiple of $\mathbf{Z}_V^{r(A)}$. Thus $U_V(A)$ is torsion, that is, $a_V(A) = 0$. □

LEMMA 4.1.5. *Let A be a minimax subgroup of $\mathbf{Q}$ and let $n \geqslant 1$. Suppose that $\ell_p(A) = 0$ for every prime p not dividing n. Then $A \leqslant \lambda\mathbf{Z}[1/n]$ for some $\lambda \in \mathbf{Q}^*$.*

Proof. Let $B = \bigoplus B_p$ be the image of A in $\mathbf{Q}/\mathbf{Z} = \bigoplus_p C_{p^\infty}$. As A is a minimax group, $B_p = 0$ for all but finitely many p, and B_p is finite when p does not divide n. Hence B is virtually contained in $\bigoplus_{p|n} C_{p^\infty} = \mathbf{Z}[1/n]/\mathbf{Z}$. Thus A is virtually contained in $\mathbf{Z}[1/n]$. As A is locally cyclic, A is generated by $A \cap \mathbf{Z}[1/n]$ and some rational u/v. Then $A \leqslant v^{-1}\mathbf{Z}[1/n]$. □

LEMMA 4.1.6. *Let A be an abelian group with $r(A) < \infty$. The maps $S \mapsto a_V(A/S)$ and $S \mapsto \gamma_V(A/S)$ are upper semi-continuous on $\mathcal{S}(A)$.*

Proof. Take $S_0 \in \mathcal{S}(A)$. By the same argument as in the proof of Lemma 3.2.4, we can suppose that S_0 is torsion.
By the first and second inequalities in Lemma 4.1.2, we see that $S \mapsto \gamma_V(A/S)$ and $S \mapsto a_V(A/S)$ are maximal at S_0, respectively, and so are upper semi-continuous at S_0. □

4.2. *More maps on $\mathcal{S}(A)$*

Let A be a minimax group, and recall that $\mathrm{cr}(A)$ or $\mathrm{ncr}(A)$ denotes the set of critical primes of A or the set of non-critical primes of A, respectively. We define three maps $\mathcal{S}(A) \to \mathbf{N}$.

(i) The *level* of a subgroup S is given by

$$\lambda_A(S) = a_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{cr}(A)}(S) + \kappa_{\mathrm{cr}(A)}(A/S).$$

(ii) The *levelled weight* is given by

$$d_A(S) = \gamma_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{ncr}(A)}(S) + \kappa_{\mathrm{ncr}(A)}(A/S).$$

Note that $w_A = \lambda_A + d_A$.

LEMMA 4.2.1. *Let A be a minimax group. The maps λ_A and d_A are constant on each commensurability class in $\mathcal{S}(A)$.*

Proof. This is clear for the maps $S \mapsto \kappa_p(A/S)$ and $S \mapsto \ell_p(S)$. For $S \mapsto \gamma_V(A/S)$, let S and S' be commensurable subgroups of A. We can deduce from Lemma 4.1.2 that $\gamma_V(A/S) = \gamma_V(A/(S \cap S')) = \gamma_V(A/S')$. Finally $a_V(A/S) = r(A/S) - \gamma_V(A/S)$ is settled. □

LEMMA 4.2.2. *Let A be a minimax group. The maps λ_A and d_A are upper semi-continuous on $\mathcal{S}(A)$.*

Proof. Observe that

$$\lambda_A(S) = a_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{cr}(A)}(A) - \tau_{\mathrm{cr}(A)}(A/S),$$

$$d_A(S) = \gamma_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{ncr}(A)}(A) - \tau_{\mathrm{ncr}(A)}(A/S);$$

hence they are upper semi-continuous as consequences of Lemmas 3.2.4 and 4.1.6. □

PROPOSITION 4.2.3. *In restriction to the open subset $\mathcal{I} = \{\lambda_A = 0\}$ of $\mathcal{S}(A)$, the map w_A $(= d_A)$ is strictly upper semi-continuous.*

Proof. If $\lambda_A(S) = 0$ and $\ell_p(S) = 0$ for every critical prime p, then S is not strongly p-critical for any p. By Lemma 3.1.4, we see that S is idle. □

4.3. *Hereditary properties (second part)*

If A is a minimax group, in view of Corollary 4.1.4, the set $\mathcal{I}$ of Proposition 3.4.1 coincides with $\{S \in \mathcal{S}(A) | \lambda_A(S) = 0\}$. Accordingly, Proposition 3.4.1 states that, on $\mathcal{I}$, the function w_A ($= d_A$) is strictly hereditary.

Lemma 4.3.1. *Let A be a minimax group and let $S \in \mathcal{S}(A)$ with $\lambda_A(S) \leqslant 1$. Then $a_{\mathrm{cr}(A)}(A/S) = \kappa_{\mathrm{cr}(A)}(A/S) = 0$.*

Proof. We have $\lambda_A(S) = a_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{cr}(A)}(S) + \kappa_{\mathrm{cr}(A)}(A/S) \leqslant 1$. By Corollary 4.1.4, we see that $\kappa_{\mathrm{cr}(A)}(A/S) = 0$ if and only if $a_{\mathrm{cr}(A)}(A/S) = 0$, and so they are both zero. □

Using this, Lemma 3.4.2 can be restated in the following form.

Lemma 4.3.2. *Let A be a minimax group and let $S \leqslant A$. Suppose that $\lambda_A(S) \leqslant 1$ and $d_A(S) \geqslant 1$. Then there exists H with commensurable convergence $H \to S$, with $d_A(H) = d_A(S) - 1$ and $\lambda_A(H) = \lambda_A(S)$.*

Proof. By Lemma 3.4.2, there exists H with commensurable convergence $H \to S$, with $\ell_{\mathrm{cr}(A)}(H) = \ell_{\mathrm{cr}(A)}(S)$ and $w_A(H) = w_A(S) - 1$. By upper semi-continuity of λ_A, we have $\lambda_A(H) \leqslant \lambda_A(S)$. By Lemma 4.3.1, we have $\lambda_A(S) = \ell_{\mathrm{cr}(A)}(S) = \ell_{\mathrm{cr}(A)}(H) \leqslant \lambda_A(H)$. Thus $\lambda_A(H) = \lambda_A(S)$, and hence $d_A(H) = d_A(S) - 1$. □

Lemma 4.3.3. *Suppose that $\lambda_A(S) \geqslant 1$. Then there exists H with commensurable convergence $H \to S$, with $\lambda_A(H) = \lambda_A(S) - 1$ and $d_A(H) = d_A(S)$.*

Proof. We can suppose that S is torsion (arguing as in the beginning of the proof of Lemma 3.4.2). We easily check from the definition that $\kappa_p(A/S) = \kappa_p(A)$ for any prime p. By Lemma 4.1.2, we also have $a_{\mathrm{cr}(A)}(A/S) = a_{\mathrm{cr}(A)}(A)$. Therefore $\lambda_A(S) = a_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{cr}(A)}(S) + \kappa_{\mathrm{cr}(A)}(A)$.

Suppose that $a_{\mathrm{cr}(A)}(A) \geqslant 1$, that is, that $U_{\mathrm{cr}(A)}$ is not torsion. Write $A = T_A \oplus Q$ with Q torsion-free. We have $U_{\mathrm{cr}(A)}(A) = U_{\mathrm{cr}(A)}(T_A) \oplus U_{\mathrm{cr}(A)}(Q)$, and so $U_{\mathrm{cr}(A)}(Q)$ is not torsion. Let Z be an infinite cyclic subgroup of $U_{\mathrm{cr}(A)}(Q)$ and let L be the inverse image in Q of the torsion subgroup of Q/Z. Finally set $H = S \oplus L$. We have (note that $\gamma_{\mathrm{cr}(A)}(A/S) = \gamma_{\mathrm{cr}(A)}(A)$ by Lemma 4.1.2)

$$d_A(H) - d_A(S) = \gamma_{\mathrm{cr}(A)}(A/H) - \gamma_{\mathrm{cr}(A)}(A) - \tau_{\mathrm{ncr}(A)}(A/H) + \tau_{\mathrm{ncr}(A)}(A/S).$$

We claim that:

(1) $\tau_p(A/H) = \tau_p(A/S)$ for every prime p;

(2) $\gamma_{\mathrm{cr}(A)}(A/H) = \gamma_{\mathrm{cr}(A)}(A)$ and $a_{\mathrm{cr}(A)}(A/H) = a_{\mathrm{cr}(A)}(A) - 1$.

Proof of Claim (1). As Q/L is torsion-free, we have the groups $T_{A/H}$ and $T_{A/S}$ are both isomorphic to T_A/S. Hence $\tau_p(A/H) = \tau_p(A/S)$ for every p.

Proof of Claim (2). By Lemma 4.1.2, we have $\gamma_{\mathrm{cr}(A)}(A/H) = \gamma_{\mathrm{cr}(A)}(A/Z)$ and $a_{\mathrm{cr}(A)}(A/H) = a_{\mathrm{cr}(A)}(A/Z)$. Since $Z \leqslant U_{\mathrm{cr}(A)}(A)$, we have $\mathrm{Hom}(A/Z, Z_{\mathrm{cr}(A)}) = \mathrm{Hom}(A, Z_{\mathrm{cr}(A)})$ and hence

$\gamma_{\mathrm{cr}(A)}(A/Z) = \gamma_{\mathrm{cr}(A)}(A)$, or equivalently $a_{\mathrm{cr}(A)}(A/Z) = a_{\mathrm{cr}(A)}(A) - r(Z) = a_{\mathrm{cr}(A)}(A) - 1$, which completes the proof of claim (2).

We deduce from the previous claims that $d_A(H) = d_A(S)$ and $\lambda_A(H) = \lambda_A(S) - 1$. Moreover, S is a strict limit of subgroups of finite index in H.

Finally, suppose that $a_{\mathrm{cr}(A)}(A) = 0$ (which implies $\kappa_{\mathrm{cr}(A)}(A) = 0$ and $\tau_{\mathrm{cr}(A)}(S) \geqslant 1$). Write $S = P \oplus H$, where P is isomorphic to C_{p^∞} for some critical prime p. Then $d_A(S) = d_A(H)$ and $\lambda_A(H) = \lambda_A(S) - 1$. Moreover, S is a strict limit of subgroups containing H as a subgroup of finite index. □

LEMMA 4.3.4. *Let A be a minimax group, let S be a subgroup and let $m \leqslant \lambda_A(S)$. Then there exists H with commensurable convergence $H \to S$, with $\lambda_A(H) = m$ and $d_A(H) = d_A(S)$.*

Proof. This is a straightforward induction on $\lambda_A(S) - m$, based on Lemma 4.3.3 and making use of the transitivity of commensurable convergence (Corollary 3.3.5). □

PROPOSITION 4.3.5. *Let A be a minimax group. Again, set $\mathcal{I} = \{S \in \mathcal{S}(A) \mid \lambda_A(S) = 0\}$. Then the map d_A is $\mathcal{I}$-hereditary on $\mathcal{S}(A)$.*

Proof. First, the statement presupposes that $\mathcal{I}$ is dense in $\mathcal{S}(A)$, which is a consequence of Proposition 3.3.4 and Corollary 2.1.2 (but also follows from the argument below).

Choose $S \leqslant A$. Let V be an open neighbourhood of S. Using Lemma 4.3.4 with $m = 0$, we obtain that there exists $H \in \mathcal{I} \cap V$ such that $d_A(H) = d_A(S)$. Hence d_A is $\mathcal{I}$-hereditary at S. □

LEMMA 4.3.6. *Let A be a minimax group, let S be a subgroup of A with $\lambda_A(S) \geqslant 1$ and let $1 \leqslant n \leqslant d_A(S)$. Then there exists H with commensurable convergence $H \to S$, with $\lambda_A(H) = 1$ and $d_A(H) = n$.*

Proof. By Lemma 4.3.4, there exists $H \in \mathcal{S}(A)$ with commensurable convergence $H \to S$, with $\lambda_A(H) = 1$ and $d_A(H) = d_A(S)$. Applying several times Lemma 4.3.2 (using Corollary 3.3.5), we find H with commensurable convergence $H \to S$, with $d_A(H) = n$ and $\lambda_A(H) = 1$. Again with Corollary 3.3.5, we have commensurable convergence $H \to S$. □

The following lemma gives somehow the 'smallest' examples of minimax groups whose space of subgroups is not scattered (equivalently uncountable).

LEMMA 4.3.7. *Let $A = (C_{p^\infty})^2$. The set $\mathcal{K}$ of subgroups of A isomorphic to C_{p^∞} is a Cantor space. In particular $\mathcal{S}(A)$ is uncountable.*

Proof. By direct computation $\mathrm{Aut}(A) = \mathrm{GL}_2(\mathbf{Z}_p)$, the group of 2×2 matrices with coefficients in the p-adics, acting on $(C_{p^\infty})^2$ through its identification with $(\mathbf{Q}_p/\mathbf{Z}_p)^2$. The action on the set of subgroups is easily checked to be continuous. The stabilizer of the 'line' $C_{p^\infty} \oplus \{0\}$ is the set of upper triangular matrices $T_2(\mathbf{Z}_p)$. The quotient can be identified on the one hand with the projective line $\mathbf{P}^1(\mathbf{Q}_p)$, which is known to be a Cantor space, and on the other hand with the orbit of $C_{p^\infty} \oplus \{0\}$.

Now let $P \in \mathcal{K}$. Being divisible, it has a direct complement Q in A. Necessarily, Q is isomorphic to C_{p^∞}. Therefore there exists an automorphism of A mapping $C_{p^\infty} \oplus \{0\}$ to P. Thus $\mathcal{K}$ coincides with the orbit of $C_{p^\infty} \oplus \{0\}$, which completes the proof. □

LEMMA 4.3.8. *Suppose that S is strongly p-critical. Then there exists a Cantor space $\mathcal{K} \subset \mathcal{S}(A)$ such that $S \in \mathcal{K}$, and every $K \in \mathcal{K} - \{S\}$ is parallel and non-commensurable to S.*

Proof. Let W be the kernel of a map of S onto C_{p^∞}. Working inside A/W, we can suppose that $W = 0$, that is, S is isomorphic to C_{p^∞}. As S is strongly p-critical, there exists another subgroup P which is isomorphic to C_{p^∞} and such that $P \cap S = 0$. Then the set of subgroups of $P \oplus S$ that are isomorphic to C_{p^∞} is a Cantor space, by Lemma 4.3.7. They are all parallel to S and pairwise non-commensurable. □

LEMMA 4.3.9. *Suppose that $\lambda_A(S) = 1$. Then S belongs to a Cantor space whose points are subgroups of H with $d_A(S) = d_A(H)$ and $\lambda_A(H) = 1$.*

Proof. By Lemma 4.3.1, we have $a_{\mathrm{cr}(A)}(A/S) = \kappa_{\mathrm{cr}(A)}(A/S) = 0$. So $\ell_{\mathrm{cr}(A)}(S) = 1$ and $\tau_{\mathrm{cr}(A)}(A/S) = \ell_{\mathrm{cr}(A)}(A/S) \geqslant 1$. Let p be the critical prime such that $\ell_p(S) = 1$. Then S is strongly p-critical. Thus we conclude by Lemma 4.3.8. □

Let A be a minimax group. For $n \geqslant 0$, define the subset $\mathcal{C}_n(A) = \{S |\, \lambda_A(S) \geqslant 1,\, d_A(S) \geqslant n\}$. By upper semi-continuity (Lemma 4.2.2), it is closed.

PROPOSITION 4.3.10. *Let A be a minimax group and let $n \geqslant 0$. The subset $\mathcal{C}_n(A) \subset \mathcal{S}(A)$ is perfect.*

Proof. By Lemma 4.3.4 with $m = 1$, its subset $\mathcal{C}_n^1(A) = \{S \in \mathcal{C}_n \mid \lambda_A(S) = 1\}$ is dense in $\mathcal{C}_n(A)$. By Lemma 4.3.9, we see that $\mathcal{C}_n^1(A)$ is perfect, and so $\mathcal{C}_n(A)$ is perfect as well. □

PROPOSITION 4.3.11. *Let A be a minimax group and let $n \geqslant 0$. Then $\mathcal{C}_{n+1}(A)$ has empty interior in $\mathcal{C}_n(A)$.*

Proof. Let S belong to $\mathcal{C}_{n+1}(A)$. By Lemma 4.3.6 there exists $H \in \mathcal{S}(A)$ with commensurable convergence $H \to S$, with $\lambda_A(H) = 1$ and $d_A(H) = n$. Hence H and its commensurable subgroups belong to $\mathcal{C}_n(A)$ but not to $\mathcal{C}_{n+1}(A)$, and therefore S is not in the interior of $\mathcal{C}_{n+1}(A)$. □

4.4. *Maximal values*

LEMMA 4.4.1. *Let A be a minimax group. On the set $\mathcal{I} = \{S \mid \lambda_A(S) = 0\}$, the maximal value of $w_A (= d_A)$ is $\gamma_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{ncr}(A)}(A)$.*

Proof. We have

$$d_A(S) = \gamma_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{ncr}(A)}(A) - \tau_{\mathrm{ncr}(A)}(A/S) \leqslant \gamma_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{ncr}(A)}(A)$$

and

$$\lambda_A(S) = a_{\mathrm{cr}(A)}(A/S) + \ell_{\mathrm{cr}(A)}(A) - \tau_{\mathrm{cr}(A)}(A/S) \leqslant a_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{cr}(A)}(A).$$

Both inequalities are sharp, as they become equalities when $S = T_A$.

Moreover by Lemma 4.3.4, we get the existence of a subgroup S of A with $\lambda_A(S) = 0$ and $d_A(S) = \gamma_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{ncr}(A)}(A)$. □

4.5. *Conclusion in the non-scattered case*

Define

$$\sigma(A) = \gamma_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{ncr}(A)}(A) = h(A) - (a_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{cr}(A)}(A)).$$

THEOREM 4.5.1. *Let A be a minimax group. If $S \in \mathcal{S}(A)$, then we have the following:*

(i) *S belongs to the scattered part of $\mathcal{S}(A)$ if and only if $\lambda_A(S) = 0$;*
(ii) *the extended Cantor–Bendixson rank of S in $\mathcal{S}(A)$ is $d_A(S)$;*
(iii) *the Cantor–Bendixson rank of $\mathcal{S}(A)$ is $\sigma(A) + 1$.*

Moreover, if A has at least one critical prime, then $\mathcal{S}(A)$ is homeomorphic to $D^{\sigma(A)} \times W$.

Proof. If $\mathcal{I} = \{\lambda_A = 0\}$ and $\mathcal{C} = \{\lambda_A \geqslant 1\}$, then, by Proposition 4.3.10 (for $n = 0$), Lemma 4.2.2, Propositions 4.3.5, 4.2.3 and 3.4.1, we have $\mathcal{I}$, $\mathcal{C}$ and d_A satisfy the hypotheses of Lemma 1.2.3. This settles the first two assertions, and the third follows from Lemma 4.4.1.

For the last statement, we appeal to Proposition 1.3.2, with $C_i = \mathcal{C}_i(A)$ as above. By Proposition 4.3.10, we see that C_i is perfect. As $d_A(T_A) = \sigma(A)$ and $\lambda_A(T_A) = a_{\mathrm{cr}(A)}(A) + \ell_{\mathrm{cr}(A)}(A) \geqslant 1$ (see the proof of Lemma 4.4.1), we have $T_A \in C_{\sigma(A)}$. Thus C_i is a Cantor space for all $i \leqslant \sigma(A)$. Finally C_{i+1} has empty interior in C_i for all i, by Proposition 4.3.11. □

4.6. *Example: Artinian groups*

For an Artinian group A, the invariant γ_V vanishes for every set of primes V. Thus, by definition

$$\sigma(A) = \ell_{\mathrm{ncr}(A)}(A),$$

and, for $S \in \mathcal{S}(A)$, we have

$$w_A(S) = \ell(S); \quad \lambda_A(S) = \ell_{\mathrm{cr}(A)}(S); \quad d_A(S) = \ell_{\mathrm{ncr}(A)}(S).$$

The properties of these maps established above are even easier to obtain in this particular case. Indeed, if A is decomposed as a direct sum of its p-components: $A = \bigoplus A_p$, then $\mathcal{S}(A) = \bigoplus \mathcal{S}(A_p)$. Then, given that $W \times [n]$ for $n \geqslant 1$ and $W \times W$ are homeomorphic to W, it is enough to study $\mathcal{S}(A)$ when A is an Artinian p-group.

If $\ell(A) = 0$, then A is finite and $\mathcal{S}(A)$ is homeomorphic to $[n]$ for $n = n(A)$, the number of subgroups of A.

If $\ell(A) = 1$, then the finite subgroups of A are isolated, and form a dense subset, while there are only finitely many infinite subgroups, namely finite index subgroups of A. If there are n many such subgroups, then $\mathcal{S}(A)$ is homeomorphic to $D \times [n]$.

If $\ell(A) = \ell_p(A) \geqslant 2$, again finite subgroups form a dense subset consisting of isolated points. If C is the set of infinite subgroups, then it is closed. Now C contains a dense subset, namely the set L_1 of subgroups S with $\ell(S) = 1$. The set L_1 is perfect, by Lemma 4.3.9. Thus C is perfect. Accordingly, $\mathcal{S}(A)$ is homeomorphic to W.

References

1. B. Boyer, 'Enumeration theorems in infinite Abelian groups', *Proc. Amer. Math. Soc.* 7 (1956) 565–570. doi:10.2307/2033351.

2. C. Chabauty, 'Limite d'ensembles et géométrie des nombres', *Bull. Soc. Math. France* 78 (1950) 143–151.

3. C. Champetier, 'L'espace des groupes de type fini', *Topology* 39 (2000) 657–680.

4. C. Champetier and V. Guirardel, 'Limit groups as limits of free groups', *Israel J. Math.* 146 (2005) 1–75.

5. Y. de Cornulier, L. Guyot and W. Pitsch, 'On the isolated points in the space of groups', *J. Algebra* 307 (2007) 254–277.

6. R. Grigorchuk, 'Degrees of growth of finitely generated groups and the theory of invariant means', *Izv. Akad. Nauk SSSR Ser. Mat.* 48 (1984) 939–985.

7. M. Gromov, 'Groups of polynomial growth and expanding maps', *Publ. Math. Inst. Hautes Études Sci.* 53 (1981) 53–73.

8. A. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics 156 (Springer, New York, 1995).

9. B. Knaster and M. Reichbach, 'Notion d'homogénéité et prolongements des homéomorphies', *Fund. Math.* 40 (1953) 180–193.

10. S. Mazurkiewicz and W. Sierpinski, 'Contribution à la topologie des ensembles dénombrables', *Fund. Math.* 1 (1920) 17–27.

11. R. S. Pierce, 'Existence and uniqueness theorems for extensions of zero-dimensional compact metric spaces', *Trans. Amer. Math. Soc.* 148 (1970) 1–21.

12. S. Thomas, 'On the complexity of the classification problem for torsion-free abelian groups of finite rank', *Bull. Symbolic Logic*, 7 (2001) 329–344.

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