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TREE LEVEL LIE ALGEBRA STRUCTURES OF PERTURBATIVE INVARIANTS

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ABSTRACT

We study two different Lie algebra structures on the space of Feynman diagrams at tree level. We show that each such structure arises naturally from a tower of automorphism groups of nilpotent quotients of a free group

Keywords: Tree Lie algebra, Feynman diagram, pertubative invariant,

1. Introduction and Statement of Results

Let H denote a free abelian group of rank l and let $L(H) = \oplus_{n=1}^{\infty} L_n(H)$ denote the free Lie algebra on the $\mathbf{Z}$-module H. Define $D_n(H)$ to be the kernel of the Lie bracket map

$$[\ ,\ ]\colon H \otimes L_n(H) \to L_{n+1}(H).$$

$D_n(H)$ is the universal module for Milnor invariants of degree n. (See [1].) We will see that $D(H) = \oplus_{n=1}^{\infty} D_n(H)$ is equipped with a graded Lie algebra structure and a compatible structure as a module over the symmetric group $\Sigma(l)$ on l letters.

Rationally, $D(H) \otimes \mathbf{Q}$ is isomorphic to the module $C^t(l)$, the $\mathbf{Q}$-module of (total) degree n, l-labeled, tree Feynman diagrams (see [2]). $C^t(l)$ is equipped with a Lie algebra structure and a compatible structure as a module over $\Sigma(l)$. $C^t(l)$ is graded by the degree, where the degree is half the number of vertices.

Theorem 1. The isomorphism

$$D(H) \otimes \mathbf{Q} \simeq C^t(l)$$

is an isomorphism of Lie algebras and $\Sigma(l)$ modules.

Now suppose that $l = 2g$. We set

$$D_n^{sp}(H) = D_{n+1}(H)$$

and $D^{sp}(H) = \oplus_{n=1}^{\infty} D_n^{sp}(H)$. We will see that $D^{sp}(H)$ has a natural structure of a graded Lie algebra and a module structure over the discrete group $Sp(2g; \mathbf{Z})$.

Rationally, $D_n^{sp}(H) \otimes \mathbf{Q}$ is isomorphic to the module $P^t(H \otimes \mathbf{Q}) = \oplus_{n=1}^{\infty} P_n^t(H \otimes \mathbf{Q})$ of tree Feynman diagrams with labels in $H \otimes \mathbf{Q}$. Here, the degree is the 'internal' degree, i.e., the number of 3-valent vertices. (One has $P_n^t(H \otimes \mathbf{Q}) \simeq C_{n+1}^t(2g)$.) $P^t(H \otimes \mathbf{Q})$ comes equipped with a natural structure of a graded Lie algebra and a module over the algebraic group $Sp(2g; \mathbf{Q})$.

Theorem 2. The isomorphism

$$D^{sp}(H) \otimes \mathbf{Q} \simeq P^t(H \otimes \mathbf{Q})$$

is an isomorphism of Lie algebras, $Sp(2g; \mathbf{Q})$ and $\Sigma(l)$ modules.

2. Preliminary results

In this section we show that the graded group associated to the tower of the automorphisms of free nilpotent groups has a natural structure of a Lie algebra over $\mathbf{Z}$. Let $F = F(l)$ be a free group on l generators, denote by F_n its lower central series, given by $F_1 = F$ and $F_{n+1} = [F, F_n]$. If we set $H = F/F_2$, then it is well known that $\oplus_{n=1}^{\infty} L_n(H)$, the free graded lie algebra Lie algebra on the $\mathbf{Z}$-module H is isomorphic to $\oplus_{n=1}^{\infty} F_n/F_{n+1}$ with Lie bracket induced from the commutator. Recall that in any group we have the Hall identities among commutators :

1. $[ab, c] = [b, c]^a [a, c]$ and $[a, bc] = [a, b][a, c]^b$;

2. $[a^b, [b, c]][b^c, [c, a]][c^a, [b, a]] = 1$.

Where $[a, b] = aba^{-1}b^{-1}$ and $c^a = aca^{-1}$. There are canonical extensions :

$$0 \longrightarrow \mathrm{Hom}(H, F_n/F_{n+1}) \longrightarrow \mathrm{Aut}(F/F_{n+1}) \longrightarrow \mathrm{Aut}(F/F_n) \longrightarrow 1,$$

where the action of $\mathrm{Aut}(F/F_n)$ on $\mathrm{Hom}(H, F_n/F_{n+1})$ is given as follows : Hopf's theorem shows that $F_n/F_{n+1} = H_2(F/F_n)$, the second homology with trivial $\mathbf{Z}$ coefficients of the group F/F_n, and $H = H_1(F/F_n)$. If $\phi \in \mathrm{Aut}(F/F_n)$ and $f \in \mathrm{Hom}(H, F_n/F_{n+1})$, then $\phi \cdot f = H_2(\phi) \circ f \circ H_1(\phi)^{-1}$. This action clearly factors through the projection $\mathrm{Aut}(F/F_n) \to \mathrm{Aut}(F/F_2)$, so that if we denote by $\mathrm{Aut}^h(F/F_3)$ the kernel of the projection $\mathrm{Aut}(F/F_3) \to \mathrm{Aut}(F/F_2)$ and by $\mathrm{Aut}^h(F/F_{n+1})$ the preimage of $\mathrm{Aut}^h(F/F_n)$ in $\mathrm{Aut}(F/F_{n+1})$ for $n \geq 2$ we get central extensions :

$$0\mathrm{Hom}(H, F_n/F_{n+1}) \longrightarrow \mathrm{Aut}^h(F/F_{n+1}) \longrightarrow \mathrm{Aut}^h(F/F_n) \longrightarrow 1 \quad (v_n).$$

Proposition 1. The graded group $\oplus_{n=2}^{\infty}\mathrm{Hom}(H, F_n/F_{n+1})$ has the structure of a graded Lie algebra, with $\mathrm{Hom}(H, F_n/F_{n+1})$ in degree $n-1$ and with Lie bracket induced form the commutators in the groups $\mathrm{Aut}^h(F/F_N)$ and the natural action of $\mathrm{Aut}(F/F_2)$ on $\oplus_{n=2}^{\infty}\mathrm{Hom}(H, F_n/F_{n+1})$.

Proof. Let $a, b \in \mathrm{Hom}(H, F_n/F_{n+1}) \times \mathrm{Hom}(H, F_m/F_{m+1})$ be two elements and lift them to $f, g \in \mathrm{Aut}^h(F/F_N)$, for $N \geq \max(n+m)$. We claim that the commutator $[f, g]$ is a uniquely and well defined element in the group $\mathrm{Hom}(H, F_{n+m-1}/F_{n+m})$. As the extensions v_n are central, this commutator is independent from the lifts.

Lemma 1. Suppose $f : F/F_N \to F/F_N$ satisfies $f(x) = x \bmod F_n/F_N$. If $x \in F_j/F_N$ then $f(x) = x \bmod F_{n+j-1}/F_N$

Proof. For $j = 1$ this is true by hypothesis. Suppose $x = [x_1, x_2]$ with $x_i \in F_{j_i}/F_N$. Set $f(x_i) = x_i d_i$ with $d_i \in F_{n+j_i-1}/F_N$. Then

$$\begin{aligned} f(x) &= [f(x_1), f(x_2)] \\ &= [x_1 d_1, x_2 d_2] \end{aligned}$$

using Hall identities this is equal to $[d_1, x_2 d_2]^{x_1}[x_1, x_2][x_1, d_d]^{x_2}$. □

Lemma 2. Suppose $f(x) = x \bmod F_n/F_N$ and $g(x) = x \bmod F_m/F_N$. Then $[f, g](x) = x \bmod F_{n+m-1}/F_N$.

Proof. We have $f^{-1}(g^{-1}(x)) = g^{-1}(x)d$ for some element $d \in F_n/F_N$ and $g^{-1}(x) = xd'$ for some element $d' \in F_m/F_N$. Then by definition :

$$\begin{aligned} xd' &= g^{-1}(x) &&= f(g^{-1}(x)d) \\ &= f(g^1(x))f(d) &&= f(xd')f(d) \\ &= f(x)d'f(d) &&= f(x)f(d)d' \\ &= f(xd)d' \bmod F_{n+m-1}/F_N. \end{aligned}$$

Finally we get $x = f(xd) \bmod F_{n+m-1}/F_N$. It follows that $[f,g](x) = fg(g^{-1}(x)d) = f(xg(d)) = f(xd) = x \bmod F_{n+m-1}/F_N$, since by the preceding lemma $g(d) = d \bmod F_{n+m-1}/F_N$. $\square$

We proceed now with the proof of proposition 1. By construction our lifts f, g satisfy the conditions of the preceding lemma, that is $[f,g] \in \ker(\mathrm{Aut}^h(F/F_N) \to \mathrm{Aut}^h(F/F_{n+m-1}))$.

Set $Q_{N,r} = \ker(\mathrm{Aut}^h(F/F_N) \twoheadrightarrow \mathrm{Aut}^h(F/F_r))$. The following commutative diagram shows that the image of $[f,g]$ in $\mathrm{Aut}^h(F/F_{n+m})$ is a well defined element in $\mathrm{Hom}(H, F_{n+m-1}/F_{n+m})$ and independent of the choice of N.

$$\begin{array}{ccccc}
Q_{M,n+m-1} & \hookrightarrow & \mathrm{Aut}^h(F/F_M) & \twoheadrightarrow & \mathrm{Aut}^h(F/F_{n+m-1}) \\
\downarrow & & \downarrow & & \| \\
Q_{N,n+m-1} & \hookrightarrow & \mathrm{Aut}^h(F/F_N) & \twoheadrightarrow & \mathrm{Aut}^h(F/F_{n+m-1}) \\
\downarrow & & \downarrow & & \| \\
\mathrm{Hom}(H, F_{n+m-1}/F_{n+m}) & \hookrightarrow & \mathrm{Aut}^h(F/F_{n+m}) & \twoheadrightarrow & \mathrm{Aut}^h(F/F_{n+m-1}).
\end{array}$$

The Jacobi identity follows immediately from Hall identities and the fact that the extensions v_h are central.

The action of $\mathrm{Aut}(F/F_2)$ is induced by the conjugation in $\mathrm{Aut}(F/F_N)$ and is therefore compatible with the Lie bracket induced from the commutators. $\square$

Remark : Set $\mathrm{Aut}(F)[k]$ for the kernel of the canonical automorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(F/F_{k+1})$. These groups form a decreasing filtration $\cdots \subset \mathrm{Aut}(F)[k] \subset \mathrm{Aut}(F)[k-1] \subset \mathrm{Aut}(F)[1] \subset \mathrm{Aut}(F)$, and there is a canonical injective morphism of graded groups $\oplus_{n=1}^{\infty} \mathrm{Aut}(F)[n]/\mathrm{Aut}(F)[n+1] \to \oplus_{n=2}^{\infty} \mathrm{Hom}(H, F_n/F_{n+1})$ which is in fact a morphism of graded Lie algebras where the Lie bracket in the first group is induced from the commutator in $\mathrm{Aut}(F)$.

3. Milnor Invariants and Trees

This section is a review of Milnor invariants from the point of view of [1], [2]. Recall that by a theorem of Artin, the pure braid group on l strands, $P(l)$, is isomorphic to the subgroup of the automorphism group of the free group $F = F(l)$ on generators $x_1, \ldots, x_l$, which satisfy the conditions

1. $x_i \longmapsto x_i^{\lambda_i}$;
2. $x_1 \cdots x_l \longmapsto x_1 \cdots x_l$.

Each generator is conjugated and the product $x_1 \cdots x_l$ is fixed. Geometrically the λ_i represent longitudes of the braid and $x_1 \cdots x_l$ is the boundary curve. Denote this group by $\mathrm{Aut}_0(F)$ where the subscript indicates the imposition of conditions 1 and 2. By the ending remark of the preceding section, filtering $\mathrm{Aut}_0(F)$ by the kernel of its images in the groups $\mathrm{Aut}(F/F_n)$ gives rise to a graded Lie algebra monomorphism

$$\oplus_{n=1}^{\infty} \mathrm{Aut}_0(F)[n]/\mathrm{Aut}_0(F)[n+1] \hookrightarrow \oplus_{n=2}^{\infty} \mathrm{Hom}(H, F_n/F_{n+1}).$$

One has approximations of the images of the groups $\mathrm{Aut}_0(F)[n]$ by the group $\mathrm{Aut}_0(F/F_n)$ of automorphisms of F/F_n subject to conditions 1 and 2. Then $\mathrm{Aut}_0(F/F_2)$ is trivial and $\mathrm{Aut}(F/F_{n+2})$ is a central extension of $\mathrm{Aut}_0(F/F_{n+1})$ by the group $\mu_n(l)$ (denoted by $K_n(l)$ in [3]) so that we have a commutative diagram of central extensions for each $n \geq 2$:

$$\begin{array}{ccccc}
\mu_n(l) & \hookrightarrow & \mathrm{Aut}_0(F/F_{n+2}) & \twoheadrightarrow & \mathrm{Aut}_0(F/F_{n+1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(H, F_{n+1}/F_{n+3}) & \hookrightarrow & \mathrm{Aut}^h(F/F_{n+2}) & \twoheadrightarrow & \mathrm{Aut}^h(F/F_{n+1}).
\end{array}$$

We have adopted the notation $\mu_n(l)$ here to signify that this group is the universal collection of Milnor invariants of degree n (length $n+1$) of sting links whose Milnor invariants of degree $n-1$ vanish. One can think of $\mathrm{Aut}_0(F/F_{n+2})$ as being the universal collection of Milnor invariants of degree less than or equal to n and of $\oplus_{n=1}^{\infty}\mu_n(l)$ as being an approximation of $\oplus_{n=2}^{\infty}\mathrm{Aut}_0(F)[n]/\mathrm{Aut}_0(F)[n+1]$. Our next step is to show that $\oplus_{n=1}^{\infty}\mu_n(l)$ is a Lie sub-algebra of the Lie algebra given in proposition 1 and give the geometrical interpretation of the Lie structure given in theorem 1.

The group $\mu_n(l)$ is a free abelian group of known rank [1] and can be computed as follows. Define $D_n(H)$ to be kernel of the map $[\ ,\ ] : H \otimes L_n(H) \longrightarrow L_{n+1}(H)$. Then a map $D_n(H) \longrightarrow \mu_n(H)$ is given by sending $\sum_i x_i \otimes \lambda_i \in H \otimes L_n(H)$ to the automorphism which sends x_i to $x_i^{\lambda_i}$, where it is to be noted that $x_i^{\lambda_i}$ depends mod F_{n+2} only on λ_i mod F_{n+1} and we have identified $L_n(H)$ with F_n/F_{n+1}. Condition 2 is then satisfied since $\sum_i x_i \otimes \lambda_i$ is assumed to be in the kernel of $[\ ,\ ]$.

The above map $D_n(H) \longrightarrow \mu_n(l)$ is actually an isomorphism for $n > 1$ and has a small kernel generated by $x_i \otimes x_i$ for $n = 1$, these correspond to "framings" and may be incorporated by multiplying $\mathrm{Aut}_0(F/F_n)$ by a free abelian group of rank l. Set $H^{\mathbf{Q}} = H \otimes \mathbf{Q}$ and let $L(H^{\mathbf{Q}})$ denote the free Lie algebra on the $\mathbf{Q}$-vector space $H^{\mathbf{Q}}$. Set $D_n(H^{\mathbf{Q}}) = \ker\left(H^{\mathbf{Q}} \otimes L_n(H^{\mathbf{Q}}) \xrightarrow{[\ ,\ ]} L_{n+1}(H^{\mathbf{Q}})\right)$. Then $D_n(H^{\mathbf{Q}}) = D_n(H) \otimes \mathbf{Q}$ has an interpretation as the module $P^t_{n-1}(H^{\mathbf{Q}})$ of internal degree $n-1$ tree Feynman diagrams with labels in $H^{\mathbf{Q}}$.

Recall that a Feynman diagram with labels in $H^{\mathbf{Q}}$ is a vertex-oriented uni-trivalent graph with univalent vertices labelled by an element of $H^{\mathbf{Q}}$. These graphs are subject to the classical AS and IHX relations and multilinearity of the labels. When a basis $x_1, \ldots, x_l$ of $H^{\mathbf{Q}}$ is given it is enough to take the labels in the set $\{x_1, \ldots, x_l\}$ or equivalently in the set $\{1, \ldots, l\}$ and dispense with multilinearity. When the labels are restricted to the set $\{1, \ldots, l\}$ we will write $C^t_n(l)$ for $P^t_{n-1}(H^{\mathbf{Q}})$. Here P stands for primitives and t for trees. The shift from $n-1$ to n indicates that we are using the degree and not the internal degree, where the degree of a diagram is half the number of internal and external vertices. The isomorphism $P^t_{n-1}(H^{\mathbf{Q}}) \longrightarrow D_n(H^{\mathbf{Q}})$ is given as follows :

Observe [2] that $L_n(H^{\mathbf{Q}})$ has a diagrammatic representation as the module of the diagrams having a distinguished univalent vertex called the root, without a label. Call such a tree a rooted tree. Then the map is given by $\Gamma \longmapsto \sum_{v\in u(\Gamma)} x_v \otimes \Gamma_v$ where $u(\Gamma)$ denotes the set of univalent vertices of Γ, $x_v \in H^{\mathbf{Q}}$ denotes the label of the vertex v and Γ_v is the rooted tree obtained from Γ by removing the label x_v and considering v as the distinguished vertex.

The isomorphism $C^t_n(l) \overset{\sim}{\to} D_n(H^{\mathbf{Q}}) \simeq \mu_n(l) \otimes \mathbf{Q}$ actually is more than just a vector space isomorphism.

Theorem 3. The $\mathbf{Q}$-vector spaces $\mu(l) = \oplus_{n=1}^{\infty} \mu_n(l)$ and $C^t(l) = \oplus_{n=1}^{\infty} C^t_n(l)$ are naturally graded Lie algebras and the morphism $\mu(l) \longrightarrow C^t(l)$ is an isomorphism of graded Lie algebras. Moreover, both $\mu(l)$ and $C^t(l)$ are equipped with an action of the symmetric group on l letters and these actions correspond.

Remark : One can deduce this result from the theory of Milnor invariants of string links and the Konsevich integral since both the Konsevich integral at tree level and the Artin braid representation are representations of the group of string links up to concordance, and the induced filtrations on this group are the same [2]. One can nevertheless give an entirely algebraic and elementary proof as follows.

Proof. Our first goal is to show that $\mu(l)$ is a Lie sub-algebra of the Lie algebra $\oplus_{n=2}^{\infty}\mathrm{Hom}(H, F_{n+1}/F_{n+2})$. The computation of the Lie bracket amounts to a computation of commutators in the automorphism group $\mathrm{Aut}_0(F/F_N)$. The following result is a consequence of lemmas 1 and 2.

Lemma 3. If $f, g \in \mathrm{Aut}^h(F/F_N)$ satisfies $f(x) = x^{\lambda}$ and $g(x) = x^{\eta}$ with $\lambda \in F_n/F_N$ and $\eta \in F_m/F_N$, then if $x \in F_j/F_N$, $f(x) = x \bmod F_{n+j}/F_N$ and $\forall y \in F/F_N$ $[f, g](y) = y \bmod F_{n+m+1}/F_N$.

Proposition 2. If $a = \sum_{i=1}^{l} x_i \otimes \lambda_i \in \mu_n(l)$ and $b = \sum_{i=1}^{l} x_i \otimes \eta_i \in \mu_m$ are represented respectively by f and g in $\mathrm{Aut}_0(F/F_N)$ for N sufficiently large then

$$[a,b] = \sum_{i=1}^{l} x_i \otimes \left([\eta_i^{-1}, \lambda_i^{-1}] + \lambda_i g(\lambda_i^{-1}) + \eta_i^{-1} f(\eta_i)\right).$$

Proof. We have $f(x_i) = x_i^{\lambda_i}$, $g(x_i) = x_i^{\eta_i}$. Then $f^{-1}(x_i) = x_i^{\lambda_i'}$, so $x_i = f(f^{-1}(x_i)) = x_i^{f(\lambda_i')\lambda_i}$ and $[f(\lambda_i')\lambda_i, x_i] = 1 \mod F_N$. It follows that $f(\lambda_i')\lambda_i = x_i^k \mod F_{N-1}$ (see [1]). For $N \geq 2$ $f(\lambda_i')\lambda_i \in F_2$ so k must be 0, when $N = 1$ we still have $k = 0$ since we may suppose λ_i to be x_i-reduced. Thus $f(\lambda_i') = \lambda_i^{-1} \mod F_{N-1}$ and similarly $g(\eta_i') = \eta_i^{-1} \mod F_{N-1}$. We have that

$$fgf^{-1}g^{-1}(x_i) = x_i^{fgf^{-1}(\eta_i')fg(\lambda_i')f(\eta_i)\lambda_i},$$

which shows that $\mu(l)$ is a Lie sub-algebra. Computing modulo F_{n+m+1}, we have :

$$\begin{aligned}
fgf^{-1}(\eta_i') \cdot fg(\lambda_i') \cdot f(\eta_i)\lambda_i &= [f,g]g(\eta_i') \cdot fgf^{-1}f(\lambda_i') \cdot f(\eta_i) \cdot \lambda_i \\
&= [f,g](\eta_i^{-1}) \cdot fgf^{-1}(\lambda_i^{-1}) \cdot f(\eta_i) \cdot \lambda_i \\
&= \eta_i^{-1} \cdot g(\lambda_i^{-1}) \cdot f(\eta_i) \cdot \lambda_i \\
&= [\eta_i^{-1}, \lambda_i^{-1}] \cdot \left(\lambda_i g(\lambda_i^{-1})\right) \cdot \left(\eta_i^{-1} f(\eta_i)\right).
\end{aligned}$$

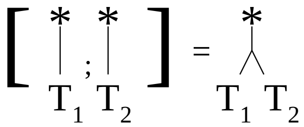
Since the three terms are in F_{n+m}, we may write them additively mod F_{n+m+1} as in the proposition. □

We now recall the Lie algebra structure on $C^t(l)$. It arises from the algebra structure on $A^t(l)$ induced from the algebra structure on $A(l)$, the algebra of Feynman diagrams with support on a collection of l disjoint intervals, $A^t(l)$ is obtained form $A(l)$ by setting diagrams having homology equal to zero. The algebra structure is given by stacking diagrams , the primitives form then a Lie algebra via the algebra commutator $[u,v] = uv - vu$. This is compatible using the STU relation and at the tree level is given as follows :

Proposition 3. Let T_1, T_2 be labelled trees. Then $[T_1, T_2] = \sum T_{v_1,v_2}$, where the sum ranges over the set $u(T_1) \times u(T_2)$, where $u(T_i)$ denotes the set of univalent vertices of T_i and for $v_i \in u(T_i)$, T_{v_1,v_2} is the "bracketing" of the trees T_1, T_2.

Here by the "bracketing" of two trees T_1, T_2 we mean the rooted tree corresponding to the Lie bracket in the free Lie algebra, it is given as follows:

where the orientation of the new 3-valent vertex is the trigonometric one. Then T_{v_1,v_2} is zero if the labels of v_1 and v_2 are different and is the tree obtained by removing the common label of v_1 and v_2, bracketing the two



rooted trees and labeling the new root by the common label. The new edge ending at the root will be called the "bracketing edge".

Using the above two propositions we now finish the proof of the theorem, that is we show the two Lie algebra structures are the same. Let us call the map $C_n^t(l) \to H^{\mathbf{Q}} \otimes L_n(H^{\mathbf{Q}})$ the expansion map. Let $T_1 \in C_n^t(l), T_2 \in C_m^t(l)$ be two tree diagrams. The expansion of T_1 is an element $a = \sum_{i=1}^l x_i \otimes \lambda_i$ where λ_i is the sum of rooted trees obtained from T_1 by replacing one of the i-labeled vertices by a root. A similar remark applies to the expansion of T_2 into an element $b = \sum_{i=1}^l x_i \otimes \eta_i$. We must show that the expansion of $[T_1, T_2]$ is given by proposition 2, i.e. $\sum_{i=1}^l x_i \otimes \big([\eta_i^{-1}, \lambda_i^{-1}] + \lambda_i g(\lambda_i)^{-1} + \eta_i^{-1} f(\eta_i)\big)$.

The term $[\eta_i^{-1}, \lambda_i^{-1}]$ of $[a, b]$ is the sum of the bracketings of the rooted trees obtained from T_1 with those obtained from T_2 by rooting an i-labelled vertex v_1 of T_1 and v_2 of T_2. This corresponds to the terms in the expansion of $[T_1, T_2]$ arising from T_{v_1,v_2} by rooting the univalent vertex of the "bracketing edge".

The remaining terms in the expansion of $[T_1, T_2]$ correspond to $\lambda_i g(\lambda_i^{-1})$ and $\eta_i f(\eta_i^{-1})$. Indeed η_i is a product (sum) of terms $\eta_i^1 \ldots \eta_i^k$ and $\eta_i f(\eta_i^{-1})$ is the product

$$(\eta_i^1 \ldots \eta_i^k)^{-1} f(\eta_i^1 \ldots \eta_i^k) = (\eta_i^1)^{-1} f(\eta_i^1) \ldots (\eta_i^k)^{-1} f(\eta_i^k).$$

To compute each term, note that if the rooted tree T with labels x_j corresponds to η then $f(\eta)$ corresponds to the tree with labels $x_j^{\lambda_j}$. As this equals $[\lambda_j, x_j] x_j$ (using $[ab, c] = [b, c]^a [a, c]$ and $[a, bc] = [a, b][a, c]^b$), $\eta^{-1} f(\eta)$ is a sum of terms obtained from T by gluing to some edge labelled x_j the labelled tree corresponding to the bracketing $[\lambda_j, x_j]$. Each such tree in the expansion of $[T_1, T_2]$ corresponds to rooting a vertex of T_{v_1,v_2} at an edge labelled x_j lying in $T_2 \subset T_{v_1,v_2}$. Similarly, the terms of $\lambda_i^{-1} g(\lambda_i)$ correspond to vertices lying in $T_1 \subset T_{v_1,v_2}$. $\square$

The action of the symmetric group comes via the action of conjugation in the group $\widetilde{\mathrm{Aut}_0}(F/F_n)$, where the tilde denotes that condition 1 is replaced by the condition :

1bis) $x_i \mapsto x_{\sigma(i)}^{\lambda_i}$

where σ denotes a permutation in the symmetric group $\Sigma(l)$. $\mathrm{Aut}_0(F/F_n)$

is then the kernel of $\widetilde{\mathrm{Aut}_0}(F/F_n) \to \Sigma(l)$.

4. The Johnson homomorphism and trees

In this section we review the point of view of Garoufalidis and Levine [4] regarding the Johnson homomorphism. This is in many ways analogous to the results of section 3, in fact the analogy can be made very explicit geometrically, see [5].

Fix an oriented surface of genus g with one boundary component, $\Sigma_{g,1}$, and fix a basis of $\pi_1(\Sigma_{g,1}) \simeq F(2g)$, say $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ which projects to a symplectic basis $x_1, x_2, \ldots, y_1, y_2 \ldots$ with respect to the intersection pairing ω in $H_1(\Sigma_{g,1})$. Recall that by a theorem of Nielsen, the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ is isomorphic to the subgroup $\mathrm{Aut}_{Sp}(F) \subset \mathrm{Aut}(F)$, $F = F(2g)$ where we use the subscript to denote the condition :

1. $[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] \longmapsto [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]$.

Setting $\mathcal{M}_{g,1}[n] = \mathcal{M}_{g,1} \cap \mathrm{Aut}(F)[n]$ we obtain a deceasing filtration of the mapping class group, $\mathcal{M}_{g,1} = \mathcal{M}_{g,1}[0] \supset \mathcal{T}_{g,1} = \mathcal{M}_{g,1}[1] \supset \mathcal{K}_{g,1} = \mathcal{M}_{g,1}[2] \ldots$ where $\mathcal{T}_{g,1}$ stands for the Torelli group and $\mathcal{K}_{g,1}$ for the Johnson group i.e. the group generated by Dehn twists along separating curves (the result $\mathcal{K}_{g,1} = \mathcal{M}_{g,1}[2]$ was settled by Johnson [6]). The embedding $\mathcal{M}_{g,1} \to \mathrm{Aut}(F)$ induces a family of homomorphisms $\mathcal{M}_{g,1}[n] \to \mathrm{Aut}(F/F_{n+1})$ whose images can be approximated by the subgroups $\mathrm{Aut}_{Sp}(F/F_n) \subset \mathrm{Aut}(F/F_n)$ where we require condition (1) to hold in a slightly stronger sense : an isomorphism of F/F_n lifts to a self map of F and we require the lift to satisfy (1) mod F_{n-1} and not just mod F_n. Since two lifts differ on the generators by elements of F_n, commutators differ by an element of F_{n+1}, so this condition is independent of the lift. Now $\mathrm{Aut}_{Sp}(F/F_2) = \mathrm{Sp}(H) = \mathrm{Sp}(2g, \mathbf{Z})$, where $\mathrm{Sp}(H)$ denotes the group of automorphisms preserving the intersection pairing. $\mathrm{Aut}_{Sp}(F/F_{n+2}) \to \mathrm{Aut}_{Sp}(F/F_{n+1})$ is an extension by an abelian group $J_n(2g)$ and by construction it restricts to a map $\mathcal{M}_{g,1}[n] \to J_n(2g)$ known as the Johnson homomorphism. It is a fundamental problem to describe its image. Here we used the letter J to indicate that J_n is the universal group in which the generalized Johnson invariants of order n take values.

The generalized Johnson invariants. These are h-cobordism (or homology cobordism) invariants from $\Sigma_{g,1}$ to itself (see [4]). By an h-cobordism from $\Sigma_{g,1}$ to itself we mean a connected 3-manifold M whose boundary is identified with $\partial(\Sigma_{g,1} \times [0,1])$ such that the inclusions $\Sigma_{g,1} \times \{0\} \hookrightarrow M$ and $\Sigma_{g,1} \times \{1\} \hookrightarrow M$ induce isomorphisms on H_1, and hence on all of H_*. By a theorem of Stallings, these inclusions induce isomorphisms on $\pi_1(\Sigma_{g,1})/(\pi_1(\Sigma_{g,1}))_{n+1}$ for all $n \geq 1$, thus producing an automorphism of

F/F_n for all n satisfying (1).

Set $\mathrm{Aut}^h_{Sp}(F/F_n)$ for $\mathrm{Aut}_{Sp}(F/F_n) \cap \mathrm{Aut}^h(F/F_n)$, and remark that by definition $\mathcal{T}_{g,1}$ maps into $\mathrm{Aut}^h_{Sp}(F/F_3)$, so that if we set $\mathcal{T}_{g,1}[n] = \mathcal{T}_{g,1} \cap \mathrm{Aut}(F)[n]$ we have for all $n \geq 1$, $\mathcal{T}_{g,1}[n] = \mathcal{M}_{g,1}[n]$ and the Johnson homomorphism maps $\mathcal{T}_{g,1}[n]$ into $\mathrm{Aut}^h_{Sp}(F/F_n)$. The group $\mathrm{Aut}^h_{Sp}(F/F_{n+1})$ is a central extension of $\mathrm{Aut}^h_{Sp}(F/F_n)$ by the abelian group $J_{n-1}(2g)$, in particular it is a nilpotent group of class $\leq n-1$ and thus $(\mathcal{T}_{g,1})_n \subset \mathcal{T}_{g,1}[n]$. It is an important problem to study the map $(\mathcal{T}_{g,1})_n/(\mathcal{T}_{g,1})_{n+1} \to \mathcal{T}_{g,1}[n]/\mathcal{T}_{g,1}[n+1]$.

By construction we have a commutative diagram

$$\begin{array}{ccccc}
J_n(2g) & \hookrightarrow & \mathrm{Aut}^h_{Sp}(F/F_{n+2}) & \twoheadrightarrow & \mathrm{Aut}^h_{Sp}(F/F_{n+1}) \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(H, F_{n+1}/F_{n+2}) & \hookrightarrow & \mathrm{Aut}(F/F_{n+2}) & \twoheadrightarrow & \mathrm{Aut}(F/F_{n+1}).
\end{array}$$

If we compose the leftmost vertical map with the isomorphism

$$\mathrm{Hom}(H, F_{n+1}/F_{n+2}) \longrightarrow H \otimes L_{n+1}(H)$$

induced by the intersection pairing we get an isomorphism $J_n(2g) \xrightarrow{\sim} D_{n+1}(H)$ which inverse is given as follows : an element $\sum_{i=1}^g \alpha_i \otimes \mu_i^\beta - \beta_i \otimes \mu_i^\alpha$ such that $\sum_{i=1}^g [\alpha_i, \mu_i^\beta] - [\beta_i, \mu_i^\alpha]$ is sent to the automorphism of F/F_{n+1} which sends α_i to $\alpha_i \mu_i^\alpha$ and β_i to $\beta_i \mu_i^\beta$. Condition $1bis$ is then an immediate consequence of Hall identities. Combining this with the isomorphism $D_n(H^{\mathbf{Q}}) \to P^t_{n-1}(H^{\mathbf{Q}})$ we obtain an isomorphism $J_{n-1}(2g) \otimes \mathbf{Q} \to P^t_{n-1}(H^{\mathbf{Q}})$.

In [4] Garoufalidis and Levine defined a Lie algebra structure on $P^t(H^{\mathbf{Q}})$ as follows. Set $\omega\ : H^{\mathbf{Q}} \otimes H^{\mathbf{Q}} \to \mathbf{Q}$ for the intersection pairing. Then if T_1, T_2 are two labeled trees, then :

$$[T_1, T_2] = \sum_{v_1 \in u(T_1), v_2 \in u(T_2)} \omega(x_{v_1}, x_{v_2}) T_{v_1, v_2}$$

where $u(T_i)$ denotes the set of univalent vertices of T_i, $x_v \in H^{\mathbf{Q}}$ denotes the label of the vertex v and T_{v_1,v_2} denotes the tree resulting of gluing vertices v_1 to v_2. The analogue of theorem of the last section is:

Theorem 4.

1. $J(2g) = \oplus_{n=1}^\infty J_n(2g)$ and $P^t(H^{\mathbf{Q}}) = \oplus_{n=1}^\infty P^t_n(H^{\mathbf{Q}})$ have natural Lie algebra structures and the isomorphism $J(2g) \otimes \mathbf{Q} \simeq P^t(H^{\mathbf{Q}})$ is an isomorphism of Lie algebras.

2. Furthermore, $J(2g)$ has the structure of a $Sp(2g, \mathbf{Z})$-module and $P^t(H^{\mathbf{Q}})$ has a $Sp(2g, \mathbf{Q})$-module structure and the above isomorphism respects

the $Sp(2g,\mathbf{Z})$-module structure. In particular the $Sp(2g,\mathbf{Z})$-module structure on $J(2g)$ extends to an $Sp(2g,\mathbf{Q})$-module over $J(2g)\otimes\mathbf{Q}$.

Remark: The Lie algebra structures on $P^t(H^{\mathbf{Q}})$ described in this section and on $P^t(2g)$ described in the last section are different. Note that the degrees are shifted by 1 and that $P^t(H^{\mathbf{Q}})$ does not contain $P_1^t(2g)$.

Remark : This theorem is needed in the proof of theorem of [4], when it was assumed that the map was a morphism of Lie algebras.

Proof. The Lie algebra structure on $J(2g)$ comes by restriction of the Lie algebra structure on $\oplus_{n=2}^{\infty}\mathrm{Hom}(H,L_n(H))$, we proceed to compute the Lie bracket in $\oplus_{n=2}^{\infty}\mathrm{Hom}(H,L_n(H))$, and the show that $D(H)$ is a Lie subalgebra.

For $f\in\mathrm{Hom}(H,L_n(H))$ and $l=[[\dots[l_1,l_2],\dots,l_{k-1}],l_k]$ an elementary commutator in $L_k(H)$ set $f\{l\}$ for $\displaystyle\sum_{i=1}^{k}[[[\dots[l_1,l_2],\dots,f(l_i)]\dots,]l_k]$. This extends to a well defined derivation of degree $n-1$ in the Lie algebra $L(H)$. Define a bilinear pairing $\mathrm{Hom}(H,L_n(H))\otimes\mathrm{Hom}(H,L_m(H))\to\mathrm{Hom}(H,L_{n+m-1}(H))$ by setting for all $u\in H$ $[f,g](u)=f\{g(u)\}-g\{f(u)\}$.

Remark : The map which sends the couple $(f,g)\in\mathrm{Hom}(H,L_n(H))\times\mathrm{Hom}(H,L_m(H))$ to $f\{g\}$ is not a an associative product, instead we have the following lemma which shows that the bracket defined above is a Lie bracket.

Lemma 4. For all

$$(f,g,h)\in\mathrm{Hom}(H,L_n(H))\times\mathrm{Hom}(H,L_m(H))\times\mathrm{Hom}(H,L_r(H))$$

we have $f\{g\{h\}\}-f\{g\}\{h\}=g\{f\{h\}\}-g\{f\}\{h\}$.

Proof. Direct computation. □

Proposition 4. The Lie algebra structures on $\oplus_{n=2}^{\infty}\mathrm{Hom}(H,L_n(H))$ given in prop 1 and given by the above bracket are equal.

Proof. Let $a\in\mathrm{Hom}(H,L_n(H))$ and $b\in\mathrm{Hom}(H,L_m(H))$ be represented by $f,g\in\mathrm{Aut}^h_{Sp}(F/F_N)$. The first Lie algebra structure is given by the projection of the commutator $[f,g]$ in $\mathrm{Aut}^h_{Sp}(F/F_{n+m})$ .For $x_i\in\{\alpha_i,\beta_i\mid 1\le i\le g\}$ set $f(x_i)=x_i\lambda_i$, $f^{-1}(x_i)=x_i\lambda_i'$ with $\lambda_i,\lambda_i'\in F_n/F_N$ and $g(x_i)=x_i\eta_i$, $g^{-1}(x_i)=x_i\eta_i'$ with $\eta_i,\eta_i'\in F_m/F_N$. It is immediate that $g(\eta_i')=\eta_i^{-1}$ and

$f(\lambda_i') = \lambda_i^{-1}$. We can now compute :

$$\begin{array}{rcl} [f,g](x_i) & = & fgf^{-1}(x_i\eta_i') \\ & = & fg(x_i\lambda_i') \cdot fgf^{-1}(eta_i') \\ & = & x_i \cdot \lambda_i \cdot f(\eta_i) \cdot fg(\lambda_i') \cdot fgf^{-1}(\eta_i') \\ & = & x_i \cdot \lambda_i \cdot f(\eta_i) \cdot fgf^{-1}f(\lambda_i') \cdot [f,g]g(\eta_i') \\ & = & x_i \cdot \lambda_i \cdot f(\eta_i) \cdot [f,g]g(\lambda_i^{-1}) \cdot [f,g](\eta_i^{-1}) \\ & = & x_i \cdot \lambda_i \cdot g(\lambda_i^{-1}) \cdot f(\eta_i)\eta_i^{-1}) \bmod F_{n+m}/F_N \text{ by lemma 2} \\ & = & x_i(f\{g(x_i)\} - g\{f(x_i)\}) \bmod F_{n+m}/F_N \text{by Hall's identities.} \end{array}$$

□

For $a \otimes [\ldots[u_1,u_2],\ldots,u_n] \in H \otimes L_n(H)$ and $b \otimes [\ldots[v_1,v_2],\ldots,v_m] \in H \otimes L_m(H)$ the Lie bracket writes

$$\begin{array}{rl} [a\otimes[u_1,\ldots,u_n], b\otimes[v_1,\ldots,v_m]] & = \\ \sum_{i=1}^{m} \omega(a,v_i)\ b\otimes[v_1,\ldots,[u_1,\ldots,u_n],v_{i+1},\ldots,v_n] & - \\ \sum_{i=1}^{n} \omega(b,u_i)\ a\otimes[u_1,\ldots,[v_1,\ldots,v_n],u_{i+1},\ldots,u_n] & \end{array}$$

Let T_1, T_2 be two trees in $P_n^T(H^{\mathbf{Q}}), P_m^T(H^{\mathbf{Q}})$ and expand them respectively into
$\sum_{v_1\in u(T_1)} x_{v_1}\otimes T_{v_1}$ and $\sum_{v_2\in u(T_2)} x_{v_2}\otimes T_{v_2}$. The Lie bracket of the expansions is then :

$$\sum_{v_1\in u(T_1)}\ \sum_{v_2\in u(T_2)\ v_2'\in u(T_2)} \omega(x_{v_1},x_{v_2'})\ x_{v_2}\otimes(T_{v_1}\sharp T_{v_2})_{v_2'}$$
$$-\sum_{v_1'\in u(T_1)} \omega(x_{v_2},x_{v_1'})\ x_{v_1}\otimes(T_{v_1}\sharp T_{v_2})_{v_1'}$$

where $(T_{v_1}\sharp T_{v_2})_{v_2'}$ stands for the result of gluing the root of T_{v_1} to the leg v_2' of T_{v_2}.

Now, the expansion of the Lie bracket $[T_1,T_2]$ is

$$\sum_{v_1\in u(T_1)\ v_2\in u(T_2)} \omega(x_{v_1},x_{v_2}) \sum_{v_1'\in u(T_1)\ v_1'\neq v_1} x_{v_1'}\otimes T_{v_1,v_2}^{v_1'} + \sum_{v_2'\in u(T_2)\ v_2'\neq v_2} x_{v_2'}\otimes T_{v_1,v_2}^{v_2'}$$

where $x_{v_i'}\otimes T_{v_1,v_2}^{v_i'}$ denotes the expansion of the tree T_{v_1,v_2} rooted by its leg v_i'. It is clear that $x_{v_1'}\otimes T_{v_1,v_2}^{v_1'} = x_{v_1'}\otimes(T_{v_1'}\sharp T_{v_2})_{v_1}$, then by the antisymmetry of ω the two expressions coincide.

As the image of the canonical map $\mathcal{M}_{g,1} \to \mathrm{Aut}(F/F_2)$ can be identified with the group $\mathrm{Sp}(2g, \mathbf{Z})$, the natural action of $\mathrm{Aut}(F/F_2)$ on the graded group $\oplus_{n=2}^{\infty}\mathrm{Hom}(H, F_n/F_{n+1})$ restricts to an $\mathrm{Sp}(2g, \mathbf{Z})$-action on $D(H)$ (notice that the isomorphism $\mathrm{Hom}(H, F_n/F_{n+1}) \simeq H \otimes L_n(H)$ is $\mathrm{Sp}(2g, \mathbf{Z})$-equivariant but not $\mathrm{Aut}(F/F_2)$-equivariant).

More precisely, an element $\phi \in \mathrm{Sp}(2g, \mathbf{Z})$ acts on $\sum_i u_i \otimes l_i \in D_n(H)$ by

$$\phi \cdot \left(\sum_i u_i \otimes l_i \right) = \sum_i \phi(u_i) \otimes \phi(l_i)$$

where

$$\phi([\dots[u_1, u_2], \dots, u_n]) = [\dots[\phi(u_1), \phi(u_2)], \dots, \phi(u_n)].$$

The $\mathrm{Sp}(2g, \mathbf{Q})$ structure on $P^t(H^{\mathbf{Q}})$ is given by applying the element ψ to each label of a tree. The map $P^t(H^{\mathbf{Q}}) \to H^{\mathbf{Q}} \otimes L(H^{\mathbf{Q}})$ is then obviously equivariant. $\square$

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