Coarse geometry of isometry groups of symmetric spaces

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- joint with Misha Kapovich and Bernhard Leeb, arXiv:1403.7671
- Geometry on Groups and Spaces, August 7–12, 2014, KAIST.

Motivation

- Which properties of convex cocompact groups in rank one can be generalized to higher rank symmetric spaces?
- For $X = G/K$ symmetric space of noncompact type, $\mathrm{rank}(X) \geq 2$, look for discrete $\Gamma < G$ with *"rank-1 behavior"*
- Plan of the talk:

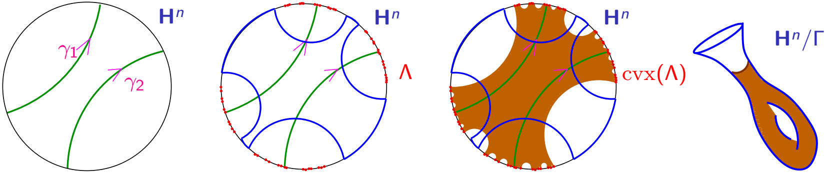
I. Convex cocompact groups in rank one *(through one example: Schottky groups)*

II. Higher rank symmetric spaces of non compact type $X = G/K$ (through one example: $SL(3, \mathbf{R})/SO(3)$)

III. Morse group actions on higher rank symmetric spaces.

- Get the image of Anosov representations.
- Take the point of view of discrete subgroups $\Gamma < G$ acting on $X = G/K$.

Review on Schottky groups in $\mathbf{H}^n$



Schottky: For $n_1, n_2 \gg 1$, $\Gamma = \langle \gamma_1^{n_1}, \gamma_2^{n_2} \rangle$ discrete & free (ping-pong)

Limit set $\Lambda = \overline{\Gamma x} \cap \partial_\infty \mathbf{H}^n$. Discontinuity domain: $\Omega = \partial_\infty \mathbf{H}^n \setminus \Lambda$

(A) There is a compact fundamental domain in $\mathbf{H}^n \cup \Omega$

(B) $(\mathbf{H}^n \cup \Omega)/\Gamma$ is compact

(C) (convex hull of Λ)$/\Gamma$ is compact (Convex cocompact)

(D) Γ is undistorted (Γ quasi-isometric to Γx)

(E) Γ is word hyperbolic and $\partial_\infty \Gamma \cong_\Gamma \Lambda$

(F) Every $\xi \in \Lambda$ is conical ($\gamma_n x \to \xi$, $d(\gamma_n x, l) < c$ for some line l)

Thm: For $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ discrete, all properties are equivalent

Convex cocompact groups in rank 1

Thm: For $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ discrete, the following are equivalent

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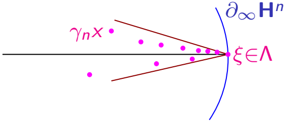
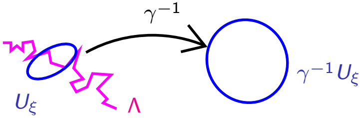




(G) Γ expanding at every $\xi \in \Lambda$: there exist $\gamma \in \Gamma$, $U_\xi \subset \partial_\infty \mathbf{H}^n$, and $c > 1$ so that:

$$d(\gamma^{-1}\xi_1, \gamma^{-1}\xi_2) > c \cdot d(\xi_1, \xi_2), \qquad \forall \xi_1, \xi_2 \in U_\xi.$$

Question Which properties generalize to higher rank? (D), (E), (F), (G)

Symmetric spaces of noncompact type: $X = G/K$

- $X = G/K$ symmetric space of noncompact type, without Euclidean factors
  e.g. $\mathbf{H}^n = SO(n,1)/SO(n)$, $SL(n)/SO(n)$
- X is a Cartan-Hadamard manifold ($\sec \leq 0$ and 1-connected)
- $\mathrm{rank}(X) =$ dim maximal flat in X.
  $\mathrm{rank}(X) = 1$ iff $\sec(X) < 0$.
  Higher rank means $\mathrm{rank}(X) \geq 2$
- Ideal (visual) boundary:
  $\partial_\infty X = \{r : [0, +\infty) \to X \text{ geodesic } \}/\sim$
  where $r_1 \sim r_2$ if $d(r_1(t), r_2(t)) \leq C$

  $\partial_\infty X \cong (T_x X)^1 \cong S^{\dim X - 1}$ but $G \curvearrowright \partial_\infty X$ is *not transitive*.
- $\partial_\infty X$ has a structure of spherical Tits building induced by the ideal boundary of flats.
- To simplify, here assume $X = SL(3)/SO(3)$, but everything works for general $X = G/K$ symmetric space noncompact type

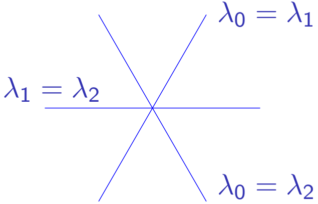
Higher rank example: $X = SL(3)/SO(3)$

- $\dim X = 5$ and $\partial_\infty X \cong S^4$
- $\operatorname{rank} X = 2$.

  Maximal flats ($\cong \mathbf{R}^2$ tot. geod.) through x_0:

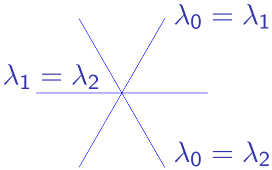
  $$F = \left\{ g \exp \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix} g^{-1} x_0 \mid \lambda_0 + \lambda_1 + \lambda_2 = 0 \right\}$$
- A geodesic is regular if contained in a *unique* maximal flat.

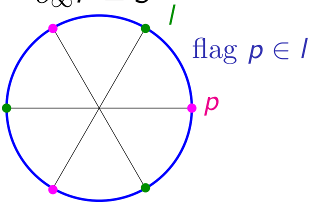
  Singular geodesics through x_0: $\lambda_i = \lambda_j$



$\partial_\infty(SL(3)/SO(3))$

$F \cong \mathbf{R}^2$ $\qquad$ $\partial_\infty F \cong S^1$

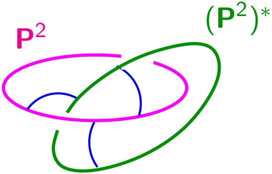




$$\text{As } t \to +\infty,\ \exp t \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix} \to \begin{cases} p \in \mathbf{P}^2 & \text{if } \lambda_0 > \lambda_1 = \lambda_2 \\ l \in (\mathbf{P}^2)^* & \text{if } \lambda_0 = \lambda_1 > \lambda_2 \\ (p, l),\ p \in l & \text{if } \lambda_0 > \lambda_1 > \lambda_2 \end{cases}$$

$$\mathrm{Flag}(\mathbf{P}^2) = \{(p, l) \in \mathbf{P}^2 \times (\mathbf{P}^2)^* \mid p \in l\}$$

$$\partial_\infty X = \mathrm{Flag}(\mathbf{P}^2) \times (0, \tfrac{\pi}{3}) \sqcup \mathbf{P}^2 \sqcup (\mathbf{P}^2)^* \cong S^4 \ (SL(3)\text{-inv})$$



Chamber:$= (p, l) \times (0, \frac{\pi}{3})$

$\Gamma \subset SL(3)$ *discrete regular subgroup*

- Assume Γ regular, i.e. $\Lambda(\Gamma) \subset (\partial_\infty X)^{reg} \xrightarrow{\pi} \mathrm{Flag}(\mathbf{P}^2)$
  Chamber limit set: $\Lambda_{Ch}(\Gamma) = \pi(\Lambda(\Gamma)) \subset \mathrm{Flag}(\mathbf{P}^2)$

Thm (KLP 2014) $\Gamma \subset SL(3)$ discrete regular subgroup. TFAE

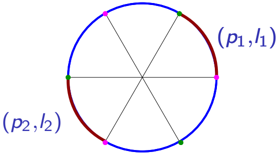
(1) Γ word hyperbolic and "Morse" q.isom. embedded.

(2) Γ word hyperbolic, antipodal, and $\partial_\infty \Gamma \cong_\Gamma \Lambda_{Ch}(\Gamma)$

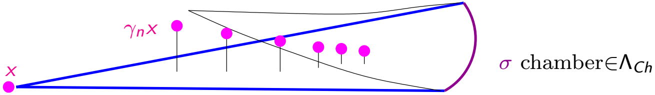
(3) Γ antipodal and $\Lambda_{Ch}(\Gamma)$ is chamber conical

(4) Γ antipodal and expanding at $\Lambda_{Ch}(\Gamma) \subset \mathrm{Flag}(\mathbf{P}^2)$

- Antipodal: $(p_1, l_1) \neq (p_2, l_2) \in \Lambda_{Ch}(\Gamma) \Rightarrow p_2 \notin l_1$ & $p_1 \notin l_2$.

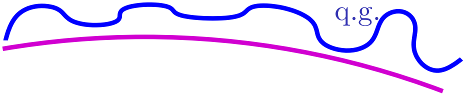


- Chamber conical:



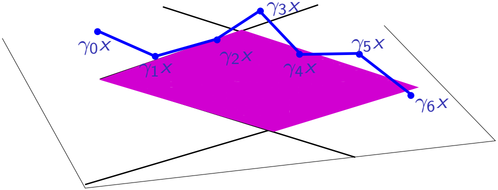
Morse orbits and diamonds

- Morse lemma in rank 1: Uniform quasi-geodesics in $\mathbf{H}^n$ are uniformly close to a geodesic.



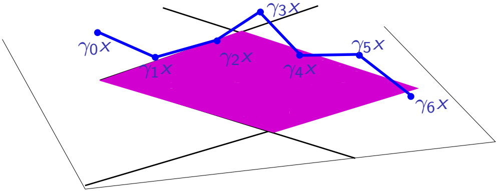
It fails in higher rank (it fails in $\mathbf{R}^2$)

Def $\Gamma < SL(3)$, $\Gamma \curvearrowright X$ is Morse if it is regular, and orbits of uniform quasi-geodesic segments in Γ are uniform quasi-geodesics, uniformly close to a diamond (intersection of cones on opposite chambers in a flat)



Morse group actions

Def $\Gamma < SL(3)$, $\Gamma \curvearrowright X$ is Morse if it is regular, and orbits of uniform quasi-geodesic segments in Γ are uniform quasi-geodesics, uniformly close to a diamond (intersection of cones on opposite chambers in a flat)



Thm (KLP 2014) $\Gamma \subset SL(3)$ discrete regular subgroup. TFAE:

(1) Γ word hyperbolic and Morse (hence undistorted)

(2) Γ antipodal and $\Lambda_{Ch}(\Gamma)$ is chamber conical

(3) Γ antipodal and expanding at $\Lambda_{Ch}(\Gamma) \subset \mathrm{Flag}(\mathbf{P}^2)$

(4) Γ word hyperbolic, antipodal, and $\partial_\infty \Gamma \cong_\Gamma \Lambda_{Ch}(\Gamma)$

Local to global

- Assume Γ nonelementary word hyperbolic, $\rho \in \hom(\Gamma, SL(3))$.

Def: $\rho \in \hom(\Gamma, SL(3))$ is locally Morse if $\forall q : \{0, 1, \ldots, n\} \to \Gamma$ (C, A)-quasi-geodesic, $q(0) = 1$, of length $n \leq N$, for fixed N:

1. the orbit $\{\rho(q(0))x, \rho(q(1))x, \ldots, \rho(q(n))x\}$ is D-close to a diamond.
2. the segment $\overline{\rho(q(0))x,\ \rho(q(n))x} \subset X$ is uniformly regular (its direction in a fixed compact set of the open chamber)

Thm: (Local to global). If $\rho \in \hom(\Gamma, SL(3))$ is *locally* Morse for suf. large N, then $\rho(\Gamma)$ is Morse ($\Rightarrow$ discrete & $\ker(\rho)$ finite)

Cor: (Openness) $\hom_{\mathrm{Morse}}(\Gamma, SL(3))$ is *open* in $\hom(\Gamma, SL(3))$.

Thm: (Structural stability) The homeo $\partial_\infty \Gamma \cong_\Gamma \Lambda_{\rho(\Gamma)}$ changes *continuously* on $\rho \in \hom_{\mathrm{Morse}}(\Gamma, SL(3))$.

Cor: (Algorithmic recognition) There is an algorithm that stops iff $\rho \in \hom(\Gamma, SL(3))$ is Morse.

Anosov representations

- Assume Γ nonelementary word hyperbolic, $\rho \in \mathrm{hom}(\Gamma, SL(3))$.

Def: ρ is Anosov if

(i) there exists $\beta : \partial_\infty \Gamma \to \mathrm{Flag}(\mathbf{P}^2)$ antipodal equiv. embedding (for $\xi_1 \neq \xi_2 \in \partial_\infty \Gamma$, $\beta(\xi_1)$ opposite (generic) to $\beta(\xi_2)$) and

(ii) for every $q : \mathbf{N} \to \Gamma$ discrete geodesic ray, with $q(0) = 1$ and $q(+\infty) = \xi \in \partial_\infty \Gamma$, $\rho(q(n))^{-1}$ acts as an expansion on $T_{\beta(\xi)}\mathrm{Flag}(\mathbf{P}^2)$ with unbounded expansion factor.

Thm:

(i) ρ is Morse iff it is Anosov

(ii) ρ is Anosov in this sense iff it is Anosov in the sense of Labourie, and Guichard and Wienhard (using geodesic flow and uniform exponential expansion factors).

Question: Is there a coarse Lipschitz retraction $X \to \Gamma x$?

THANKS FOR YOUR ATTENTION