Spherical cone structures on 2-bridge links

Joan Porti (UAB)

ICTP Trieste, June 24, 2005

Cone 3-manifolds

- A Euclidean cone 3-manifold is locally isometic to Euclidean space except at the singular locus Σ. Σ is a graph locally isometric to

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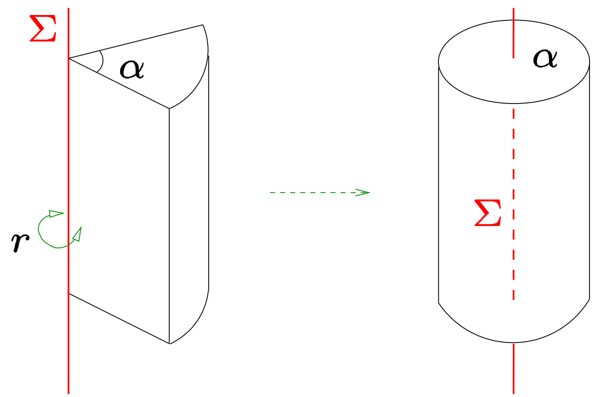
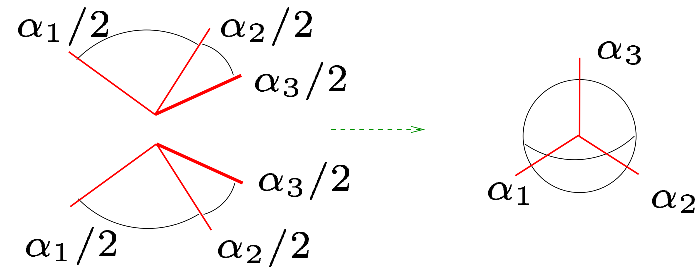


or



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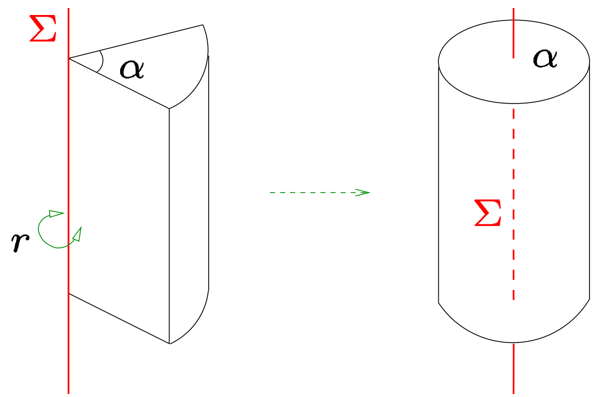

$$ds^2 = dr^2 + \frac{\alpha}{2\pi} r^2 d\theta^2 + dh^2$$

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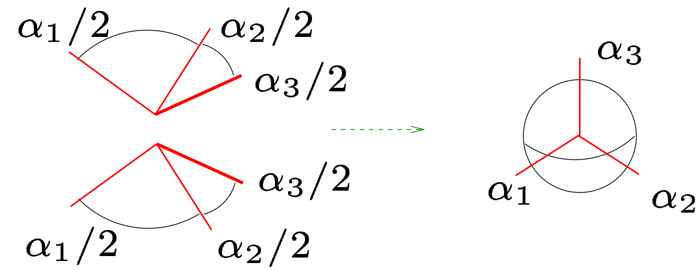

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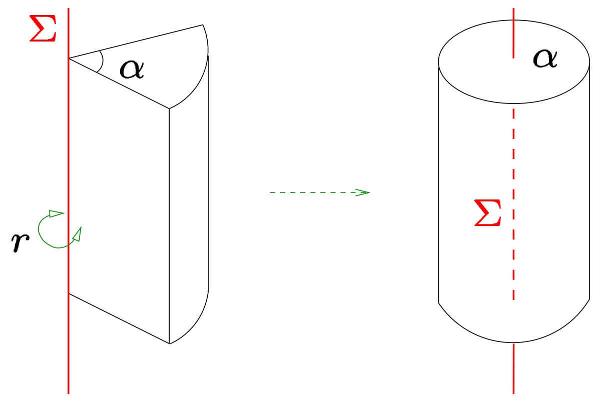

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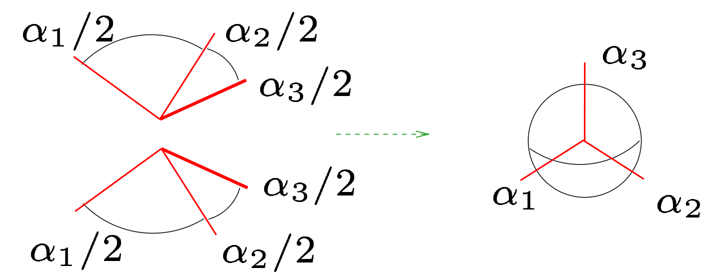

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- *Euclidean* can be replaced by *spherical* or *hyperbolic*.

hyperbolic: $$ds^2 = dr^2 + \frac{\alpha}{2\pi}\sinh^2(r)\, d\theta^2 + \cosh^2(r) dh^2$$

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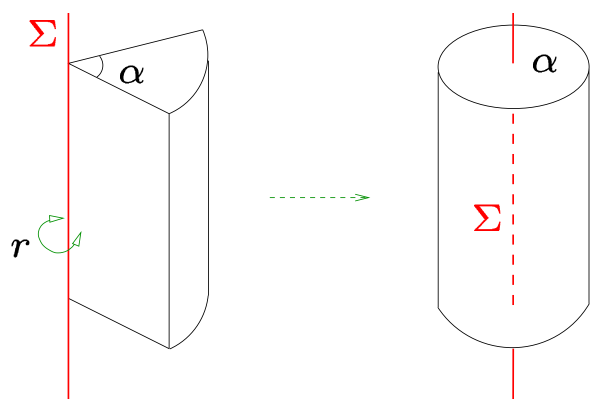

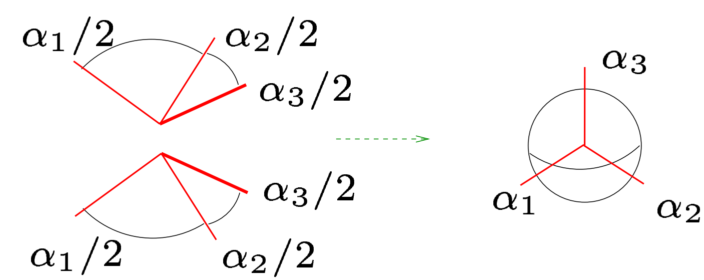

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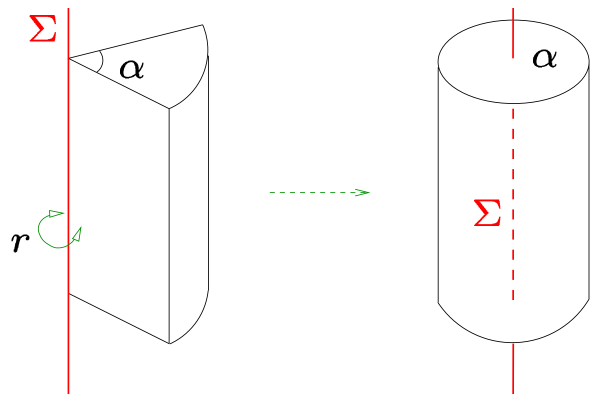

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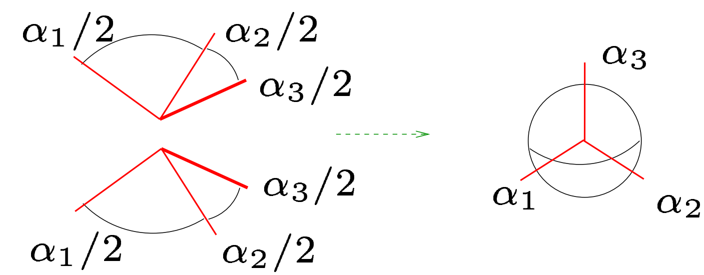

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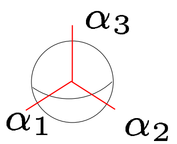
- *Euclidean* can be replaced by *spherical* or *hyperbolic*.

- Locally defined as metric cone on spherical $(n-1)$- cone manifolds

- Cone 3-manifolds are well understood when cone angles are $\leq \pi$ (in the proof of the orbifold theorem)



e.g. $\sum_i (2\pi - \alpha_i) < 4\pi$ at vertices implies that

- for cone angles $< 2\pi/3$ singular vertices do not occur
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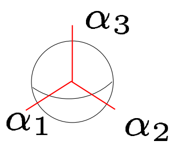

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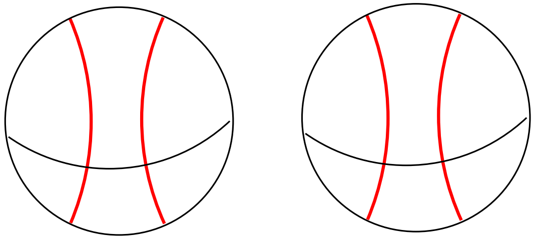
GOAL: study examples with cone angle $\geq \pi$

S^3 with singular locus $\Sigma =$ two bridge knots and links

2-bridge knots and links

$\Sigma = L \subset S^3$ is obtained by gluing two trivial 2-tangles:



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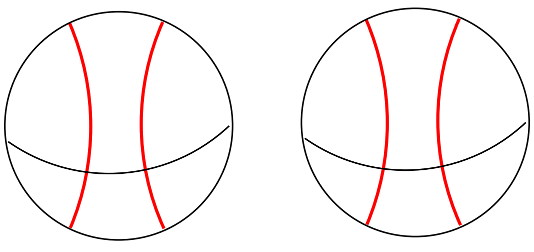

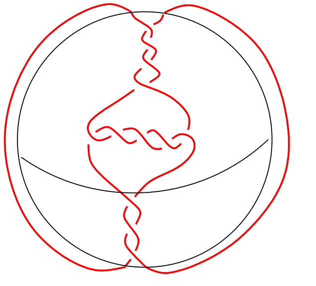

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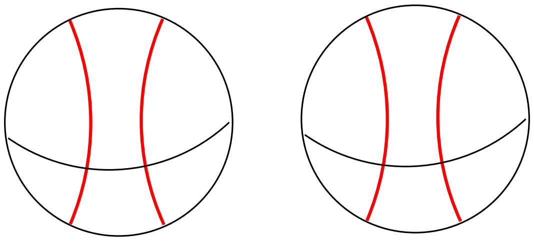

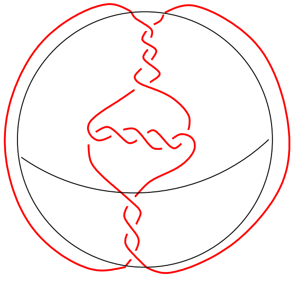

$L \subset S^3$ has at most two components and it is:

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- or a torus link $t(2, n)$ (L is made of fibres of a Seifert fibration of S^3)

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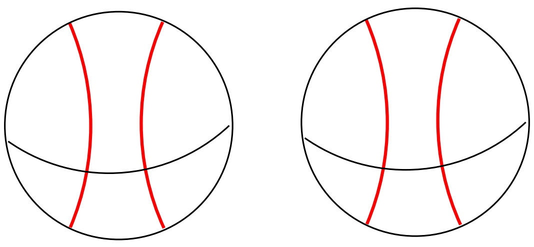

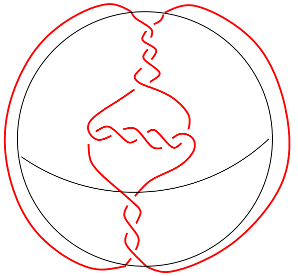

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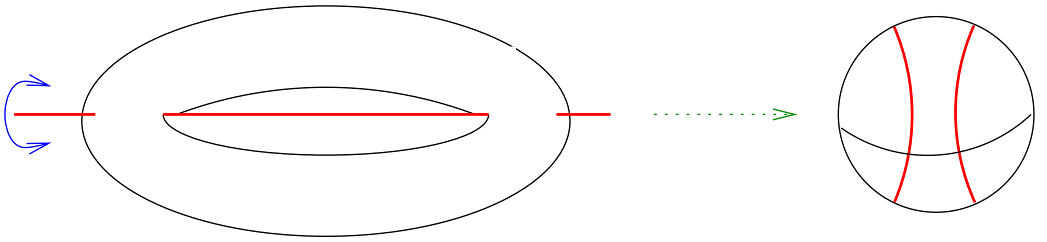
The double cover of S^3 branched along L is a (generalized) lens space



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  (by W. Thurston's hyperbolic Dehn filling).

Cone angles $\leq \pi$

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- From the proof of the orbifold thm:

> If L hyperbolic, then there exists $\frac{2\pi}{3} \leq \alpha_0 < \pi$ such that $C(\alpha_0)$ Euclidean.

($\alpha_0 = \frac{2\pi}{3}$ iff $L =$ figure eigth)

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Question: what happens for $\alpha > \pi$?

Theorem

If L is a hyperbolic two bridge link, $C(\alpha_0)$ Euclidean, then

$C(\alpha)$ is spherical for $\alpha \in (\alpha_0, 2\pi - \alpha_0)$.

- When $\alpha \to 2\pi - \alpha_0$, $C(\alpha) \to$ spherical suspension of sphere with 4 cone points and the tunnels shrink to a point.

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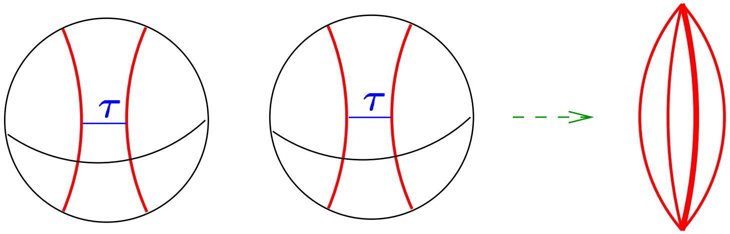

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- When $\alpha \to \alpha_0$, rescale $\frac{1}{\sqrt{\alpha - \alpha_0}} C(\alpha) \to C(\alpha_0)$ Euclidean.

A tool for the proof: variety of representations

Want to deform incomplete metrics on $S^3 - L$ that complete to $C(\alpha)$

$$Dev: \quad \widetilde{S^3 - L} \to S^3 \qquad \text{(local isometry)}$$

$$hol: \quad \pi_1(S^3 - L) \to SO(4) \qquad \text{(representation)}$$

$$\boxed{Dev(\gamma \cdot x) = hol(\gamma)(Dev(x))}$$

- Step 1 Study $\hom(\pi_1(S^3 - L), SO(4))/SO(4)$
- Step 2 Show that some points in $\hom(\pi_1(S^3 - L), SO(4))/SO(4)$ give cone structures $C(\alpha)$.

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Easier to work with $\begin{cases} Spin(4) = \widetilde{SO(4)} \cong S^3 \times S^3 \\ Spin(3) = \widetilde{SO(3)} \cong S^3 \quad \text{(diagonal in } Spin(4)) \end{cases}$

Spin(3) and Spin(4)

- $S^3 \cong SU(2)$ $\qquad (a,b) \in S^3 \subset \mathbf{C}^2 \mapsto \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix} \right) \in SU(2)$

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- (p,q) is a rotation of angle $\theta \iff tr(p) = tr(q) = \pm 2\cos(\theta/2)$.

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}^{-1} = \begin{pmatrix} e^{i(\alpha-\beta)}a & e^{i(\alpha+\beta)}b \\ -e^{i(-\alpha-\beta)}\bar{b} & e^{i(-\alpha+\beta)}\bar{a} \end{pmatrix}$$

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- $X(\Gamma) = \hom(\Gamma, SU(2))/SU(2)$, $\Gamma = \pi_1(S^3 - L)$
  Holonomy reps. of $C(\alpha) - L$ viewed in:

$$\{(\rho^+, \rho^-) \in X(\Gamma) \times X(\Gamma) \mid tr(\rho^+(\mu)) = tr(\rho^-(\mu)),\ \mu \text{ meridian}\}$$

Representations in $SU(2)$ **with** $\alpha \leq \pi$

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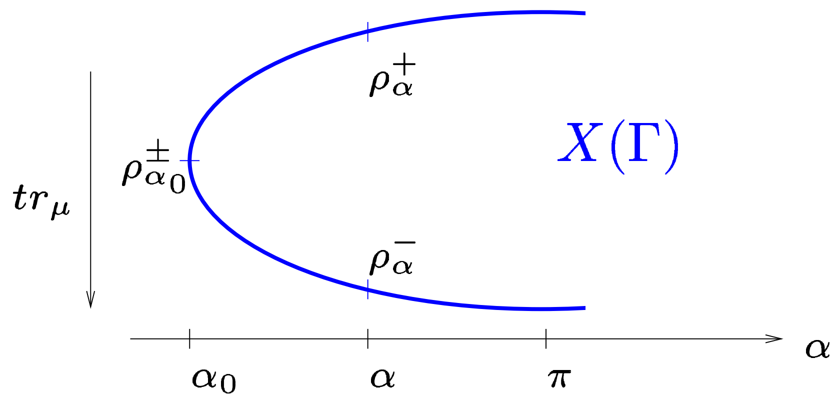

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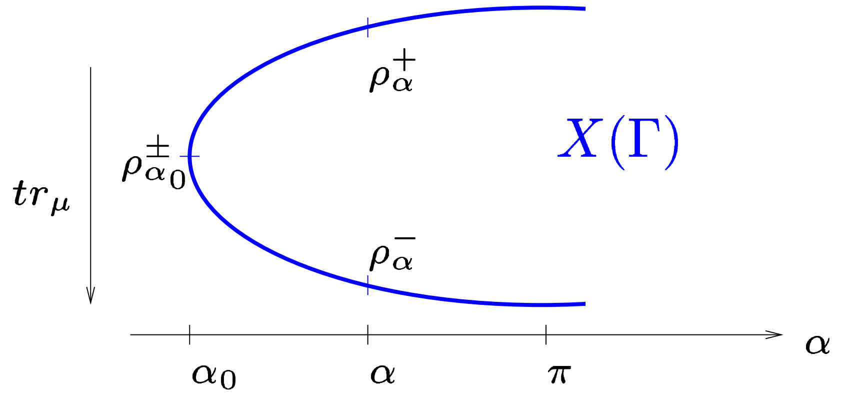

- $X(\Gamma)$ well understood for $\alpha \le \pi$.

- $C(\alpha_0)$ Euclidean $\Rightarrow \rho^+_{\alpha_0} = \rho^-_{\alpha_0}$. $(\rho^+_{\alpha_0}, \rho^-_{\alpha_0})$ diagonal, in $Spin(3)$.

  $(\rho^+_{\alpha_0}, \rho^-_{\alpha_0})$ comes from $\mathrm{Isom}(\mathbf{R}^3) \to SO(3)$

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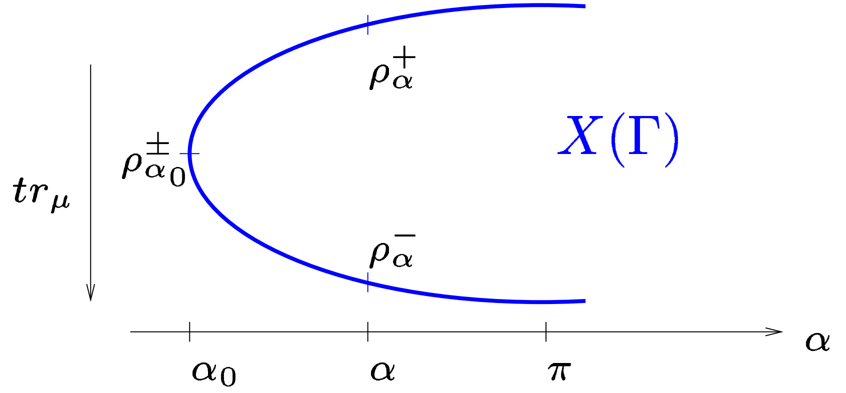

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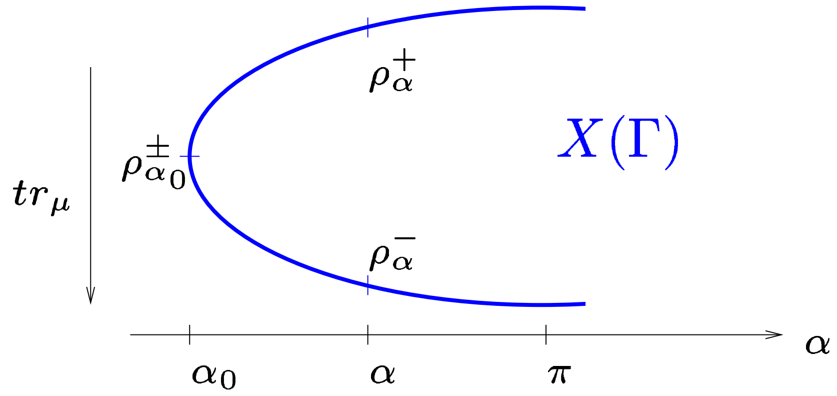
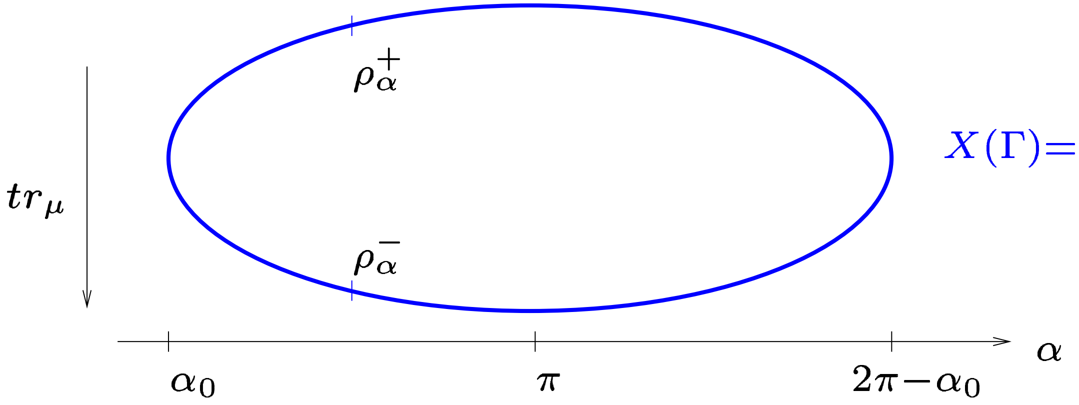


- How to find reps. in $(\pi, 2\pi - \alpha_0)$?
  - reach $\alpha = \pi + \varepsilon$ for some $\varepsilon > 0$ (local paramet./rigidity)
  - $(\rho^+_{\pi+\varepsilon}, \rho^-_{\pi+\varepsilon})$ and $(\rho^+_{\pi-\varepsilon}, \rho^-_{\pi-\varepsilon})$ project to the same rep. in $SO(4)$.

  $\Rightarrow$ Can complete the ellipse symmetrically.

Representations in $SU(2)$ ***with*** $\alpha > \pi$

Using the structure of $X(\Gamma)$ when $\alpha \leq \pi$ and the symmetry:



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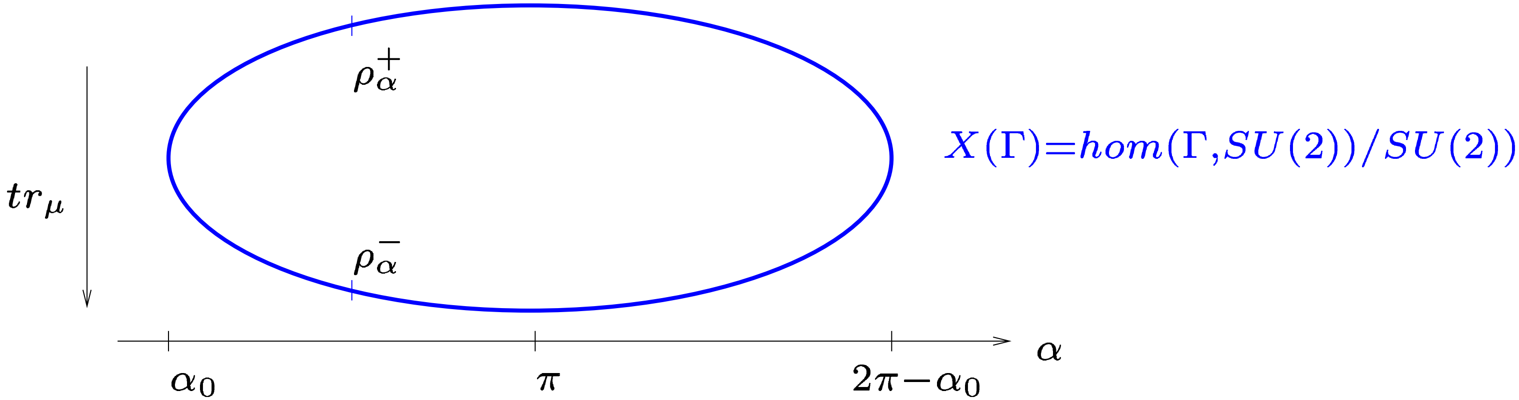

- $(\rho_\alpha^+, \rho_\alpha^-) = \pm(\rho_{2\pi-\alpha}^+, \rho_{2\pi-\alpha}^-)$
  i.e. $(\rho_\alpha^+, \rho_\alpha^-)$ and $(\rho_{2\pi-\alpha}^+, \rho_{2\pi-\alpha}^-)$ induce the same rep. in $SO(4)$.

Notice that $tr(\rho_\alpha^\pm(\mu)) = \pm 2\cos(\alpha/2)$ local parameter at $\alpha = \pi$.
thus the angle $\alpha > \pi$ makes sense.

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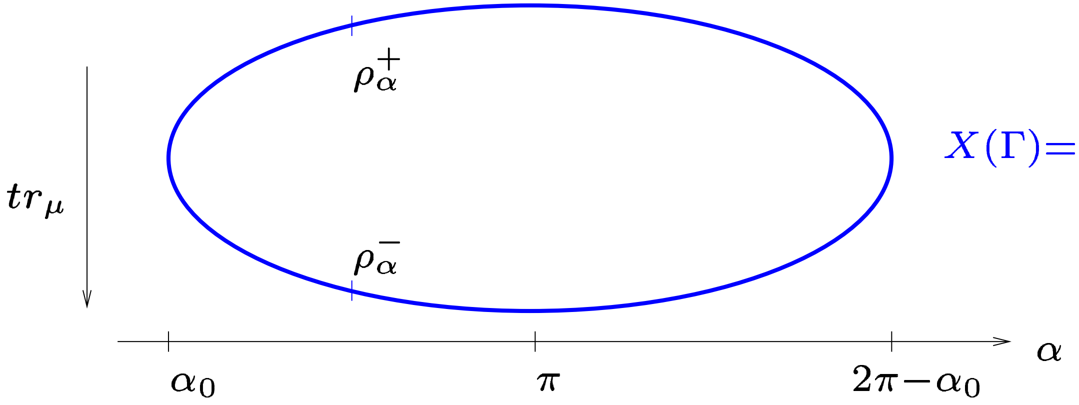

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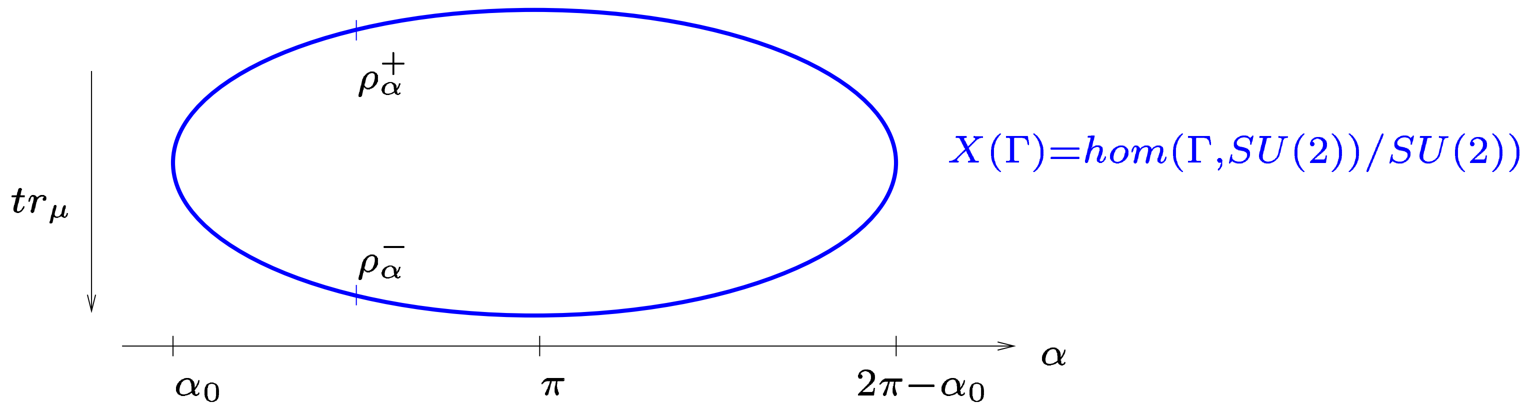

Next step: why those reps. correspond to spherical structures?

- $C(\alpha_0)$ Euclidean: $\rho^+_{\alpha_0} = \rho^-_{\alpha_0}$ (in $Spin(3)$)
- $\rho^+_{2\pi-\alpha_0} = \rho^-_{2\pi-\alpha_0}$ (also in $Spin(3)$). Also want to show:

$$\lim_{\alpha \to 2\pi-\alpha_0} C(\alpha) = \text{ spherical suspension of } S^2 \text{ with } 4 \text{ cone points}$$

Realizing reps. as holonomy of cone manifolds

$$A = \left\{ \alpha \in [\pi, 2\pi - \alpha_0) \;\middle|\; \begin{array}{l} (\rho^+_\alpha, \rho^-_\alpha) \in X(\Gamma) \times X(\Gamma) \text{ holonmy of a sph.} \\ \text{metric on } S^3 - L \text{ that completes to } C(\alpha) \end{array} \right\}$$

- A is open (deformations of holonomy $\Rightarrow$ defs. of structure)

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- A is open (deformations of holonomy $\Rightarrow$ defs. of structure)

- To show A is closed take $\alpha_n \in A$, $\alpha_n \nearrow \alpha_\infty$ and look at $\lim C(\alpha_n) =?$

  Need to bound:
  1. bound above the diameter of $C(\alpha_n)$
  2. radius of an embedded metric tube of $\Sigma \subset C(\alpha_n)$ $(\geq r > 0)$
  3. injectivity radius on $C(\alpha_n) - N_r(\Sigma)$ $(\geq \varepsilon > 0)$

  With those bounds, $\Longrightarrow \lim C(\alpha_n) = C(\alpha_\infty)$ and $\alpha_\infty \in A$.

Finding bounds

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- $\mathrm{diam}(C(\alpha_n)) \leq \pi$ bc. it is an Alexandrov space with curv. ≥ 1.
- $r(\alpha) = \sup\{\delta > 0 \mid N_\delta(\Sigma) \subset C(\alpha)$ embedded metric tube$\}$

Need to bound $r(\alpha) > 0$ uniformly for $\pi \leq \alpha \leq c < 2\pi - \alpha_0$
(i.e. the singular locus does not cross with itself before $2\pi - \alpha_0$).

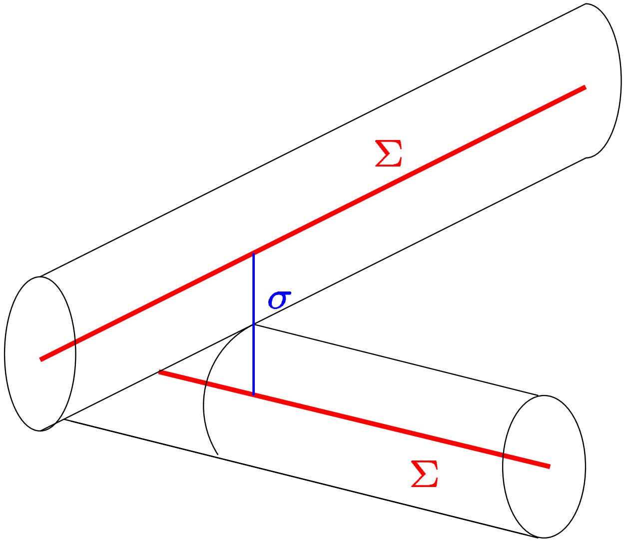
- $\mathrm{vol}(C(\alpha)) \leq 2\pi r(\alpha) + 2\pi(\alpha - \pi)$
- $\mathrm{vol}(C(\alpha)) = \mathrm{vol}(C(2\pi - \alpha)) + 2\pi(\alpha - \pi)$

Hence $r(\alpha) \geq \frac{1}{2\pi}\,\mathrm{vol}(C(2\pi - \alpha))$

- Proof of $\boxed{\mathrm{vol}(C(\alpha)) \leq 2\pi r(\alpha) + 2\pi(\alpha - \pi)}$

$$r = r(\alpha) = \sup\{\delta > 0 \mid N_\delta(\Sigma) \subset C(\alpha) \text{ embedded metric tube}\}$$

σ segment of length $2r$ perpendicular to Σ.



- Proof of $\boxed{\mathrm{vol}(C(\alpha)) \leq 2\pi r(\alpha) + 2\pi(\alpha - \pi)}$

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$$D(\sigma) = \{x \in C(\alpha) \,|\, x \text{ has a unique minimizing segment to } \sigma\}$$

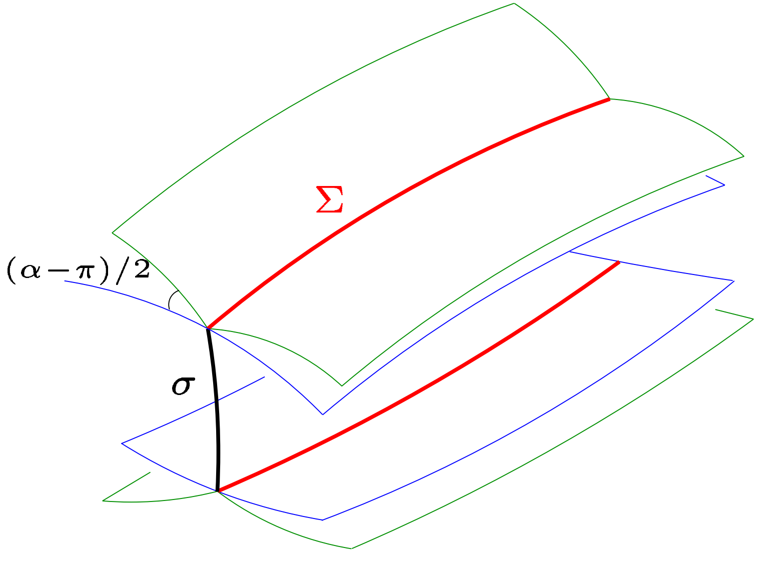
- $D(\sigma)$ not convex but star-shaped!

- Proof of $\boxed{\mathrm{vol}(C(\alpha)) \leq 2\pi r(\alpha) + 2\pi(\alpha - \pi)}$

σ segment of length $2r$ perpendicular to Σ.

$$D(\sigma) = \{x \in C(\alpha) \,|\, x \text{ has a unique minimizing segment to } \sigma\}$$

- $D(\sigma) \subset$ a lens of width $2r$ and 4 lenses of width $\frac{\alpha-\pi}{2}$ in S^3.



vol(lens)$=\pi\cdot$width(lens)

- Proof of $\boxed{\mathrm{vol}(C(\alpha)) = \mathrm{vol}(C(2\pi - \alpha)) + 2\pi(\alpha - \pi)}$ $\forall \alpha \in [\pi, 2\pi - \alpha_0)$

$l(\alpha) =$ total length of $\Sigma \subset C(\alpha)$

- Schläfli's formula: $d\ \mathrm{vol}\, C(\alpha) = \frac{1}{2} l(\alpha)\, d\, \alpha$

$$\boxed{\mathrm{vol}\, C(\alpha) = \int_{\alpha_0}^{\alpha} \tfrac{1}{2} l(\theta)\, d\, \theta}$$

Thus $\mathrm{vol}\, C(\alpha)$ increases whith α.

Volume and symmetry

- Proof of $\boxed{\mathrm{vol}(C(\alpha)) = \mathrm{vol}(C(2\pi - \alpha)) + 2\pi(\alpha - \pi)}$ $\forall \alpha \in [\pi, 2\pi - \alpha_0)$

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\mathrm{vol}(C(\alpha)) &= \int_{\alpha_0}^{\pi} \frac{1}{2} l(\theta)\, d\,\theta + \int_{\pi}^{\alpha} \frac{1}{2} l(\theta)\, d\,\theta \\
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&= \int_{\alpha_0}^{2\pi-\alpha} \frac{1}{2} l(\theta)\, d\theta + (\alpha - \pi) 2\pi
\end{aligned}
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More bounds (final)

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Remarks

- $\mathrm{vol}(C(\alpha_0)) = 0$. Thus $r(2\pi - \alpha_0) \geq 0$ trivial bound.
- $2\pi(\alpha - \pi) = \mathrm{vol}$(spherical suspension of $S^2(\alpha, \alpha, \alpha, \alpha)$)
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  In particular $\mathrm{vol}(C(2\pi - \alpha_0))$=vol. spherical suspension

- The injectivity radius in $C(\alpha) - N_r(\Sigma)$ is bounded because:

$$\left\{\begin{array}{l} \bullet\ \mathrm{vol}\, C(\alpha) \text{ increases whith } \alpha \text{ (Schläfli's).} \\ \bullet\ \mathrm{diam}(C(\alpha)) \leq \pi \text{ (Alexandrov space).} \end{array}\right.$$

The crossing of Σ when $\alpha \to 2\pi - \alpha_0$

- The length of tunnels $|\tau_i| \to 0$ as $\alpha \to 2\pi - \alpha_0$, by symmetry of the variety of reps.

The crossing of Σ ***when*** $\alpha \to 2\pi - \alpha_0$

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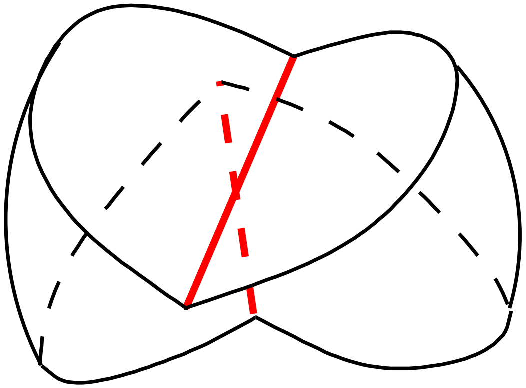
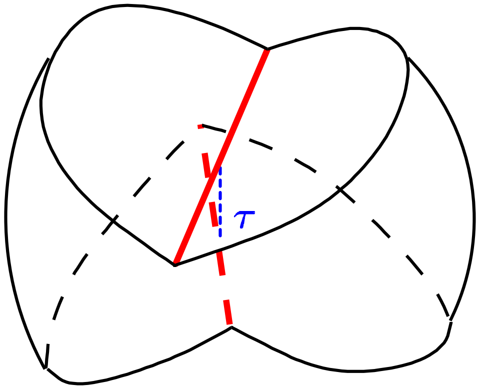
- We construct the metric on $C(2\pi - \alpha_0)$ from the tangles and we deform them decreasing α explicitely.



$\alpha = 2\pi - \alpha_0$



$\alpha < 2\pi - \alpha_0$

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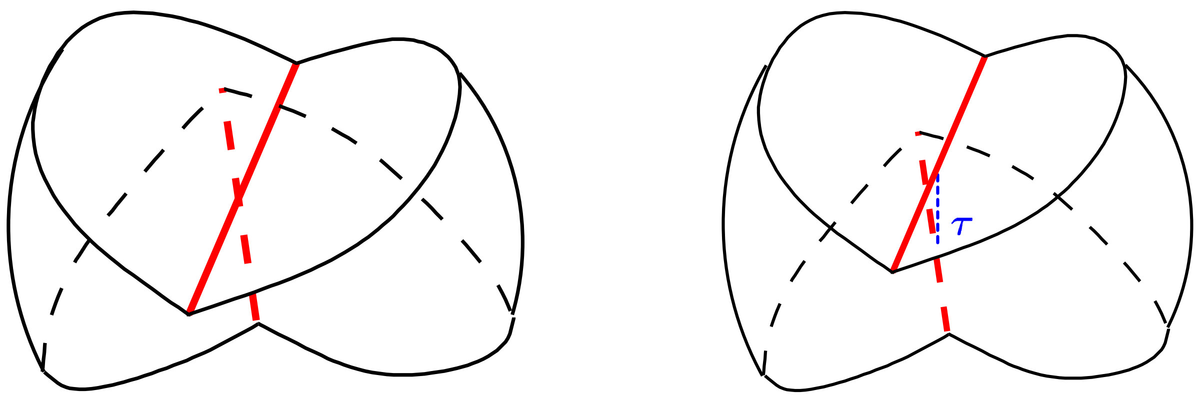

$\alpha = 2\pi - \alpha_0$ $\alpha < 2\pi - \alpha_0$

- At $2\pi - \alpha_0$ angle of crossing is $\neq 0$ (bc. $(\rho^+_{\alpha_0}, \rho^-_{\alpha_0})$ non abelian)

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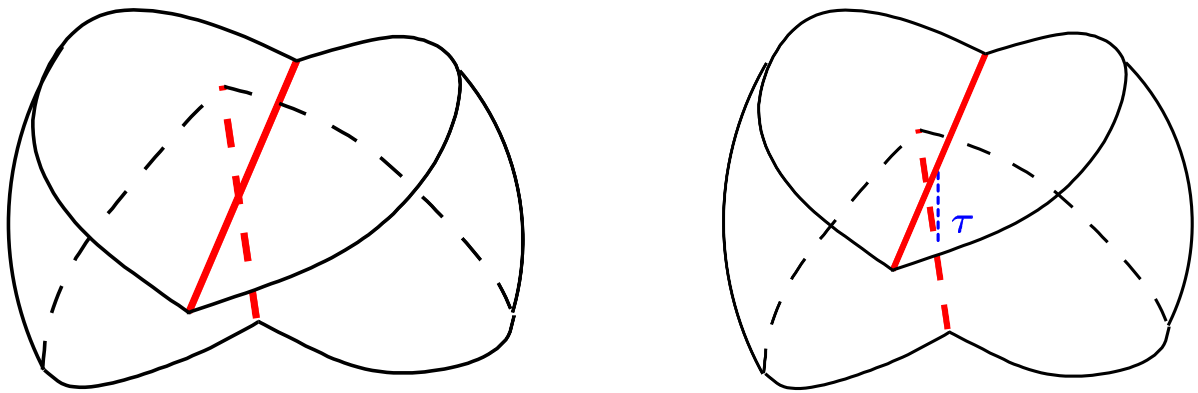

$\alpha = 2\pi - \alpha_0$ $\qquad\qquad$ $\alpha < 2\pi - \alpha_0$

- At $2\pi - \alpha_0$ angle of crossing is $\neq 0$ (bc. $(\rho^+_{\alpha_0}, \rho^-_{\alpha_0})$ non abelian)
- Why this deformation is the same as the previous one?
  By the same volume calculations, can decrease α to π and apply de Rham's global rigidity for orbifolds

Torus links

Cone structures described by the basis of the Seifert fibration.

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$L = t(2,n)$, n odd
(i.e. L a knot).

$$\Rightarrow C(\alpha) \begin{cases} PSL_2(\mathbf{R}) & \text{for } \alpha \in (0, \pi - \frac{2\pi}{n}) \\ Nil & \text{for } \alpha = \frac{2\pi}{n} \\ \text{spherical} & \text{for } \alpha \in (\pi - \frac{2\pi}{n}, \pi + \frac{2\pi}{n}) \end{cases}$$

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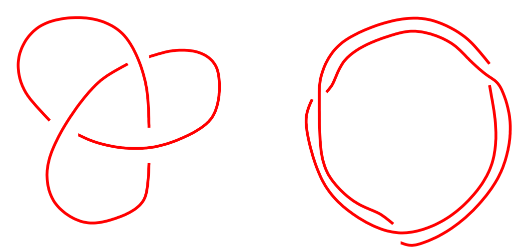
When $\alpha \to \pi + \dfrac{2\pi}{n} \Rightarrow \begin{cases} \Sigma \text{ intersectes itself tangentially} \\ \text{and get a round circle with cone angle } \frac{4\pi}{n} \end{cases}$

- When $n = 3$



Addendum

During my talk I forgot to mention that A. Mednykh and A. Rasskazov had obtained the same result for the figure eigth knot. Mednykh was attending the talk and complained about my omission.
The referee of my paper let me know about that (so I should have mentioned it), but I was not aware that this paper was available on the web. Google found the preprint in http://cis.paisley.ac.uk/research/reports/tr22.zip

The paper of my talk can be found in
http://mat.uab.es/~porti/twobridge040127.pdf
and it just appeared in Kobe J. of Math. **21** (2004), 61-70