Deforming Euclidean cone 3-manifolds into hyperbolic and spherical ones

Joan Porti (UAB) and Hartmut Weiss (LMU)

Trois Journées de Topologie à Orsay

7 décembre 2005

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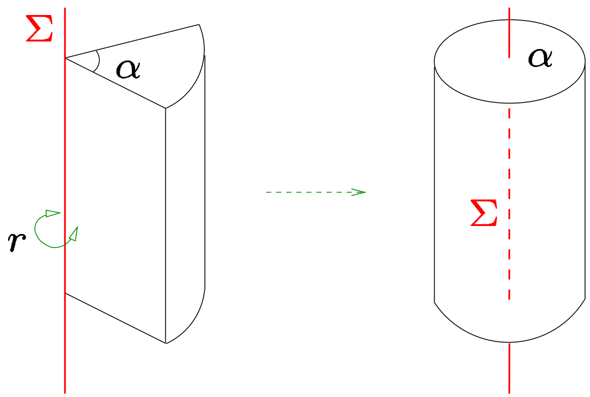

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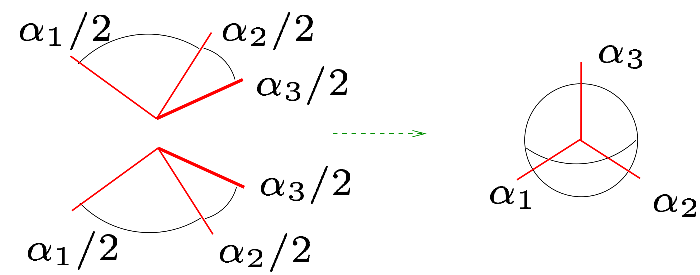

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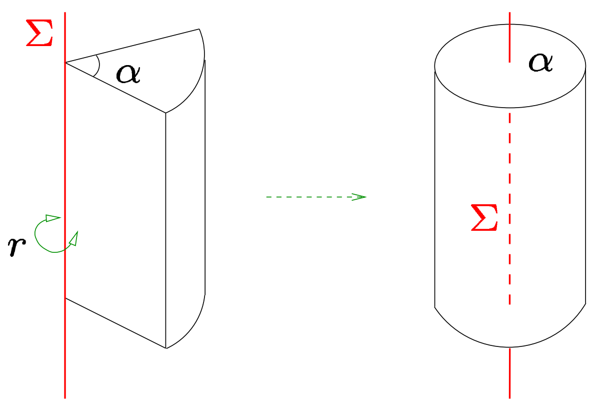

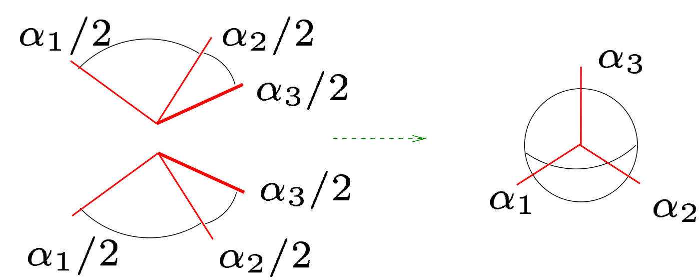

$$d\,s^2 = d\,r^2 + \left(\frac{\alpha}{2\pi}r\right)^2 d\,\theta^2 + d\,h^2$$

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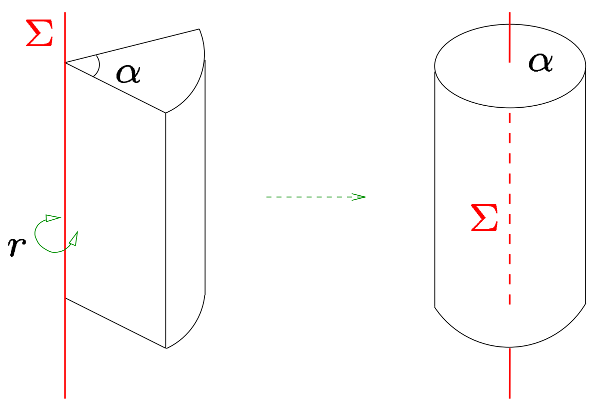

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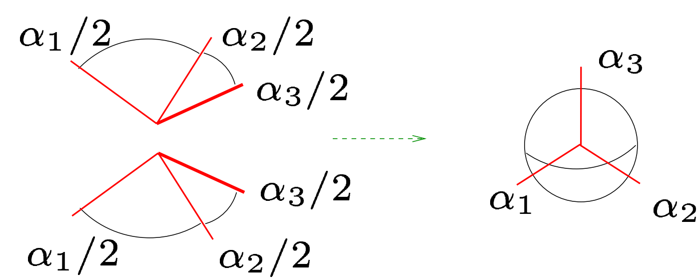

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- *Euclidean* can be replaced by *spherical* or *hyperbolic*.

hyperbolic: $$d\,s^2 = d\,r^2 + \left(\tfrac{\alpha}{2\pi}\right)^2 \sinh^2(r)\,d\,\theta^2 + \cosh^2(r) d\,h^2$$

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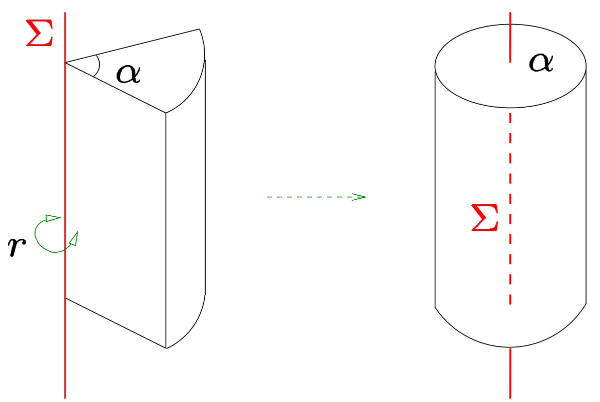

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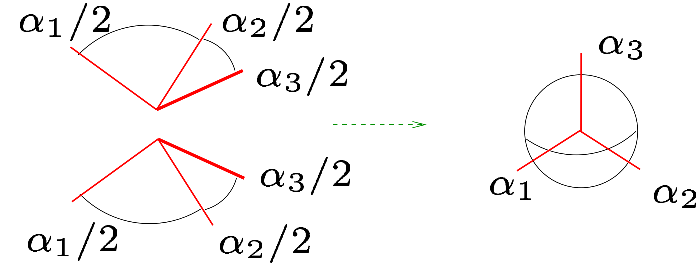

$$d\,s^2 = d\,r^2 + \left(\tfrac{\alpha}{2\pi} r\right)^2 d\,\theta^2 + d\,h^2 \qquad \sum_i (2\pi - \alpha_i) < 4\pi \text{ at vertices}$$

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spherical: $d\,s^2 = d\,r^2 + \left(\tfrac{\alpha}{2\pi}\right)^2 \sin^2(r)\,d\,\theta^2 + \cos^2(r) d\,h^2$

Cone 3-manifolds

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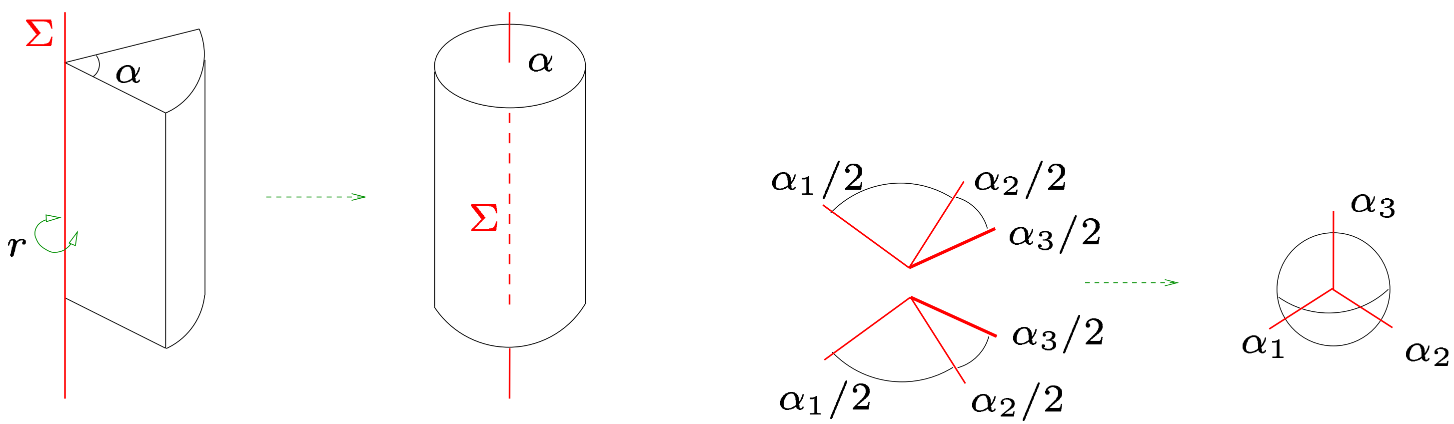
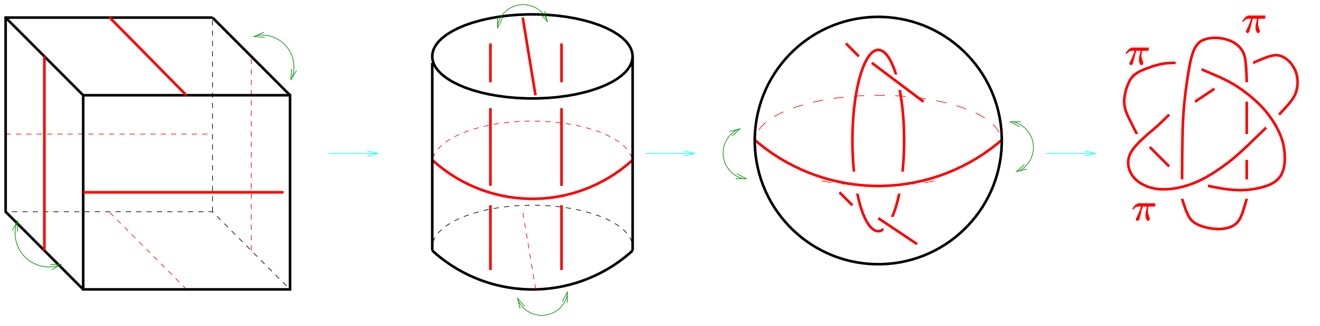


$ds^2 = dr^2 + \left(\frac{\alpha}{2\pi} r\right)^2 d\theta^2 + dh^2$ $\quad \sum_i (2\pi - \alpha_i) < 4\pi$ at vertices

- *Euclidean* can be replaced by *spherical* or *hyperbolic*.

- Locally defined as the metric cone of $(n-1)$-spherical cone manifolds.

Examples



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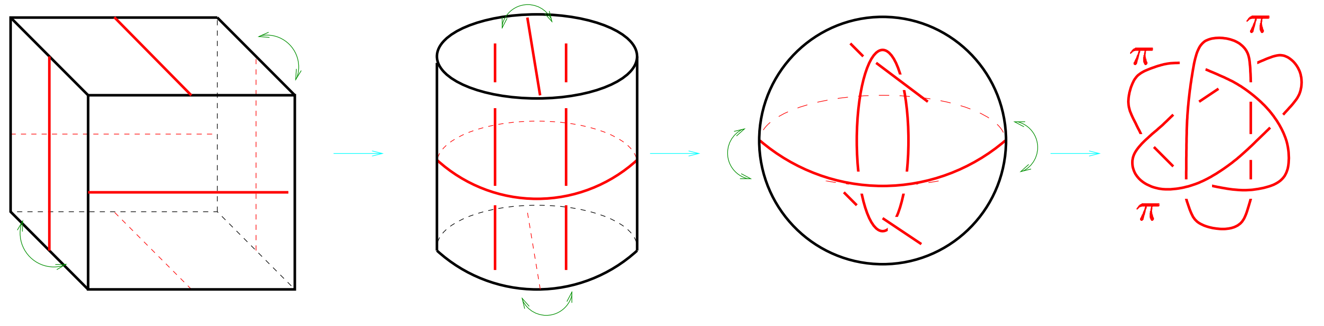

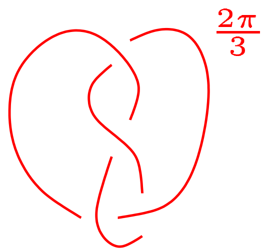

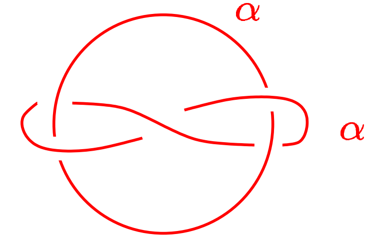

$$\alpha \approx 2.507077676$$

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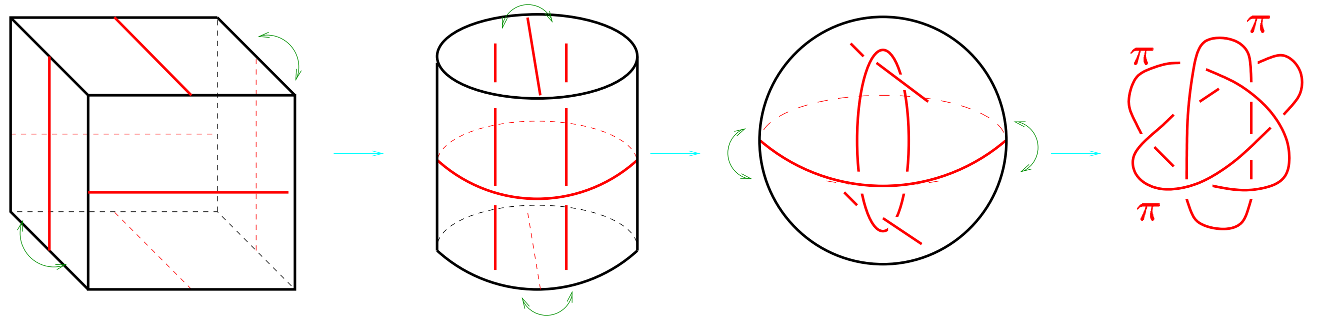

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Definition: C^3 is almost product if $C^3 = E^2 \times S^1/G$ with $G < \mathrm{Isom}(E^2) \times \mathrm{Isom}(S^1)$ finite.

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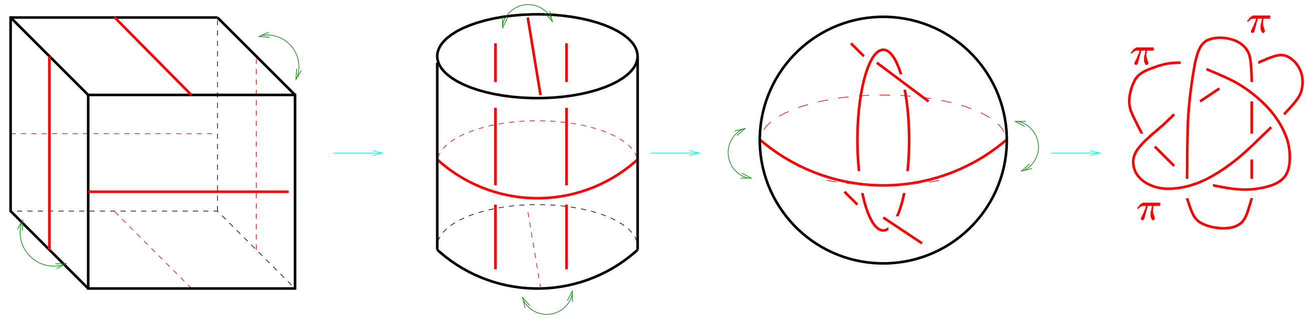

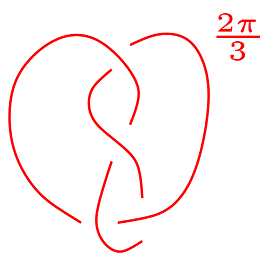

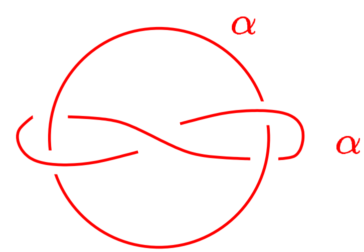

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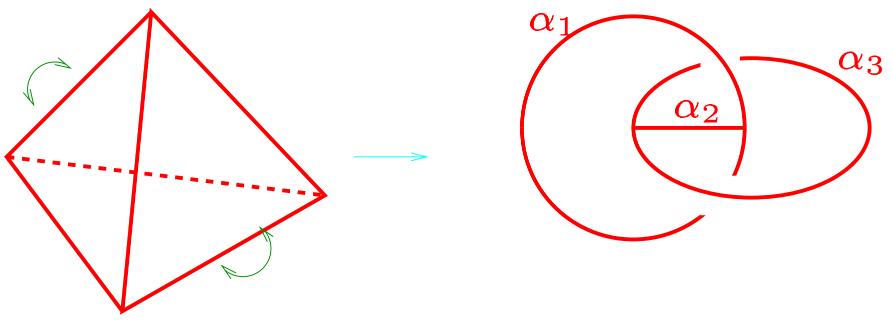
- almost product $\Leftrightarrow$ fibered by S^1 and S^1/C_2.

  The Borromean rings example is almost product

More examples



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  $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ multiangle of cone angles.

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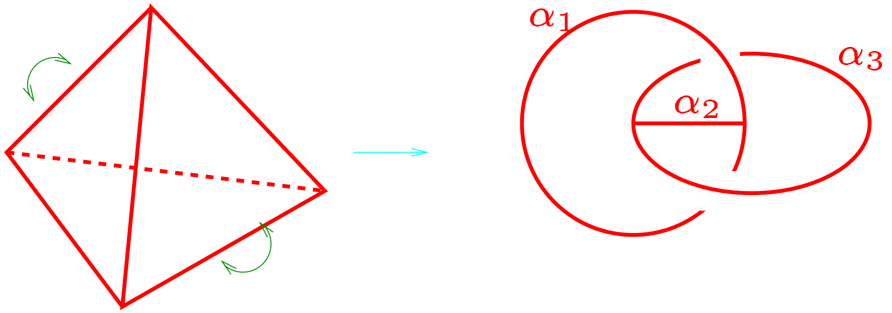

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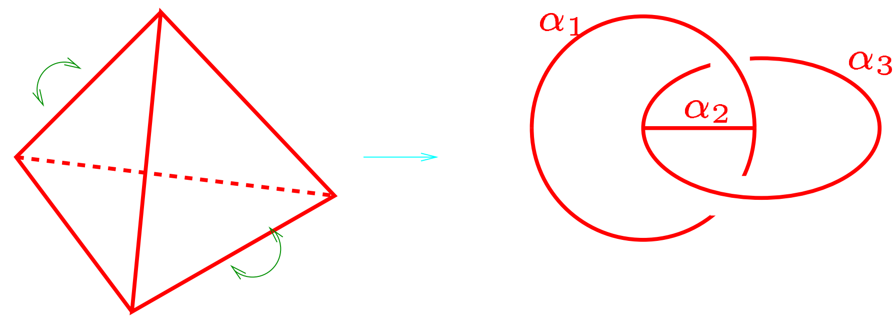

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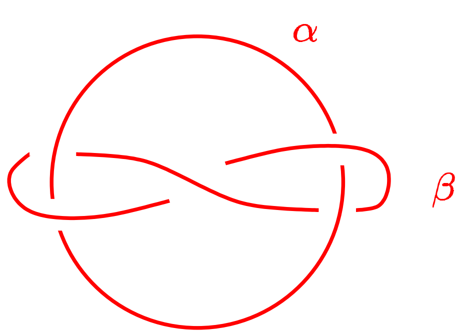

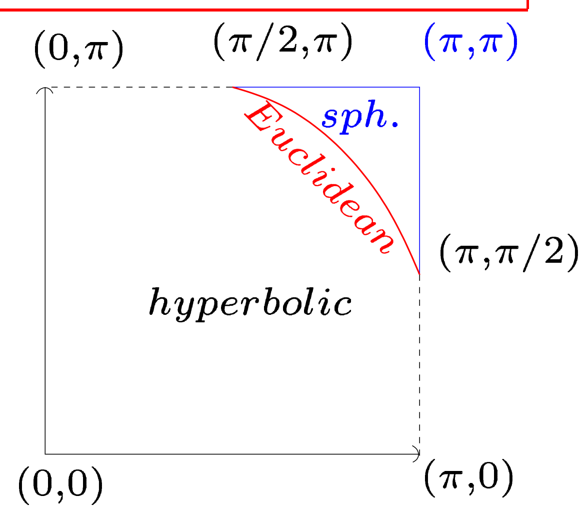

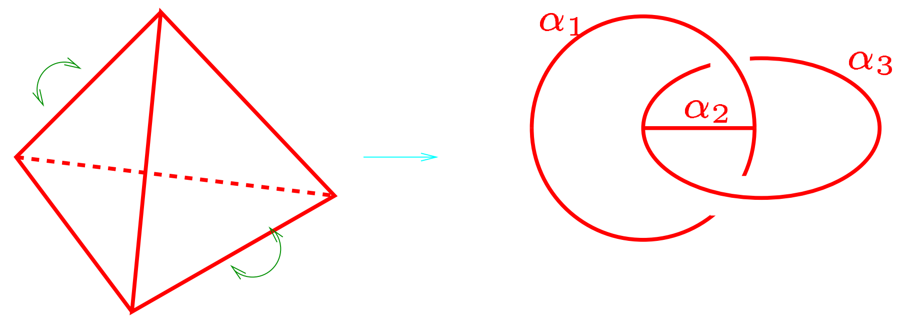

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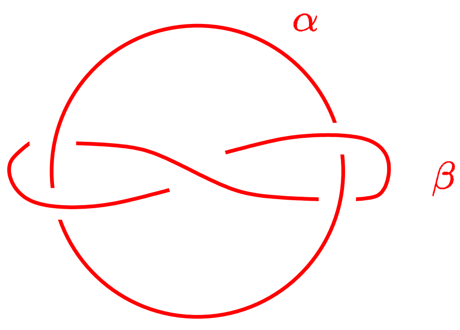
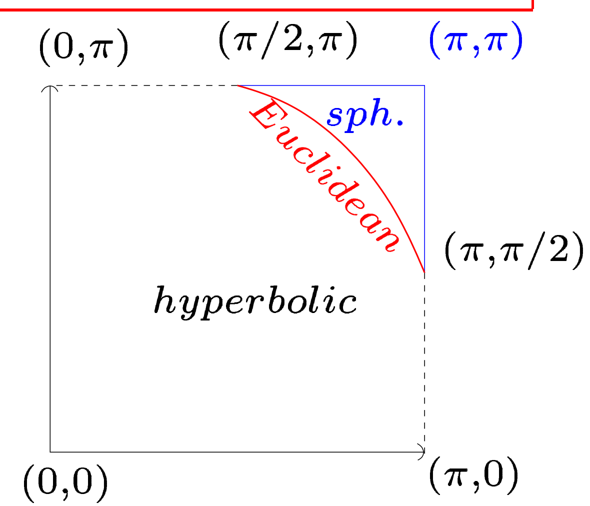




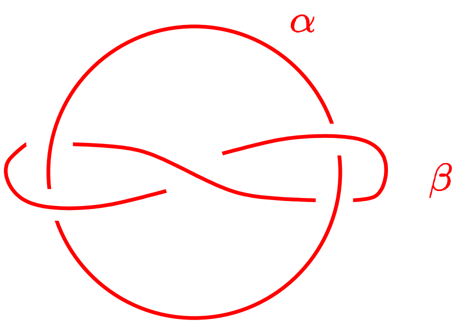
(Mednyhk et al.)

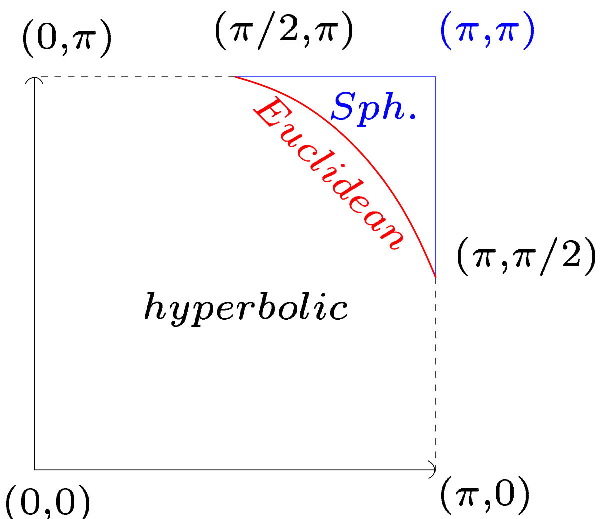
C closed, or., Euclidean cone 3-mfld. with cone angles $\leq \pi$

Theorem (P. & Weiss)

If C is not almost product, then for every $\overline{\alpha} \in (0, \pi)^N$ there is a unique cone metric on C of curvature 0, -1 o 1.

- In addition $E = \{\overline{\alpha}$ es multiangule of Euclidean metric$\}$ is a hypersurface that separates the hyperbolic and spherical compts.





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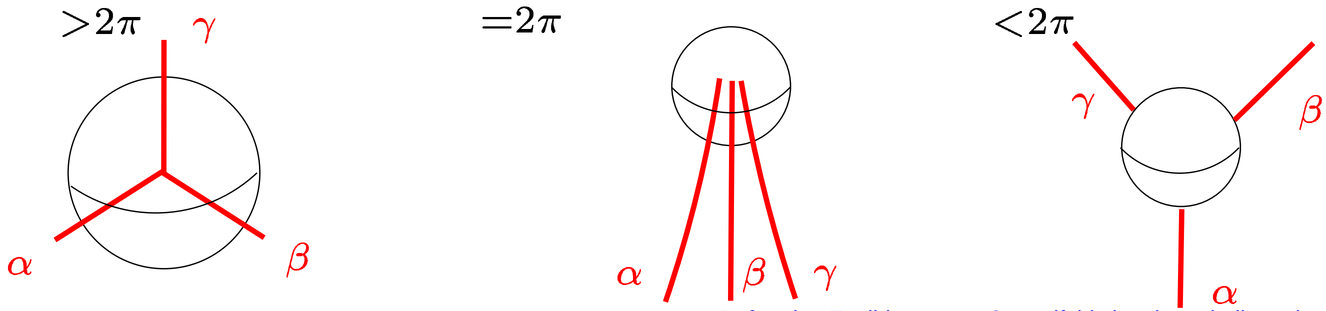
In a vertex $\alpha + \beta + \gamma > 2\pi$

- If $\alpha + \beta + \gamma = 2\pi$, the metric is hyperbolic and has a cusp.
- If $\alpha + \beta + \gamma < 2\pi$, hyp. metric and tot. geodesic boundary.



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- (Schläfli: For curvature K, $K \, d\,Vol(C_t) = \Sigma \frac{1}{2} l_i \, d\,\alpha_i$)

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- More natural transition Euclidean-spherical in the proof of the orbifold theorem (in particular without Ricci flow...).

Steps of the proof:

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A tool: the variety of representations

Want to deform incomplete metrics on $M = C - \Sigma$
that completes in a cone manifold.

$$\mathbf{X}^3 = \mathbf{H}^3,\ \mathbf{R}^3 \text{ or } \mathbf{S}^3$$

$Dev: \quad \widetilde{M} \to \mathbf{X}^3$ *(loc. isometry)*

$hol: \quad \pi_1(M) \to \mathrm{Isom}^+(\mathbf{X}^3)$ *(representation)*

$$\boxed{Dev(\gamma \cdot x) = hol(\gamma)(Dev(x))}$$

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- 1 Study $X(M, G) = \hom(\pi_1(M), G)/G$ for $G = \widetilde{\mathrm{Isom}^+}(\mathbf{X}^3)$
- 2 From representations, deform the structures in M that complete in cone manifolds.

Isometries of S^3: Spin(4)

- $S^3 \cong SU(2)$ $\qquad (a,b) \in S^3 \subset \mathbf{C}^2 \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$

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- (p,q) is a rotation of angle $\theta \iff tr(p) = tr(q) = \pm 2\cos(\theta/2)$.

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}^{-1} = \begin{pmatrix} e^{i(\alpha-\beta)}a & e^{i(\alpha+\beta)}b \\ -e^{i(-\alpha-\beta)}\bar{b} & e^{i(-\alpha+\beta)}\bar{a} \end{pmatrix}$$

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$\{\mu_1, \ldots, \mu_N\}$ = meridians of circles and axis of Σ

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- $\mathrm{Isom}^+(\mathbf{H}^3) \cong PSL_2(\mathbf{C})$ $(\partial \mathbf{H}^3 \cong \mathbf{P}^1(\mathbf{C}))$. $\widetilde{\mathrm{Isom}^+}(\mathbf{H}^3) \cong SL_2(\mathbf{C})$

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- $X(M, SL_2(\mathbf{C})) \subset \mathbf{C}^q$ algeb. and $X(M, SU(2)) \subset X(M, SL_2(\mathbf{C})) \cap \mathbf{R}^q$

Summary

	$\widetilde{\text{Isom}^+}(\mathbf{X}^3)$	point stabilizer	image of μ_i is a rotation
$\mathbf{H}^3$	$SL_2(\mathbf{C})$	$SU(2)$	$tr(\rho)(\mu_i) \in (-2, 2)$
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What about $\mathrm{Isom}^+(\mathbf{R}^3)$?

Isometries of R^3: Rotational part and translations

- $$\begin{array}{ccccc} \mathbf{R}^3 & \to & \mathrm{Isom}^+(\mathbf{R}^3) & \to & SO(3) \\ & & \phi(x) = A\,x + b & \mapsto & A \end{array}$$

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- $\mathbf{R}^3 \to \widetilde{\mathrm{Isom}^+(\mathbf{R}^3)} \xrightarrow{ROT} SU(2)$

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- A. Weil: $T^{Zar}_{[\rho_0]} X(M, SU(2)) \cong H^1(\pi_1 M, su(2)_{Ad_{\rho_0}}) \quad su(2) \cong \mathbf{R}^3$

  $\rho_t(\gamma) = \rho_0(\gamma)(Id + t\, d(\gamma) + O(t^2)) \qquad d\colon \pi_1 M \to su(2)_{Ad_{\rho_0}}$ cocycle

Euclidean holonomy: (ρ_0, v), with $v \in T_{[\rho_0]} X(M, SU(2))$

Summary

	$\widetilde{\mathrm{Isom}^+}(\mathbf{X}^3)$	point stabilizer	image of μ_i is a rotation
$\mathbf{H}^3$	$SL_2(\mathbf{C})$	$SU(2)$	$tr(\rho)(\mu_i) \in (-2, 2)$
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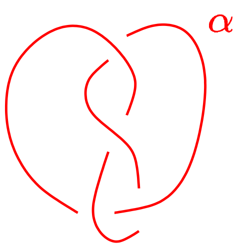
- Objects to be studied: $\begin{cases} X(M, SU(2)) \subset X(M, SL_2(\mathbf{C})) \\ tr_{\mu_i} \end{cases}$

Example: the figure eight knot

- (Hilden-Lozano-Montesinos) there is a cone metric on S^3 with singular locus the figure eight knot and cone angle α

$$\begin{cases} \text{hyperbolic if } \alpha < \frac{2\pi}{3} \\ \text{Euclidean if } \alpha = \frac{2\pi}{3} \\ \text{spherical if } \frac{2\pi}{3} < \alpha \leq \pi \end{cases}$$



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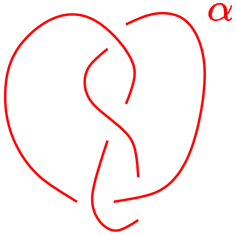

$M = S^3 - K$

$X(M, SL_2(\mathbf{C})) = \{(x, y) \in \mathbf{C}^2 \mid x^2 = (y^2 - 1)(x - 1)\}$ with $y = tr_\mu$

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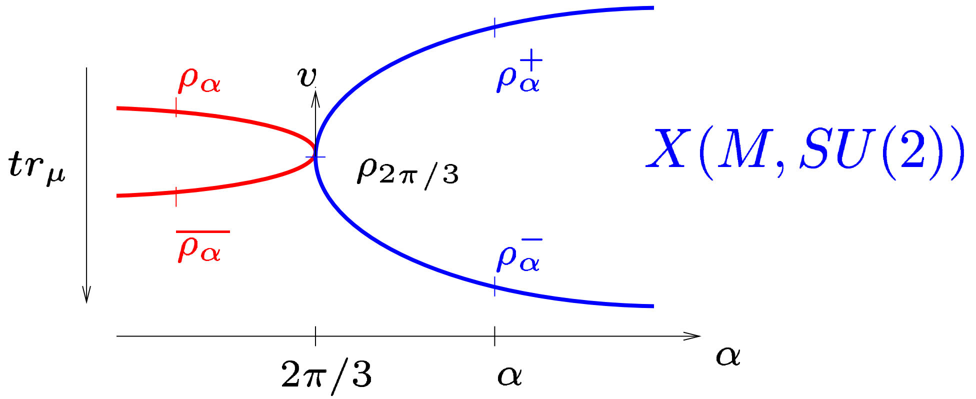
- Set $y = \pm 2\cos(\alpha/2)$ and look at:

$$y^{-1}(\mathbf{R}) = \{(x,y) \in X(M, SL_2(\mathbf{C})) \mid y \in \mathbf{R}\}$$



Locally, $y = 1 - \frac{x^2}{2} + \cdots$. $\Rightarrow y^{-1}(\mathbf{R}) \approx \mathbf{R} \cup \sqrt{-1}\mathbf{R}$

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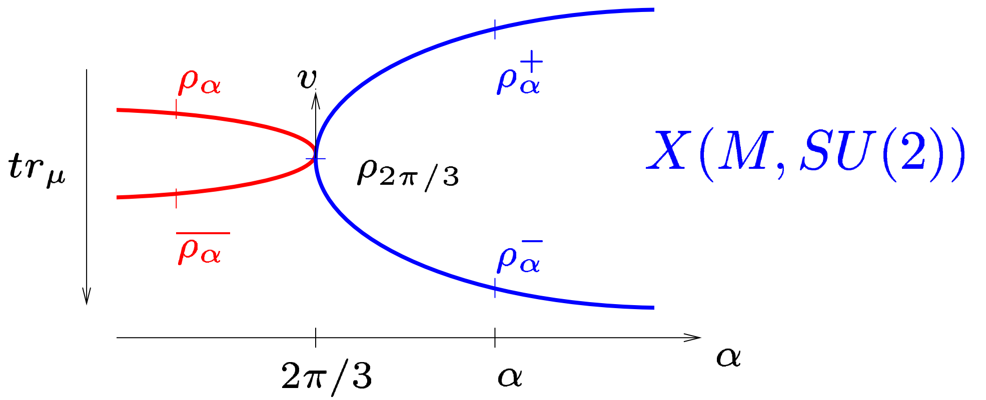

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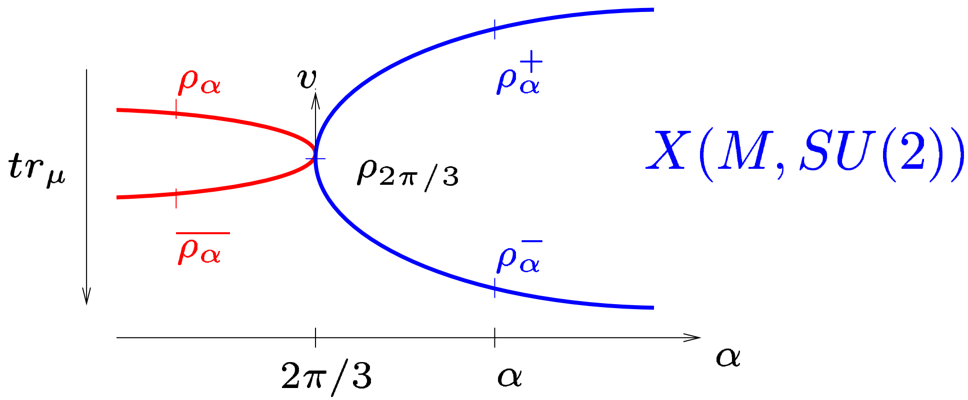

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Steps to build local deformations.

1. $\rho_0 = ROT \circ hol$ smooth point of $X(M, SU(2))$ and $X(M, SL_2(\mathbf{C}))$.
   Moreover $X(M, SL_2(\mathbf{C})) \cap \mathbf{R}^q = X(M, SU(2))$ locally.
2. $(tr_{\mu_1}, \dots, tr_{\mu_N})$ loc. equiv. to $(z_1, \dots, z_N) \mapsto (z_1^2, z_2, \dots, z_N)$.
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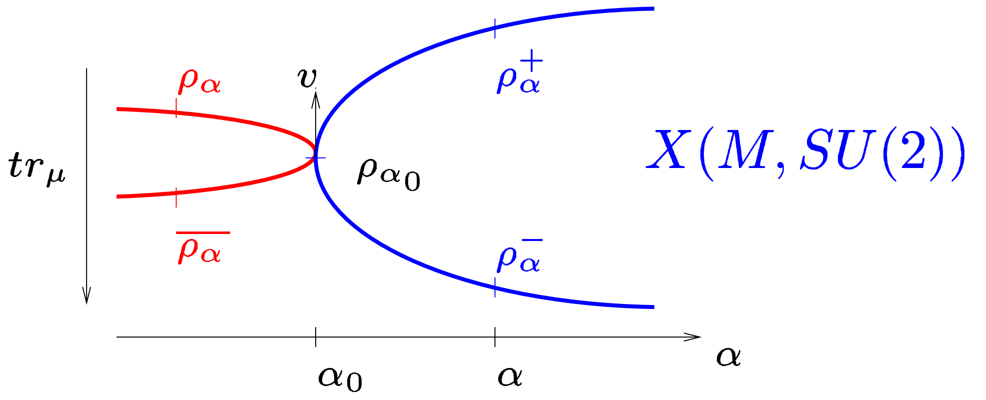

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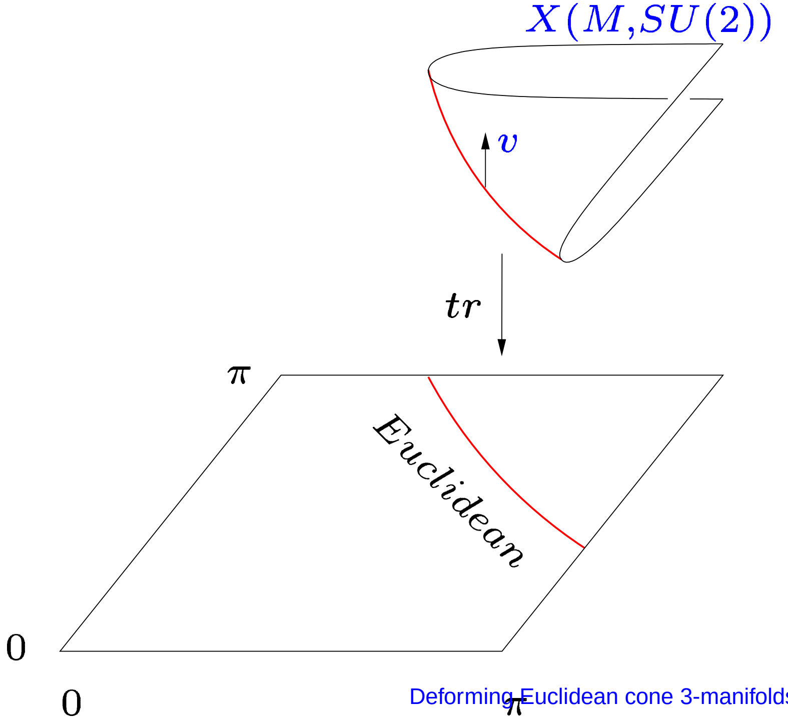

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3. Deformations of representations induce deformations of structures. (Conditions on infinitessimal isometries)

L^2 **cohomology**

$$T_{[\rho_0]}X(M, SU(2)) \cong H^1(\pi_1 M; su(2)_{Ad_{\rho_0}}) \cong H^1(M; TM)$$
$$v \quad \longleftrightarrow \quad [TRANS \circ hol] \quad \longleftrightarrow \quad [id]$$

(Weil and $su(2)_{Ad_\rho} = \mathbf{R}^3_\rho$)

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Theorem (Weiss): $\boxed{H^1_{L_2}(M; TM) = \{w \in \Omega^1(M; TM) \mid \nabla w = 0\}}$

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Consequences:

- $H^1_{L^2}(M;TM) = \langle id \rangle \cong \mathbf{R}$ (C not almost-product)
- $0 \to H^1(M;TM) \to H^1(\partial M;TM)$ ($\partial M := \partial\overline{\mathcal{N}(\Sigma)}$)

$$\begin{array}{ccccc} H^1(M,\partial M;TM) & \longrightarrow & H^1(M;TM) & \to & H^1(\partial M;TM) \\ & \searrow & \nearrow & & \\ & & H^1_{L^2}(M;TM) & & \end{array}$$

id non-trivial in $H^1(\partial M;TM)$.

$[\rho_0] = ROT \circ hol$ **is a smooth point of dim** N

- $\dim_{\mathbf{R}} T^{Zar}_{[\rho_0]} X(M, SU(2)) = \dim_{\mathbf{R}} \mathbf{H}^1(M; TM) = \frac{1}{2} \dim_{\mathbf{R}} \mathbf{H}^1(\partial M; TM) = N$

$$0 \to H^1(M; TM) \to H^1(\partial M; TM) \to H^2(M, \partial M; TM) \to 0$$

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- Elements in $T^{Zar}_{[\rho]} X(M, SU(2))$ could be non-integrable. There is an infinite sequence of obstructions in $H^2(\pi_1 M, su(2))$.

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Obstructions are natural and vanish for surfaces
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- As $sl_2(\mathbf{C}) = su(2) \times_{\mathbf{R}} \mathbf{C}$, the same argument gives:
  - smooth point of $X(M, SL_2(\mathbf{C}))$
  - locallly $X(M, SU(2))$ is the real part of $X(M, SL_2(\mathbf{C}))$

- Want to see:

  $(tr_{\mu_1}, \ldots, tr_{\mu_N})$ loc. equiv. to $(z_1, \ldots, z_N) \mapsto (z_1^2, z_2, \ldots, z_N)$.

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Lema $\ker\langle d\,tr_{\mu_1}, \ldots, d\,tr_{\mu_N}\rangle = \langle v\rangle = \langle [TRANS \circ hol]\rangle = \langle id\rangle \cong \mathbf{R}$.

$\supseteq$ because $hol(\mu_i)$ is a rotation.

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$\Rightarrow$ Up to permutation, $d\,tr_{\mu_2}, \ldots, d\,tr_{\mu_N}$ linearly independent.

$\mathcal{C} = \{[\rho] \in X(SL_2(\mathbf{C})) \mid tr_{\mu_i}(\rho) = ct_i,\ i = 2, \ldots, N\}$ complex curve

$$d\,tr_{\mu_1}|_{\mathcal{C}} = 0 \Rightarrow tr_{\mu_1}|_{\mathcal{C}} \text{ constant or } z \mapsto z^k, k \geq 2$$

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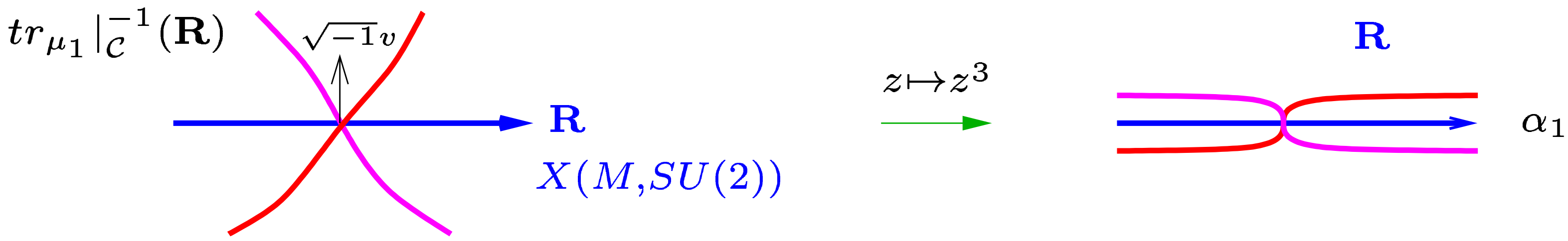

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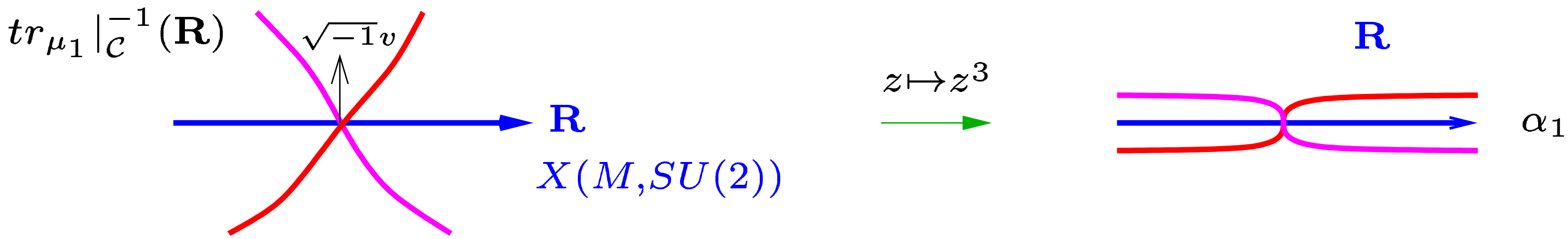

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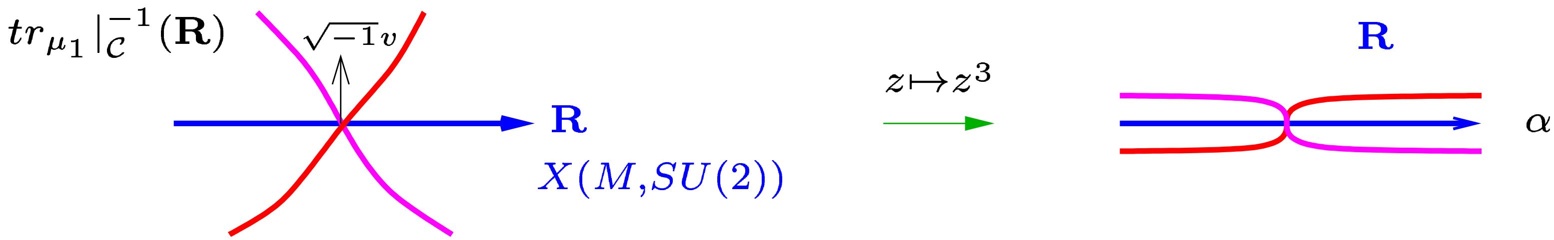

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- Schläfli: $K\,d\,Vol(C_t) = \Sigma \frac{1}{2} l_i\, d\,\alpha_i \Rightarrow$ α_1 must decrease because we go from $K\ Vol = 0$ to $K\ Vol < 0$. !

Regeneration conditions

Fix $p \in \mathbf{X}^3$ $(= \mathbf{H}^3$ or $\mathbf{S}^3)$. $SU(2) = \widetilde{SO(3)}$ stabilizer of p.

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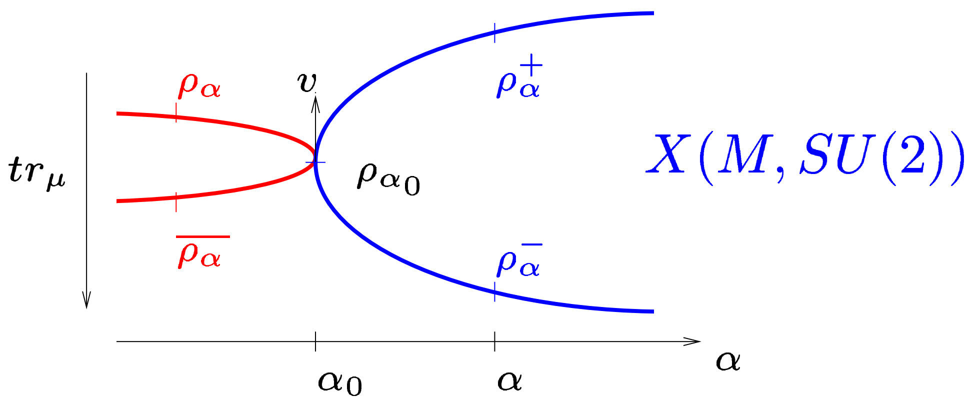

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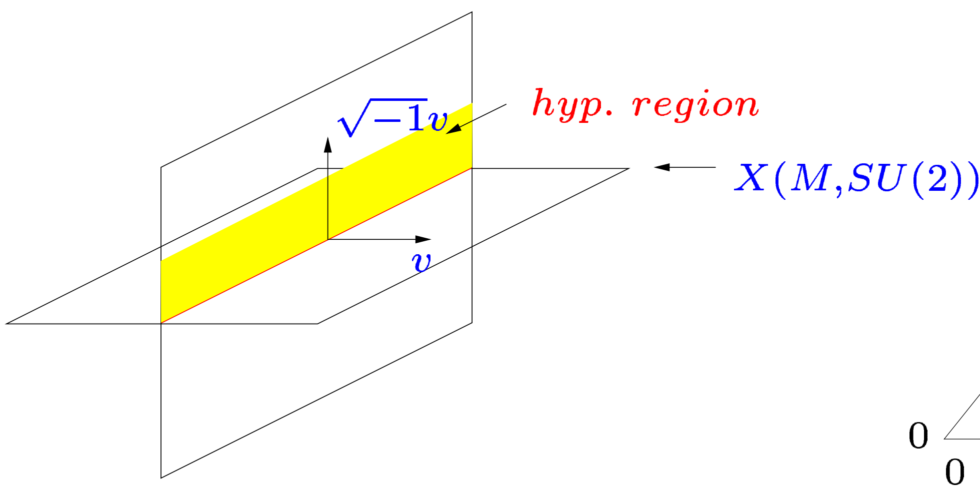
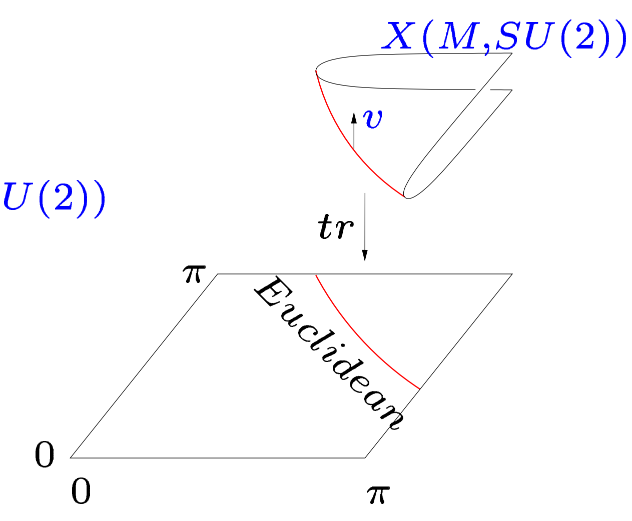




Fix $p \in \mathbf{X}^3$ ($= \mathbf{H}^3$ or $\mathbf{S}^3$). $SU(2) = \widetilde{SO(3)}$ stabilizer of p.

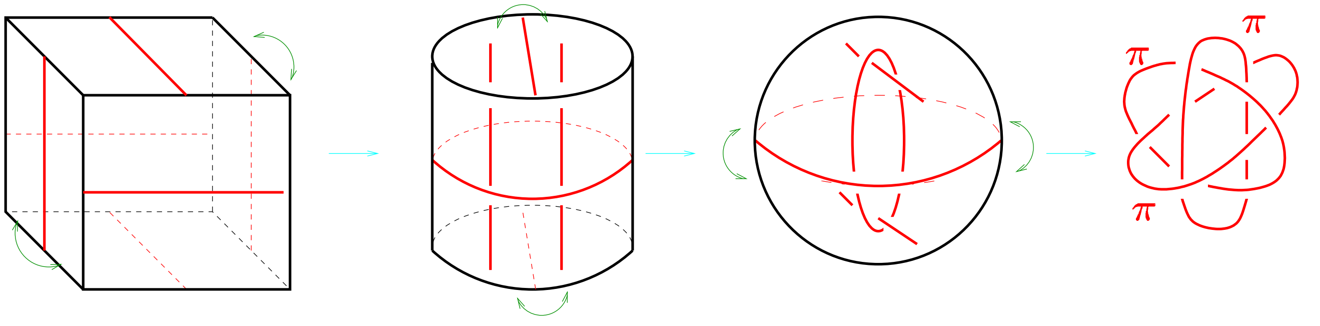
$v = [TRANS \circ hol]$

$$isom(\mathbf{X}^3) = so(3) \oplus T_p X^3$$

$$\begin{cases} Im(\frac{d}{d\alpha}\rho_{\alpha_0}) = v \\ \frac{d}{d\alpha}\rho_{\alpha_0^+} - \frac{d}{d\alpha}\rho_{\alpha_0^-} = v \end{cases} \qquad \begin{cases} sl_2(\mathbf{C}) = su(2) \oplus su(2)\sqrt{-1} \\ su(2) \times su(2) = \text{diagonal} \oplus \text{antidiagonal.} \end{cases}$$

- Identify $T_p\mathbf{X}^3 \cong \mathbf{R}^3$ and use $exp_p\colon T_p\mathbf{X}^3 \to \mathbf{X}^3$ and homotheties to pass from $\mathbf{R}^3$ to $\mathbf{X}^3$.

What happens to the Borromean rings?



The theorem doesn't apply because it is almost-product: can deform in product geometries $\mathbf{H}^2 \times \mathbf{R}$ and $\mathbf{S}^2 \times \mathbf{R}$ ($\alpha = \beta = \pi$, deform γ).

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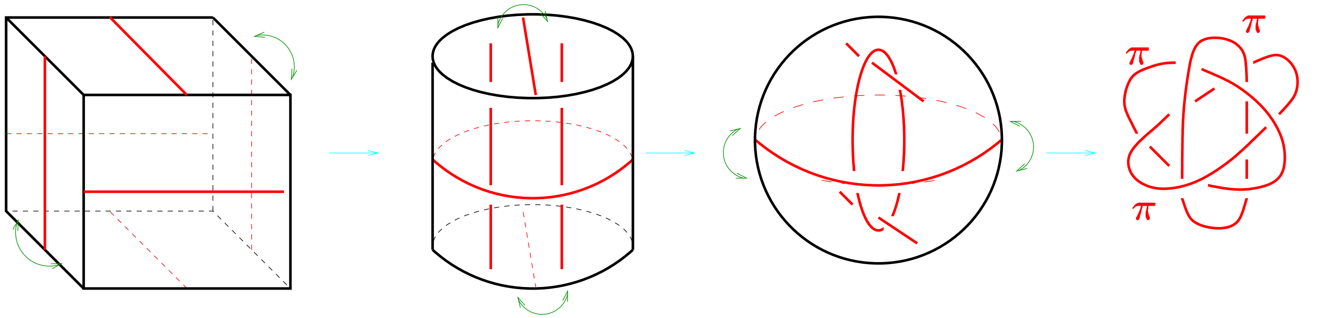

The theorem doesn't apply because it is almost-product: can deform in product geometries $\mathbf{H}^2 \times \mathbf{R}$ and $\mathbf{S}^2 \times \mathbf{R}$ ($\alpha = \beta = \pi$, deform γ).

If we impose $\alpha = \beta = \gamma$, then we have deformations in $\mathbf{H}^3$ and $\mathbf{S}^3$, because the theorem applies to the quotient by an action of $\mathbf{Z}/3\mathbf{Z}$